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Advanced Copula-Based Models for Type II Censored Data: Applications in Industrial and Medical Settings

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Abstract: Copula models are increasingly recognized for their ability to capture complex dependencies among random variables. In this study, we introduce three innovative bivariate models utilizing copula functions: the XLindley (XL) distribution with Frank, Gumbel, and Clayton copulas. The results highlight the fundamental characteristics and effectiveness of these newly introduced bivariate models. Statistical inference for the distribution parameters is conducted using a Type II censored sampling design. This employs maximum likelihood and Bayesian estimation techniques. Asymptotic and credible confidence intervals are calculated, and numerical analysis is performed using the Markov Chain Monte Carlo method. The proposed methodology's applicability is illustrated by analyzing several real-world datasets. The initial dataset examines burr formation occurrences and consists of two observation sets. Additionally, the second and third datasets contain medical information. The second dataset focuses on diabetic nephropathy, while the third dataset explores infection and recurrence time among kidney patients.

Keywords: XLindley distribution; Censoring scheme; kidney patients; Frank copula; type II censored samples; Clayton copula; Markov Chain Monte Carlo; Gumbel copula; Bayesian estimation; maximum likelihood estimation

MSC: 62F10; 62F15; 6N02; 62P10; 62P30; 62H05; 62H20

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1. Introduction

The construction of several bivariate distributions has attracted considerable attention. Bivariate data find applications in a wide range of fields, including engineering, industry, finance, economics, technology, sports, drought, weather, medical sciences, and many more. In recent years, various methods have been explored to generate new bivariate distributions based on different approaches. These methods include the marginal transformation method [1], copula method [2], conditional specification method [3], frailty approach [4], and the Marshall–Olkin methodology [5]. For more in-depth information on these methods and their application in establishing bivariate distributions, one can refer to the works of Ashkar and Aucoin [1], Nelsen [2], Vincent Raja and Gopalakrishnan [4], and Pathak and Vellaisamy [6].

In recent decades, copula theories have witnessed substantial growth in response to the increasing importance placed on modeling and describing diverse relationships among random variables. The emergence of new copula families has been motivated by the need to explore the structure of dependence in various fields, including the insurance industry [7], actuarial science [8,9], finance [10], hydrology [11], and public health and medicine [12], among others. This highlights the recognition of the significance of investigating the patterns of dependence between random variables in these domains.

In the field of statistical literature, copula structures have been widely employed by numerous authors as a method for establishing and modeling bivariate distributions. Flores [13] presented various bivariate Weibull models based on different copula functions, such as the Farlie–Gumbel–Morgenstern (FGM), Clayton, Ali–Mikhail–Haq (AMH), Gumbel–Hougaard, Gumbel–Barnett, and Nelsen Ten copulas. Abd Elaal et al. [14] introduced a bivariate generalized exponential distribution based on the FGM and Plackett copulas. Baharith et al. [15] proposed two bivariate Pareto Type II distributions; one is derived from the Gaussian copula (BPIIG) and the other is based on the mixture and Gaussian copula (BPIImG). El-sherpieny et al. [16] introduced the bivariate generalized Rayleigh distribution, denoted as Clayton-BGR, which is based on the Clayton copula. The study also derived the likelihood function for a progressive Type II censoring scheme with random removal and applied it to the Clayton-BGR distribution. Usman and Aliyu [17] proposed a new bivariate Nadarajah–Haghighi distribution using two copula functions: the Gumbel–Barnett and Clayton copula functions. Qura et al. [18] introduced a bivariate power Lomax distribution, named BFGMPLx, which is based on Farlie–Gumbel–Morgenstern (FGM) copulas. Fayomi et al. [19] presented a novel family of bivariate continuous Lomax generators called the BFGMLG family. This family is constructed by utilizing univariate Lomax generator (LG) families and the Farlie–Gumbel–Morgenstern (FGM) copula. Fayomi et al. [20] introduced two innovative continuous bivariate families, the FGM bivariate Kavya–Manoharan transformation and the Clayton bivariate Kavya–Manoharan transformation families, which are derived from the univariate Kavya–Manoharan transformation family and incorporate FGM and Clayton copulas, respectively.

In numerous life-testing experiments, it is typical for some test units' lifetimes to remain unobserved. Censored observations arise when a sample is drawn from a complete population, but either the initial or final observations remain unknown. Type I and Type II censoring are the two predominant methods employed. Consider a scenario where n items are simultaneously subjected to a life test, planned to end upon observing the r-th failure (where r is predetermined). Consequently, only the initial r failures are recorded. This kind of data collected from a restricted life test is denoted as a Type II censored sample. In the context of bivariate censoring samples, Balakrishnan and Kim [21] introduced the likelihood function for a bivariate distribution based on Type II censoring. Kim et al. [22] have explored both maximum likelihood and Bayesian estimation methods for determining the unknown parameters in the bivariate generalized exponential distribution under a Type II censoring scheme. Almetwally [23] has explored the estimation of parameters in bivariate models within a specific censoring framework. Bai et al. [24] introduced the Type I progressive interval censoring schemes for the Marshall–Olkin bivariate Weibull (MOBW) distribution and discussed how confidence intervals depend on the percentile bootstrap of the unknown parameters. El-Morshedy et al. [25] have examined both maximum likelihood and Bayesian methods for estimating the parameters of the bivariate Burr X-G family, considering various distributions and utilizing Type II censored data. Lin et al. [26] used the Pareto type I distribution for the joint frailty–copula model, a well-known heavy-tailed distribution. They devised statistical inference techniques encompassing bivariate random censoring, semi-competing risks, and competing risks, alongside maximum likelihood estimation procedures. Haj Ahmed et al. [27] introduced a novel bivariate model based on the FGM copula and the univariate modified extended exponential distribution, named the bivariate modified extended exponential distribution. They estimated the unknown parameters using both maximum likelihood and Bayesian estimation methods within a Type II censored sampling scheme.

The bivariate model, based on a censored sample, holds significant importance in various fields of data science. It facilitates the analysis of the relationship between two variables through copula functions, effectively reducing data collection costs while maintaining overall efficiency comparable to that of using complete samples. In recent times, the analysis of data has become increasingly challenging due to the prevalence of large and diverse datasets. A notable characteristic of these extensive datasets is their frequent redundancy. For further information on data science, refer to the works of Gramaje et al. [28] and Tien [29].

In this study, three novel bivariate distributions were developed using a special mixture of exponential and Lindley distributions, as detailed in Chouia and Zeghdoudi [30]. The specific mixture of exponential and Lindley distributions mentioned is referred to as the XLindley (XL) distribution. This new model features a single parameter, making it easy to use, granting significant flexibility to its density function and suitability for modeling different data patterns. Moreover, it demonstrates an increasing hazard function, making it a valuable tool in various statistical analyses and modeling, as discussed in Chouia and Zeghdoudi [30]. Furthermore, other papers have utilized the XLindley distribution to create new extended distributions, as seen in Ahsan-ul-Haq et al. [31], Etaga et al. [32], Musekwa and Makubate [33], and Gemeay et al. [34]. In the latter, Alotaibi et al. [35] employed an adaptive Type II progressive hybrid censoring plan to tackle estimation challenges related to the XL distribution using both classical and Bayesian methods. Their study illustrated that datasets from chemical engineering could be effectively modeled using the XL distribution instead of traditional distributions such as gamma and Weibull distributions. Additionally, Alotaibi et al. [36] delved into the estimation of constant-stress accelerated life tests (CSALT) for XL distribution based on progressive.

Type II censoring, applying it to model specific engineering domains. Based on previous studies investigating this distribution, it was proven that it is more efficient than many famous competing distributions, such as the Lindley, Xgamma, exponential, Zeghdoudi, log-normal, and Pareto distributions. Therefore, this was a strong motivation to use this distribution to create the best bivariate model.

Considering the above, the main objectives of this paper are as follows:

1. Introducing three innovative models derived from the XLindley distribution and copula functions:
    - The Frank Bivariate XL (FBXL) Distribution;
    - The Gumbel Bivariate XL (GBXL) Distribution;
    - The Clayton Bivariate XL (CBXL) Distribution.

    These novel models offer enhanced flexibility and suitability for application across diverse fields, including the medical and industrial domains.
2. Sub-objectives of this paper include:
    - Examining multiple scenarios based on an XLindley distribution with a single parameter, enabling a more thorough comparison, particularly across various datasets.
    - Estimating the parameters of bivariate XLindley models involves utilizing both maximum likelihood estimation (MLE) and Bayesian estimation methods. This consideration accounts for both complete and censored samples within a Type II framework.
    - Evaluating the estimators of novel bivariate models involves employing numerical techniques such as the Metropolis–Hastings algorithm.
    - Investigating two types of confidence intervals, namely asymptotic intervals and Bayesian credible intervals, for estimating the unknown parameters.
    - Assessing the fit of the bivariate model involves the use of various goodness-of-fit measures.

This study holds significant importance for two critical sectors in any country: the industrial and medical sectors.

In the industrial sector, tasks involving metal sheet manipulation, such as perforating procedures and measurements related to hole diameter and sheet thickness, are essential in metalworking and manufacturing processes. Burr formation, a common occurrence in these processes, is particularly significant for engineers and scientists due to its implications for product quality, manufacturing efficiency, innovation, and cost reduction.

In the healthcare sector, our research focuses on two key areas: diabetic nephropathy and infection recurrence in kidney patients. Understanding the progression of diabetic nephropathy is crucial, as it provides insights into key risk factors and aids in predicting renal complications, thus enabling the optimization of treatment protocols. Similarly, in the analysis of infection recurrence, survival analysis plays a pivotal role in evaluating infections at catheter insertion sites, guiding the optimization of infection management. This knowledge empowers healthcare professionals to refine treatment strategies, make well-informed decisions, and enhance patient outcomes in both diabetic nephropathy and infection recurrence scenarios.

The rest of this paper is organized as follows: Section 2 provides a review of the copula function, along with discussions on the Frank, Gumbel, and Clayton copulas. The FBXL, GBXL, and CBXL distributions are defined in Section 3. In Section 4, maximum likelihood estimation and Bayesian estimation were performed. Section 5 delves into the discussion of confidence intervals. In Section 6, the suitability of the new models is demonstrated through a simulation study. Section 7 discusses the applications of iron material jobs data, diabetic nephropathy data, and kidney patient data. Finally, in Section 8, the conclusion and some remarks regarding the FBXL, GBXL, and CBXL distributions are presented.

2. Copula Function

In this section, the general definition of Archimedean copulas is presented, and the Frank, Gumbel, and Clayton copulas are discussed.

The copula is a convenient approach to describe a bivariate distribution with a dependence structure. The utilization of copulas is motivated by their numerous advantages in analyzing the interdependence between random variables. The copula functions present a flexible framework that allows for capturing diverse dependence patterns, encompassing both linear and non-linear relationships, tail dependencies, and asymmetries. By adopting a "vine copula" approach, copulas effectively separate the modeling of marginal distributions from the modeling of dependence. Additionally, the copulas facilitate the estimation of conditional dependencies, enabling the investigation of the interdependence among the remaining variables. This feature is valuable in risk management, portfolio optimization, and the pricing of complex financial products. Compared to traditional correlation-based approaches, copula models offer a more robust alternative by providing a comprehensive and accurate representation of dependence, particularly in the presence of non-linear relationships or non-normal distributions.

The Sklar theorem, introduced by Sklar [37], plays a crucial role in copula theory. According to this theorem, for two random variables X and Y with marginal cumulative distribution functions (cdfs) denoted as $u = F_1(x)$ and $v = F_2(y)$, and marginal probability density functions (pdfs) represented as $f_1(x)$ and $f_2(y)$, respectively, let $C_\theta(u,v)$ represent the copula cdf, and $c_\theta(u,v)$ denote the copula pdf. Then, the joint cdf and joint pdf of (X, Y) based on the copula can be expressed as follows:

$$F(x,y) = C_\theta\Big(F_1(x;\Gamma_1), F_2(y;\Gamma_2)\Big), \tag{1}$$

and

$$f(x,y) = f_1(x;\Gamma_1) f_2(y;\Gamma_2) c_\theta\Big(F_1(x;\Gamma_1), F_2(y;\Gamma_2)\Big), \tag{2}$$

respectively, where θ is a parameter that measures the dependence between the marginals. For further information regarding copulas, please refer to Nelsen [2] and Joe [38]. Vectors Γ_1 and Γ_2 represent the parameters for variables X and Y, respectively. Various copula functions have been developed using Equations (1) and (2), including the Frank, Gumbel, Clayton, and other copula models.

An Archimedean copula with a strict generator is represented by the following form:

$$C(u,v) = \phi\Big(\phi^{-1}(u), \phi^{-1}(v)\Big), \tag{3}$$

where the generator function ϕ in an Archimedean copula must adhere to specific conditions outlined by Nelsen [2]. Firstly, ϕ should be a continuous, strictly decreasing, and convex function that maps the interval $[0,1]$ onto $[0,\infty)$. Secondly, it is required that $\phi(0)$ equals infinity, and, finally, $\phi(1)$ should be equal to zero. Archimedean copulas have been widely applied due to several factors:

- Their straightforward construction process.
- The extensive assortment of copula families within this category.
- The numerous favorable properties exhibited by copulas in this class.

For additional information, refer to Nelsen [2]. Other well-known completely monotone generators of Archimedean copulas include the Frank, Gumbel, and Clayton copulas.

In this paper, three families of Archimedean copulas are considered: the Frank copula, the Gumbel copula, and the Clayton copula. These copulas are utilized for constructing bivariate distributions and are among the most commonly used Archimedean copulas.

2.1. Frank Copula

The Frank copula was introduced by Frank [39], and several statistical properties of this copula family were discussed in Nelsen [40] and Genest [41]. It belongs to the class of Archimedean copulas, and its generator function is represented as

$$\phi(t) = -\log\left[\left(e^{-\theta t} - 1\right) / \left(e^{-\theta} - 1\right)\right], \quad \theta \in (-\infty,\infty)\backslash, t \in [0,1],$$

where θ is the dependence parameter. Consequently, the cdf of the Frank copula with copula parameter $\theta \in (-\infty,\infty) \setminus \{0\}$, where the symbol $\setminus$ denotes set minus, is obtained as follows:

$$C_\theta(u,v) = -\frac{1}{\theta}\log\left[1 + \frac{\left(e^{-\theta u} - 1\right)\left(e^{-\theta v} - 1\right)}{e^{-\theta} - 1}\right], \quad \theta \in (-\infty,\infty) \setminus \{0\}. \tag{4}$$

The copula density function for the Frank copula is given as follows:

$$c_\theta(u,v) = \frac{-\theta e^{-\theta(u+v)}\left(e^{-\theta} - 1\right)}{\left[e^{-\theta} - e^{-\theta u} - e^{-\theta v} + e^{-\theta(u+v)}\right]^2}, \quad \theta \in (-\infty,\infty) \setminus \{0\}, \tag{5}$$

where $u, v \in [0,1]$. The Frank copula achieves the Frechet–Hoeffding upper bound as θ approaches infinity and the Frechet–Hoeffding lower bound as θ approaches negative infinity. Additionally, it has the capability to represent both negative and positive dependence, as highlighted by Nelsen [2]. The Frank copula is renowned for its flexibility in modeling both positive and negative dependencies, making it a versatile choice for various applications, as discussed in Joe [38]. Its parameter θ directly governs the strength of tail dependence, offering intuitive interpretation, as described by Nelsen [2]. However, its symmetric nature may not accurately capture the asymmetrical dependencies present in some datasets. Additionally, while it can model tail dependence to some extent, extreme tail dependence may not be fully captured by the Frank copula, as observed in Ashkar and Aucoin [11].

2.2. Gumbel Copula

The Gumbel copula, initially introduced by Gumbel [42], is commonly referred to as the Gumbel family by many authors. However, to avoid confusion with another Archimedean copula family associated with Gumbel's name and its presence in Hougaard's work [43], Hutchinson and Lai [44] named it the Gumbel–Hougaard family. This copula is classified as an Archimedean copula and is characterized by its generator function,

$\phi(t) = (-\ln t)^{\theta}$, with $t \in [0,1]$ and $\theta \geq 1$. The cdf of the Gumbel copula with copula parameter $\theta \geq 1$ is defined as follows:

$$C_{\theta}(u,v) = \exp\left\{-\left[(-\ln u)^{\theta} + (-\ln v)^{\theta}\right]^{\frac{1}{\theta}}\right\}, \quad \theta \geq 1. \tag{6}$$

Setting θ to 1 indicates the presence of the independence copula, since $C_{\theta}(u,v) = uv$. When $\theta \to \infty$, the Gumbel copula attains the Frechet–Hoeffding lower bound, implying a higher level of dependence in the upper tail of the copula distribution. The applicability of this copula function is limited to scenarios involving positive dependence. Furthermore, all members of this copula family are absolutely continuous.

The copula density function for the Gumbel copula is given as follows:

$$\begin{aligned} c_{\theta}(u,v) = &\frac{C_{\theta}(u,v)}{uv} \frac{[(-\ln u)(-\ln v)]^{\theta-1}}{\left[(-\ln u)^{\theta} + (-\ln v)^{\theta}\right]^{2-\frac{1}{\theta}}} \times \\ &\left\{\left[(-\ln u)^{\theta} + (-\ln v)^{\theta}\right]^{\frac{1}{\theta}} + \theta - 1\right\}, \quad \theta \geq 1. \end{aligned} \tag{7}$$

The Gumbel copula is a widely used copula model that is commonly employed to capture extreme value dependence, as discussed in Joe [38].

2.3. Clayton Copula

The Clayton copula is a member of the Archimedean copula class and is characterized by its generator function, $\phi(t) = \frac{1}{\theta}\left(t^{-\theta} - 1\right)$, where $t \in [0,1]$ and $\theta \in (0,\infty)$. This copula is commonly known as the Clayton copula and has been introduced by Clayton [45], Cook and Johnson [46], and Oakes [47]. The cdf of the Clayton copula with copula parameter $\theta \in (0,\infty)$ is as follows:

$$C_{\theta}(u,v) = \left\{u^{-\theta} + v^{-\theta} - 1\right\}^{-\frac{1}{\theta}}, \quad \theta \in (0,\infty). \tag{8}$$

As θ approaches infinity, the Clayton copula achieves the Frechet–Hoeffding upper bound. This indicates that the Clayton copula exhibits lower tail dependence, implying a stronger dependence in the lower tail of the copula distribution.The copula density function for the Clayton copula is given as follows:

$$c_{\theta}(u,v) = (1+\theta)(uv)^{-1-\theta}\left(u^{-\theta} + v^{-\theta} - 1\right)^{-2-\frac{1}{\theta}}, \quad \theta \in (0,\infty). \tag{9}$$

It is important to note that, as the parameter θ approaches infinity, the Clayton copula exhibits perfect positive dependence. It boasts a straightforward parameterization, enhancing ease of interpretation, as noted in Nelsen [2]. However, its ability to capture extreme tail dependence may be limited, and it may not be as flexible as other copula functions in modeling complex dependence structures, as discussed by Genest et al. [11].

3. Bivariate XLindley Distribution

In this section, we introduce three bivariate XLindley distributions that are based on the Frank, Gumbel, and Clayton copulas.

The XLindley distribution was first studied by Chouia and Zeghdoudi [30], in which the pdf and cdf, each with one parameter $a > 0$, are given as follows:

$$f(x;a) = \frac{a^2(2+a+x)e^{-ax}}{(1+a)^2}, x > 0, a > 0, \tag{10}$$

and

$$F(x;a) = 1 - \left(1 + \frac{ax}{(1+a)^2}\right)e^{-ax}, x > 0, a > 0. \quad (11)$$

The reliability and hazard functions of the XL distribution are given by

$$S(x;a) = \left(1 + \frac{ax}{(1+a)^2}\right)e^{-ax}, x > 0, a > 0, \quad (12)$$

and

$$h(x;a) = \frac{a^2(x+\theta+2)}{(1+\theta)^2 + x\theta}, x > 0, a > 0. \quad (13)$$

3.1. Frank Bivariate XL (FBXL) Distribution

The joint pdf and cdf of the bivariate XL distribution with the Frank copula function are presented as follows:

$$\begin{aligned} f_{FBXL}(x,y) = & \frac{a_1^2(2+a_1+x)e^{-a_1 x}}{(1+a_1)^2}\frac{a_2^2(2+a_2+y)e^{-a_2 y}}{(1+a_2)^2} \times \\ & \frac{-\theta e^{-\theta(u(x)+v(y))}\left(e^{-\theta}-1\right)}{\left\{e^{-\theta} - e^{-\theta u(x)} - e^{-\theta v(y)} + e^{-\theta(u(x)+v(y))}\right\}^2}, \quad a_1, a_2 > 0, \theta \in (-\infty, \infty) \setminus \{0\}, \end{aligned} \quad (14)$$

$$F_{FBXL}(x,y) = -\frac{1}{\theta}\log\left\{1 + \frac{\left[e^{-\theta u(x)} - 1\right]\left[e^{-\theta v(y)} - 1\right]}{e^{-\theta} - 1}\right\}, \quad a_1, a_2 > 0, \theta \in (-\infty, \infty) \setminus \{0\}, \quad (15)$$

where

$$u(x) = \left[1 - \left(1 + \frac{a_1 x}{(1+a_1)^2}\right)e^{-a_1 x}\right],$$

and

$$v(y) = \left[1 - \left(1 + \frac{a_2 y}{(1+a_2)^2}\right)e^{-a_2 y}\right].$$

The joint pdf of the FBXL distribution combines the individual characteristics of the XLindley marginal distributions with the dependency structure encoded by the Frank copula function. The joint cdf of the FBXL distribution is expressed in terms of the XLindley marginal cdfs and the Frank copula function, allowing us to quantify the cumulative probabilities associated with various combinations of x and y.

The reliability functions of the marginal distributions are defined as follows:

$$S_{XL}(x;a_1) = \left(1 + \frac{a_1 x}{(1+a_1)^2}\right)e^{-a_1 x}, x > 0, a_1 > 0, \quad (16)$$

and

$$S_{XL}(y;a_2) = \left(1 + \frac{a_2 y}{(1+a_2)^2}\right)e^{-a_2 y}, y > 0, a_2 > 0. \quad (17)$$

The joint survival function for copula is expressed as

$$S(x,y) = C(S(x), S(y)). \quad (18)$$

Hence, the reliability function of the FBXL distribution is

$$S_{FBXL}(x,y) = -\frac{1}{\theta}\log\left[1+\frac{\left(e^{-\theta(1-u(x))}-1\right)\left(e^{-\theta(1-v(y))}-1\right)}{e^{-\theta}-1}\right], \quad a_1,a_2>0, \theta\in(-\infty,\infty)\setminus\{0\}. \tag{19}$$

In the context of the FBXL distribution, the reliability function captures the probability of survival beyond given thresholds, offering insights into the reliability and persistence of the system represented by X and Y.

Basu [48] was the pioneer in defining the bivariate failure rate function, which is expressed as follows:

$$h(x,y) = \frac{f(x,y)}{S(x,y)}. \tag{20}$$

Hence, the hazard rate function of the FBXL distribution is

$$\begin{aligned} h_{FBXL}(x,y) = & \frac{a_1^2(2+a_1+x)e^{-a_1x}}{(1+a_1)^2}\frac{a_2^2(2+a_2+y)e^{-a_2y}}{(1+a_2)^2}\times \\ & \frac{\theta^2 e^{-\theta(u(x)+v(y))}\left(e^{-\theta}-1\right)}{\left\{e^{-\theta}-e^{-\theta u(x)}-e^{-\theta v(y)}+e^{-\theta(u(x)+v(y))}\right\}^2}\times \\ & \left[\log\left[1+\frac{\left(e^{-\theta(1-u(x))}-1\right)\left(e^{-\theta(1-v(y))}-1\right)}{e^{-\theta}-1}\right]\right]^{-1}, a_1,a_2>0, \theta\in(-\infty,\infty)\setminus\{0\}. \end{aligned} \tag{21}$$

The hazard rate function measures the likelihood of experiencing an event (failure) per unit time, given that the system has survived up to that point. By examining changes in the hazard rate over time or across different values of x and y, we can assess the dynamic behavior and risk profile of the FBXL distribution, see Figures 1 and 2.

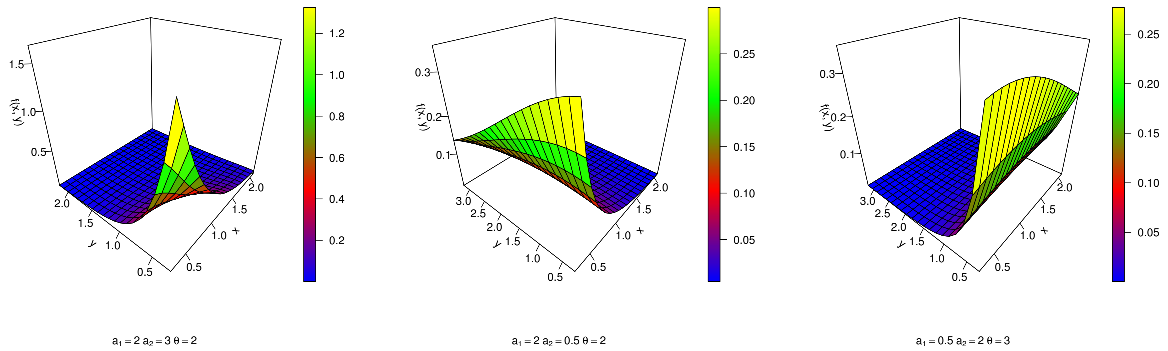

Figure 1. Plots of joint density FBXL distribution for some parameter values.

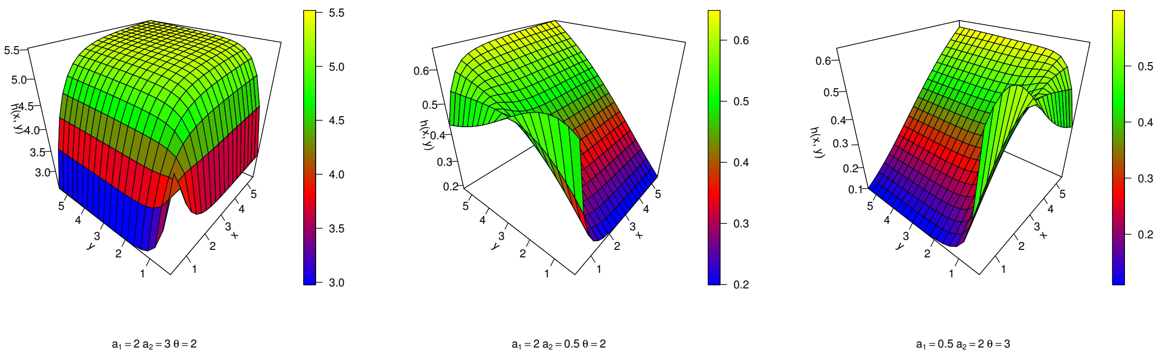

Figure 2. Plots of hazard rate function of the FBXL distribution for some parameter values.

3.2. Gumbel Bivariate XLindley (GBXL) Distribution

The joint pdf and cdf of the bivariate XL distribution with the Gumbel copula function are presented as follows:

$$
\begin{aligned}
f_{GBXL}(x,y) =& \frac{a_1^2(2+a_1+x)e^{-a_1x}}{(1+a_1)^2}\frac{a_2^2(2+a_2+y)e^{-a_2y}}{(1+a_2)^2}\times \\
& \frac{e^{\left\{-\left[(-\ln u(x))^\theta+(-\ln v(y))^\theta\right]^{\frac{1}{\theta}}\right\}}}{u(x)v(y)}\frac{[(-\ln u(x))(-\ln v(y))]^{\theta-1}}{\left[(-\ln u(x))^\theta+(-\ln v(y))^\theta\right]^{2-\frac{1}{\theta}}}\times \\
& \left\{\left[(-\ln u(x))^\theta+(-\ln v(y))^\theta\right]^{\frac{1}{\theta}}+\theta-1\right\}, \quad a_1,a_2>0,\theta\geq 1,
\end{aligned}
\tag{22}
$$

$$
F_{GBXL}(x,y)=\exp\left\{-\left[(-\ln u(x))^\theta+(-\ln v(y))^\theta\right]^{\frac{1}{\theta}}\right\}, \quad a_1,a_2>0,\theta\geq 1. \tag{23}
$$

This distribution captures the dependence between x and y through the Gumbel copula function, which is characterized by its tail behavior and the parameter θ. The marginal distributions, represented by $u(x)$ and $v(y)$, determine the individual behavior of x and y, respectively. The parameter $\theta \geq 1$ controls the strength of dependence between x and y.

The reliability function of the GBXL distribution is

$$
S_{GBXL}(x,y)=\exp\left\{-\left[(-\ln(1-u(x)))^\theta+(-\ln(1-v(y)))^\theta\right]^{\frac{1}{\theta}}\right\}, \quad a_1,a_2>0,\theta\geq 1. \tag{24}
$$

The hazard rate function of the GBXL distribution is

$$
\begin{aligned}
h_{GBXL}(x,y) =& \frac{a_1^2(2+a_1+x)e^{-a_1x}}{(1+a_1)^2}\frac{a_2^2(2+a_2+y)e^{-a_2y}}{(1+a_2)^2}\times \\
& \frac{e^{\left\{-\left[(-\ln u(x))^\theta+(-\ln v(y))^\theta\right]^{\frac{1}{\theta}}\right\}}}{u(x)v(y)}\frac{[(-\ln u(x))(-\ln v(y))]^{\theta-1}}{\left[(-\ln u(x))^\theta+(-\ln v(y))^\theta\right]^{2-\frac{1}{\theta}}}\times \\
& \frac{\left\{\left[(-\ln u(x))^\theta+(-\ln v(y))^\theta\right]^{\frac{1}{\theta}}+\theta-1\right\}}{exp\left\{-\left[(-\ln(1-u(x)))^\theta+(-\ln(1-v(y)))^\theta\right]^{\frac{1}{\theta}}\right\}}, \quad a_1,a_2>0,\theta\geq 1.
\end{aligned}
\tag{25}
$$

By examining changes in the joint density and hazard rate over time or across different values of x and y, we can assess the dynamic behavior and risk profile of the GBXL distribution, see Figures 3 and 4.

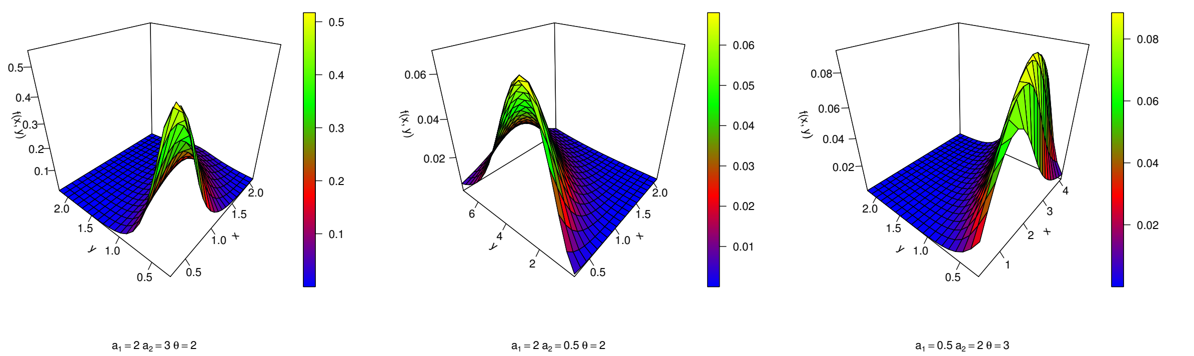

Figure 3. Plots of joint density GBXL distribution for some parameter values.

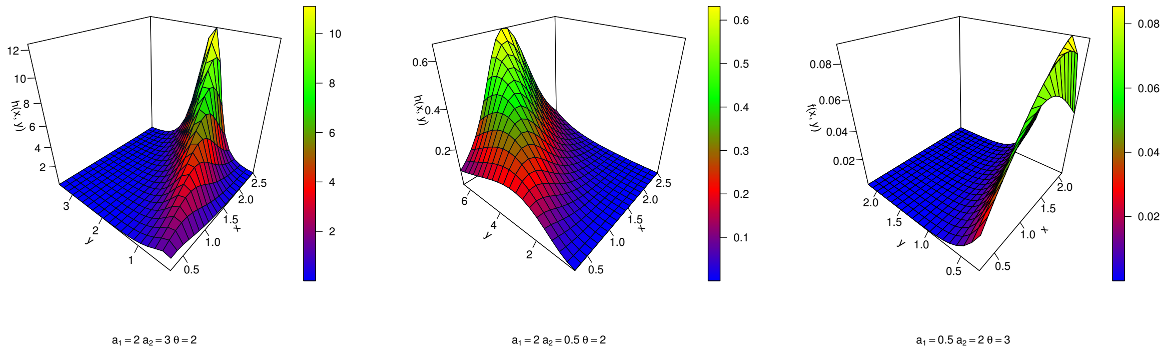

Figure 4. Plots of hazard rate function of the GBXL distribution for some parameter values.

3.3. Clayton Bivariate XLindley (CBXL) Distribution

The joint pdf and cdf of the bivariate XL distribution with the Clayton copula function are presented as follows:

$$f_{CBXL}(x,y) = \frac{a_1^2(2+a_1+x)e^{-a_1 x}}{(1+a_1)^2}\frac{a_2^2(2+a_2+y)e^{-a_2 y}}{(1+a_2)^2} \times (1+\theta)[u(x)v(y)]^{-1-\theta}\left[u(x)^{-\theta}+v(y)^{-\theta}-1\right]^{-2-\frac{1}{\theta}}, \quad a_1, a_2 > 0, \theta \in (0,\infty), \tag{26}$$

$$F_{CBXL}(x,y) = \left\{u(x)^{-\theta}+v(y)^{-\theta}-1\right\}^{-\frac{1}{\theta}}, \quad a_1, a_2 > 0, \theta \in (0,\infty). \tag{27}$$

The joint pdf of the CBXL distribution enables the modeling of the joint behavior of two random variables x and y. The parameters a_1, a_2, and θ play crucial roles in shaping the marginal distributions and determining the degree of dependence between x and y.

The survival function of the CBXL distribution is as follows:

$$S_{CBXL}(x,y) = \left\{(1-u(x))^{-\theta}+(1-v(y))^{-\theta}-1\right\}^{-\frac{1}{\theta}}, \quad a_1, a_2 > 0, \theta \in (0,\infty). \tag{28}$$

The hazard rate function of the CBXL distribution is

$$h_{CBXL}(x,y) = \frac{a_1^2(2+a_1+x)e^{-a_1 x}}{(1+a_1)^2}\frac{a_2^2(2+a_2+y)e^{-a_2 y}}{(1+a_2)^2} \times \frac{(1+\theta)[u(x)v(y)]^{-1-\theta}\left[u(x)^{-\theta}+v(y)^{-\theta}-1\right]^{-2-\frac{1}{\theta}}}{\left\{(1-u(x))^{-\theta}+(1-v(y))^{-\theta}-1\right\}^{-\frac{1}{\theta}}}, \quad a_1, a_2 > 0, \theta \in (0,\infty). \tag{29}$$

By examining changes in the joint density and hazard rate over time or across different values of x and y, we can assess the dynamic behavior and risk profile of the GBXL distribution, see Figures 5 and 6.

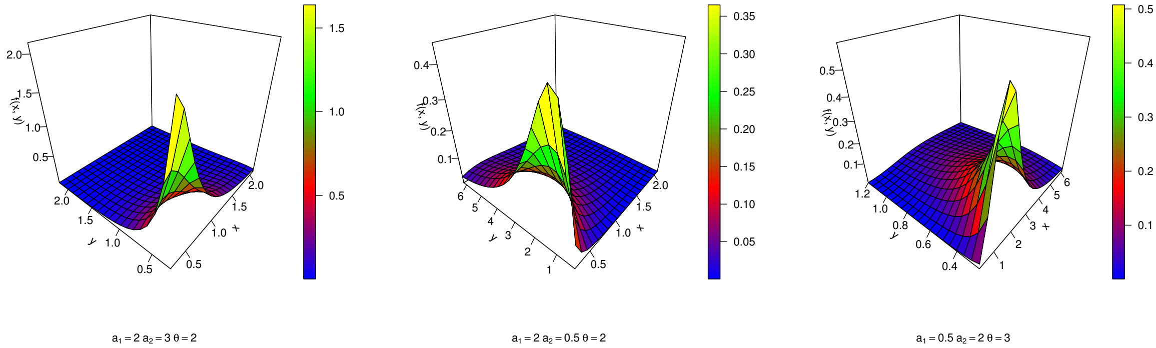

Figure 5. Plots of joint density CBXL distribution for some parameter values.

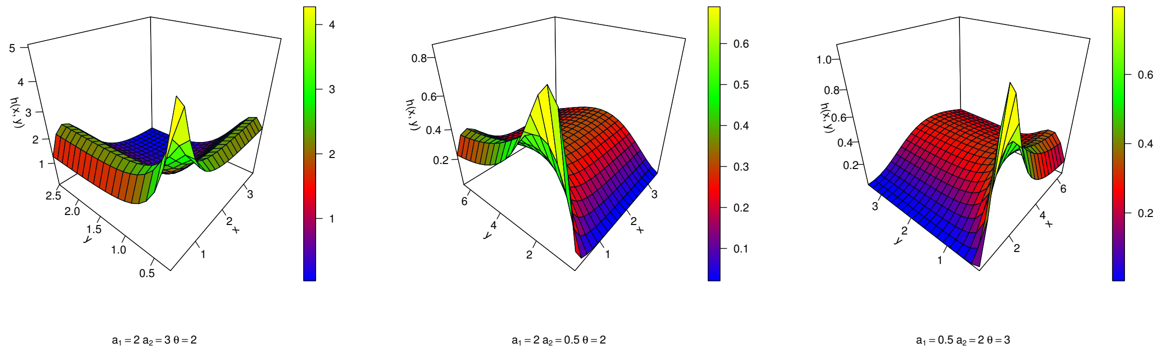

Figure 6. Plots of hazard rate function of the CBXL distribution for some parameter values.

4. Inference under Complete and Censored Samples

In this section, we explored both classical and non-classical estimation problems. Within the classical framework, we utilized maximum likelihood estimation to estimate the unknown parameters of the BXL distribution, considering various copula functions such as the Frank, Gumbel, and Clayton copulas. We also delved into point and interval estimation techniques. In contrast, Bayesian estimation emerged as a robust non-classical approach, enabling us to derive point estimates and credible intervals for the unknown parameters. These estimation challenges were tackled for both complete samples and Type II censored samples.

4.1. Maximum Likelihood Estimation (MLE)

In this subsection, we focused on the maximum likelihood estimation (MLE) technique applied to determine the unknown parameters, denoted as $\psi = (a_1, a_2, \theta)$, using both complete samples and Type II censored samples.

Suppose we have a random sample $(x_{1:n}, y_{[1:n]}), (x_{2:n}, y_{[2:n]}), \ldots, (x_{n:n}, y_{[n:n]})$ from a bivariate XL distribution with cumulative distribution function $F(x, y)$ and probability density function $f(x, y)$. Let x_i, for $i = 1, \ldots, n$, be arranged in ascending order of magnitude such that $x_{1:n} < x_{2:n} < \ldots < x_{n:n}$, and let $y_{[i:n]}$ denote the concomitant of the ith order statistic. For further details on this setup, refer to David and Nagaraja [49].

4.1.1. MLE under Complete Samples

The likelihood function for a bivariate model has been discussed in prior work by Kim et al. [50] and Elaal and Jarwan [14]. For a complete sample, the likelihood function for estimating φ is defined as follows:

$$L(\psi) = \prod_{i=1}^{n} f_{X,Y}\left(x_{i:n}, y_{[i:n]}\right) = \prod_{i=1}^{n} f_X(x_{i:n}) f_Y(y_{[i:n]}) c(F(x_{i:n}), F(y_{[i:n]})). \tag{30}$$

Then,

$$L(\psi) = \prod_{i=1}^{n} \frac{a_1^2(2 + a_1 + x_{i:n})e^{-a_1 x_{i:n}}}{(1 + a_1)^2} \frac{a_2^2\left(2 + a_2 + y_{[i:n]}\right)e^{-a_2 y_{[i:n]}}}{(1 + a_2)^2} c(F(x_{i:n}), F(y_{[i:n]})), \tag{31}$$

where the expression $c(F(x_{i:n}), F(y_{[i:n]}))$ represents the density function of the Frank copula, Gumbel copula, and Clayton copula for the XL distribution, defined as follows:

$$c_{FBLX}(F(x_{i:n}), F(y_{[i:n]})) = \frac{-\theta e^{-\theta\left(u(x_{i:n}) + v(y_{[i:n]})\right)}\left(e^{-\theta} - 1\right)}{\left\{e^{-\theta} - e^{-\theta u(x_{i:n})} - e^{-\theta v(y_{[i:n]})} + e^{-\theta\left(u(x_{i:n}) + v(y_{[i:n]})\right)}\right\}^2}, \tag{32}$$

$$\begin{aligned} c_{GBLX}(F(x_{i:n}), F(y_{[i:n]})) = & \frac{e^{\left\{-\left[(-\ln u(x_{i:n}))^\theta + \left(-\ln v(y_{[i:n]})\right)^\theta\right]^{\frac{1}{\theta}}\right\}}}{u(x_{i:n})v(y_{[i:n]})} \frac{\left[(-\ln u(x_{i:n}))\left(-\ln v(y_{[i:n]})\right)\right]^{\theta-1}}{\left[(-\ln u(x_{i:n}))^\theta + \left(-\ln v(y_{[i:n]})\right)^\theta\right]^{2-\frac{1}{\theta}}} \times \\ & \left\{\left[(-\ln u(x_{i:n}))^\theta + \left(-\ln v(y_{[i:n]})\right)^\theta\right]^{\frac{1}{\theta}} + \theta - 1\right\}, \end{aligned} \tag{33}$$

$$c_{CBLX}(F(x_{i:n}), F(y_{[i:n]})) = (1 + \theta)\left[u(x_{i:n})v(y_{[i:n]})\right]^{-1-\theta}\left[u(x_{i:n})^{-\theta} + v(y_{[i:n]})^{-\theta} - 1\right]^{-2-\frac{1}{\theta}}, \tag{34}$$

where

$$u(x_{i:n}) = \left[1 - \left(1 + \frac{a_1 x_{i:n}}{(1 + a_1)^2}\right)e^{-a_1 x_{i:n}}\right],$$

and

$$v(y_{[i:n]}) = \left[1 - \left(1 + \frac{a_2 y_{[i:n]}}{(1 + a_2)^2}\right)e^{-a_2 y_{[i:n]}}\right].$$

The log-likelihood function of the FBXL distribution, denoted as $\ell_{FBXL}(\psi)$, can be represented as follows:

$$\begin{aligned} \ell_{FBXL}(\psi) = & \varkappa(x_{i:n}) + \eta(y_{[i:n]}) - n\ln\theta - \theta\sum_{i=1}^{n}\left[u(x_{i:n}) + v(y_{[i:n]})\right] + n\ln\left[e^{-\theta} - 1\right] \\ & - 2\sum_{i=1}^{n}\ln\left[e^{-\theta} - e^{-\theta u(x_{i:n})} - e^{-\theta v(y_{[i:n]})} + e^{-\theta\left(u(x_{i:n}) + v(y_{[i:n]})\right)}\right], \end{aligned} \tag{35}$$

where

$$\varkappa(x_{i:n}) = \sum_{i=1}^{n}\ln f_X(x_{i:n}) = 2n\ln a_1 + \sum_{i=1}^{n}\ln(2 + a_1 + x_{i:n}) - \sum_{i=1}^{n} a_1 x_{i:n} - 2n\ln(1 + a_1) \tag{36}$$

$$\eta(y_{[i:n]}) = \sum_{i=1}^{n}\ln f_Y(y_{[i:n]}) = 2n\ln a_2 + \sum_{i=1}^{n}\ln\left(2 + a_2 + y_{[i:n]}\right) - \sum_{i=1}^{n} a_2 y_{[i:n]} - 2n\ln(1 + a_2). \tag{37}$$

We derived Equation (35) by taking partial derivatives with respect to the parameters of the model. The detailed derivation can be found in Appendix A.

The log-likelihood function of the GBXL distribution, denoted as $l_{GBXL}(\psi)$, can be represented as follows:

$$\begin{aligned}\ell_{GBXL}(\psi) &= \varkappa(x_{i:n}) + \eta(y_{[i:n]}) - \sum_{i=1}^{n}\Big[\ln u(x_{i:n}) + \ln v(y_{[i:n]})\Big] - \sum_{i=1}^{n}\Big[(-\ln u(x_{i:n}))^{\theta} + \Big(-\ln v(y_{[i:n]})\Big)^{\theta}\Big]^{\frac{1}{\theta}} \\ &+ (\theta-1)\sum_{i=1}^{n}\Big[\ln\ln u(x_{i:n}) + \ln\ln v(y_{[i:n]})\Big] - \Big(2-\frac{1}{\theta}\Big)\sum_{i=1}^{n}\ln\Big[(-\ln u(x_{i:n}))^{\theta} + \Big(-\ln v(y_{[i:n]})\Big)^{\theta}\Big] \\ &+ \sum_{i=1}^{n}\ln\left\{\Big[(-\ln u(x_{i:n}))^{\theta} + \Big(-\ln v(y_{[i:n]})\Big)^{\theta}\Big]^{\frac{1}{\theta}} + \theta - 1\right\}.\end{aligned} \tag{38}$$

We derived Equation (38) by taking partial derivatives with respect to the parameters of the model. The detailed derivation can be found in Appendix A.

The log-likelihood function of the CBXL distribution, denoted as $l_{CBXL}(\psi)$, can be represented as follows:

$$\begin{aligned}\ell_{CBXL}(\psi) &= \varkappa(x_{i:n}) + \eta(y_{[i:n]}) + n\ln(1+\theta) - (1+\theta)\sum_{i=1}^{n}\Big[\ln u(x_{i:n}) + \ln v(y_{[i:n]})\Big] \\ &- \Big(2+\frac{1}{\theta}\Big)\sum_{i=1}^{n}\ln\Big[u(x_{i:n})^{-\theta} + v(y_{[i:n]})^{-\theta} - 1\Big].\end{aligned} \tag{39}$$

We derived Equation (39) by taking partial derivatives with respect to the parameters of the model. The detailed derivation can be found in Appendix A.

By equating them to zero, we obtain a system of non-linear normal equations. Solving this system of equations numerically enables us to estimate the maximum likelihood estimators (MLEs) for each distribution.

4.1.2. MLE under Censored Samples

In the scenario of a Type II censored sample, the experimenter may choose to terminate the experiment after observing the first r failures. This constitutes a Type II censored sample, where we observe only the number of uncensored items r from the X-sample, where $r \le n$, along with their associated values from the Y-sample, denoted as $\left(x_{i:n}, y_{[i:n]}\right)$ for $i = 1, \ldots, r$. Here, n and r are fixed and predetermined, while the time T is random. The likelihood function for a bivariate distribution based on Type II censoring has been introduced by Balakrishnan and Kim [21] and Kim et al. [22]. They discussed the maximum likelihood estimation for the unknown parameters of the bivariate generalized exponential distribution under Type II censored samples. The likelihood function can be expressed as follows:

$$L^{C}(\psi) = \frac{n!}{(n-r)}[1 - F(x_{r:n})]^{n-r}\prod_{i=1}^{r} f_{X,Y}\left(x_{i:n}, y_{[i:n]}\right), \tag{40}$$

where $F(x_{r:n})$ represents the cdf of the X variable evaluated at the rth order statistic, denoted as $x_{r:n}$. This notation denotes the probability that the X variable is less than or equal to $x_{r:n}$ in the sample. Based on a Type II censored sample $\left(x_{i:n}, y_{[i:n]}\right)$, for $i = 1, \ldots, r$ from a bivariate XL distribution, the likelihood function can be expressed as follows:

$$\begin{aligned}L^{C}(\psi) =& \frac{n!}{(n-r)}\left[\left(1 + \frac{a_1 x_{r:n}}{(1+a_1)^2}\right)e^{-a_1 x_{r:n}}\right]^{n-r} \\ &\prod_{i=1}^{r}\frac{a_1^2(2 + a_1 + x_{i:n})e^{-a_1 x_{i:n}}}{(1+a_1)^2}\frac{a_2^2\left(2 + a_2 + y_{[i:n]}\right)e^{-a_2 y_{[i:n]}}}{(1+a_2)^2}c(F(x_{i:n}), F(y_{[i:n]}), \psi).\end{aligned} \tag{41}$$

In the context of the XL distribution, the expression $c(F(x_{i:n}), F(y_{[i:n]}))$ represents the density function of the order statistics of the variables X and Y for the Frank copula, Gumbel copula, and Clayton copula. It is explicitly defined in Equations (32), (33), and (34), respectively.

By substituting Equations (11) and (14) into Equation (41), the log-likelihood function for the Frank copula function, denoted as $\ell^{C}_{FBLX}(\varphi)$, can be formulated in the following manner:

$$\begin{aligned}\ell^{C}_{FBXL}(\psi) \propto{} & (n-r)\ln\left(1+\frac{a_1 x_{r:n}}{(1+a_1)^2}\right) - a_1 x_{r:n} + \varkappa^{C}(x_{i:n}) + \eta^{C}(y_{[i:n]}) - n\ln\theta - \theta\sum_{i=1}^{r}\left[u(x_{i:n}) + v(y_{[i:n]})\right] \\ & + n\ln\left[e^{-\theta}-1\right] - 2\sum_{i=1}^{r}\ln\left[e^{-\theta} - e^{-\theta u(x_{i:n})} - e^{-\theta v(y_{[i:n]})} + e^{-\theta\left(u(x_{i:n})+v(y_{[i:n]})\right)}\right],\end{aligned} \tag{42}$$

where

$$\varkappa^{C}(x_{i:n}) = \sum_{i=1}^{r}\ln f_X(x_{i:n}) = 2n\ln a_1 + \sum_{i=1}^{r}\ln(2+a_1+x_{i:n}) - \sum_{i=1}^{r} a_1 x_{i:n} - 2n\ln(1+a_1) \tag{43}$$

$$\eta^{C}(y_{[i:n]}) = \sum_{i=1}^{r}\ln f_Y(y_{[i:n]}) = 2n\ln a_2 + \sum_{i=1}^{r}\ln\left(2+a_2+y_{[i:n]}\right) - \sum_{i=1}^{r} a_2 y_{[i:n]} - 2n\ln(1+a_2). \tag{44}$$

We derived Equation (42) by taking partial derivatives with respect to the parameters of the model. The detailed derivation can be found in Appendix B.

By substituting Equations (11) and (22) into Equation (41), the log-likelihood function for the Frank copula function, denoted as $l^{C}_{GBLX}(\varphi)$, can be formulated in the following manner:

$$\begin{aligned}\ell^{C}_{GBXL}(\psi) \propto{} & (n-r)\ln\left(1+\frac{a_1 x_{r:n}}{(1+a_1)^2}\right) - a_1 x_{r:n} + \varkappa^{C}(x_{i:n}) + \eta^{C}(y_{[i:n]}) - \sum_{i=1}^{r}\left[\ln u(x_{i:n}) + \ln v(y_{[i:n]})\right] \\ & - \sum_{i=1}^{r}\left[(-\ln u(x_{i:n}))^{\theta} + \left(-\ln v(y_{[i:n]})\right)^{\theta}\right]^{\frac{1}{\theta}} + (\theta-1)\sum_{i=1}^{r}\left[\ln\ln u(x_{i:n}) + \ln\ln v(y_{[i:n]})\right] \\ & - \left(2-\frac{1}{\theta}\right)\sum_{i=1}^{r}\ln\left[(-\ln u(x_{i:n}))^{\theta} + \left(-\ln v(y_{[i:n]})\right)^{\theta}\right] + \sum_{i=1}^{r}\ln\left\{\left[(-\ln u(x_{i:n}))^{\theta} + \left(-\ln v(y_{[i:n]})\right)^{\theta}\right]^{\frac{1}{\theta}} + \theta - 1\right\}.\end{aligned} \tag{45}$$

We derived Equation (45) by taking partial derivatives with respect to the parameters of the model. The detailed derivation can be found in Appendix B.

By substituting Equations (11) and (26) into Equation (41), the log-likelihood function for the Frank copula function, denoted as $l^{C}_{CBLX}(\varphi)$, can be formulated in the following manner:

$$\begin{aligned}\ell^{C}_{CBXL}(\psi) \propto{} & (n-r)\ln\left(1+\frac{a_1 x_{r:n}}{(1+a_1)^2}\right) - a_1 x_{r:n} + \varkappa^{C}(x_{i:n}) + \eta^{C}(y_{[i:n]}) + n\ln(1+\theta) \\ & - (1+\theta)\sum_{i=1}^{r}\left[\ln u(x_{i:n}) + \ln v(y_{[i:n]})\right] - \left(2+\frac{1}{\theta}\right)\sum_{i=1}^{r}\ln\left[u(x_{i:n})^{-\theta} + v(y_{[i:n]})^{-\theta} - 1\right].\end{aligned} \tag{46}$$

We derived Equation (46) by taking partial derivatives with respect to the parameters of the model. The detailed derivation can be found in Appendix B.

By equating these partial derivative equations to zeros, we get a system of non-linear (normal) equations. Since there is no closed-form expression for the MLE $\hat{a}_1, \hat{a}_2$, and $\hat{\theta}$, their values were computed numerically using a non-linear optimization algorithm. All numerical calculations and their results were obtained and are summarized in Section 6.

4.2. Bayesian Estimation

The parameters are viewed as random variables in the Bayesian approach, and their ambiguity is characterized as a joint prior distribution that is visible prior to the gathered failure times. The Bayesian approach is highly beneficial in reliability analysis since it allows for the collection of prior knowledge during the analysis. One of the major issues with the reliability analysis is the limited availability of data. Prior distribution choice is crucial in Bayesian inference. See Muhammed and Almetwally [51] for further information on the gamma density prior's varied shapes depending on its parameter values. As a result, it is adaptable and is a good prior for the bivariate XL parameters. When choosing a prior distribution for a parameter, it is important to consider the natural range of that parameter's values. For example: normal distributions are commonly used for data that can range from negative to positive infinity (because that is the range of values a normal distribution can take). Beta distributions are often chosen for data between 0 and 1 (the domain of the beta distribution). Similarly, gamma distributions are a natural fit for non-negative continuous data (from 0 to positive infinity) due to their own domain. Gamma distributions are popular across various fields because of this compatibility. However, copula parameters have different ranges depending on the specific copula function used (like Frank, Gumbel, or Clayton). These ranges can vary significantly.

Since there's no single "same" range for copula parameters across different functions, the text suggests using a uniform distribution as a non-informative prior. A uniform distribution assigns equal weight to all values within a specified range, making it a reasonable choice when we have little prior knowledge about the specific parameter value within its valid range for a particular copula function.

Since the unknown model parameters, a_1 and a_2, are independent and follow gamma (G) distributions, we use the Bayesian estimate method, while the copula parameter θ has a non-informative prior.

Regarding the squared error loss function (SELF), Bayesian estimates of the unknown parameters are obtained as $\varphi = (a_1, a_2, \theta)$. The parameters a_1 and a_2 have independent gamma distribution while the copula parameter has uniform distribution as a non-informative prior:

$$\pi_j(\varphi) \propto \varphi_j^{-\alpha_j - 1} e^{-\beta_j \varphi_j}, \quad \alpha_j, \beta_j > 0, j = 1, 2. \tag{47}$$

Then, the joint prior distribution is

$$\pi(\varphi) \propto a_1^{-\alpha_1 - 1} a_2^{-\alpha_2 - 1} e^{-(\beta_1 a_1 + \beta_2 a_2)}, \quad \alpha_j, \beta_j > 0, j = 1, 2, \;\; \pi(\varphi_3) \propto 1. \tag{48}$$

This part explains how to set up informative priors for the model's hyperparameters. Informative priors incorporate some existing knowledge about the parameters. Here, they are obtained using the values we already found for the regular parameters (a_1 and a_2) using MLE. We have multiple samples (k) from the bivariate XL distribution based on different copula functions. For each sample ($j = 1, 2, \ldots, k$), we take the MLEs of a_1 and a_2, and match their average and spread (mean and variance) to those of pre-chosen gamma distributions. These gamma distributions will act as our informative priors. By forcing these alignments, we obtain the following equations:

$$\frac{1}{k}\sum_{j=1}^{k} \hat{a_1}^j = \frac{\alpha_1}{\beta_1} \quad , \quad \frac{1}{k-1}\sum_{j=1}^{k}\left(\hat{a_1}^j - \frac{1}{k}\sum_{j=1}^{k}\hat{a_1}^j\right)^2 = \frac{\alpha_1}{\beta_1^2},$$
$$\frac{1}{k}\sum_{j=1}^{k} \hat{a_2}^j = \frac{\alpha_2}{\beta_2} \quad \text{and} \quad \frac{1}{k-1}\sum_{j=1}^{k}\left(\hat{a_2}^j - \frac{1}{k}\sum_{j=1}^{k}\hat{a_2}^j\right)^2 = \frac{\alpha_2}{\beta_2^2}.$$

By solving the specified set of equations, the inferred hyperparameters can be expressed as follows:

$$\alpha_1 = \frac{\left(\frac{1}{k}\sum_{j=1}^{k}\hat{a_1}^j\right)^2}{\frac{1}{k-1}\sum_{j=1}^{k}\left(\hat{a_1}^j - \frac{1}{k}\sum_{j=1}^{k}\hat{a_1}^j\right)^2}, \qquad \beta_1 = \frac{\frac{1}{k}\sum_{j=1}^{k}\hat{a_1}^j}{\frac{1}{k-1}\sum_{j=1}^{k}\left(\hat{a_1}^j - \frac{1}{k}\sum_{j=1}^{k}\hat{a_1}^j\right)^2}$$
$$\alpha_2 = \frac{\left(\frac{1}{k}\sum_{j=1}^{k}\hat{a_2}^j\right)^2}{\frac{1}{k-1}\sum_{j=1}^{k}\left(\hat{a_2}^j - \frac{1}{k}\sum_{j=1}^{k}\hat{a_2}^j\right)^2}, \qquad \beta_2 = \frac{\frac{1}{k}\sum_{j=1}^{k}\hat{a_2}^j}{\frac{1}{k-1}\sum_{j=1}^{k}\left(\hat{a_2}^j - \frac{1}{k}\sum_{j=1}^{k}\hat{a_2}^j\right)^2}. \tag{49}$$

The joint posterior distribution of a_1 and a_2 is represented by the notation $pi^*(a_1, a_2, \theta | data)$. The likelihood function and the priors can be combined to produce, which can be expressed as follows:

$$\pi^*(a_1, a_2, \theta|\text{data}) \propto a_1^{-\alpha_1-1} a_2^{-\alpha_2-1} e^{-(\beta_1 a_1 + \beta_2 a_2)} \left[\left(1 + \frac{a_1 x_{r:n}}{(1+a_1)^2}\right) e^{-a_1 x_{r:n}}\right]^{n-r} \times$$
$$\prod_{i=1}^{n} \frac{a_1^2 (2 + a_1 + x_{i:n}) e^{-a_1 x_{i:n}}}{(1+a_1)^2} \frac{a_2^2 \left(2 + a_2 + y_{[i:n]}\right) e^{-a_2 y_{[i:n]}}}{(1+a_2)^2} c(F(x_{i:n}), F(y_{[i:n]}), \psi). \tag{50}$$

The term $c(F(x_{i:n}), F(y_{[i:n]}))$ denotes the density copula function of the order statistics of the variables X and Y for the Frank copula, Gumbel copula, and Clayton copula in the context of the XL distribution.

Because it can be used to characterize overestimation and underestimation in the research, the loss function is essential to Bayesian analysis. There are two common loss functions: symmetric loss and asymmetric loss. Overestimation and underestimation are treated equally by the symmetric loss function. The square error (SE) loss function is one of the most well-liked symmetric loss functions. The posterior mean is typically accepted to be the Bayes estimate for the SE loss function. As a result, the following are the parameter Bayes estimators based on SE loss functions:

$$\int_0^\infty \int_0^\infty \int_0^\infty a_1\, a_2\, \theta\, \pi^*(a_1, a_2, \theta \mid \text{data})\, da_1\, da_2\, d\theta. \tag{51}$$

It is clear that the complicated forms of the integration make it difficult to analytically construct the Bayes estimators using Equation (51). We recommend utilizing the MCMC technique in this situation to obtain the Bayes estimates of the unknown parameters and the accompanying HPD credible ranges. Before using the MCMC technique, we must first determine the entire conditional distributions. Given Equation (50), it is possible to determine the required full conditional distributions as follows:

$$\pi^*(a_1|a_2, \theta, \text{data}) \propto a_1^{-\alpha_1-1} e^{-\beta_1 a_1} \left[\left(1 + \frac{a_1 x_{r:n}}{(1+a_1)^2}\right) e^{-a_1 x_{r:n}}\right]^{n-r} \times$$
$$\prod_{i=1}^{n} \frac{a_1^2 (2 + a_1 + x_{i:n}) e^{-a_1 x_{i:n}}}{(1+a_1)^2} c(F(x_{i:n}), F(y_{[i:n]}), \psi), \tag{52}$$

$$\pi^*(a_2|a_1, \theta, \text{data}) \propto a_2^{-\alpha_2-1} e^{-\beta_2 a_2} \prod_{i=1}^{n} \frac{a_2^2 \left(2 + a_2 + y_{[i:n]}\right) e^{-a_2 y_{[i:n]}}}{(1+a_2)^2} c(F(x_{i:n}), F(y_{[i:n]}), \psi). \tag{53}$$

$$\pi^*(\theta|a_1, a_2, \text{data}) \propto \prod_{i=1}^{n} c(F(x_{i:n}), F(y_{[i:n]}), \psi). \tag{54}$$

Despite the fact that we have the complete conditional distributions for each parameter, it is challenging to directly pick samples from them because they lack a defined form.

Therefore, we create samples from these distributions using the Metropolis–Hasting (MH) algorithm. We use the normal distribution to be the proposal distribution for the MH sampling in order to calculate the Bayesian estimates and the HPD credible ranges.

5. Confidence Intervals

In this section, we explore two methods for establishing confidence intervals (CIs) for the unspecified parameters of the BXL distribution. The first method is the asymptotic confidence interval (ACI), which relies on the asymptotic properties of the estimators. The second method is credible confidence intervals (CCI) based on the highest probability density (HPD) for Bayesian estimation. These approaches are specifically used for estimating the parameters a_1, a_2, and θ. The ACI and CCI provide different techniques for estimating the confidence intervals, and their details will be discussed in this section.

5.1. Asymptotic Confidence Intervals

A frequently employed technique for establishing confidence intervals for parameters relies on the asymptotic normality of the MLE. This method involves utilizing the Fisher information matrix, represented as $I(\varphi)$, which stems from the negative second derivatives of the natural logarithm of the likelihood function assessed at the estimated parameter values $\hat{\varphi} = (\hat{a}_1, \hat{a}_2, \hat{\theta})$. The asymptotic variance–covariance matrix of the parameter vector φ can be formulated as follows:

$$I(\hat{\varphi}) = -\begin{bmatrix} I_{\hat{a}_1\hat{a}_1} & & \\ I_{\hat{a}_2\hat{a}_1} & I_{\hat{a}_2\hat{a}_2} & \\ I_{\hat{\theta}\hat{a}_1} & I_{\hat{\theta}\hat{a}_2} & I_{\hat{\theta}\hat{\theta}} \end{bmatrix}. \quad (55)$$

The variance–covariance matrix of the estimated parameters, denoted as $V(\hat{\varphi})$, is calculated as the inverse of the Fisher information matrix, $I^{-1}(\hat{\varphi})$. CIS for the parameter vector φ can be formulated by leveraging the asymptotic normality of the MLE. Specifically, a $100(1-\alpha)\%$ CIS for each parameter can be computed in the following manner:

For a_1: $\hat{a}_1 \pm Z_{\frac{\alpha}{2}} \sqrt{V_{\hat{a}_1\hat{a}_1}}$.

For a_2: $\hat{a}_2 \pm Z_{\frac{\alpha}{2}} \sqrt{V_{\hat{a}_2\hat{a}_2}}$.

For θ: $\hat{\theta} \pm Z_{\frac{\alpha}{2}} \sqrt{V_{\hat{\theta}\hat{\theta}}}$.

Here, $Z_{\frac{\alpha}{2}}$ represents the percentile of the standard normal distribution with a right tail probability of $\frac{\alpha}{2}$. The values $V_{\hat{a}_1\hat{a}_1}$, $V_{\hat{a}_2\hat{a}_2}$, and $V_{\hat{\theta}\hat{\theta}}$ correspond to the diagonal elements of the variance–covariance matrix $V(\hat{\varphi})$.

5.2. Credible Confidence Intervals

According to Chen and Shao [52], Bayes credible intervals for the parameter φ are obtained as follows:

1. Arrange $\varphi_l^{\langle j \rangle}$ as $a_1^{[1]}$, $a_1^{[2]}$, ..., $a_1^{[L]}$, $a_2^{[1]}$, $a_2^{[2]}$, ..., $a_2^{[L]}$, and $\theta^{[1]}$, $\theta^{[2]}$, ..., $\theta^{[L]}$, where L is the length of the generated simulation.
2. The $100(1-\alpha)\%$ symmetric credible intervals are as follows: $\left(\tilde{a}_1^{[L\frac{\alpha}{2}]}, \tilde{a}_1^{[L(1-\frac{\alpha}{2})]}\right)$, $\left(\tilde{a}_2^{[L\frac{\alpha}{2}]}, \tilde{a}_2^{[L(1-\frac{\alpha}{2})]}\right)$, $\left(\tilde{\theta}^{[L\frac{\alpha}{2}]}, \tilde{\theta}^{[L(1-\frac{\alpha}{2})]}\right)$.

6. Simulation Study

In this section, a Monte Carlo simulation is conducted to compare bivariate XL distribution under different copulas. The simulation involves point estimation, interval estimation, and dependence measures. To generate random variables, we utilize the technique outlined by Nelsen [2] for sampling from a designated joint bivariate distribution through the conditional distribution method. We will explore the generation of bivariate XL distribution under various copulas using both the conditional distribution method and iterative algorithms.

Firstly, we generate random variables Q and W independently from a uniform distribution with a lower bound of 0 and an upper bound of 1.

Secondly, we derive the conditional distribution of X given Y for each distribution using copulas:

In the Frank copula,

$$C(v|u) = \frac{-e^{-\theta u}\left(e^{-\theta v} - 1\right)}{\left(e^{-\theta u} - 1\right)\left(e^{-\theta v} - 1\right) + e^{-\theta} - 1}, \tag{56}$$

and

$$C(u|v) = \frac{-e^{-\theta v}\left(e^{-\theta u} - 1\right)}{\left(e^{-\theta u} - 1\right)\left(e^{-\theta v} - 1\right) + e^{-\theta} - 1}. \tag{57}$$

In the Gumbel copula,

$$C(v|u) = \frac{1}{u} e^{\left\{-\left[(-\ln u)^{\theta} + (-\ln v)^{\theta}\right]^{\frac{1}{\theta}}\right\}} \left[(-\ln u)^{\theta} + (-\ln v)^{\theta}\right]^{\frac{1}{\theta}-1} (-\ln u)^{\theta-1}, \tag{58}$$

and

$$C(u|v) = \frac{1}{v} e^{\left\{-\left[(-\ln u)^{\theta} + (-\ln v)^{\theta}\right]^{\frac{1}{\theta}}\right\}} \left[(-\ln u)^{\theta} + (-\ln v)^{\theta}\right]^{\frac{1}{\theta}-1} (-\ln v)^{\theta-1}. \tag{59}$$

In the Clayton copula,

$$C(v|u) = \left\{u^{-\theta} + v^{-\theta} - 1\right\}^{-\frac{1}{\theta}-1} u^{-\theta-1}, \tag{60}$$

$$C(u|v) = \left\{u^{-\theta} + v^{-\theta} - 1\right\}^{-\frac{1}{\theta}-1} v^{-\theta-1}. \tag{61}$$

Here u represents the cumulative distribution function (cdf) of the XL distribution with parameter a_1, while v represents the cdf of another XL distribution with parameter a_2. When the real value of the parameters (a_1, a_2, θ) is considered as $a_1 = 2, a_2 = 2, \theta = 1.5$, $a_1 = 0.5, a_2 = 1.5, \theta = 2$, $a_1 = 3, a_2 = 1.5, \theta = 2$, $a_1 = 0.75, a_2 = 1.5, \theta = 1.2$, we replicated Type II censored samples 5,000 times using various n (number of test items) and r (number of effective test items) options, for different choices of n such as n = 35 and 100, as well as different censored samples such as r = 25, 32, 35 for n = 35 and r = 70, 85, 100 for n = 100. The number of iterations (sample) is 5000 iteration in the simulation study. The acquired mean squared error (MSE) and mean bias (MB) values are used to compare the likelihood and Bayes estimates of a_1, a_2, and θ.

R software (version 4.3.0) was used to find the MLEs and 95% CIs for unknown parameters by the Newton Raphson iterative algorithm. This likely involved installing specific packages called "maxLik" and "AdequacyModel" in R, with the "maxlike" and "goodness.fit" functions, respectively. To estimate the unknown parameters using a Bayesian approach by the Metropolis–Hasting iterative algorithm, the "coda" and "HDInterval" packages in R (also version 4.3.0) were used. This method involved simulating 12,000 samples using MCMC to obtain Bayesian point estimates and their HPD intervals. It is important to note that the initial 2000 simulations were discarded (considered "burn-in") to avoid including them in the analysis since they might not be representative of the final converged state. Tables 1–4 show the obtained results based on the complete and censored samples (if $r = n$ this is a complete sample and if $r < n$ then these results are censored samples).

Table 1. MB, MSE, and LACI for MLE and Bayesian based on TIICS for Frank, Clayton, and Gumbel copula: $a_1 = 2, a_2 = 2, \theta = 1.5$.

			MLE												Bayesian								
			Frank				Clayton				Gumbel				Frank			Clayton			Gumbel		
n	***r***		**MB**	**MSE**	**LACI**	**CP**	**MB**	**MSE**	**LACI**	**CP**	**MB**	**MSE**	**LACI**	**CP**	**MB**	**MSE**	**LCCI**	**MB**	**MSE**	**LCCI**	**MB**	**MSE**	**LCCI**
35	25	a_1	0.4123	0.2776	1.2865	93.40%	−0.0621	0.0722	1.0253	94.50%	1.9413	4.4589	3.2589	94.65%	0.0184	0.0020	0.1555	−0.0206	0.0025	0.1749	0.0328	0.0027	0.1502
		a_2	0.4081	0.2725	1.2766	95.13%	−0.0666	0.0682	0.9905	94.67%	2.7251	8.3765	3.8232	94.60%	0.0227	0.0011	0.0859	0.0097	0.0009	0.1023	0.1628	0.0275	0.1144
		θ	1.7567	4.4440	4.5706	94.17%	−0.7839	1.3946	3.4642	94.13%	−0.0104	0.0423	0.8056	94.00%	0.2274	0.4661	2.3989	−0.6778	0.6723	1.4376	0.3190	0.2279	1.2894
		τ	0.1579	0.0339	0.3707	94.00%	−0.2132	0.0732	0.6530	94.65%	−0.2841	0.0898	0.3733	94.75%	0.0214	0.0046	0.1555	−0.1536	0.0341	0.1749	−0.1707	0.0414	0.1502
		ρ	0.2208	0.0652	0.5026	95.33%	−0.2925	0.1355	0.8767	94.77%	−0.3387	0.1311	0.5026	94.96%	0.0302	0.0095	0.0859	−0.2031	0.0600	0.1023	−0.1963	0.0574	0.1144
	32	a_1	0.3957	0.2043	0.8563	94.67%	−0.0809	0.0384	0.6998	94.67%	1.8775	3.9174	2.4561	95.33%	0.0148	0.0011	0.1106	−0.0003	0.0012	0.1341	0.0233	0.0016	0.1162
		a_2	0.3978	0.2126	0.9146	95.00%	−0.0809	0.0430	0.7491	96.33%	2.6315	7.4247	2.7727	96.97%	0.0124	0.0005	0.0730	0.0023	0.0005	0.0839	0.1001	0.0106	0.0940
		θ	1.6228	3.2458	3.0692	95.67%	−0.9679	1.1392	1.7640	95.00%	−0.0522	0.0261	0.5996	95.30%	0.0347	0.1410	1.3746	−0.3565	0.2569	1.2752	0.1900	0.0918	0.8801
		τ	0.1510	0.0272	0.2613	94.87%	−0.2386	0.0707	0.4602	94.97%	−0.2699	0.0820	0.3157	97.33%	0.0030	0.0015	0.1106	−0.0730	0.0106	0.1341	−0.1620	0.0409	0.1162
		ρ	0.2134	0.0539	0.3584	95.30%	−0.3226	0.1306	0.6386	96.73%	−0.3577	0.1240	0.4329	96.67%	0.0041	0.0032	0.0730	−0.0939	0.0176	0.0839	−0.1823	0.0547	0.0940
	35	a_1	0.3775	0.1650	0.5891	95.97%	−0.0951	0.0243	0.4848	96.33%	1.7163	3.2932	2.3122	94.00%	0.0147	0.0010	0.1058	−0.0002	0.0011	0.1306	0.0089	0.0014	0.1055
		a_2	0.3783	0.1682	0.6220	95.17%	−0.0937	0.0263	0.5198	95.93%	2.4259	6.3686	2.7275	95.67%	0.0069	0.0004	0.0729	0.0022	0.0004	0.1021	0.0606	0.0044	0.1078
		θ	1.5751	2.7628	2.0824	95.50%	−1.0398	1.1418	0.9664	96.97%	−0.0669	0.0175	0.4480	93.67%	0.0207	0.0284	0.6143	−0.0736	0.0402	0.6912	0.0766	0.0283	0.5833
		τ	0.1489	0.0243	0.1795	96.67%	−0.2492	0.0681	0.3035	96.93%	−0.3025	0.0810	0.2356	94.00%	0.0020	0.0003	0.1576	−0.0140	0.0012	0.1358	−0.1403	0.0406	0.1549
		ρ	0.2116	0.0487	0.2467	96.67%	−0.3347	0.1242	0.4329	96.83%	−0.3261	0.1137	0.3198	95.67%	0.0029	0.0007	0.0920	−0.0175	0.0019	0.1021	−0.1796	0.0485	0.1078
100	70	a_1	0.2110	0.1748	1.4176	95.33%	−0.1776	0.1468	1.3334	95.00%	1.7452	5.9998	6.7515	94.33%	0.0306	0.0032	0.1823	−0.0589	0.0061	0.2081	0.0930	0.0113	0.2049
		a_2	0.2134	0.1755	1.4161	95.27%	−0.1889	0.1340	1.2315	94.33%	2.7704	7.2951	6.3582	94.37%	0.0575	0.0041	0.1050	0.0283	0.0019	0.1222	0.4138	0.1726	0.1443
		θ	1.4596	2.2247	2.0362	95.67%	−0.8674	1.5850	2.0798	95.33%	−0.2267	3.0648	6.8188	94.80%	0.3290	0.4740	2.1841	−1.0054	1.0958	0.7988	0.1750	0.0755	0.8011
		τ	0.1420	0.0325	0.2787	96.33%	−0.5725	0.3524	0.6168	95.20%	−0.2031	0.0932	0.8953	96.33%	0.0319	0.0047	0.1823	−0.2390	0.0629	0.2081	−0.2066	0.0486	0.2049
		ρ	0.2152	0.0647	0.3206	95.47%	−0.7475	0.6086	0.8768	95.33%	−0.2124	0.1210	1.0819	94.67%	0.0457	0.0099	0.1050	−0.3203	0.1137	0.1222	−0.2374	0.0660	0.1443
	85	a_1	0.2081	0.0955	0.8997	96.33%	−0.2036	0.0714	0.6812	96.33%	1.5159	4.6263	6.0098	96.00%	0.0280	0.0019	0.1292	−0.0278	0.0021	0.1410	0.0565	0.0046	0.1400
		a_2	0.1674	0.0828	0.9217	96.43%	−0.2107	0.0937	0.8749	96.33%	2.3743	6.7096	4.0787	96.53%	0.0338	0.0016	0.0824	0.0054	0.0006	0.0883	0.2447	0.0607	0.1174
		θ	1.3193	2.0014	2.0155	95.60%	−0.9251	1.4608	0.8919	96.33%	−0.8549	1.6254	3.7251	97.00%	0.1237	0.1395	1.3874	−0.6480	0.5285	1.1496	0.1623	0.0636	0.7992
		τ	0.1399	0.0304	0.1906	96.63%	−0.5747	0.3404	0.3955	96.53%	−0.1933	0.0918	1.0576	96.60%	0.0122	0.0014	0.1292	−0.1387	0.0252	0.1410	−0.1863	0.0396	0.1400
		ρ	0.2052	0.0606	0.2194	96.73%	−0.7490	0.5830	0.5840	96.73%	−0.2362	0.1024	1.0033	96.73%	0.0176	0.0031	0.0824	−0.1808	0.0434	0.0883	−0.2115	0.0525	0.1174
	100	a_1	0.2114	0.0578	0.4678	96.90%	−0.1953	0.0431	0.2873	96.67%	0.8209	1.3707	3.4178	97.50%	0.0278	0.0018	0.1636	−0.0059	0.0019	0.1667	0.0409	0.0035	0.1572
		a_2	0.1840	0.0456	0.4433	97.67%	−0.2143	0.0595	0.4767	96.85%	1.6764	3.5865	3.6074	96.83%	0.0207	0.0013	0.1138	−0.0016	0.0005	0.0801	0.1621	0.0273	0.1282
		θ	1.0718	1.9890	2.0908	97.67%	−0.7990	1.3952	0.4296	97.67%	−0.7201	1.5963	2.9560	97.67%	0.0623	0.0355	0.6594	−0.1877	0.0672	0.6874	0.1471	0.0404	0.4827
		τ	0.1893	0.0302	0.1523	97.60%	−0.5598	0.3360	0.2113	97.67%	−0.1797	0.0821	0.9687	97.85%	0.0063	0.0004	0.1636	−0.0342	0.0023	0.1667	−0.1721	0.0347	0.1357
		ρ	0.2409	0.0598	0.1722	97.97%	−0.7083	0.5619	0.3162	96.95%	−0.2144	0.1003	0.9524	97.33%	0.0092	0.0008	0.1138	−0.0425	0.0035	0.1017	−0.2042	0.0506	0.1028

Table 2. MB, MSE, and LACI for MLE and Bayesian based on TIICS for Frank, Clayton, and Gumbel copula: $a_1 = 0.5, a_2 = 1.5, \theta = 2$.

			MLE												**Bayesian**								
			Frank				**Clayton**				**Gumbel**				**Frank**			**Clayton**			**Gumbel**		
n	***r***		**MB**	**MSE**	**LACI**	**CP**	**MB**	**MSE**	**LACI**	**CP**	**MB**	**MSE**	**LACI**	**CP**	**MB**	**MSE**	**LACI**	**MB**	**MSE**	**LACI**	**MB**	**MSE**	**LACI**
		a_1	0.0657	0.0122	0.3487	93.50%	−0.0717	0.0100	0.2726	94.50%	0.2022	0.1105	1.0346	94.60%	0.0335	0.0043	0.2303	−0.0315	0.0040	0.2162	0.0634	0.0106	0.2922
		a_2	0.4360	0.4570	2.0262	95.03%	0.2281	0.1912	1.4632	95.33%	5.3051	40.2317	13.6354	94.56%	0.0356	0.0019	0.0966	−0.0001	0.0011	0.1216	0.2031	0.0430	0.1593
	25	θ	1.6094	6.5608	7.8151	93.33%	−2.0317	4.2879	1.5700	94.33%	0.5845	3.5907	7.0693	94.58%	0.0912	0.4445	2.3307	−1.1366	1.4619	1.5329	0.1768	0.3584	2.0608
		τ	0.1218	0.0388	0.6068	94.50%	−0.5457	0.3213	0.6016	95.00%	−0.1056	0.0225	0.4185	95.16%	0.0066	0.0041	0.2303	−0.2110	0.0523	0.2162	−0.1600	0.0440	0.2922
		ρ	0.1605	0.0695	0.8208	95.33%	−0.7507	0.6124	0.8663	95.43%	−0.1093	0.0212	0.3763	95.04%	0.0083	0.0083	0.0966	−0.2671	0.0848	0.1216	−0.1708	0.0543	0.1593
		a_1	0.0704	0.0104	0.2888	94.67%	−0.0491	0.0058	0.2299	95.67%	0.1948	0.1071	0.9135	94.83%	0.0318	0.0022	0.1321	−0.0081	0.0017	0.1616	0.0489	0.0052	0.2047
		a_2	0.3162	0.1856	1.1477	95.00%	−0.0327	0.0591	0.9444	96.00%	3.0779	15.5639	9.6790	95.33%	0.0203	0.0009	0.0836	−0.0011	0.0006	0.0900	0.1320	0.0183	0.1106
35	32	θ	1.8363	5.3256	5.4820	96.33%	−1.8465	3.6026	1.7229	95.63%	0.2890	0.7460	3.1923	95.30%	0.1236	0.1574	1.4883	−0.6363	0.5575	1.4606	0.1433	0.1628	1.4535
		τ	0.1503	0.0337	0.4129	94.97%	−0.4586	0.2344	0.6086	95.77%	−0.0993	0.0213	0.4065	95.83%	0.0114	0.0015	0.1321	−0.1023	0.0150	0.1616	−0.1485	0.0300	0.2047
		ρ	0.2034	0.0609	0.5474	96.25%	−0.6230	0.4381	0.8763	96.90%	−0.1465	0.0204	0.3507	96.53%	0.0160	0.0030	0.0836	−0.1244	0.0225	0.0900	−0.1513	0.0333	0.1106
		a_1	0.0746	0.0101	0.2774	96.44%	−0.0416	0.0046	0.2056	96.14%	0.1835	0.0917	0.8404	96.00%	0.0303	0.0020	0.1309	0.0007	0.0016	0.1590	0.0275	0.0038	0.2283
		a_2	0.2568	0.1245	0.9489	96.47%	−0.1185	0.0554	0.7976	95.90%	1.9293	4.6206	3.7178	97.33%	0.0131	0.0008	0.0750	−0.0008	0.0006	0.0880	0.0798	0.0072	0.1121
	35	θ	1.6230	4.5502	4.9269	96.53%	−1.7261	3.2100	1.8835	97.17%	−0.0428	0.3373	2.2716	96.67%	0.0222	0.0288	0.6094	−0.1449	0.0551	0.7166	−0.0085	0.0388	0.7459
		τ	0.1482	0.0309	0.3459	96.57%	−0.4104	0.1920	0.6025	97.40%	−0.0818	0.0207	0.4006	96.90%	0.0020	0.0003	0.1391	−0.0200	0.0010	0.1901	−0.1390	0.0298	0.2003
		ρ	0.2005	0.0601	0.4459	96.75%	−0.5531	0.3541	0.8607	96.50%	−0.1394	0.0209	0.3187	97.53%	0.0029	0.0006	0.1075	−0.0233	0.0014	0.1088	−0.1376	0.0315	0.1102
		a_1	0.0581	0.0061	0.2031	94.57%	−0.0827	0.0081	0.1396	94.38%	0.1343	0.0669	0.8670	95.33%	0.0344	0.0025	0.1308	−0.0589	0.0043	0.1099	0.1042	0.0137	0.2101
		a_2	0.3780	0.2300	1.1579	94.53%	0.1893	0.0829	0.8507	94.33%	5.5661	38.6952	10.8930	95.62%	0.0800	0.0074	0.1250	0.0145	0.0018	0.1558	0.4640	0.2183	0.2119
	70	θ	1.3510	3.1166	4.4568	94.67%	−2.0870	4.3800	0.6126	94.33%	0.1664	0.2302	1.7651	94.33%	0.3882	0.5049	2.1920	−1.5369	2.3908	0.5395	0.3038	0.4412	2.1553
		τ	0.1153	0.0218	0.3614	94.70%	−0.5518	0.3107	0.3091	94.83%	−0.1412	0.0241	0.2527	95.65%	0.0356	0.0044	0.1308	−0.3154	0.1020	0.1099	−0.1299	0.0310	0.2101
		ρ	0.1582	0.0406	0.4894	95.33%	−0.7598	0.5909	0.4564	94.93%	−0.1405	0.0237	0.2482	96.17%	0.0500	0.0088	0.1250	−0.4103	0.1734	0.1558	−0.1357	0.0359	0.2119
		a_1	0.0533	0.0047	0.1706	96.00%	−0.0691	0.0060	0.1370	95.67%	0.2521	0.1329	1.0328	95.80%	0.0344	0.0022	0.1245	−0.0337	0.0021	0.1238	0.0889	0.0100	0.1804
		a_2	0.2893	0.1231	0.7780	95.67%	−0.0119	0.0273	0.6465	95.67%	3.4695	17.6470	9.2887	96.00%	0.0475	0.0028	0.0868	−0.0099	0.0009	0.1086	0.3047	0.0945	0.1472
100	85	θ	1.0629	2.9373	3.3302	95.67%	−1.9754	3.9344	0.7036	95.46%	0.1615	0.1975	1.6241	96.67%	0.1841	0.1646	1.3594	−1.1575	1.3908	0.8309	0.2563	0.1882	1.4448
		τ	0.1412	0.0244	0.2610	96.50%	−0.4952	0.2521	0.3268	96.67%	−0.1437	0.0269	0.3098	96.80%	0.0173	0.0015	0.1245	−0.2078	0.0460	0.1238	−0.1218	0.0212	0.1804
		ρ	0.1947	0.0458	0.3479	95.97%	−0.6757	0.4718	0.4850	96.67%	−0.1447	0.0284	0.3389	96.80%	0.0245	0.0031	0.0868	−0.2604	0.0728	0.1086	−0.1210	0.0229	0.1472
		a_1	0.0683	0.0062	0.1557	96.00%	−0.0545	0.0044	0.1462	95.80%	0.3316	0.1364	0.6380	96.84%	0.0304	0.0020	0.0961	−0.0144	0.0013	0.1302	0.0481	0.0037	0.1432
		a_2	0.2424	0.0773	0.5334	96.84%	−0.1043	0.0273	0.5083	96.95%	1.7636	3.2677	1.5560	97.37%	0.0239	0.0012	0.0998	−0.0076	0.0008	0.1002	0.2149	0.0473	0.1190
	100	θ	2.0631	4.7039	2.6242	96.67%	−1.8331	3.3930	0.7112	96.93%	−0.2561	0.2218	1.5501	97.16%	0.0386	0.0296	0.6280	−0.3634	0.1592	0.6119	0.0044	0.0283	0.5686
		τ	0.1762	0.0334	0.1917	96.90%	−0.4290	0.1894	0.2871	97.50%	−0.2730	0.0977	0.5971	97.43%	0.0037	0.0003	0.0961	−0.0511	0.0032	0.1302	−0.1169	0.0203	0.1432
		ρ	0.2416	0.0624	0.2483	97.43%	−0.5773	0.3450	0.4239	97.85%	−0.3049	0.1330	0.7846	97.47%	0.0052	0.0006	0.0998	−0.0599	0.0045	0.1178	−0.1070	0.0203	0.1019

Table 3. MB, MSE, and LACI for MLE and Bayesian based on TIICS for Frank, Clayton, and Gumbel copula: $a_1 = 3, a_2 = 1.5, \theta = 2$.

			MLE												**Bayesian**								
			Frank				**Clayton**				**Gumbel**				**Frank**			**Clayton**			**Gumbel**		
n	***r***		**MB**	**MSE**	**LACI**	**CP**	**MB**	**MSE**	**LACI**	**CP**	**MB**	**MSE**	**LACI**	**CP**	**MB**	**MSE**	**LACI**	**MB**	**MSE**	**LACI**	**MB**	**MSE**	**LACI**
35	25	a_1	0.5221	0.7851	2.8077	94.50%	−0.1879	0.5412	2.7895	95.00%	0.5928	1.7817	4.6904	93.67%	0.0119	0.0015	0.1448	−0.0185	0.0019	0.1550	0.0295	0.0030	0.1864
		a_2	0.4323	0.4536	2.0254	94.53%	0.3207	0.2885	1.6897	92.67%	5.6837	33.2247	12.2878	94.67%	0.0348	0.0019	0.0945	0.0070	0.0015	0.1481	0.1955	0.0410	0.2119
		θ	1.6136	6.5899	7.8304	94.57%	−1.9332	4.0726	2.2718	95.17%	0.2386	1.1762	4.1493	94.63%	0.2029	0.4824	2.5531	−1.1036	1.4193	1.6075	0.1907	0.4039	2.1943
		τ	0.1221	0.0389	0.6072	95.10%	−0.5112	0.2903	0.6672	95.60%	−0.1448	0.0266	0.2931	95.37%	0.0174	0.0043	0.1448	−0.2045	0.0504	0.1550	−0.1600	0.0460	0.1864
		ρ	0.1607	0.0697	0.8210	95.33%	−0.7017	0.5497	0.9393	95.27%	−0.1460	0.0254	0.2522	96.67%	0.0238	0.0087	0.0945	−0.2588	0.0816	0.1481	−0.1718	0.0580	0.2119
	32	a_1	0.5489	0.6331	2.2593	94.90%	−0.0989	0.3485	2.2826	95.96%	0.5240	1.2976	4.5558	96.38%	0.0091	0.0012	0.1326	−0.0016	0.0011	0.1340	0.0202	0.0016	0.1258
		a_2	0.3140	0.1832	1.1408	95.33%	0.0574	0.1275	1.3821	96.33%	4.1576	29.1054	12.1074	96.67%	0.0192	0.0009	0.0808	0.0035	0.0008	0.1084	0.1193	0.0158	0.1464
		θ	1.8291	5.2954	5.4767	96.20%	−1.7565	3.3600	2.0555	96.00%	0.2932	0.6400	2.9193	96.83%	0.0449	0.1418	1.4456	−0.5896	0.4983	1.4803	0.1396	0.1797	1.5614
		τ	0.1498	0.0335	0.4130	95.40%	−0.4253	0.2054	0.6147	96.15%	−0.1303	0.0260	0.3721	96.80%	0.0037	0.0014	0.1326	−0.0939	0.0132	0.1340	−0.1533	0.0351	0.1258
		ρ	0.2027	0.0606	0.5477	95.73%	−0.5751	0.3804	0.8739	96.56%	−0.1325	0.0247	0.2381	97.67%	0.0049	0.0028	0.0808	−0.1140	0.0198	0.1084	−0.1584	0.0418	0.1464
	35	a_1	0.5089	0.6047	2.1460	96.30%	−0.0970	0.3161	2.1721	96.67%	0.4834	1.0684	3.2157	97.03%	0.0096	0.0011	0.1419	0.0006	0.0010	0.1259	0.0134	0.0012	0.1826
		a_2	0.2576	0.1240	0.9414	96.67%	−0.0624	0.0817	1.0943	96.33%	2.0554	5.7698	4.8752	97.53%	0.0113	0.0009	0.1074	−0.0035	0.0007	0.1001	0.0769	0.0069	0.1271
		θ	1.2153	4.4774	4.9143	96.53%	−1.6553	3.0258	2.0966	96.95%	0.1456	0.4995	2.7123	97.86%	0.0375	0.0306	0.6566	−0.1308	0.0504	0.6467	0.0062	0.0330	0.6777
		τ	0.1381	0.0304	0.3458	96.67%	−0.3864	0.1735	0.6100	97.27%	−0.1273	0.0250	0.3560	98.03%	0.0036	0.0003	0.1419	−0.0181	0.0009	0.1588	−0.1693	0.0309	0.1826
		ρ	0.1824	0.0573	0.4462	96.87%	−0.5189	0.3176	0.8625	97.63%	−0.1287	0.0231	0.2266	98.15%	0.0050	0.0006	0.1074	−0.0211	0.0013	0.1115	−0.1704	0.0320	0.1271
100	70	a_1	0.4451	0.3671	1.6123	94.35%	0.0065	0.4146	2.5252	94.50%	0.1460	0.3288	2.1749	96.20%	0.0202	0.0020	0.1594	−0.0589	0.0053	0.1696	0.1120	0.0160	0.2334
		a_2	0.3722	0.2257	1.1580	94.07%	0.4076	0.2919	1.3908	92.67%	3.1725	15.0972	6.3881	95.40%	0.0760	0.0069	0.1284	0.0236	0.0024	0.1576	0.4073	0.1712	0.2789
		θ	1.3574	3.1455	4.4767	94.00%	−2.0184	4.1005	0.6418	94.67%	0.0132	0.0243	0.6095	95.73%	0.4512	0.6105	2.4706	−1.5099	2.3099	0.5563	0.1614	0.3148	2.0776
		τ	0.1158	0.0220	0.3626	94.63%	−0.5150	0.2704	0.2830	95.00%	−0.1649	0.0276	0.0778	96.30%	0.0412	0.0052	0.1594	−0.3066	0.0964	0.1696	−0.1578	0.0395	0.2334
		ρ	0.1588	0.0409	0.4908	94.80%	−0.7051	0.5084	0.4156	96.00%	−0.1636	0.0271	0.0724	95.67%	0.0578	0.0103	0.1284	−0.3977	0.1629	0.1576	−0.1661	0.0468	0.2789
	85	a_1	0.4004	0.2723	1.3125	95.33%	−0.1111	0.3453	2.2629	95.67%	0.1267	0.3050	4.6857	95.80%	0.0151	0.0011	0.1197	−0.0218	0.0016	0.1248	0.0753	0.0073	0.1483
		a_2	0.2850	0.1202	0.7739	95.67%	0.1475	0.1296	1.2879	96.80%	2.8948	13.0779	8.1607	96.67%	0.0405	0.0021	0.0758	−0.0017	0.0010	0.1175	0.2640	0.0723	0.1947
		θ	1.1629	3.0374	3.2333	94.97%	−1.9203	3.7227	0.7359	96.80%	0.0131	0.0215	0.5886	95.73%	0.2031	0.1522	1.2686	−1.1113	1.3206	1.0180	0.2394	0.2113	1.5752
		τ	0.1041	0.0214	0.2612	95.64%	−0.4685	0.2255	0.3034	95.73%	−0.1570	0.0260	0.1443	95.98%	0.0193	0.0014	0.1197	−0.1987	0.0435	0.1248	−0.1286	0.0250	0.1483
		ρ	0.1495	0.0406	0.3482	96.25%	−0.6361	0.4177	0.4477	97.66%	−0.1556	0.0256	0.1445	97.17%	0.0274	0.0028	0.0758	−0.2489	0.0689	0.1175	−0.1298	0.0281	0.1947
	100	a_1	0.3302	0.2575	1.1818	96.36%	−0.1388	0.2161	1.7400	96.97%	0.1138	0.2589	4.1748	96.03%	0.0141	0.0009	0.1052	−0.0020	0.0015	0.1178	0.0544	0.0053	0.1884
		a_2	0.2429	0.0772	0.5293	96.65%	−0.0702	0.0583	0.9063	96.33%	1.7890	3.4226	1.8480	96.25%	0.0224	0.0011	0.0958	−0.0096	0.0009	0.1183	0.1805	0.0342	0.1616
		θ	1.0507	2.0651	2.6172	97.15%	−1.7893	3.2441	0.8091	97.92%	−0.0118	0.0166	0.5298	97.22%	0.0585	0.0342	0.6971	−0.3571	0.1606	0.6972	0.0987	0.0401	0.6252
		τ	0.0918	0.0203	0.1916	97.26%	−0.4119	0.1758	0.3074	97.27%	−0.1945	0.0252	0.1347	97.20%	0.0056	0.0003	0.1519	−0.0504	0.0033	0.1776	−0.1466	0.0232	0.1884
		ρ	0.1240	0.0362	0.2484	97.35%	−0.5522	0.3181	0.4509	97.63%	−0.1521	0.0250	0.1259	97.53%	0.0080	0.0007	0.0958	−0.0591	0.0046	0.1183	−0.1447	0.0232	0.1616

Table 4. MB, MSE, and LACI for MLE and Bayesian based on TIICS for Frank, Clayton, and Gumbel copula: $a_1 = 0.75, a_2 = 1.5, \theta = 1.2$.

			MLE												**Bayesian**								
			Frank				**Clayton**				**Gumbel**				**Frank**			**Clayton**			**Gumbel**		
n	***r***		**MB**	**MSE**	**LACI**	**CP**	**MB**	**MSE**	**LACI**	**CP**	**MB**	**MSE**	**LACI**	**CP**	**MB**	**MSE**	**LACI**	**MB**	**MSE**	**LACI**	**MB**	**MSE**	**LACI**
35	25	a_1	0.1370	0.0298	0.4113	94.27%	−0.0035	0.0074	0.3365	94.70%	0.4083	0.2039	0.7569	94.57%	0.0662	0.0072	0.1996	0.0072	0.0033	0.2143	0.0788	0.0093	0.2097
		a_2	0.3059	0.1477	0.9119	95.33%	0.0047	0.0346	0.7295	95.33%	1.8100	3.8357	2.9343	94.26%	0.0315	0.0018	0.1032	−0.0033	0.0011	0.1284	0.2316	0.0559	0.1754
		θ	1.4723	3.4928	4.5147	94.60%	−0.5171	0.7885	2.8310	96.00%	0.0839	0.0256	0.5340	94.80%	0.1205	0.4229	2.3276	−0.4696	0.3761	1.3633	0.0809	0.0237	0.5077
		τ	0.1388	0.0295	0.3979	94.57%	−0.1605	0.0510	0.6230	95.27%	−0.3326	0.1169	0.3107	95.67%	0.0110	0.0044	0.1996	−0.1209	0.0235	0.2143	−0.2201	0.0567	0.2097
		ρ	0.1971	0.0589	0.5548	95.67%	−0.2261	0.0977	0.8464	95.67%	−0.4217	0.1901	0.4350	94.96%	0.0154	0.0093	0.1032	−0.1646	0.0434	0.1284	−0.2697	0.0873	0.1754
	32	a_1	0.1323	0.0224	0.2750	96.00%	−0.0094	0.0034	0.2249	96.60%	0.4171	0.1885	0.4724	96.33%	0.0544	0.0042	0.1328	−0.0037	0.0014	0.1483	0.0678	0.0060	0.1327
		a_2	0.2993	0.1176	0.6557	94.67%	−0.0035	0.0176	0.5202	95.67%	1.7279	3.1439	1.5609	96.33%	0.0313	0.0015	0.0874	−0.0029	0.0005	0.0868	0.2273	0.0527	0.1172
		θ	1.3485	2.4187	3.0384	96.00%	−0.6820	0.6147	1.5169	95.84%	0.0587	0.0124	0.3702	95.37%	0.1155	0.1280	1.2541	−0.4038	0.2423	1.0340	0.0478	0.0120	0.4550
		τ	0.1316	0.0224	0.2798	96.60%	−0.1856	0.0463	0.4265	96.93%	−0.3444	0.1022	0.2340	96.76%	0.0118	0.0014	0.1328	−0.0972	0.0143	0.1483	−0.2042	0.0506	0.1327
		ρ	0.1890	0.0458	0.3945	97.47%	−0.2563	0.0889	0.5979	96.67%	−0.4369	0.1810	0.3329	96.53%	0.0172	0.0031	0.0874	−0.1304	0.0259	0.0868	−0.2498	0.0810	0.1172
	35	a_1	0.1269	0.0184	0.1893	96.53%	−0.0138	0.0018	0.1584	96.67%	0.4086	0.1744	0.3376	97.43%	0.0463	0.0040	0.1206	−0.0029	0.0009	0.1107	0.0583	0.0057	0.1092
		a_2	0.2858	0.0946	0.4462	96.95%	−0.0135	0.0085	0.3581	97.53%	1.6573	2.8359	1.1716	97.67%	0.0304	0.0012	0.0884	0.0024	0.0004	0.0984	0.2083	0.0481	0.1186
		θ	1.3067	1.9841	2.0629	97.53%	−0.7355	0.5931	0.8959	97.47%	0.0519	0.0072	0.2637	97.86%	0.0738	0.0331	0.6684	−0.2491	0.0912	0.7060	0.0318	0.0065	0.3507
		τ	0.1296	0.0192	0.1921	97.85%	−0.1932	0.0426	0.2842	97.47%	−0.3465	0.1019	0.1685	97.94%	0.0077	0.0004	0.1206	−0.0551	0.0047	0.1107	−0.1928	0.0482	0.1092
		ρ	0.1871	0.0398	0.2715	97.47%	−0.2647	0.0808	0.4060	97.95%	−0.4390	0.1796	0.2406	97.87%	0.0114	0.0008	0.0884	−0.0723	0.0081	0.0984	−0.2341	0.0812	0.1186
100	70	a_1	0.3113	0.1522	0.9224	94.70%	0.0106	0.0346	0.7287	94.07%	0.8883	1.0246	1.9036	94.53%	0.0460	0.0039	0.1560	0.0016	0.0019	0.1658	0.0657	0.0061	0.1545
		a_2	0.3076	0.1486	0.9113	95.33%	0.0096	0.0364	0.7476	94.00%	1.8685	4.1825	3.2609	94.53%	0.0305	0.0017	0.1143	0.0007	0.0009	0.1245	0.2257	0.0531	0.1649
		θ	1.4637	3.4694	4.5181	96.00%	−0.5265	0.7745	2.7657	94.63%	0.0810	0.0245	0.5253	94.83%	0.1704	0.3507	1.9366	−0.4306	0.3363	1.4122	0.0732	0.0201	0.4815
		τ	0.1380	0.0294	0.3990	95.67%	−0.1617	0.0507	0.6142	94.80%	−0.3342	0.1178	0.3076	95.33%	0.0167	0.0037	0.1560	−0.1102	0.0210	0.1658	−0.2150	0.0544	0.1545
		ρ	0.1960	0.0586	0.5569	95.67%	−0.2275	0.0972	0.8359	95.33%	−0.4238	0.1917	0.4312	95.33%	0.0240	0.0080	0.1143	−0.1498	0.0387	0.1245	−0.2630	0.0835	0.1649
	85	a_1	0.2986	0.1136	0.6131	96.60%	−0.0005	0.0159	0.4940	95.67%	0.9339	0.9606	1.1662	96.33%	0.0429	0.0030	0.1227	−0.0008	0.0013	0.1455	0.0609	0.0049	0.1453
		a_2	0.3001	0.1180	0.6562	95.83%	0.0000	0.0170	0.5109	94.97%	1.7510	3.2766	1.7992	95.40%	0.0292	0.0013	0.0807	0.0038	0.0007	0.0992	0.2233	0.0510	0.1370
		θ	1.3416	2.4003	3.0389	96.14%	−0.6784	0.6149	1.5426	96.67%	0.0590	0.0117	0.3560	95.73%	0.0802	0.1424	1.4092	−0.3469	0.2230	1.1051	0.0593	0.0109	0.4111
		τ	0.1310	0.0223	0.2803	96.90%	−0.1846	0.0457	0.4228	96.67%	−0.3438	0.1021	0.2232	96.67%	0.0080	0.0016	0.1227	−0.0846	0.0132	0.1455	−0.2372	0.0526	0.1453
		ρ	0.1881	0.0456	0.3954	96.53%	−0.2548	0.0876	0.5906	96.80%	−0.4360	0.1797	0.3169	96.67%	0.0115	0.0034	0.0807	−0.1139	0.0240	0.0992	−0.2909	0.0819	0.1370
	100	a_1	0.2858	0.0932	0.4204	96.40%	−0.0113	0.0081	0.3509	96.95%	0.9249	0.8917	0.7455	96.67%	0.0405	0.0024	0.1519	0.0023	0.0012	0.1460	0.0774	0.0043	0.1458
		a_2	0.2864	0.0950	0.4457	97.16%	−0.0093	0.0088	0.3667	97.16%	1.6612	2.8527	1.1976	97.47%	0.0361	0.0012	0.0980	0.0034	0.0006	0.1016	0.2750	0.0477	0.1252
		θ	1.3008	1.9677	2.0589	97.40%	−0.7404	0.6014	0.9054	97.57%	0.0511	0.0066	0.2479	97.50%	0.0619	0.0324	0.6198	−0.2321	0.0790	0.5806	0.0417	0.0061	0.3393
		τ	0.1290	0.0190	0.1919	96.90%	−0.1950	0.0435	0.2906	97.53%	−0.3468	0.1002	0.1589	97.57%	0.0065	0.0004	0.1519	−0.0508	0.0039	0.1460	−0.2788	0.0800	0.1458
		ρ	0.1863	0.0395	0.2712	97.40%	−0.2674	0.0828	0.4160	97.57%	−0.4393	0.1633	0.2271	97.75%	0.0095	0.0008	0.0980	−0.0666	0.0068	0.1016	−0.3455	0.1234	0.1252

The following conclusions can be reached from Tables 1–4:

- In terms of the least MSE, MB, the length of asymptotic confidence intervals (LACI), and coverage probability (CP) values, all provided estimates of the unidentified bivariate XL parameters a_1, a_2, θ, τ, and ρ are outstanding.
- As n (or r) increases, all estimated estimations perform as expected. When $n - r$ decreases, all estimates act in the same manner.
- The MCMC estimates performed based on SE loss better against the MLE (are more favorable compared to the classical estimates) in terms of the smallest MSE and MB values when compared to the point estimation methods of a_1, a_2, and θ, because a_1, and a_2 parameters have gamma prior.
- When $n = r$, which typically yields the best results for all unknown parameters as expected, the approaches for estimating model parameters or reliability qualities perform the best among the derived estimates. This is an expected outcome because the acquired estimates were obtained using all of the data from the entire sample.
- When comparing the bivariate XL distribution based on three copula functions, we find that in most cases the result of the bivariate XL based on the Clayton copula is better.

7. Application to Real Data

Iron material jobs data: The data were sourced from Dasgupta [53]. For tasks involving iron sheets, we utilize a perforation process. Specifically, we drill four holes—two on each arm—into an L-shaped rectangular sheet with dimensions of 100 mm by 150 mm. This is accomplished swiftly using a 100-ton press operating at a speed of 250 strokes per hour, a process known as piercing. During each operation, two holes are created simultaneously. These L-shaped iron sheets with punctures are essential for use in the chassis of mini- or light-duty trucks. Following the piercing process, a burr forms around each hole on the metal sheet, creating a circular ridge. The high pressure used in piercing distorts the contact surface, causing the metal granules forming the burr to be unevenly raised around the hole's rim. The burr size varies based on the metal grain characteristics and the applied piercing load. According to Skrotzki et al. [54], factors such as composition, melting point, cooling rate, thermal and constitutional undercooling, and convection influence the grain structure and texture of metals. Subsequently, the burr is removed by chamfering with a drill. The burr measurements for the datasets were taken using a dial gauge with a minimum count of 20 microns (μm), or 0.02 millimeters. In the first dataset, which includes 50 observations of burr measurements (in millimeters), the hole diameter is 12 mm and the sheet thickness is 3.15 mm. In the second dataset, also comprising 50 observations, the hole diameter is 9 mm and the sheet thickness is 2 mm. For each set, one hole is selected and oriented according to a predetermined direction, and the diameter readings are taken with reference to that hole. These two datasets are associated with two different computers being compared. For more details on the iron material jobs data, see Dasgupta [53]. X: "0.04, 0.02, 0.06, 0.12, 0.14, 0.08, 0.22, 0.12, 0.08, 0.26, 0.24, 0.04, 0.14, 0.16, 0.08, 0.26, 0.32, 0.28, 0.14, 0.16, 0.24, 0.22, 0.12, 0.18, 0.24, 0.32, 0.16, 0.14, 0.08, 0.16, 0.24, 0.16, 0.32, 0.18, 0.24, 0.22, 0.16, 0.12, 0.24, 0.06, 0.02, 0.18, 0.22, 0.14, 0.06, 0.04, 0.14, 0.26, 0.18, 0.16". Y: "0.06, 0.12, 0.14, 0.04, 0.14, 0.16, 0.08, 0.26, 0.32, 0.22, 0.16, 0.12, 0.24, 0.06, 0.02, 0.18, 0.22, 0.14, 0.22, 0.16, 0.12, 0.24, 0.06, 0.02, 0.18, 0.22, 0.14, 0.02, 0.18, 0.22, 0.14, 0.06, 0.04, 0.14, 0.22, 0.14, 0.06, 0.04, 0.16, 0.24, 0.16, 0.32, 0.18, 0.24, 0.22, 0.04, 0.14, 0.26, 0.18, 0.16".

Diabetic nephropathy data: Grover et al. [55] have discussed this data. Two datasets were used in our investigation. The first dataset comes from Dr. Lal's Path Lab, a reputable NABL-certified path lab, and contains retrospective data on 132 type 2 diabetic patients who had their diabetes diagnosed in accordance with ADA guidelines. A house-to-house survey was used to reach out to these patients, and current pathology reports from the time of the diabetes diagnosis to the end of the study, in November 2007, were gathered. Each patient had information about the length of their diabetes as well as their Fasting Blood Glucose (FBG), Diastolic Blood Pressure (DBP), Systolic Blood Pressure (SBP), Low Density Lipoprotein (LDL), and SrCr readings recorded. The study automatically disregards the impact on the eyes, heart, and other organs because it exclusively examines renal complications caused by type 2 diabetes. We have also eliminated instances in which the development of diabetes was preceded by renal complications. Patients in our study who have had diabetes for the same amount of time have varying levels of renal health. SrCr is used to assess a patient's renal health since it serves as a key indicator for DN risk prediction due to its rapid rise in value. Thus, using SrCr values, the data were divided into two groups: DN (SrCr 1.4mg/dl) and non-diabetic nephropathy (NDN) (SrCr 1.4 mg/dL). It was discovered at the conclusion of the study that only 60 (45.45%) of the 132 patients were DN cases and 72 (54.55%) were NDN cases. These data are: X: "7.4, 9, 10, 11, 12, 13,13.75, 14.92, 15.8286, 16.9333, 18, 19, 20, 21, 22, 23, 24, 26, 26.6" Y: "1.925, 1.5, 2, 1.6, 1.7, 1.7533, 1.54, 1.694, 1.8843, 1.8433, 1.832, 1.59, 1.7833, 1.2, 1.792, 1.5, 1.5033, 2, 2.14".

Kidney patients: For 30 kidney patients using a portable dialysis unit, McGilchrist and Aisbett [56] provided a dataset that included the times (in days) between the first and second recurrences of infection at the site of catheter insertion. The recurrence time is the period of time between infections. The first recurrence of infection is measured at the time a catheter is implanted, and the second recurrence of infection is measured as the time between the second catheter insertion and the second infection. This information illustrates the frequency of infection recurrence in kidney patients, where X denotes the frequency of the first occurrence and Y that of the second occurrence. The following data set is as follows: X: "8, 23, 22, 447, 30, 24, 7, 511, 53, 15, 7, 141, 96, 149, 536, 152, 402, 13, 39, 12, 113, 132, 34, 2, 130, 17, 185, 292, 22, 15". Y: "16, 13, 28, 318, 12, 245, 9, 30, 196, 154, 333, 8, 38, 70, 25, 362, 24, 66, 46, 40, 201, 156, 30, 25, 26, 4, 117, 114, 159, 108".

Figure 7 shows the correlation matrix for each dataset, kidney, iron material jobs, and diabetic nephropathy, respectively. All the data under study have a correlation value between 0.04 and 0.16. In addition, Table 5 shows goodness-of-fit measures such as Anderson–Darling (AD), Cramer–von Mises criterion (CVM), and the estimated copula parameter by the non-parametric method to test whether or not these copula are a good fit for this data. By the results of goodness of fit test for each copula, we note that p-values are more than 0.05, then these copula functions are a good fit for these datasets.

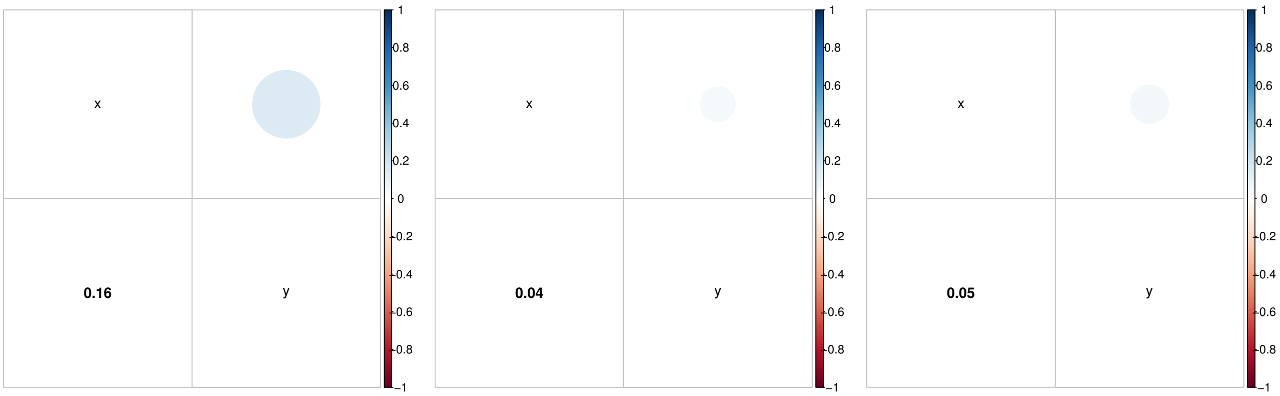

Figure 7. Correlation matrix for each dataset.

Figures 8, 9 and 10 discussed scatter, contour, and density plots for Frank, Gumbel, and Clayton for iron material jobs, diabetic nephropathy, and kidney patients data, respectively.

These included the Akaike information criterion (AIC), the Kolmogorov–Smirnov test, the corrected Akaike information criterion (CAIC), the Bayesian information criterion (BIC), and the Hannan–Quinn information criterion (HQIC). The three real data examples illustrate the suitability of the proposed distributions. Table 6 discussed MLE, StEr, and goodness of fit for different datasets.

Table 5. Goodness-of-fit measures for each dataset.

Data		Frank	Gumbel	Clayton
Kidney	CVM	4.0179	4.0756	4.0425
	AD	27.2411	27.1873	27.2226
	parameter	1.0137	1.1166	0.3033
	statistic	0.2680	0.3035	0.1442
	p-Value	0.4330	0.3913	0.7540
Iron material jobs	CVM	7.3449	7.3410	7.3162
	AD	42.4138	42.4532	42.2753
	parameter	1.1605	1.1125	0.3302
	statistic	0.2014	0.2725	0.1675
	p-Value	0.5813	0.3672	0.7003
Diabetic nephropathy	CVM	0.3352	0.1675	0.2884
	AD	1.5927	0.8188	1.3654
	parameter	0.2612	1.2549	0.0359
	statistic	0.4531	0.2721	0.4602
	p-Value	0.1572	0.3307	0.1302

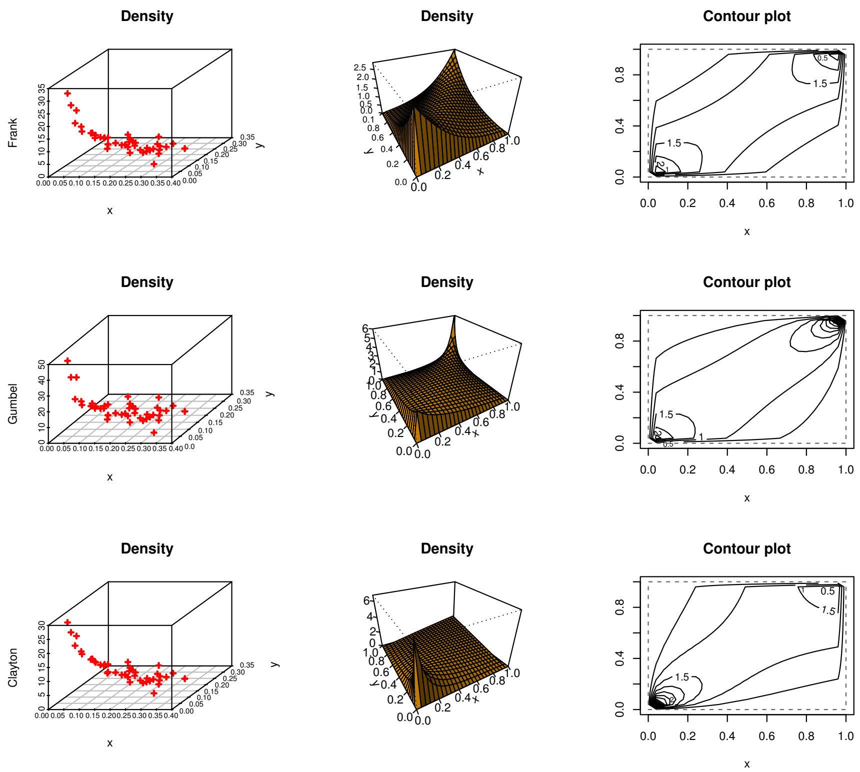

Figure 8. Scatter, contour, and density plots for Frank, Gumbel, and Clayton with iron material jobs data.

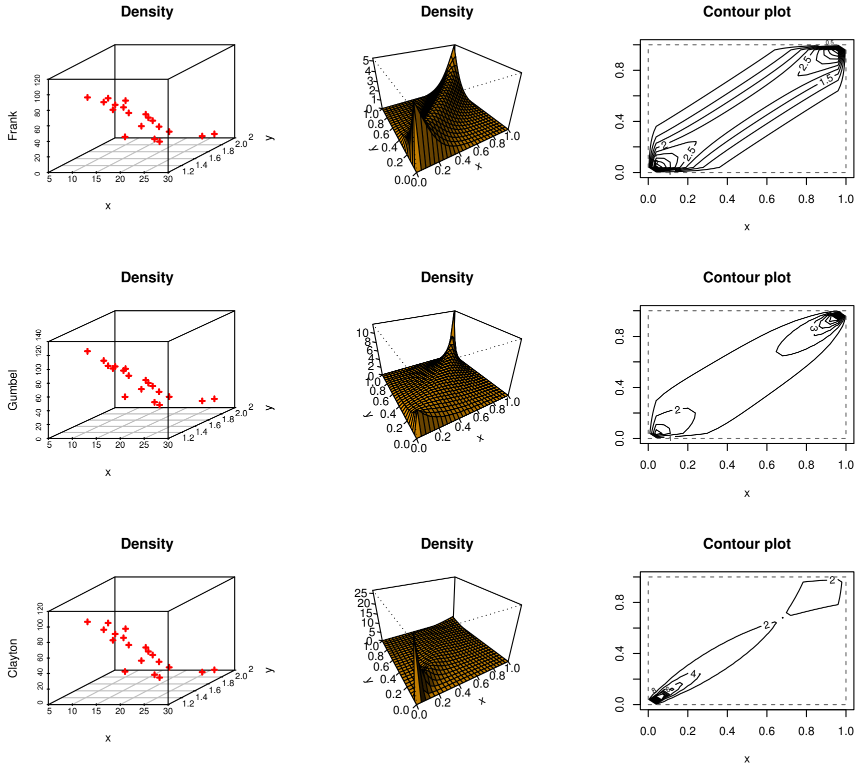

Figure 9. Scatter, contour, and density plots for Frank, Gumbel, and Clayton with diabetic nephropathy data.

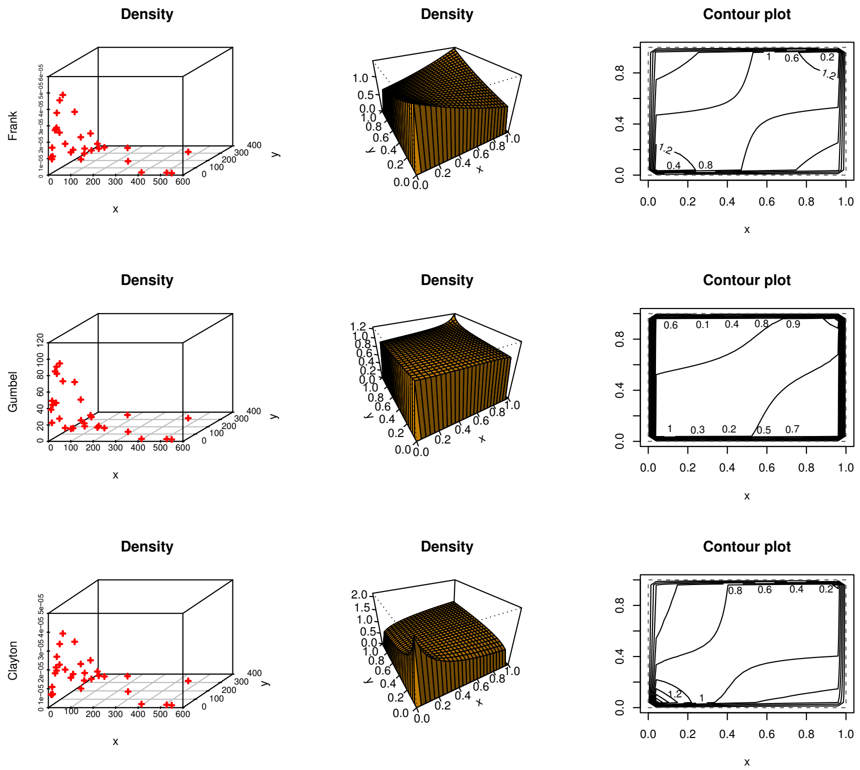

Figure 10. Scatter, contour, and density plots for Frank, Gumbel, and Clayton with kidney patients data.

Table 6. MLE, StEr, and goodness of fit for different datasets.

Data		FBXL		CBXL		GBXL	
		Estimates	**StEr**	**Estimates**	**StEr**	**Estimates**	**StEr**
Kidney	a_1	0.0162	0.0021	0.0157	0.0020	0.0181	0.0024
	a_2	0.0193	0.0026	0.0181	0.0025	0.0365	0.0031
	θ	0.8682	0.7539	0.2298	0.1159	1.0219	0.0242
	AIC	728.7794		**724.2553**		870.3779	
	CAIC	729.7025		**725.1784**		871.3010	
	BIC	732.9830		**728.4589**		874.5815	
	HQIC	730.1242		**725.6001**		871.7227	
Iron material jobs	a_1	6.6625	0.9158	6.8219	0.8809	9.0018	0.9731
	a_2	7.2263	0.9969	7.4392	0.9640	11.7440	1.0177
	θ	3.5215	1.1619	1.0095	0.3782	1.5524	0.1622
	AIC	−170.5505		**−171.7684**		−84.1483	
	CAIC	−170.0288		**−171.2466**		−83.6265	
	BIC	−164.8144		**−166.0323**		−78.4122	
	HQIC	−168.3662		**−169.5841**		−81.9639	
Diabetic nephropathy	a_1	0.1113	0.0168	0.1241	0.0175	0.1510	0.0168
	a_2	0.7480	0.0983	0.7871	0.1000	1.1998	0.1253
	θ	8.8648	2.1219	4.0309	1.1338	2.3474	0.4139
	AIC	**186.6374**		186.8399		207.6029	
	CAIC	**188.2374**		188.4399		209.2029	
	BIC	**189.4708**		189.6732		210.4362	
	HQIC	**187.1170**		187.3194		208.0824	

Here, we fit the FBXL, CBXL, and GBXL models on each dataset based on Type II censored samples. In Tables 7–9, we provide the estimates and Standard Error (StEr) values for all parameters of each model. The Bayesian estimations have smaller StEr than MLE, Therefore, the Bayesian estimation is better than MLE. Also, when r decreases then the StEr of the parameter model increases.

Table 7. MLE and Bayesian based on Type II censored for diabetic nephropathy patients.

			MLE				Bayesian			
r			**Estimates**	**StEr**	**Lower**	**Upper**	**Estimates**	**StEr**	**Lower**	**Upper**
19	FBXL	a_1	0.1113	0.0168	0.0783	0.1443	0.1121	0.0140	0.0847	0.1395
		a_2	0.7480	0.0983	0.5554	0.9406	0.7530	0.0813	0.5937	0.9124
		θ	8.8648	2.1219	4.7058	13.0239	8.6660	2.1006	4.5489	12.7831
	CBXL	a_1	0.1241	0.0175	0.0898	0.1585	0.1254	0.0129	0.1001	0.1507
		a_2	0.7871	0.1000	0.5911	0.9832	0.8181	0.0810	0.6593	0.9769
		θ	4.0309	1.1338	1.8086	6.2531	3.4572	1.1218	1.2584	5.6561
	GBXL	a_1	0.1510	0.0168	0.1181	0.1840	0.1509	0.0146	0.1223	0.1794
		a_2	1.1997	0.1253	0.9540	1.4454	1.2133	0.1073	1.0030	1.4236
		θ	2.3474	0.4139	1.5361	3.1587	2.1632	0.3573	1.4629	2.8635
15	FBXL	a_1	0.0938	0.0177	0.0592	0.1285	0.0943	0.0154	0.0641	0.1245
		a_2	0.5735	0.1127	0.3526	0.7943	0.5807	0.0964	0.3918	0.7697
		θ	11.0966	2.7971	5.6144	16.5789	11.0205	2.5005	6.1196	15.9215
	CBXL	a_1	0.1018	0.0189	0.0648	0.1387	0.1016	0.0156	0.0709	0.1322
		a_2	0.5899	0.1204	0.3539	0.8259	0.5808	0.1009	0.3831	0.7786
		θ	4.3271	1.2762	1.8258	6.8285	4.1589	1.1367	1.9309	6.3868
	GBXL	a_1	0.1450	0.0177	0.1103	0.1797	0.1457	0.0153	0.1157	0.1756
		a_2	1.0353	0.1338	0.7732	1.2975	1.0568	0.1179	0.8256	1.2880
		θ	2.6859	0.6150	1.4806	3.8912	2.5021	0.4979	1.5262	3.4779
12	FBXL	a_1	0.0856	0.0214	0.0437	0.1275	0.0864	0.0201	0.0470	0.1258
		a_2	0.4727	0.1424	0.1936	0.7518	0.4758	0.1361	0.2090	0.7426
		θ	15.1779	4.2963	6.7571	23.5986	14.8527	4.2138	6.5937	23.1117
	CBXL	a_1	0.0893	0.0209	0.0484	0.1302	0.0884	0.0182	0.0526	0.1241
		a_2	0.4686	0.1428	0.1887	0.7485	0.4703	0.1180	0.2391	0.7015
		θ	5.1132	1.4427	2.2854	7.9409	5.4655	1.3403	2.8385	8.0926
	GBXL	a_1	0.1415	0.0187	0.1048	0.1781	0.1395	0.0165	0.1072	0.1719
		a_2	0.8907	0.1412	0.6141	1.1674	0.9123	0.1223	0.6725	1.1521
		θ	3.5137	0.9680	1.6165	5.4109	3.0379	0.7618	1.5448	4.5310

Table 8. MLE and Bayesian based on Type II censored for kidney patients.

			MLE				Bayesian			
r			**Estimates**	**StEr**	**Lower**	**Upper**	**Estimates**	**StEr**	**Lower**	**Upper**
30	FBXL	a_1	0.0162	0.0021	0.0121	0.0203	0.0162	0.0021	0.0121	0.0203
		a_2	0.0193	0.0026	0.0143	0.0243	0.0194	0.0025	0.0145	0.0243
		θ	0.8681	0.3914	0.1010	1.6353	0.9626	0.2615	0.4500	1.4752
	CBXL	a_1	0.0157	0.0020	0.0117	0.0197	0.0158	0.0020	0.0118	0.0198
		a_2	0.0181	0.0025	0.0132	0.0230	0.0184	0.0024	0.0136	0.0231
		θ	0.2298	0.1159	0.0027	0.4569	0.2283	0.0691	0.0929	0.3636
	GBXL	a_1	0.0181	0.0024	0.0133	0.0228	0.0174	0.0022	0.0132	0.0216
		a_2	0.0365	0.0031	0.0304	0.0425	0.0368	0.0030	0.0309	0.0426
		θ	1.0219	0.0242	0.9745	1.0694	1.0279	0.0195	0.9898	1.0661
25	FBXL	a_1	0.0219	0.0030	0.0160	0.0279	0.0222	0.0029	0.0164	0.0279
		a_2	0.0189	0.0028	0.0134	0.0244	0.0192	0.0027	0.0140	0.0245
		θ	1.4408	0.6359	0.1946	2.6871	1.3444	0.4699	0.4235	2.2654
	CBXL	a_1	0.0211	0.0029	0.0154	0.0269	0.0215	0.0028	0.0161	0.0269
		a_2	0.0179	0.0027	0.0126	0.0232	0.0181	0.0026	0.0129	0.0232
		θ	0.2942	0.1451	0.0097	0.5786	0.3274	0.1287	0.0752	0.5796
	GBXL	a_1	0.0216	0.0041	0.0136	0.0296	0.0205	0.0032	0.0143	0.0267
		a_2	0.0378	0.0044	0.0292	0.0463	0.0387	0.0038	0.0312	0.0462
		θ	0.9596	0.1091	0.7457	1.1735	0.9285	0.0824	0.7670	1.0900
20	FBXL	a_1	0.0230	0.0034	0.0163	0.0297	0.0231	0.0034	0.0165	0.0297
		a_2	0.0204	0.0037	0.0131	0.0277	0.0210	0.0037	0.0137	0.0283
		θ	1.3207	0.5802	0.1834	2.4580	1.1552	0.4714	0.2312	2.0791
	CBXL	a_1	0.0222	0.0033	0.0157	0.0287	0.0226	0.0032	0.0164	0.0288
		a_2	0.0189	0.0034	0.0122	0.0256	0.0185	0.0029	0.0129	0.0242
		θ	0.3233	0.1266	0.0751	0.5716	0.2883	0.1210	0.0512	0.5254
	GBXL	a_1	0.0228	0.0058	0.0116	0.0341	0.0201	0.0037	0.0129	0.0273
		a_2	0.0461	0.0089	0.0287	0.0635	0.0522	0.0079	0.0367	0.0678
		θ	0.8753	0.1720	0.5381	1.2125	0.7453	0.1088	0.5321	0.9585

Table 9. MLE and Bayesian based on Type II censored for iron material jobs data.

			MLE				Bayesian			
r			**Estimates**	**StEr**	**Lower**	**Upper**	**Estimates**	**StEr**	**Lower**	**Upper**
50	FBXL	a_1	6.6625	0.9158	4.8675	8.4575	6.6772	0.8604	4.9907	8.3637
		a_2	7.2263	0.9969	5.2725	9.1801	7.2302	0.9316	5.4041	9.0562
		θ	3.5215	1.1619	1.2442	5.7989	3.4021	1.1448	1.1583	5.6459
	CBXL	a_1	6.8219	0.8809	5.0953	8.5485	6.7526	0.8356	5.1149	8.3903
		a_2	7.4392	0.9640	5.5497	9.3286	7.5439	0.9348	5.7116	9.3762
		θ	1.0095	0.3782	0.2683	1.7507	1.1466	0.3542	0.4524	1.8408
	GBXL	a_1	9.0018	0.9731	7.0945	10.9090	8.9193	0.9199	7.1163	10.7223
		a_2	11.7440	1.0177	9.7492	13.7388	11.9008	0.9177	10.1022	13.6995
		θ	1.5524	0.1622	1.2345	1.8704	1.5190	0.1418	1.2410	1.7970
45	FBXL	a_1	6.0350	0.9164	4.2390	7.8311	6.0132	0.8564	4.3347	7.6917
		a_2	6.5701	1.0853	4.4429	8.6974	6.5936	0.9873	4.6585	8.5286
		θ	3.8559	1.4141	1.0842	6.6276	3.6040	1.2300	1.1932	6.0148
	CBXL	a_1	6.3301	0.8959	4.5741	8.0861	6.5277	0.8372	4.8868	8.1687
		a_2	7.1817	1.0034	5.2150	9.1484	7.2556	0.9480	5.3975	9.1137
		θ	0.9416	0.4005	0.1565	1.7266	0.8141	0.3746	0.0800	1.5482
	GBXL	a_1	8.2747	1.0513	6.2142	10.3352	8.2288	1.0484	6.1740	10.2836
		a_2	11.2780	1.0448	9.2301	13.3258	11.3505	1.0005	9.3895	13.3115
		θ	1.4914	0.2031	1.0934	1.8895	1.4680	0.1998	1.0765	1.8596
38	FBXL	a_1	5.1757	0.9866	3.2421	7.1094	5.2021	0.8631	3.5105	6.8937
		a_2	5.8298	1.4696	2.9495	8.7102	5.9074	1.1712	3.6119	8.2029
		θ	3.2946	1.3430	0.6624	5.9268	2.9904	1.0619	0.9091	5.0717
	CBXL	a_1	5.4001	0.9981	3.4439	7.3564	5.4202	0.8743	3.7066	7.1339
		a_2	6.9066	1.1158	4.7197	9.0935	6.7647	1.1114	4.5864	8.9430
		θ	0.6184	0.2587	0.1113	1.1256	0.4840	0.1383	0.2129	0.7550
	GBXL	a_1	3.5359	1.4648	0.6648	6.4070	3.4446	1.1417	1.2069	5.6823
		a_2	30.3453	4.8823	20.7760	39.9145	31.3004	4.1654	23.1362	39.4646
		θ	0.3539	0.1307	0.0979	0.6100	0.3575	0.1209	0.1206	0.5944

8. Conclusions

In this paper, we introduce the utilization of bivariate models employing copula functions to examine industry data associated with iron material jobs, medical data related to diabetic nephropathy, and the recurrence process of infections among kidney patients. Our analysis of iron material jobs data reveals a significant relationship between the hole diameter and sheet thickness with the size of burrs. Burrs, when present, can adversely impact product performance and reliability by causing friction and wear, potentially leading to premature failure. Hence, it is crucial to manage burr size during the manufacturing process to ensure product quality. Additionally, our investigation into diabetic nephropathy data demonstrates a probabilistic correlation between the duration of diabetes and SrCr levels. Through the analysis of medical data, we establish this relationship in a probabilistic manner. Furthermore, our study of kidney patient data focuses on analyzing infection recurrence times in patients utilizing portable dialysis machines. These empirical findings provide the basis for our approach to the bivariate modeling of iron material jobs, diabetic nephropathy, and infection recurrence processes among kidney patients using various copula functions.Three different copula models were utilized, and their accuracy was assessed by fitting them to real-world data. We have introduced a novel category of BXL distributions derived from Frank, Gumbel, and Clayton copulas. Furthermore, we derived the reliability functions for the BXL distributions, utilizing various copulas under investigation. Consequently, these findings can be effectively applied in life testing data analysis. We employed both the maximum likelihood method and the Bayesian method utilizing the Metropolis–Hastings (M-H) algorithm for parameter estimation, considering both complete and censored samples. Therefore, our findings lead to the conclusion that Bayesian estimation emerges as the optimal estimator for the BXL distribution under various copulas. Monte Carlo simulation was employed to compare distributions, and the results indicate that, in the majority of cases, the bivariate XL based on the Clayton copula outperforms other models. The CBXL distribution has been identified as the most suitable model for fitting the kidney data and iron material jobs, whereas the FBXL distribution demonstrates the best fit for diabetic nephropathy.

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Appendix A

By taking partial derivatives of Equation (35) with respect to the parameters of the model, we obtain the following:

$$\begin{aligned}\frac{\partial \ell_{FBXL}(\varphi)}{\partial a_1} &= \frac{2n}{a_1} + \sum_{i=1}^{n} [2 + a_1 + x_{i:n}]^{-1} - \sum_{i=1}^{n} x_{i:n} - \frac{2n}{1 + a_1} \\ &- \theta \sum_{i=1}^{n} A(x_{i:n}) - 2 \sum_{i=1}^{n} \frac{\theta A(x_{i:n}) e^{-\theta u(x_{i:n})} \left(1 - e^{-\theta v(y_{[i:n]})}\right)}{\left[e^{-\theta} - e^{-\theta u(x_{i:n})} - e^{-\theta v(y_{[i:n]})} + e^{-\theta\left(u(x_{i:n}) + v(y_{[i:n]})\right)}\right]},\end{aligned} \tag{A1}$$

$$\begin{aligned}\frac{\partial \ell_{FBXL}(\varphi)}{\partial a_2} &= \frac{2n}{a_2} + \sum_{i=1}^{n} \left[2 + a_2 + y_{[i:n]}\right]^{-1} - \sum_{i=1}^{n} y_{[i:n]} - \frac{2n}{1 + a_2} \\ &- \theta \sum_{i=1}^{n} B(y_{[i:n]}) - 2 \sum_{i=1}^{n} \frac{\theta B(y_{[i:n]}) e^{-\theta v(y_{[i:n]})} \left(1 - e^{-\theta u(x_{i:n})}\right)}{\left[e^{-\theta} - e^{-\theta u(x_{i:n})} - e^{-\theta v(y_{[i:n]})} + e^{-\theta\left(u(x_{i:n}) + v(y_{[i:n]})\right)}\right]},\end{aligned} \tag{A2}$$

$$\frac{\partial \ell_{FBXL}(\varphi)}{\partial \theta} = -\frac{n}{\theta} - \sum_{i=1}^{n}\Big[u(x_{i:n}) + v(y_{[i:n]})\Big] - \frac{ne^{-\theta}}{e^{-\theta}-1}$$
$$- 2\sum_{i=1}^{n} \frac{-e^{-\theta} + u(x_{i:n})e^{-\theta u(x_{i:n})} + v(y_{[i:n]})e^{-\theta v(y_{[i:n]})} - \Big[u(x_{i:n}) + v(y_{[i:n]})\Big]e^{-\theta\left(u(x_{i:n})+v(y_{[i:n]})\right)}}{\Big[e^{-\theta} - e^{-\theta u(x_{i:n})} - e^{-\theta v(y_{[i:n]})} + e^{-\theta\left(u(x_{i:n})+v(y_{[i:n]})\right)}\Big]}, \tag{A3}$$

where

$$A(x_{i:n}) = \frac{\partial}{\partial a_1}u(x_{i:n}) = x_{i:n}e^{-a_1 x_{i:n}}\left\{1 + \frac{1}{(1+a_1)^2}\left[1 + a_1 x_{i:n} - \frac{2a_1}{1+a_1}\right]\right\}, \tag{A4}$$

and

$$B(y_{[i:n]}) = \frac{\partial}{\partial a_2}v\Big(y_{[i:n]}\Big) = y_{[i:n]}e^{-a_2 y_{[i:n]}}\left\{1 + \frac{1}{(1+a_2)^2}\left[1 + a_2 y_{[i:n]} - \frac{2a_2}{1+a_2}\right]\right\}. \tag{A5}$$

By taking partial derivatives of Equation (38) with respect to the parameters of the model, we obtain the following:

$$\frac{\partial \ell_{GBXL}(\varphi)}{\partial a_1} = \frac{2n}{a_1} + \sum_{i=1}^{n}[2 + a_1 + x_{i:n}]^{-1} - \sum_{i=1}^{n} x_{i:n} - \frac{2n}{1+a_1} - \sum_{i=1}^{n}\frac{A(x_{i:n})}{u(x_{i:n})}$$
$$+ \sum_{i=1}^{n}\frac{A(x_{i:n})}{u(x_{i:n})}(-\ln u(x_{i:n}))^{\theta-1}\left[(-\ln u(x_{i:n}))^{\theta} + \Big(-\ln v(y_{[i:n]})\Big)^{\theta}\right]^{\frac{1}{\theta}-1}$$
$$+ (\theta-1)\sum_{i=1}^{n}\frac{A(x_{i:n})}{u(x_i)\ln u(x_{i:n})} + \left(2 - \frac{1}{\theta}\right)\sum_{i=1}^{n}\frac{\theta A(x_{i:n})(-\ln u(x_{i:n}))^{\theta-1}}{u(x_{i:n})\left[(-\ln u(x_{i:n}))^{\theta} + \Big(-\ln v(y_{[i:n]})\Big)^{\theta}\right]} \tag{A6}$$
$$- \sum_{i=1}^{n}\frac{\theta A(x_{i:n})(-\ln u(x_{i:n}))^{\theta-1}}{u(x_{i:n})\left\{\left[(-\ln u(x_{i:n}))^{\theta} + \Big(-\ln v(y_{[i:n]})\Big)^{\theta}\right]^{\frac{1}{\theta}} + \theta - 1\right\}},$$

$$\frac{\partial \ell_{GBXL}(\varphi)}{\partial a_2} = \frac{2n}{a_2} + \sum_{i=1}^{n}\Big[2 + a_2 + y_{[i:n]}\Big]^{-1} - \sum_{i=1}^{n} y_{[i:n]} - \frac{2n}{1+a_2} - \sum_{i=1}^{n}\frac{B(y_{[i:n]})}{v(y_{[i:n]})}$$
$$+ \sum_{i=1}^{n}\frac{B(y_{[i:n]})}{v(y_i)}\Big(-\ln v(y_{[i:n]})\Big)^{\theta-1}\left[(-\ln u(x_{i:n}))^{\theta} + \Big(-\ln v(y_{[i:n]})\Big)^{\theta}\right]^{\frac{1}{\theta}-1}$$
$$+ (\theta-1)\sum_{i=1}^{n}\frac{B(y_{[i:n]})}{v(y_i)\ln v(y_{[i:n]})} + \left(2 - \frac{1}{\theta}\right)\sum_{i=1}^{n}\frac{\theta B(y_{[i:n]})\Big(-\ln v(y_{[i:n]})\Big)^{\theta-1}}{u(x_{i:n})\left[(-\ln u(x_{i:n}))^{\theta} + \Big(-\ln v(y_{[i:n]})\Big)^{\theta}\right]} \tag{A7}$$
$$- \sum_{i=1}^{n}\frac{\theta B(y_{[i:n]})\Big(-\ln v(y_{[i:n]})\Big)^{\theta-1}}{v(y_{[i:n]})\left\{\left[(-\ln u(x_{i:n}))^{\theta} + \Big(-\ln v(y_{[i:n]})\Big)^{\theta}\right]^{\frac{1}{\theta}} + \theta - 1\right\}},$$

$$
\begin{aligned}
\frac{\partial \ell_{GBXL}(\varphi)}{\partial \theta} =& -\sum_{i=1}^{n} G(x_{i:n}, y_{[i:n]}) + \sum_{i=1}^{n}\Big[\ln\ln u(x_{i:n}) + \ln\ln v(y_{[i:n]})\Big] \\
&+ \frac{1}{\theta}\sum_{i=1}^{n} \frac{(-\ln u(x_{i:n}))^{\theta}\ln(-\ln u(x_{i:n})) + \Big(-\ln v(y_{[i:n]})\Big)^{\theta}\ln\Big(-\ln v(y_{[i:n]})\Big)}{(-\ln u(x_{i:n}))^{\theta} + \Big(-\ln v(y_{[i:n]})\Big)^{\theta}} \\
&- \frac{1}{\theta^2}\sum_{i=1}^{n} \ln\left[(-\ln u(x_{i:n}))^{\theta} + \Big(-\ln v(y_{[i:n]})\Big)^{\theta}\right] \\
&+ \sum_{i=1}^{n} \frac{G(x_{i:n}, y_{[i:n]}) + 1}{\left[(-\ln u(x_{i:n}))^{\theta} + \Big(-\ln v(y_{[i:n]})\Big)^{\theta}\right]^{\frac{1}{\theta}} + \theta - 1},
\end{aligned} \tag{A8}
$$

where

$$
\begin{aligned}
G(x_{i:n}, y_{[i:n]}) =& \frac{\partial}{\partial \theta}\left[(-\ln u(x_{i:n}))^{\theta} + \Big(-\ln v(y_{[i:n]})\Big)^{\theta}\right]^{\frac{1}{\theta}} \\
=& \frac{1}{\theta}\left[(-\ln u(x_{i:n}))^{\theta}\ln(-\ln u(x_{i:n})) + \Big(-\ln v(y_{[i:n]})\Big)^{\theta}\ln\Big(-\ln v(y_{[i:n]})\Big)\right] \\
&\left[(-\ln u(x_{i:n}))^{\theta} + \Big(-\ln v(y_{[i:n]})\Big)^{\theta}\right]^{\frac{1}{\theta}-1}
\end{aligned} \tag{A9}
$$

By taking partial derivatives of Equation (39) with respect to the parameters of the model, we obtain the following:

$$
\begin{aligned}
\frac{\partial \ell_{CBXL}(\varphi)}{\partial a_1} =& \frac{2n}{a_1} + \sum_{i=1}^{n}[2 + a_1 + x_{i:n}]^{-1} - \sum_{i=1}^{n} x_{i:n} - \frac{2n}{1 + a_1} \\
&- (1+\theta)\sum_{i=1}^{n}\frac{A(x_{i:n})}{u(x_{i:n})} + \left(2 + \frac{1}{\theta}\right)\sum_{i=1}^{n}\frac{\theta A(x_{i:n})\left[u(x_{i:n})^{-\theta-1}\right]}{\left[u(x_{i:n})^{-\theta} + v(y_{[i:n]})^{-\theta} - 1\right]},
\end{aligned} \tag{A10}
$$

$$
\begin{aligned}
\frac{\partial \ell_{CBXL}(\varphi)}{\partial a_2} =& \frac{2n}{a_2} + \sum_{i=1}^{n}\Big[2 + a_2 + y_{[i:n]}\Big]^{-1} - \sum_{i=1}^{n} y_{[i:n]} - \frac{2n}{1 + a_2} \\
&- (1+\theta)\sum_{i=1}^{n}\frac{B(y_{[i:n]})}{v\Big(y_{[i:n]}\Big)} + \left(2 + \frac{1}{\theta}\right)\sum_{i=1}^{n}\frac{\theta B(y_{[i:n]})\Big[v(y_{[i:n]})^{-\theta-1}\Big]}{\Big[u(x_{i:n})^{-\theta} + v(y_{[i:n]})^{-\theta} - 1\Big]},
\end{aligned} \tag{A11}
$$

$$
\begin{aligned}
\frac{\partial \ell_{CBXL}(\varphi)}{\partial \theta} =& \frac{n}{1+\theta} - \sum_{i=1}^{n}\Big[\ln u(x_{i:n}) + \ln v(y_{[i:n]})\Big] + \frac{1}{\theta^2}\sum_{i=1}^{n}\ln\Big[u(x_{i:n})^{-\theta} + v(y_{[i:n]})^{-\theta} - 1\Big] \\
&+ \frac{1}{\theta}\sum_{i=1}^{n}\frac{u(x_{i:n})^{-\theta}\ln u(x_{i:n}) + v(y_{[i:n]})^{-\theta}\ln v(y_{[i:n]})}{\Big[u(x_{i:n})^{-\theta} + v(y_{[i:n]})^{-\theta} - 1\Big]},
\end{aligned} \tag{A12}
$$

Appendix B

By taking partial derivatives of Equation (42) with respect to the parameters of the model, we obtain the following:

$$
\begin{aligned}
\frac{\partial \ell^{C}_{FBXL}(\varphi)}{\partial a_1} =& (n-r)D(x_{r:n}) - x_{r:n} + \frac{2n}{a_1} + \sum_{i=1}^{r}[2 + a_1 + x_{i:n}]^{-1} - \sum_{i=1}^{r} x_{i:n} - \frac{2n}{1 + a_1} \\
&- \theta\sum_{i=1}^{r}A(x_{i:n}) - 2\sum_{i=1}^{r}\frac{\theta A e^{-\theta u(x_{i:n})}\Big(1 - e^{-\theta v(y_{[i:n]})}\Big)}{\Big[e^{-\theta} - e^{-\theta u(x_{i:n})} - e^{-\theta v(y_{[i:n]})} + e^{-\theta\big(u(x_{i:n}) + v(y_{[i:n]})\big)}\Big]},
\end{aligned} \tag{A13}
$$

$$\frac{\partial \ell^{C}_{FBXL}(\varphi)}{\partial a_2} = \frac{2n}{a_2} + \sum_{i=1}^{r}\left[2 + a_2 + y_{[i:n]}\right]^{-1} - \sum_{i=1}^{r} y_{[i:n]} - \frac{2n}{1+a_2} - \theta\sum_{i=1}^{r} B(y_{[i:n]}) - 2\sum_{i=1}^{r}\frac{\theta B e^{-\theta v(y_{[i:n]})}\left(1 - e^{-\theta u(x_{i:n})}\right)}{\left[e^{-\theta} - e^{-\theta u(x_{i:n})} - e^{-\theta v(y_{[i:n]})} + e^{-\theta\left(u(x_{i:n}) + v(y_{[i:n]})\right)}\right]}, \tag{A14}$$

$$\frac{\partial \ell^{C}_{FBXL}(\varphi)}{\partial \theta} = -\frac{n}{\theta} - \sum_{i=1}^{r}\left[u(x_{i:n}) + v(y_{[i:n]})\right] - \frac{ne^{-\theta}}{e^{-\theta} - 1} - 2\sum_{i=1}^{r}\frac{-e^{-\theta} + u(x_{i:n})e^{-\theta u(x_{i:n})} + v(y_{[i:n]})e^{-\theta v(y_{[i:n]})} - \left[u(x_{i:n}) + v(y_{[i:n]})\right]e^{-\theta\left(u(x_{i:n}) + v(y_{[i:n]})\right)}}{\left[e^{-\theta} - e^{-\theta u(x_{i:n})} - e^{-\theta v(y_{[i:n]})} + e^{-\theta\left(u(x_{i:n}) + v(y_{[i:n]})\right)}\right]}, \tag{A15}$$

where

$$D(x_{r:n}) = \frac{\partial}{\partial a_1}\ln\left(1 + \frac{a_1 x_{r:n}}{(1+a_1)^2}\right) = \frac{x_{r:n(1-a_1)}}{(1-a_1)\left[(1-a_1)^2 + a_1 x_{r:n}\right]}, \tag{A16}$$

By taking partial derivatives of Equation (45) with respect to the parameters of the model, we obtain the following:

$$\begin{aligned}\frac{\partial \ell^{C}_{GBXL}(\varphi)}{\partial a_1} &= (n-r)D(x_{r:n}) - x_{r:n} + \frac{2n}{a_1} + \sum_{i=1}^{r}[2 + a_1 + x_{i:n}]^{-1} - \sum_{i=1}^{r} x_{i:n} - \frac{2n}{1+a_1} - \sum_{i=1}^{r}\frac{A(x_{i:n})}{u(x_{i:n})} \\ &+ \sum_{i=1}^{r}\frac{A(x_{i:n})}{u(x_{i:n})}(-\ln u(x_{i:n}))^{\theta-1}\left[(-\ln u(x_{i:n}))^{\theta} + \left(-\ln v(y_{[i:n]})\right)^{\theta}\right]^{\frac{1}{\theta}-1} \\ &+ (\theta-1)\sum_{i=1}^{r}\frac{A(x_{i:n})}{u(x_{i:n})\ln u(x_{i:n})} + \left(2 - \frac{1}{\theta}\right)\sum_{i=1}^{r}\frac{\theta A(x_{i:n})(-\ln u(x_{i:n}))^{\theta-1}}{u(x_{i:n})\left[(-\ln u(x_{i:n}))^{\theta} + \left(-\ln v(y_{[i:n]})\right)^{\theta}\right]} \\ &- \sum_{i=1}^{r}\frac{\theta A(x_{i:n})(-\ln u(x_{i:n}))^{\theta-1}}{u(x_{i:n})\left\{\left[(-\ln u(x_{i:n}))^{\theta} + \left(-\ln v(y_{[i:n]})\right)^{\theta}\right]^{\frac{1}{\theta}} + \theta - 1\right\}},\end{aligned} \tag{A17}$$

$$\begin{aligned}\frac{\partial \ell^{C}_{GBXL}(\varphi)}{\partial a_2} &= \frac{2n}{a_2} + \sum_{i=1}^{r}\left[2 + a_2 + y_{[i:n]}\right]^{-1} - \sum_{i=1}^{r} y_{[i:n]} - \frac{2n}{1+a_2} - \sum_{i=1}^{r}\frac{B(y_{[i:n]})}{v(y_{i:n})} \\ &+ \sum_{i=1}^{r}\frac{B(y_{[i:n]})}{v(y_{[i:n]})}\left(-\ln v(y_{[i:n]})\right)^{\theta-1}\left[(-\ln u(x_{i:n}))^{\theta} + \left(-\ln v(y_{[i:n]})\right)^{\theta}\right]^{\frac{1}{\theta}-1} \\ &+ (\theta-1)\sum_{i=1}^{r}\frac{B(y_{[i:n]})}{v(y_{[i:n]})\ln v(y_{[i:n]})} + \left(2 - \frac{1}{\theta}\right)\sum_{i=1}^{r}\frac{\theta B(y_{[i:n]})\left(-\ln v(y_{[i:n]})\right)^{\theta-1}}{u(x_{i:n})\left[(-\ln u(x_{i:n}))^{\theta} + \left(-\ln v(y_{[i:n]})\right)^{\theta}\right]} \\ &- \sum_{i=1}^{r}\frac{\theta B(y_{[i:n]})\left(-\ln v(y_{[i:n]})\right)^{\theta-1}}{v(y_{[i:n]})\left\{\left[(-\ln u(x_{i:n}))^{\theta} + \left(-\ln v(y_{[i:n]})\right)^{\theta}\right]^{\frac{1}{\theta}} + \theta - 1\right\}},\end{aligned} \tag{A18}$$

$$\begin{aligned}\frac{\partial \ell^{C}_{GBXL}(\varphi)}{\partial \theta} &= -\sum_{i=1}^{r} G(x_{i:n}, y_{[i:n]}) + \sum_{i=1}^{r}\Big[\ln\ln u(x_{i:n}) + \ln\ln v(y_{[i:n]})\Big] \\ &+ \frac{1}{\theta}\sum_{i=1}^{r}\frac{(-\ln u(x_{i:n}))^{\theta}\ln(-\ln u(x_{i:n})) + \Big(-\ln v(y_{[i:n]})\Big)^{\theta}\ln\Big(-\ln v(y_{[i:n]})\Big)}{(-\ln u(x_{i:n}))^{\theta} + \Big(-\ln v(y_{[i:n]})\Big)^{\theta}} \\ &- \frac{1}{\theta^2}\sum_{i=1}^{r}\ln\Big[(-\ln u(x_{i:n}))^{\theta} + \Big(-\ln v(y_{[i:n]})\Big)^{\theta}\Big] \\ &+ \sum_{i=1}^{r}\frac{G(x_{i:n}, y_{[i:n]}) + 1}{\Big[(-\ln u(x_{i:n}))^{\theta} + \Big(-\ln v(y_{[i:n]})\Big)^{\theta}\Big]^{\frac{1}{\theta}} + \theta - 1},\end{aligned} \tag{A19}$$

where

$$\begin{aligned}G(x_{i:n}, y_{[i:n]}) =& \frac{\partial}{\partial \theta}\Big[(-\ln u(x_{i:n}))^{\theta} + \Big(-\ln v(y_{[i:n]})\Big)^{\theta}\Big]^{\frac{1}{\theta}} \\ =& \frac{1}{\theta}\Big[(-\ln u(x_{i:n}))^{\theta}\ln(-\ln u(x_{i:n})) + \Big(-\ln v(y_{[i:n]})\Big)^{\theta}\ln\Big(-\ln v(y_{[i:n]})\Big)\Big] \\ & \Big[(-\ln u(x_{i:n}))^{\theta} + \Big(-\ln v(y_{[i:n]})\Big)^{\theta}\Big]^{\frac{1}{\theta}-1}\end{aligned} \tag{A20}$$

By taking partial derivatives of Equation (46) with respect to the parameters of the model, we obtain the following:

$$\begin{aligned}\frac{\partial \ell^{C}_{CBXL}(\varphi)}{\partial a_1} &= (n-r)D(x_{r:n}) - x_{r:n} + \frac{2n}{a_1} + \sum_{i=1}^{r}[2 + a_1 + x_{i:n}]^{-1} - \sum_{i=1}^{r} x_i - \frac{2n}{1+a_1} \\ &- (1+\theta)\sum_{i=1}^{r}\frac{A(x_{i:n})}{u(x_{i:n})} + \left(2 + \frac{1}{\theta}\right)\sum_{i=1}^{r}\frac{\theta A(x_{i:n})\big[u(x_{i:n})^{-\theta-1}\big]}{\big[u(x_{i:n})^{-\theta} + v(y_{[i:n]})^{-\theta} - 1\big]},\end{aligned} \tag{A21}$$

$$\begin{aligned}\frac{\partial \ell^{C}_{CBXL}(\varphi)}{\partial a_2} &= \frac{2n}{a_2} + \sum_{i=1}^{r}\Big[2 + a_2 + y_{[i:n]}\Big]^{-1} - \sum_{i=1}^{r} y_{[i:n]} - \frac{2n}{1+a_2} \\ &- (1+\theta)\sum_{i=1}^{r}\frac{B(y_{[i:n]})}{v\Big(y_{[i:n]}\Big)} + \left(2 + \frac{1}{\theta}\right)\sum_{i=1}^{r}\frac{\theta B(y_{[i:n]})\Big[v(y_{[i:n]})^{-\theta-1}\Big]}{\Big[u(x_{i:n})^{-\theta} + v(y_{[i:n]})^{-\theta} - 1\Big]},\end{aligned} \tag{A22}$$

$$\begin{aligned}\frac{\partial \ell^{C}_{CBXL}(\varphi)}{\partial \theta} &= \frac{n}{1+\theta} - \sum_{i=1}^{r}\Big[\ln u(x_{i:n}) + \ln v(y_{[i:n]})\Big] + \frac{1}{\theta^2}\sum_{i=1}^{r}\ln\Big[u(x_{i:n})^{-\theta} + v(y_{[i:n]})^{-\theta} - 1\Big] \\ &+ \frac{1}{\theta}\sum_{i=1}^{n}\frac{u(x_{i:n})^{-\theta}\ln u(x_{i:n}) + v(y_{[i:n]})^{-\theta}\ln v(y_{[i:n]})}{\Big[u(x_{i:n})^{-\theta} + v(y_{[i:n]})^{-\theta} - 1\Big]},\end{aligned} \tag{A23}$$

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