Monotonicity Results of Ratios Between Normalized Tails of Maclaurin Power Series Expansions of Sine and Cosine

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Abstract: In the paper, the authors establish the monotonicity results of the ratios between normalized tails of the Maclaurin power series expansions of the sine and cosine functions and restate them in terms of the generalized hypergeometric functions.

Keywords: monotonicity; ratio; Maclaurin power series expansion; normalized tail; sine; cosine; integral representation; generalized hypergeometric function

MSC: 41A80; 26A48; 33B10; 33C20; 41A58

1. Motivations

It is well known [1] (p. 436) that

\[
\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots \tag{1}
\]

and

\[
\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \cdots \tag{2}
\]

for \( x \in \mathbb{R} \). For our convenience, we denote the tails, or, say, the remainders, of the Maclaurin power series expansions (1) and (2) by

\[
SR_n(x) = \sin x - \sum_{k=0}^{n-1} (-1)^k \frac{x^{2k+1}}{(2k+1)!}
\]

or

\[
SR_n(x) = \sin x - \sum_{k=0}^{n-1} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = (-1)^n \int_0^x (x - t)^{2n-1} \sin t \, dt
\]

and

\[
CR_n(x) = \cos x - \sum_{k=0}^{n-1} (-1)^k \frac{x^{2k}}{(2k)!}
\]

or

\[
CR_n(x) = \cos x - \sum_{k=0}^{n-1} (-1)^k \frac{x^{2k}}{(2k)!} = (-1)^n \int_0^x (x - t)^{2n-2} \sin t \, dt
\]
for \( n \in \mathbb{N} \) and \( x \in \mathbb{R} \). This is a classical topic, and there is a long history of investigating the tails \( \text{SR}_n(x) \) and \( \text{CR}_n(x) \) defined by (3) and (4), respectively. For example, in [2] (Corollaries 1.3 and 1.4), Koumandos proved that the tails \((-1)^{n+1} \text{SR}_n(x)\) and \((-1)^{n+1} \text{CR}_n(x)\) for \( n \in \mathbb{N} \) are positive, increasing, convex, and logarithmically concave in \( x \in (0, \infty) \) and that the ratios

\[
\frac{\text{CR}_{n+1}(x)}{\text{SR}_{n+1}(x)}, \quad \frac{\text{CR}_n(x)}{\text{SR}_{n+1}(x)}, \quad \frac{\text{SR}_n(x)}{\text{CR}_{n+1}(x)} \tag{5}
\]

for \( n \in \mathbb{N} \) are decreasing in \( x \in (0, \infty) \).

In the papers [3–6], among other things, Qi introduced the functions \( \text{SinR}_n(x) \) and \( \text{CosR}_n(x) \) by

\[
\text{SinR}_n(x) = \begin{cases} 
-1)^n \frac{2n+1)!}{x^{2n+1}} \left[ \sin x - \sum_{k=0}^{n-1} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \right], & x \neq 0 \\
1, & x = 0
\end{cases}
\]

\[
= \begin{cases} 
-1)^n \frac{2n+1)!}{x^{2n+1}} \text{SR}_n(x), & x \neq 0 \\
1, & x = 0
\end{cases}
\]

\[
= (2n+1)! \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2n+2k+1)!},
\]

\[
= (2n+1) \int_0^1 (1-u)^{2n} \cos(xu) \, du
\]

and

\[
\text{CosR}_n(x) = \begin{cases} 
-1)^n \frac{2n)!}{x^{2n}} \left[ \cos x - \sum_{k=0}^{n-1} (-1)^k \frac{x^{2k}}{(2k)!} \right], & x \neq 0 \\
1, & x = 0
\end{cases}
\]

\[
= \begin{cases} 
-1)^n \frac{2n)!}{x^{2n}} \text{CR}_n(x), & x \neq 0 \\
1, & x = 0
\end{cases}
\]

\[
= (2n)! \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2n+2k)!},
\]

\[
= 2n \int_0^1 (1-u)^{2n-1} \cos(xu) \, du
\]

for \( n \in \mathbb{N} \) and \( x \in \mathbb{R} \). We call these two quantities \( \text{SinR}_n(x) \) and \( \text{CosR}_n(x) \) the \( n \)th normalized tails, or the \( n \)th normalized remainders, of the Maclaurin power series expansions (1) and (2), respectively.

In the papers [3–7], Qi and his coauthors discovered many properties of the normalized tails \( \text{SinR}_n(x) \) and \( \text{CosR}_n(x) \), such as their positivity, the Maclaurin power series expansions of their logarithms, the monotonicity of the ratios of logarithms of them, their concavity, and the like. For example

1. The normalized tail \( \text{SinR}_n(x) \) for \( n \in \mathbb{N} \) is positive and decreasing in \( x \in (0, \infty) \) and is concave in \( x \in (0, \pi) \);
2. The normalized tail \( \text{CosR}_n(x) \) for \( n \geq 2 \) is positive and decreasing in \( x \in (0, \infty) \) and is concave in \( x \in (0, \pi) \).

These two properties mean that the function

\[
f_n(x) = \int_0^1 (1-u)^n \cos(xu) \, du \tag{8}
\]
for \( \alpha \in \mathbb{N} \) such that \( \alpha \geq 2 \) is positive and decreasing in \( x \in (0, \infty) \). Consequently, it is simple to deduce that

\[
\frac{\cos R_n(x)}{\sin R_n(x)} = \frac{1}{(2n + 1)} \frac{x \cos R_n(x)}{\sin R_n(x)}
\]

\[
= \frac{2n}{2n + 1} \int_0^1 (1 - u)^{2n-1} \cos(xu) \, du
\]

\[
= \frac{n}{n + 1} \int_0^1 (1 - u)^{2n+1} \cos(xu) \, du
\]

\[
\text{and}
\]

\[
\frac{\cos R_n(x)}{\cos R_{n+1}(x)} = \frac{1}{(2n + 2)(2n + 3)} \frac{x^2 \cos R_n(x)}{\cos R_{n+1}(x)}
\]

\[
= \frac{2n + 1}{2n + 3} \int_0^1 (1 - u)^{2n+2} \cos(xu) \, du
\]

\[
\text{are defined for all } n \in \mathbb{N} \text{ and } x \in \mathbb{R}.
\]

Since

\[
f_\alpha(x) = \int_0^x (1 - u)^\alpha \cos(ux) \, du = \frac{1}{x^{\alpha+1}} \int_0^x (x - t)^\alpha \cos t \, dt
\]

we acquire

\[
\frac{\cos R_n(x)}{\sin R_n(x)} = \frac{2n}{2n + 1} \int_0^1 (x - t)^{2n-1} \cos t \, dt
\]

\[
\frac{\cos R_n(x)}{\cos R_{n+1}(x)} = \frac{n}{n + 1} \int_0^1 (x - t)^{2n+1} \cos t \, dt
\]

and

\[
\frac{\sin R_n(x)}{\sin R_{n+1}(x)} = \frac{2n + 1}{2n + 3} \int_0^1 (x - t)^{2n+2} \cos t \, dt
\]

for all \( n \in \mathbb{N} \) and \( x \in \mathbb{R} \).

Since

\[
\int_0^x (x - t)^\alpha \cos t \, dt = \alpha \int_0^x (x - t)^{\alpha-1} \sin t \, dt, \quad \alpha > 0
\]

we arrive at

\[
\frac{\cos R_n(x)}{\sin R_n(x)} = \frac{2n - 1}{2n + 1} \int_0^1 (x - t)^{2n-2} \sin t \, dt
\]

\[
\frac{\cos R_n(x)}{\cos R_{n+1}(x)} = \frac{n(2n - 1)}{(n + 1)(2n + 1)} \int_0^1 (x - t)^{2n} \sin t \, dt
\]
and
\[
\frac{\sin_n(x)}{\sin_{n+1}(x)} = \frac{n(2n+1)}{(n+1)(2n+3)} \frac{x^2 \int_0^1 (x-t)^{2n-1} \sin t \, dt}{\int_0^1 (x-t)^{2n+1} \sin t \, dt}
\]  
for all \( n \in \mathbb{N} \) and \( x \in \mathbb{R} \).

Stimulated by the above interesting discussions and conclusions, we naturally propose a question: what are the monotonic properties of the functions
\[
\frac{\cos_n(x)}{\sin_n(x)}, \quad \frac{\cos_n(x)}{\sin_{n+1}(x)}, \quad \frac{\sin_n(x)}{\sin_{n+1}(x)}
\]
in \( x \in (0, \infty) \) for \( n \in \mathbb{N} \)?

Comparing the above-mentioned results obtained in [2] (Corollaries 1.3 and 1.4) by Koumandos with previous results on normalized tails and their derivations in [6,8] and Remark 7 in [7], we believe that this question and its answers should be interesting, important, significant, and useful in mathematics.

In this paper, we will answer this question.

2. Limits

In this section, we compute the limits of the functions in (17) by taking \( x \to 0 \) and \( x \to \infty \), respectively.

**Theorem 1.** The limits
\[
\lim_{x \to 0} \frac{\cos_n(x)}{\sin_n(x)} = 1, \quad \lim_{x \to 0} \frac{\cos_n(x)}{\cos_{n+1}(x)} = 1,
\]
\[
\lim_{x \to 0} \frac{\sin_n(x)}{\sin_{n+1}(x)} = 1, \quad \lim_{x \to \infty} \frac{\cos_n(x)}{\sin_{n+1}(x)} = \frac{n(2n+1)}{(n+1)(2n+3)}
\]
are valid for \( n \in \mathbb{N} \), while the limits
\[
\lim_{x \to \infty} \frac{\cos_n(x)}{\sin_n(x)} = \frac{2n-1}{2n+1} \quad \text{and} \quad \lim_{x \to \infty} \frac{\cos_n(x)}{\cos_{n+1}(x)} = \frac{n(2n-1)}{(n+1)(2n+1)}
\]
are valid for \( n \geq 2 \).

**Proof.** By virtue of the integral representations in (9)–(11), we obtain
\[
\lim_{x \to 0} \frac{\cos_n(x)}{\sin_n(x)} = \frac{2n}{2n+1} \frac{\int_0^1 (1-u)^{2n-1} \, du}{\int_0^1 (1-u)^{2n} \, du} = 1
\]
\[
\lim_{x \to 0} \frac{\cos_n(x)}{\cos_{n+1}(x)} = \frac{n}{n+1} \frac{\int_0^1 (1-u)^{2n-1} \, du}{\int_0^1 (1-u)^{2n+1} \, du} = 1
\]
and
\[
\lim_{x \to 0} \frac{\sin_n(x)}{\sin_{n+1}(x)} = \frac{2n+1}{2n+3} \frac{\int_0^1 (1-u)^{2n} \, du}{\int_0^1 (1-u)^{2n+2} \, du} = 1
\]
for \( n \in \mathbb{N} \).
Making use of the first expressions in (6) and (7), respectively, we acquire

\[
\lim_{x \to \infty} \frac{\sin R_n(x)}{\sin R_{n+1}(x)} = \lim_{x \to \infty} \frac{(-1)^n \frac{2^{n+1}!}{2^{n+2}!} \left[ \sin x - \sum_{k=0}^{n-1} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \right]}{(-1)^{n+1} \frac{2^{n+3}!}{2^{n+2}!} \left[ \sin x - \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \right]}
\]

\[
= \frac{(2n + 1)!}{(2n + 3)!} \lim_{x \to \infty} \frac{\sin x}{\sin x} - \sum_{k=0}^{n-1} (-1)^k \frac{x^{2k+2(n+1)}}{(2k+1)!}
\]

\[
= \frac{n(2n + 1)}{(n + 1)(2n + 3)}, \quad n \in \mathbb{N}
\]

and

\[
\lim_{x \to \infty} \frac{\cos R_n(x)}{\cos R_{n+1}(x)} = \lim_{x \to \infty} \frac{(-1)^n \frac{2^{n+1}!}{2^{n+2}!} \left[ \cos x - \sum_{k=0}^{n-1} (-1)^k \frac{x^{2k}}{(2k)!} \right]}{(-1)^{n+1} \frac{2^{n+3}!}{2^{n+2}!} \left[ \cos x - \sum_{k=0}^{n} (-1)^k \frac{x^{2k+2(n+1)}}{(2k)!} \right]}
\]

\[
= \frac{(2n)!}{(2n + 2)!} \lim_{x \to \infty} \frac{\cos x}{\cos x} - \sum_{k=0}^{n-1} (-1)^k \frac{x^{2k+2(n+1)-2}}{(2k)!}
\]

\[
= \frac{n(2n - 1)}{(n + 1)(2n + 1)}, \quad n \geq 2
\]

The required proof is complete. \( \square \)

**Corollary 1.** For \( n \in \mathbb{N} \), the limit

\[
\lim_{x \to \infty} \frac{\int_0^1 (1 - u)^{2n} \cos(xu) \, du}{\int_0^1 (1 - u)^{2n+2} \cos(xu) \, du} = \frac{n}{n + 1}
\]

is valid. For \( n \in \mathbb{N} \) such that \( n \geq 2 \), the limits

\[
\lim_{x \to \infty} \frac{\int_0^1 (1 - u)^{2n-1} \cos(xu) \, du}{\int_0^1 (1 - u)^{2n} \cos(xu) \, du} = \frac{2n - 1}{2n}
\]

and

\[
\lim_{x \to \infty} \frac{\int_0^1 (1 - u)^{2n-1} \cos(xu) \, du}{\int_0^1 (1 - u)^{2n+1} \cos(xu) \, du} = \frac{2n - 1}{2n + 1}
\]

are valid.

**Proof.** This follows from combining the integral representations in (9)–(11) with Theorem 1. \( \square \)

3. Monotonicity for the Cases \( n = 1, 2 \)

In this section, we discuss the maxima and monotonicity of the functions in (17) for the cases \( n = 1, 2 \).
Theorem 2. The ratios \( \frac{\cos R_1(x)}{\sin R_1(x)} \) and \( \frac{\cos R_2(x)}{\cos R_3(x)} \) have infinitely many minima 0 at \( x = 2k\pi \) for \( k \in \mathbb{N} \), the ratios

\[
\begin{align*}
\frac{\cos R_1(x)}{\sin R_1(x)}' &= \frac{x(1 - \cos x)}{3(x - \sin x)} \\
\frac{\cos R_1(x)}{\cos R_2(x)}' &= \frac{x^2(1 - \cos x)}{6(x^2 + 2\cos x - 2)} \\
\frac{\sin R_1(x)}{\sin R_2(x)} &= \frac{3x^2(x - \sin x)}{10(x^3 - 6x + 6\sin x)} \\
\frac{\cos R_2(x)}{\cos R_3(x)} &= \frac{2x^2(2\cos x - 2)}{5(x^4 - 12x^2 - 24\cos x + 24)}
\end{align*}
\]

are not monotonic in \( x \in (0, \infty) \), and the ratio \( \frac{\sin R_1(x)}{\sin R_3(x)} \) is decreasing in \( x \in (0, \infty) \).

Proof. When \( n = 1, 2 \), we obtain

\[
\begin{align*}
\frac{\cos R_1(x)}{\sin R_1(x)} &= \frac{x(1 - \cos x)}{3(x - \sin x)} \\
\frac{\cos R_1(x)}{\cos R_2(x)} &= \frac{x^2(1 - \cos x)}{6(x^2 + 2\cos x - 2)} \\
\frac{\sin R_1(x)}{\sin R_2(x)} &= \frac{3x^2(x - \sin x)}{10(x^3 - 6x + 6\sin x)} \\
\frac{\cos R_2(x)}{\cos R_3(x)} &= \frac{2x^2(2\cos x - 2)}{5(x^4 - 12x^2 - 24\cos x + 24)}
\end{align*}
\]

and

\[
\begin{align*}
\frac{\sin R_2(x)}{\sin R_3(x)} &= \frac{10x^2(x^3 - 6x + 6\sin x)}{21(x^5 - 20x^3 + 120x - 120\sin x)}
\end{align*}
\]

for all \( x \in \mathbb{R} \). Their derivatives are

\[
\begin{align*}
\left[ \frac{\cos R_1(x)}{\sin R_1(x)} \right]' &= \frac{(x^2 - 1 + \cos x) \sin x - x(1 - \cos x)}{3(x - \sin x)^2} \\
\left[ \frac{\cos R_1(x)}{\cos R_2(x)} \right]' &= \frac{x(x^3 \sin x + 4 \sin^2 x + 8 \cos x - 8)}{6(x^2 + 2\cos x - 2)^2} \\
\left[ \frac{\sin R_1(x)}{\sin R_2(x)} \right]' &= \frac{3x[(x^2 + 24) \sin x - 12 \sin^2 x - x^2(2x \cos x + 12)]}{10(x^3 - 6x + 6\sin x)^2} \\
\left[ \frac{\cos R_2(x)}{\cos R_3(x)} \right]' &= \frac{6\left[ 3\sin(2x) - (x^4 - 15x^2 + 6) \sin x - 4x^3 - 6x - (5x^3 - 6x) \cos x \right]}{5(x^3 - 6x + 6\sin x)^2} \\
\left[ \frac{\sin R_2(x)}{\sin R_3(x)} \right]' &= \frac{-4\left[ x^5 \sin x + 10x^4 - 48x^2 + 48 + 2(x^4 + 24x^2 - 48) \cos x + 48 \cos^2 x \right]}{5(x^4 - 12x^2 + 24 - 24 \cos x)^2}
\end{align*}
\]

and

\[
\begin{align*}
\left[ \frac{\sin R_2(x)}{\sin R_3(x)} \right]' &= \frac{20x[3x^6 \cos x - 2x^2(7x^4 - 120x^2 + 360) - 3x(3x^4 + 80x^2 - 480) \sin x - 720 \sin^2 x]}{21(x^5 - 20x^3 + 120x - 120\sin x)^2}
\end{align*}
\]

for all \( x \in \mathbb{R} \). It is easy to verify that

1. the first two derivatives are equal to 0 at \( x = 2k\pi \) for \( k \in \mathbb{N} \);
2. the quantity in the bracket of the numerator in the third derivative is equal to \( \pi^2(\pi^2 - 12) < 0 \) at \( x = \pi \) and is equal to \( 27\pi^2(3\pi^2 - 4) > 0 \) at \( x = 3\pi \);
3. the quantity in the bracket of the numerator in the fourth derivative is equal to \( \pi(\pi^2 - 12) < 0 \) at \( x = \pi \) and is equal to \( 9\pi(3\pi^2 - 4) > 0 \) at \( x = 3\pi \);
4. the quantity in the bracket of the numerator in the fifth derivative is equal to \(8(24 - 12\pi^2 + \pi^4) = 23.79\ldots\) at \(x = \pi\) and is equal to \(48 - 588\pi^2 + \frac{12005\pi^4}{8} - \frac{16807\pi^6}{32} = -20308.08\ldots\) at \(x = \frac{7\pi}{2}\).

Accordingly, the functions

\[
\begin{align*}
\frac{\cos R_1(x)}{\sin R_1(x)} & \quad \frac{\cos R_1(x)}{\sin R_1(x)}' & \quad \frac{\sin R_1(x)}{\cos R_2(x)} & \quad \frac{\cos R_2(x)}{\sin R_2(x)}' & \quad \frac{\cos R_2(x)}{\sin R_2(x)}'' & \quad \frac{\cos R_2(x)}{\sin R_2(x)}'''
\end{align*}
\]

are not monotonic in \(x \in (0, \infty)\).

It is easy to see that the functions \(\frac{\cos R_1(x)}{\sin R_1(x)}\) and \(\frac{\cos R_1(x)}{\sin R_1(x)}'\) are equal to 0 at \(x = 2k\pi\) for \(k \in \mathbb{N}\) and that these values 0 are minima of them in \(x \in (0, \infty)\).

Denote the function in the bracket of the numerator in the sixth derivative by \(h(x)\) on \((0, \infty)\). The series expansion of \(h(x)\) is

\[
h(x) = -24x^{14}\sum_{k=0}^{\infty} (-1)^k \frac{H(k)}{(2k + 14)!} x^{2k}
\]

\[
= -24x^{14}\sum_{k=0}^{\infty} \frac{H(2k + 1)}{(4k + 16)!} \left[ \frac{H(2k)}{(2k + 1)(4k + 14)!} - x^2 \right] x^{4k}
\]

for \(x \in (0, \infty)\), where

\[
H(k) = 15 \times 2^{2k+14} - 8k^6 - 264k^5 - 3590k^4 - 25790k^3 - 103622k^2 - 222586k - 202860
\]  

(18)

for \(k \geq 0\). Since, by regarding \(k\) as a variable, we have

\[
\begin{align*}
H(0) &= 42900, & H'(0) &= 491520 \ln 2 - 222586 > 0, \\
H''(0) &= 983040(\ln 2)^2 - 207244 > 0, & H(3)(0) &= 1966080(\ln 2)^3 - 154740 > 0, \\
H(4)(0) &= 3932160(\ln 2)^4 - 86160 > 0, & H(5)(0) &= 7864320(\ln 2)^5 - 31680 > 0, \\
H(6)(0) &= 15728640(\ln 2)^6 - 5760 > 0, & H(7)(k) &= 15 \times 2^{2k+21}(\ln 2)^7 > 0
\end{align*}
\]

for \(k \geq 0\), the sequence \(H(k)\) defined in (18) for \(k \geq 0\) is positive.

We claim that

\[
\frac{H(2k)}{H(2k + 1)(4k + 14)!} \geq \frac{825}{34} > 7\pi, \quad k \geq 0
\]

The left-hand side of the claimed inequality is equivalent to

\[
\mathcal{H}(k) = 15(136k^2 + 1054k + 1215)2^{4k+16} - 278528k^8 - 6754304k^7 - 70619648k^6 - 41510400k^5
\]

\[
- 1497640992k^4 - 33897565176k^3 - 4691959472k^2 - 3625030080k - 1194393600
\]

\[
\geq 0, \quad k \geq 0
\]

By considering \(k\) as a variable and differentiating, we obtain

\[
\mathcal{H}(0) = 0
\]

\[
\mathcal{H}'(0) = 960(4976640 \ln 2 - 2696777)
\]

\[
> 0
\]

\[
\mathcal{H}''(0) = 32[597196800(\ln 2)^2 + 259031040 \ln 2 - 284891627]
\]

\[
> 0
\]

\[
\mathcal{H}^{(3)}(0) = 96[796262400(\ln 2)^3 + 518062080(\ln 2)^2 + 33423360 \ln 2 - 211859761]
\]

\[
> 0
\]

\[
\mathcal{H}^{(4)}(0) = 768[398131200(\ln 2)^4 + 345374720(\ln 2)^3 + 33423360(\ln 2)^2 - 46801281]
\]

\[
> 0
\]
\begin{align*}
\mathcal{H}^{(5)}(0) &= 614400[1990656(\ln 2)^5 + 2158592(\ln 2)^4 + 278528(\ln 2)^3 - 81075] \\
&> 0, \\
\mathcal{H}^{(6)}(0) &= 122880[39813120(\ln 2)^6 + 51806208(\ln 2)^5 + 8355840(\ln 2)^4 - 413787] \\
&> 0, \\
\mathcal{H}^{(7)}(0) &= 983040[19906560(\ln 2)^7 + 30220288(\ln 2)^6 + 5849088(\ln 2)^5 - 34629] \\
&> 0, \\
\mathcal{H}^{(8)}(0) &= 31457280[2488320(\ln 2)^8 + 4317184(\ln 2)^7 + 974848(\ln 2)^6 - 357] \\
&> 0.
\end{align*}

and

\begin{align*}
\mathcal{H}^{(9)}(k) &= 15 \times 2^{4k+33}(\ln 2)^7 [2(136k^2 + 1054k + 1215)(\ln 2)^2 + 153(8k + 31) \ln 2 + 1224] \\
&> 0, \quad k \geq 0.
\end{align*}

Accordingly, the function \(\mathcal{H}(k)\) is non-negative for \(k \geq 0\). Consequently, the function \(h(x)\) is negative for \(0 < x \leq \sqrt[7]{\pi} = 4.689\ldots\).

On the other hand, the function \(h(x)\) for \(x \geq 6\) can be rewritten as

\begin{align*}
h(x) &= 3x^6(\cos x - 1) - (11x^6 - 240x^4 + 720x^2) - 3x(3x^4 + 80x^2 - 480) \sin x - 720 \sin^2 x \\
&\leq 3x(3x^4 + 80x^2 - 480) - (11x^6 - 240x^4 + 720x^2) \\
&= -x(11x^5 - 9x^4 + 240x^3 - 240x^2 + 720x + 1440) \\
&= -x(11(x - 6)^5 + 321(x - 6)^4 + 3504(x - 6)^3 + 17256(x - 6)^2 + 35424(x - 6) + 19152) \\
&< 0.
\end{align*}

We now prove that the function \(h(x)\) is negative on \((\pi, 2\pi) \supset (\sqrt[7]{\pi}, 6)\). Direct computation yields

\begin{align*}
h'(x) &= -3[4(7x^5 - 80x^3 + 120x) - x(3x^4 - 80x^2 + 480) \cos x + (x^6 + 15x^4 + 240x^2 - 480) \sin x + 240 \sin(2x)] \\
h''(x) &= -3[(9x^5 - 20x^3 + 960x) \sin x + (x^6 + 480x^2 - 960) \cos x + 140x^4 - 960x^2 + 480 + 480 \cos(2x)] \\
h^{(3)}(x) &= 3[(x^6 - 45x^4 + 540x^2 - 1920) \sin x - 5x(3x^4 - 4x^2 + 384) \cos x - 80(7x^3 - 24x) + 960 \sin(2x)] \\
h^{(4)}(x) &= (63x^5 - 600x^3 + 900x) \sin x + 3(x^6 - 120x^4 + 600x^2 - 3840) \cos x + 5760 \cos(2x) - 5040x^2 + 5760 \\
h^{(5)}(x) &= -3[(x^6 - 225x^4 + 1200x^2 - 6840) \sin x - x(27x^4 - 680x^2 + 4200) \cos x + 3360x + 3840 \sin(2x)] \\
h^{(6)}(x) &= -3[(33x^5 - 1580x^3 + 6600x) \sin x + 7680 \cos(2x) + 3360 + (x^6 - 360x^4 + 3240x^2 - 11040) \cos x] \\
h^{(7)}(x) &= 3(x^6 - 525x^4 + 7980x^2 - 17640) \sin x - x(3x^4 - 3020x^2 + 13080) \cos x + 46080 \sin(2x) \\
&= 3(x^6 - 525x^4 + 7980x^2 - 17640) - 3x(39x^4 - 3020x^2 + 13080) \cot x + 92160 \cos x \sin x
\end{align*}

and

\[
\left[\frac{h^{(7)}(x)}{\sin x}\right]' = 3[x(39x^4 - 3020x^2 + 13080) \cot^2 x - 15(13x^4 - 604x^2 + 872) \cot x \\
- 30720 \sin x + 5x(9x^4 - 1024x^2 + 5808)]
\]

The function \(39x^4 - 3020x^2 + 13080\) has two positive zeros

\[
\sqrt[3]{\frac{2}{39}}(755 - \sqrt[4]{442495}) = 2.145\ldots \quad \text{and} \quad \sqrt[3]{\frac{2}{39}}(755 + \sqrt[4]{442495}) = 8.534\ldots
\]
so it is negative on \((\pi, 2\pi)\). The function \(13x^4 - 604x^2 + 872\) still has two positive zeros
\[
\sqrt{\frac{2}{39}} (755 - \sqrt{442495}) = 2.145\ldots \quad \text{and} \quad \sqrt{\frac{2}{39}} (\sqrt{442495} + 755) = 8.534\ldots
\]
so it is still negative on \((\pi, 2\pi)\). The function \(9x^4 - 1024x^2 + 5808\) has also two positive zeros
\[
2\sqrt{128 - \sqrt{13117}} = 2.446\ldots \quad \text{and} \quad 2\sqrt{128 + \sqrt{13117}} = 10.382\ldots
\]
so it is also negative on \((\pi, 2\pi)\). As a result, the derivative \(\left[\frac{h(7)(x)}{\sin x}\right]'\) is negative on the interval \([\frac{3\pi}{2}, 2\pi]\).

On the interval \((\pi, \frac{3\pi}{2})\), the derivative \(\left[\frac{h(7)(x)}{\sin x}\right]'\) can be rewritten as

\[
\left[\frac{h(7)(x)}{\sin x}\right]' = 3x(39x^4 - 3020x^2 + 13080) \left[ \cot x - \frac{15(13x^4 - 604x^2 + 872)}{x(39x^4 - 3020x^2 + 13080)} \cot x \\
+ \frac{5x(9x^4 - 1024x^2 + 5808) - 30720 \sin x}{x(39x^4 - 3020x^2 + 13080)} \right]
\]

\[
= 3x(39x^4 - 3020x^2 + 13080) \left[ \cot x - \frac{15(13x^4 - 604x^2 + 872)}{2x(39x^4 - 3020x^2 + 13080)} \right]^2 \\
- \left[ \frac{15(13x^4 - 604x^2 + 872)}{2x(39x^4 - 3020x^2 + 13080)} \right]^2 + \frac{5x(9x^4 - 1024x^2 + 5808) - 30720 \sin x}{x(39x^4 - 3020x^2 + 13080)} \right]
\]

\[
= 3x(39x^4 - 3020x^2 + 13080) \left[ \cot x - \frac{15(13x^4 - 604x^2 + 872)}{2x(39x^4 - 3020x^2 + 13080)} \right]^2 \\
+ \frac{5h(x)}{4x^2(39x^4 - 3020x^2 + 13080)^2}
\]

where

\[
h(x) = 1404x^{10} - 276069x^8 + 14453528x^6 - 141173280x^4 + 351276480x^2 \\
- 34217280 - x(958464x^4 - 74219520x^2 + 321454080) \sin x \\
\geq 1404x^{10} - 276069x^8 + 14453528x^6 - 141173280x^4 + 351276480x^2 \\
- 34217280 + x(958464x^4 - 74219520x^2 + 321454080) \\
= 1404x^{10} - 276069x^8 + 14453528x^6 + 958464x^5 - 141173280x^4 \\
- 74219520x^3 + 351276480x^2 + 321454080x - 34217280 \\
\triangleq \delta(y)
\]

and

\[
\delta'(y) = 24(585x^9 - 92023x^7 + 3613382x^5 + 199680x^4 - 23528880x^3 - 9277440x^2 + 29273040x + 13393920) \\
\delta''(y) = 24(5265x^8 - 644161x^6 + 18066910x^4 + 798720x^3 - 70586640x^2 - 18554880x + 29273040) \\
\delta^{(3)}(y) = 48(21060x^7 - 1932483x^5 + 36133820x^3 + 1198080x^2 - 70586640x - 9277440) \\
\delta^{(4)}(y) = 720(9828x^6 - 644161x^4 + 7226764x^2 + 159744x - 4705776) \\
\delta^{(5)}(y) = 2880(14742x^5 - 644161x^3 + 3613382x + 39936) \\
\delta^{(6)}(y) = 2880(73710x^4 - 1932483x^2 + 3613382)
\]
\[ = 212284800 \left[ \left( x^2 - \frac{644161 - \sqrt{29656899601}}{49140} \right) \left( x^2 - \frac{\sqrt{29656899601} + 644161}{49140} \right) \right] \]

with
\[ \frac{644161 - \sqrt{29656899601}}{49140} \approx 2.026 \ldots < \pi^2 \]

and
\[ \frac{644161 + \sqrt{29656899601}}{49140} \approx 24.190 \ldots > \left( \frac{3\pi}{2} \right)^2 \]

This means that \( f^{(6)}(x) < 0 \) and therefore the fourth derivative \( f^{(4)}(x) \) is concave in \( x \in (\pi, \frac{3\pi}{2}] \). Since
\[ f^{(4)}(\pi) = 720(9828\pi^6 - 644161\pi^4 + 7226764\pi^2 + 159744\pi - 4705776) > 0 \]

and
\[ f^{(4)} \left( \frac{3\pi}{2} \right) = 405(199017\pi^6 - 5797449\pi^4 + 28907056\pi^2 + 425984\pi - 8365824) < 0 \]

the third derivative \( f^{(3)}(x) \) has a maximum on \( (\pi, \frac{3\pi}{2}] \). From the values
\[ f^{(3)}(\pi) = 48(21060\pi^7 - 1932483\pi^5 + 36133820\pi^3 + 1198080\pi^2 - 70586640\pi - 9277440) > 0 \]

and
\[ f^{(3)} \left( \frac{3\pi}{2} \right) = \frac{9}{2}(3838185\pi^7 - 156531123\pi^5 + 130081720\pi^3 + 28753920\pi^2 - 1129386240\pi - 98959360) > 0 \]

we see that \( f^{(3)}(x) > 0 \) and the second derivative \( f''(x) \) is increasing in \( x \in (\pi, \frac{3\pi}{2}] \). Since
\[ f''(\pi) = 24(29273040 - 18554880\pi - 70586640\pi^2 + 798720\pi^3 + 18066910\pi^4 - 644161\pi^6 + 5265\pi^8) > 0 \]

the second derivative \( f''(x) \) is positive, and the first derivative \( f'(x) \) is increasing in \( x \in (\pi, \frac{3\pi}{2}] \). From
\[ f'(\pi) = 24(13393920 + 29273040\pi - 9277440\pi^2 - 23528880\pi^3 + 199680\pi^4 + 3613382\pi^5 - 92023\pi^7 + 585\pi^9) > 0 \]

we see that the first derivative \( f'(x) \) is positive and the function \( f(x) \) is increasing in \( x \in \left[ \pi, \frac{3\pi}{2} \right] \). Since
\[ f(\pi) = 1404\pi^{10} - 276069\pi^8 + 14453528\pi^6 + 958464\pi^5 - 141173280\pi^4 - 74219520\pi^3 + 351276480\pi^2 + 321454080\pi - 34217280 > 0 \]

the function \( f(x) \) is positive in \( x \in (\pi, \frac{3\pi}{2}] \). Accordingly, the function \( h(x) \) is positive and therefore the derivative \( \left[ \frac{h^{(7)}(x)}{\sin x} \right]' \) is negative in \( x \in (\pi, \frac{3\pi}{2}] \).

So far, we have concluded that the derivative \( \left[ \frac{h^{(7)}(x)}{\sin x} \right]' \) is negative in \( x \in (\pi, 2\pi) \). Therefore, the function \( \frac{h^{(7)}(x)}{\sin x} \) is decreasing in \( x \in (\pi, 2\pi) \). Since
\[ \lim_{x \to \pi^+} \frac{h^{(7)}(x)}{\sin x} = \infty \quad \text{and} \quad \lim_{x \to (2\pi)^-} \frac{h^{(7)}(x)}{\sin x} = -\infty \]
the sixth derivative $h^{(6)}(x)$ has a minimum on $(\pi, 2\pi)$. Since

$$h^{(6)}(\pi) = 3(\pi^6 - 360\pi^4 + 3240\pi^2 - 22080) < 0$$

and

$$h^{(6)}(2\pi) = -96\pi^2(2\pi^4 - 180\pi^2 + 405) > 0$$

the fifth derivative $h^{(5)}(x)$ has also a minimum on $(\pi, 2\pi)$. From

$$h^{(5)}(\pi) = -3\pi(27\pi^4 - 680\pi^2 + 7560) < 0$$

and

$$h^{(5)}(2\pi) = 48\pi(54\pi^4 - 340\pi^2 + 105) > 0$$

it follows that the fourth derivative $h^{(4)}(x)$ has also a minimum on $(\pi, 2\pi)$. Due to

$$h^{(4)}(\pi) = -3(\pi^5 - 120\pi^3 + 2280\pi - 7680) < 0$$

and

$$h^{(4)}(2\pi) = 96\pi^2(2\pi^4 - 60\pi^2 - 135) < 0$$

the fourth derivative $h^{(4)}(x)$ is negative on $(\pi, 2\pi)$. This means that the third derivative $h^{(3)}(x)$ is decreasing on $(\pi, 2\pi)$. Since

$$h^{(3)}(\pi) = 15\pi(3\pi^4 - 116\pi^2 + 768) < 0$$

the third derivative $h^{(3)}(x)$ is negative on $(\pi, 2\pi)$. Hence, the second derivative $h''(x)$ is decreasing on $(\pi, 2\pi)$. Due to

$$h''(\pi) = 3(\pi^6 - 140\pi^4 + 1440\pi^2 - 1920) < 0$$

the second derivative $h''(x)$ is negative on $(\pi, 2\pi)$. Thus, the first derivative $h'(x)$ is decreasing on $(\pi, 2\pi)$. Because

$$h'(\pi) = -3\pi(31\pi^4 - 400\pi^2 + 960) < 0$$

the first derivative $h'(x)$ is negative on $(\pi, 2\pi)$. Consequently, the function $h(x)$ is decreasing on $(\pi, 2\pi)$. Considering

$$h(\pi) = -\pi^2(17\pi^4 - 240\pi^2 + 720) < 0$$

the function $h(x)$ is negative on $(\pi, 2\pi)$.

In conclusion, on the interval $(0, \sqrt{2}\pi] \cup [6, \infty) \cup (\pi, 2\pi) = (0, \infty)$, the function $h(x)$ is negative. This implies that the ratio $\frac{\sin R(x)}{\sin S(x)}$ is decreasing for $x \in (0, \infty)$. The required proof is complete. \qed

**Remark 1.** Per the request of Qi on 24 January 2024, Chao-Ping Chen (Henan Polytechnic University, China) gave a proof of the negativity of the function $h(x)$ on $(0, \infty)$ as follows.

We expand the function $h(x)$ as

$$-h(x) = \frac{1}{84672}x^{14} - \frac{17}{34927200}x^{16} + \frac{1273}{130767436800}x^{18} - \frac{83}{65387184000}x^{20}
+ \frac{10051}{8216379589017600}x^{22} - \frac{1093}{117083409143500800}x^{24} + \sum_{n=13}^{\infty} (-1)^{n-1}u_n(x)$$

(20)
where
\[ u_n(x) = \frac{24(15 \times 4^n - 8n^6 + 72n^5 - 230n^4 + 250n^3 - 92n^2 - 112n)}{(2n)!}x^{2n} \]

For \( x^2 \leq (2\pi)^2 < 40 \) and \( n \geq 13 \), we have
\[
\frac{u_{n+1}(x)}{u_n(x)} = \frac{x^2[30 \times 4^n - (n+1)(4n^5 - 16n^4 + 11n^3 + 44n^2 + 17n + 60)]}{(2n+1)(n+1)[15 \times 4^n - 2n(4n^5 - 36n^4 + 115n^3 - 125n^2 + 46n + 56)]} < \frac{40[30 \times 4^n - (n+1)(4n^5 - 16n^4 + 11n^3 + 44n^2 + 17n + 60)]}{(2n+1)(n+1)[15 \times 4^n - 2n(4n^5 - 36n^4 + 115n^3 - 125n^2 + 46n + 56)]} = \frac{80}{1200 \times 4^n}
\]
where \( x_n = \frac{2n(4n^5 - 36n^4 + 115n^3 - 125n^2 + 46n + 56)}{15 \times 4^n} \)

Since the sequence \( x_n \) is decreasing in \( n \geq 13 \), we obtain
\[ x_n \leq x_{13} = \frac{4659}{4194304}, \quad n \geq 13 \]

Substituting this inequality into the last inequality in (21) leads to
\[
\frac{u_{n+1}(x)}{u_n(x)} < \frac{80}{(2n+1)(n+1)(1-x_n)} = \frac{67108864}{837929(2n+1)(n+1)} < 1
\]
for \( 0 < x < 2\pi \) and \( n \geq 13 \). This implies that, for fixed \( x \in [0, 2\pi] \), the functional sequence \( u_n(x) \) is decreasing in \( n \geq 13 \). By this, from (20), we deduce
\[
-\frac{h(x)}{x^{14}} > \frac{1}{84672} - \frac{17}{34927200}x^2 + \frac{1273}{10051}x^4 - \frac{83}{65383718400}x^6 - \frac{1093}{11708340914350080}x^{10}, \quad 0 \leq x \leq 2\pi
\]

Let
\[
G(t) = \frac{1}{84672} - \frac{17}{34927200}t + \frac{1273}{130767436800}t^2 - \frac{83}{65383718400}t^3 + \frac{10051}{8216379589017600}t^4 - \frac{1093}{117083409143500800}t^5
\]
for \( 0 \leq t \leq 4\pi^2 \). Direct differentiation gives
\[
G'(t) = -\frac{17}{34927200} + \frac{1273}{65383718400}t - \frac{83}{217945728000}t^2 - \frac{10051}{2054094897254400}t^3 - \frac{1093}{23416681828700160}t^4
\]
\[
G''(t) = \frac{1273}{65383718400} - \frac{83}{108972864000}t + \frac{10051}{684698299084800}t^2 - \frac{1093}{5854170457175040}t^3
\]
and
\[
G^{(3)}(t) = -\frac{83}{108972864000} + \frac{10051}{342349149542400}t - \frac{1093}{1951390152391680}t^2
\]
\[
\leq h^{(3)} \left( \frac{72907}{21860} \right)
\]
\[
\begin{align*}
\frac{26869351}{166305690888192000} & < 0 \\
\text{for } 0 \leq t \leq 4\pi^2. \text{ Hence, the second derivative } G''(t) \text{ is decreasing for } 0 \leq t \leq 4\pi^2, \text{ and} \\
G''(t) & \geq G''(4\pi^2) \\
& = \frac{1424735235 - 222943644\pi^2 + 17187210\pi^4 - 874400\pi^6}{73177130714688000} \\
& > 0 \\
\text{for } 0 \leq t \leq 4\pi^2. \text{ As a result, the first derivative } G'(t) \text{ is increasing, and} \\
G'(t) & \leq G'(4\pi^2) \\
& = -\frac{8904315420 - 1424735235\pi^2 + 111471822\pi^4 - 5729070\pi^6 + 218600\pi^8}{18294282678672000} \\
& < 0 \\
\text{for } 0 \leq t \leq 4\pi^2. \text{ Hence, the function } G(t) \text{ is decreasing, and} \\
G(t) & \geq G(4\pi^2) \\
& = \frac{108030297375 - 17808630840\pi^2 + 1424735235\pi^4 - 74314548\pi^6 + 2864535\pi^8 - 87440\pi^{10}}{9147141339336000} \\
& > 0 \\
\text{for } 0 \leq t \leq 4\pi^2. \text{ Considering this positivity in (22) reveals that the function } h(x) \text{ is negative for } 0 \leq x \leq 2\pi. \\
\text{When } x \geq 2\pi, \text{ it is easy to verify that} \\
11x^4 - 240x^2 + 720 & > 0, \quad 3x^4 + 80x^2 - 480 > 0 \\
\text{and} \\
x(11x^4 - 240x^2 + 720) - 3(3x^4 + 80x^2 - 480) & = 11x^5 - 9x^4 - 240x^3 - 240x^2 + 720x + 1440 \\
& = \sum_{k=0}^{5} a_k (x - 2\pi)^k \\
\text{where} \\
a_0 & = 352\pi^5 - 144\pi^4 - 1920\pi^3 - 960\pi^2 + 1440\pi + 1440 > 0 \\
a_1 & = 880\pi^4 - 288\pi^3 - 2880\pi^2 - 960\pi + 720 > 0 \\
a_2 & = 880\pi^3 - 216\pi^2 - 1440\pi - 240 > 0 \\
a_3 & = 440\pi^2 - 72\pi - 240 > 0 \\
a_4 & = 110\pi - 9 > 0 \\
a_5 & = 11 > 0 \\
\text{Therefore, for } x \geq 2\pi, \text{ we have} \\
x(11x^4 - 240x^2 + 720) & > 3(3x^4 + 80x^2 - 480) \\
11x^5 - 240x^3 + 720x & > 3(3x^4 + 80x^2 - 480) |\sin x|
and

\[ x(11x^5 - 240x^3 + 720x) + 3x(3x^4 + 80x^2 - 480) \sin x > 0 \]

Consequently, the function

\[ h(x) = 3x^6(\cos x - 1) - 720 \sin^2 x - [x(11x^5 - 240x^3 + 720x) + 3x(3x^4 + 80x^2 - 480) \sin x] < 0 \]

for \( x \geq 2\pi \).

In conclusion, the function \( h(x) \) is negative on \((0, \infty)\).

**Remark 2.** Figure 1 plotted by the software Wolfram Mathematica 12 shows that the functions \( \frac{\cos R_1(x)}{\sin R_1(x)} \) (see the red curve in Figure 1) and \( \frac{\cos R_1(x)}{\cos R_2(x)} \) (see the blue curve in Figure 1) have infinitely many local maxima on \((0, \infty)\), and these local maxima may be of two different values. What are these two different values for the local maxima of the functions \( \frac{\cos R_1(x)}{\sin R_1(x)} \) and \( \frac{\cos R_1(x)}{\cos R_2(x)} \) on \((0, \infty)\)? In other words, what are the local maxima of the functions \( \frac{\cos R_1(x)}{\sin R_1(x)} \) and \( \frac{\cos R_1(x)}{\cos R_2(x)} \) on \((0, \infty)\)?

Figure 1. The functions \( \frac{\cos R_1(x)}{\sin R_1(x)} \) and \( \frac{\cos R_1(x)}{\cos R_2(x)} \) on \((0, 9\pi)\).

4. Decreasing Monotonicity for the Cases \( n \geq 3 \)

In this section, we determine the decreasing monotonicity of the functions in (17) for all cases where \( n \geq 3 \).

**Theorem 3.** For \( n \geq 3 \), the even functions

\[
\frac{\cos R_n(x)}{\sin R_n(x)}, \quad \frac{\cos R_n(x)}{\cos R_{n+1}(x)}, \quad \frac{\sin R_n(x)}{\cos R_{n+1}(x)}
\]

are decreasing in \( x \in (0, \infty) \).

**Proof.** Lemma 2.1 in [9] reads that

\[
\frac{d}{dx} \int_0^x f(u, x) \, du = f(x, x) + \int_0^x \frac{\partial f(u, x)}{\partial x} \, du \tag{23}
\]

where \( f(u, x) \) is differentiable in \( x \) and continuous in \((u, x)\) in some region of the \((u, x)\) plane. See also [10] (p. 11, Entry 3.3.7) for a general form of the Formula (23). Let

\[ F_n(x) = \int_0^x (x - t)^n \sin t \, dt, \quad n \in \mathbb{N}, \quad x \geq 0 \]

By virtue of the Formula (23), it is obvious that \( F_n(x) = nF_{n-1}(x) \) for \( n \geq 2 \).

Combining the positivity of the function \( f_n(x) \) defined in (8) for \( n \geq 2 \) on \((0, \infty)\) with the equalities in (12) and (13), we deduce that the function \( F_n(x) \) is positive for \( n \geq 1 \) on \((0, \infty)\).

The ratios in (14)–(16) can be rewritten in terms of \( F_n(x) \) as
\[
\frac{\cos R_n(x)}{\cos R_{n+1}(x)} = \frac{2n - 1}{2n + 1} \frac{xF_{2n-2}(x)}{F_{2n-1}(x)}, \\
\frac{\sin R_n(x)}{\sin R_{n+1}(x)} = \frac{n(2n-1)}{(n+1)(2n+1)} \frac{xF_{2n-2}(x)xF_{2n-1}(x)}{F_{2n}(x)F_{2n+1}(x)}
\]

and

\[
\frac{\sin R_n(x)}{\sin R_{n+1}(x)} = \frac{n(2n+1)}{(n+1)(2n+3)} \frac{xF_{2n}(x)}{F_{2n+1}(x)}
\]

for all \( n \in \mathbb{N} \) and \( x \in \mathbb{R} \). Therefore, it is suffices to prove the decreasing property of the positive function

\[
F_n(x) = \frac{xF_n(x)}{F_{n+1}(x)}, \quad n \geq 4, \quad x \in (0, \infty)
\]

For \( a, b \in \mathbb{R} \) such that \( a < b \), let \( p(t) \) and \( q(t) \) be continuous on \( [a, b] \) and differentiable on \( (a, b) \), and let \( q'(t) \neq 0 \) on \( (a, b) \). The monotonicity rule for the ratio of two differentiable functions [11] (Theorem 1.25) reads that if the ratio \( \frac{p'(t)}{q'(t)} \) is decreasing on \( (a, b) \), then both \( \frac{p(t) - p(a)}{q(t) - q(a)} \) and \( \frac{b(t) - p(b)}{q(t) - q(b)} \) are decreasing in \( t \in (a, b) \). Since the function \( F_n(x) \) tends to 0 as \( x \) tends to 0+ for \( n \in \mathbb{N} \), it is sufficient to prove that the derivative ratio

\[
\frac{[xF_n(x)]'}{[F_{n+1}(x)]'} = \frac{F_n(x) + nxF_{n-1}(x)}{(n+1)F_{n}(x)} = \frac{1}{n+1} \left[ 1 + \frac{xF_{n-1}(x)}{F_{n}(x)} \right]
\]

for \( n \geq 4 \). By induction, we see that it is enough to prove the decreasing property of the function

\[
F_3(x) = \frac{xF_3(x)}{F_4(x)} \quad \text{for} \quad x \in (0, \infty)
\]

A direct differentiation and some technical simplification gives

\[
F'_3(x) = \frac{6(x^4 - 28x^2 + 72) \cos x - (7x^4 - 36x^2 - 24) \sin x - 12 \sin(2x) - 2x(x^4 - 8x^2 + 36)}{(x^4 - 12x^2 - 24 \cos x + 24)^2}
\]

\[
= \frac{6\mathfrak{F}(x)}{(x^4 - 12x^2 - 24 \cos x + 24)^2}
\]

for \( x \in (0, \infty) \), where

\[
\mathfrak{F}(x) = x(x^4 - 28x^2 + 72)(\cos x - 1) - 12 \sin(2x) - x^3(x^2 + 12) - (7x^4 - 36x^2 - 24) \sin x
\]

for \( x \in (0, \infty) \). Since the function \( 7x^4 - 36x^2 - 24 \) has a unique positive zero

\[
\sqrt{\frac{2}{7}}(\sqrt{123} + 9) = 2.395 \ldots < \pi
\]

we obtain

\[
F'_3(x) \leq \frac{6(x^4 - 28x^2 + 72)(\cos x - 1) + 12 - x^3(x^2 + 12) + (7x^4 - 36x^2 - 24)}{(x^4 - 12x^2 - 24 \cos x + 24)^2}
\]

\[
= \frac{6(x^4 - 28x^2 + 72)(\cos x - 1) - [3(x - 3)^3 + 8(x - 3)^4 + 18(x - 3)^3 + 36(x - 3)^2 + 189(x - 3) + 336]}{(x^4 - 12x^2 - 24 \cos x + 24)^2}
\]
for \( x \geq \pi \). Since the function \( x^4 - 28x^2 + 72 \) has two positive zeros

\[
\sqrt{2(7 - \sqrt{31})} = 1.692 \ldots \quad \text{and} \quad \sqrt{2(7 + \sqrt{31})} = 5.013 \ldots
\]

we obtain the negativity \( F_2'(x) < 0 \) in \( x \geq 6 > 5.013 \ldots > \pi > 2.395 \ldots \).

The function \( \tilde{g}(x) \) in (24) can be expanded into

\[
\tilde{g}(x) = 8 \sum_{k=6}^{\infty} (-1)^{k+1} \frac{3 \times 2^{k-6} - 4k^5 + 24k^4 - 47k^3 + 12k^2 + 3k - 12}{(2k+1)!} x^{2k+1}
\]

\[
= -8x^{13} \sum_{k=0}^{\infty} (-1)^{k} \frac{P(k)}{(2k+13)!} x^{2k}
\]

\[
= -8x^{13} \left( \frac{1}{2419200} - \frac{x^2}{54432000} + \frac{53x^4}{134120448000} - \frac{29x^6}{5230697472000} \right) - 8x^{13} \sum_{k=2}^{\infty} \frac{P(2k+1)}{(4k+15)!} \left( \frac{(4k+15)! P(2k)}{(4k+13)! P(2k+1)} - x^2 \right) x^{4k}
\]

where

\[ P(k) = 3 \times 2^{k+12} - 4k^5 - 96k^4 - 911k^3 - 4290k^2 - 10113k - 9714 \]

The inequality

\[ \frac{(4k+15)! P(2k)}{(4k+13)! P(2k+1)} > 36, \quad k \geq 2 \]

can be rearranged as

\[ 3(4k^2 + 19k - 21)2^{4k+14} - 8(256k^7 + 4288k^6 + 29552k^5 + 106724k^4 + 210072k^3 + 202773k^2 + 47508k - 40221) > 0 \]

for \( k \geq 2 \). This can be verified by similar arguments to the proof of the inequality in (19).

On the other hand, it is computable that the function

\[ \frac{1}{2419200} - \frac{x}{54432000} + \frac{53x^2}{134120448000} - \frac{29x^3}{5230697472000} \]

has a unique real zero

\[ \frac{1}{29} \left( \sqrt[3]{13(116\sqrt{74757965051} + 21247997)} - \frac{34939 \times 13^{2/3}}{\sqrt[3]{116\sqrt{74757965051} + 21247997}} + 689 \right) = 36.471 \ldots \]

As a result, when \( 0 < x \leq 6 \), the function \( \tilde{g}(x) \) in (24) and the first derivative \( F_3'(x) \) are negative.

In a word, the function \( F_3(x) \) is decreasing in \( x \in (0, \infty) \). Consequently, the function \( F_n(x) \) for \( n \geq 4 \) is decreasing in \( x \in (0, \infty) \). The required proof is thus complete. \( \square \)

**Remark 3.** In [2] (Theorem 1.1 and Section 2), the function \( F_\alpha(x) \) was proven to be positive for given real number \( \alpha > 0 \) and to be increasing, convex, and logarithmically concave for a fixed real number \( \alpha \geq 2 \) on \( (0, \infty) \).

**Corollary 2.** For \( x \in (0, \infty) \), the double inequalities

\[
\frac{2n - 1}{2n + 1} < \frac{\cos R_n(x)}{\sin R_n(x)} < 1, \quad n \geq 3
\]

\[
\frac{n(2n - 1)}{(n + 1)(2n + 1)} < \frac{\cos R_n(x)}{\cos R_{n+1}(x)} < 1, \quad n \geq 3
\]
and
\[ \frac{n(2n+1)}{(n+1)(2n+3)} < \frac{\SinR_n(x)}{\SinR_{n+1}(x)} < 1, \quad n \geq 2 \]
are valid.

**Proof.** This follows easily from combining Theorem 1 with the decreasing properties in Theorems 2 and 3. \( \square \)

**Remark 4.** From the first equalities in (9)–(11), we immediately see that the decreasing properties in Theorems 2 and 3 are stronger than the decreasing properties established in [2] (Corollary 1.4) for the ratios in (5).

### 5. Monotonicity of Ratios between Generalized Hypergeometric Functions

For \( \alpha_1 \in \mathbb{C} \) and \( \beta_1, \beta_2 \in \mathbb{C} \setminus \{0,-1,-2, \ldots\} \), the generalized hypergeometric function \( _1F_2(\alpha_1; \beta_1, \beta_2; z) \) is defined [12] (p. 1020) by
\[
_1F_2(\alpha_1; \beta_1, \beta_2; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n}{(\beta_1)_n(\beta_2)_n} \frac{z^n}{n!}, \quad z \in \mathbb{C}
\]
where \((z)_n\) denotes the rising factorial, also known as the Pochhammer symbol, defined by
\[
(z)_n = \prod_{\ell=0}^{n-1} (z + \ell) = \begin{cases} 
  z(z+1) \cdots (z+n-1), & n \in \mathbb{N} \\
  1, & n = 0
\end{cases}
\]

In [6] (p. 16), Qi and his coauthors derived two relations
\[
\SinR_n(x) = _1F_2 \left(1; n+1, n+\frac{3}{2}; -\frac{x^2}{4} \right) \tag{25}
\]
and
\[
\CosR_n(x) = _1F_2 \left(1; n+\frac{1}{2}, n+1; -\frac{x^2}{4} \right) \tag{26}
\]
for \( n \in \mathbb{N} \). In [7] (Remark 7), the relations (25) and (26) were applied to reformulate some of main results in [5,6] in terms of the generalized hypergeometric function
\[
_1F_2 \left(1, \frac{n}{2}, \frac{n+1}{2}; -x^2 \right), \quad n \in \mathbb{N}
\]

As performed in [7] (Remark 7), using the relations (25) and (26), we can restate main results in [3,4] as follows.

1. In [3] (Theorem 1), the function
   \[
   \ln \CosR_1(x) = \ln \left[ _1F_2 \left(1, \frac{3}{2}, 2; -\frac{x^2}{4} \right) \right]
   \]
   was expanded into a Maclaurin power series for \( |x| < 2\pi \).
2. In [3] (Theorem 2), the function
   \[
   \frac{\ln \CosR_1(x)}{\ln \cos x} = \frac{\ln \left[ _1F_2 \left(1, \frac{3}{2}, 2; -\frac{x^2}{4} \right) \right]}{\ln \cos x}
   \]
   was proven to be decreasing and to map \((0, \frac{\pi}{2})\) onto \((0, \frac{1}{2})\).
3. In [4] (Theorem 1), the function
   \[
   \ln \SinR_1(x) = \ln \left[ _1F_2 \left(1, 2, \frac{5}{2}; -\frac{x^2}{4} \right) \right]
   \]
was expanded into a Maclaurin power series for \( x \in \mathbb{R} \).

4. In [4] (Theorem 2), the function

\[
\frac{\ln \sin R_1(x)}{\ln \frac{x}{\sqrt{1-x^2}}} = \ln \left[ 1 F_2 \left( 1; 2, \frac{5}{4}; -\frac{x^2}{4} \right) \right]
\]

was proven to be decreasing and to map \((0, \pi)\) onto \((0, \frac{3}{4})\).

Theorem 1 in [5] is a generalization of [3] (Theorem 1).

In terms of the generalized hypergeometric function \( 1 F_2 \) and in view of the relations (25) and (26), we can restate main results of this paper as follows.

**Theorem 4.** The limits

\[
\lim_{x \to 0} \frac{1 F_2 \left( 1; n + \frac{1}{2}, n + 1; x \right)}{1 F_2 \left( 1; n + 1, n + \frac{3}{2}; x \right)} = 1, \quad \lim_{x \to 0} \frac{1 F_2 \left( 1; n + \frac{3}{4} \pm \frac{1}{2}, n + \frac{5}{4} \pm \frac{1}{2}; x \right)}{1 F_2 \left( 1; n + \frac{7}{4} \pm \frac{1}{2}, n + \frac{9}{4} \pm \frac{1}{2}; x \right)} = 1
\]

and

\[
\lim_{x \to -\infty} \frac{1 F_2 \left( 1; n + 1, n + \frac{3}{2}; x \right)}{1 F_2 \left( 1; n + 2, n + \frac{5}{2}; x \right)} = \frac{n(2n+1)}{(n+1)(2n+3)}
\]

are valid for \( n \in \mathbb{N} \), while the limits

\[
\lim_{x \to -\infty} \frac{1 F_2 \left( 1; n + \frac{1}{2}, n + 1; x \right)}{1 F_2 \left( 1; n + \frac{3}{4}, n + \frac{5}{4} \pm \frac{1}{2}; x \right)} = \frac{8n(2n-1)}{(4n+1 \pm 1)(4n+3 \pm 1)}
\]

are valid for \( n \geq 2 \).

**Theorem 5.** The ratios

\[
\frac{1 F_2 \left( 1; \frac{3}{2}, 2; -\frac{x^2}{4} \right)}{1 F_2 \left( 1; 2, \frac{5}{2}; -\frac{x^2}{4} \right)} \quad \text{and} \quad \frac{1 F_2 \left( 1; 2, 2; -\frac{x^2}{4} \right)}{1 F_2 \left( 1; 2, 3; -\frac{x^2}{4} \right)}
\]

have infinitely many minima 0 at \( x = 2k\pi \) for \( k \in \mathbb{N} \), the ratios

\[
\frac{1 F_2 \left( 1; 2 \pm \frac{1}{2}, \frac{5}{2} \pm \frac{1}{2}; -\frac{x^2}{4} \right)}{1 F_2 \left( 1; 3, \frac{5}{2} \pm \frac{1}{2}; -\frac{x^2}{4} \right)}, \quad \frac{1 F_2 \left( 1; 2 \pm \frac{5}{2}, \frac{7}{2} \pm \frac{1}{2}; -\frac{x^2}{4} \right)}{1 F_2 \left( 1; 3, 3; -\frac{x^2}{4} \right)}, \quad \frac{1 F_2 \left( 1; 2 \pm \frac{5}{2}, \frac{7}{2} \pm \frac{1}{2}; -\frac{x^2}{4} \right)}{1 F_2 \left( 1; 2, 4; -\frac{x^2}{4} \right)}
\]

are not monotonic in \( x \in (0, \infty) \), and the ratio

\[
\frac{1 F_2 \left( 1; 3, \frac{7}{2}; -\frac{x^2}{4} \right)}{1 F_2 \left( 1; 4, \frac{7}{2}; -\frac{x^2}{4} \right)}
\]

is decreasing in \( x \in (0, \infty) \).

**Theorem 6.** For \( n \geq 3 \), the functions

\[
\frac{1 F_2 \left( 1; n + \frac{1}{2}, n + 1; -\frac{x^2}{4} \right)}{1 F_2 \left( 1; n + 1, n + \frac{3}{2}; -\frac{x^2}{4} \right)} \quad \text{and} \quad \frac{1 F_2 \left( 1; n + \frac{3}{4} \pm \frac{1}{2}, n + \frac{5}{4} \pm \frac{1}{2}; -\frac{x^2}{4} \right)}{1 F_2 \left( 1; n + \frac{7}{4} \pm \frac{1}{2}, n + \frac{9}{4} \pm \frac{1}{2}; -\frac{x^2}{4} \right)}
\]

are decreasing in \( x \in (0, \infty) \).

**Corollary 3.** The double inequalities

\[
\frac{n(2n+1)}{(n+1)(2n+3)} < \frac{1 F_2 \left( 1; n + 1, n + \frac{3}{2}; -\frac{x^2}{4} \right)}{1 F_2 \left( 1; n + 2, n + \frac{5}{2}; -\frac{x^2}{4} \right)} < 1, \quad n \geq 2
\]
\[ \frac{8n(2n - 1)}{(4n + 1 \pm 1)(4n + 3 \pm 1)} < \frac{1}{1} F_2(1; n + \frac{1}{2}, n + 1; x) < \frac{1}{1} F_2(1; n + \frac{1}{2}, n + \frac{3}{4}; x) < 1, \quad n \geq 3 \]

are valid for \( x \in (0, \infty) \).

**Remark 5.** It is well known that the classical Bernoulli polynomials \( B_n(t) \) are generated by

\[ \frac{z e^{tz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(t) \frac{z^n}{n!}, \quad |z| < 2\pi \]

and that \( B_n(0) = B_n \) for \( n \in \mathbb{N}_0 = \{0\} \cup \mathbb{N} \) are the classical Bernoulli numbers. In Ref. [8], based on the generating function in (27) with \( t = 0 \), the authors introduced the normalized tails

\[ T_n(z) = \begin{cases} 
\frac{(2n + 2)!}{B_{2n+2}} \left( 1 + \frac{1}{2} \left( 1 - z \right) - \sum_{\ell=1}^{n} \frac{B_{2\ell} z^{2\ell}}{(2\ell)!} \right), & z \neq 0 \\
1, & z = 0
\end{cases} \]

for \( n \in \mathbb{N} \) and discovered the following properties of the normalized tail \( T_n(z) \) and the Bernoulli polynomials \( B_n(t) \) for \( n \in \mathbb{N} \):

1. The normalized tail \( T_n(z) \) for \( n \in \mathbb{N} \) is positive and decreasing in \( z \in (0, \infty) \).
2. The ratio \( \frac{T_n(z)}{t^n} \) for \( n \in \mathbb{N} \) is increasing in \( z \in (0, \infty) \).
3. The ratio \( \frac{B_{2n}}{B_{2n+1}(t)} \) for \( n \in \mathbb{N} \) is increasing in \( t \in (0, \frac{1}{2}) \) and decreasing in \( t \in (\frac{1}{2}, 1) \).

**Remark 6.** In the electronic preprint at the site https://doi.org/10.48550/arxiv.2405.05280 (accessed on 6 May 2024), Yang and Qi gave an alternative proof of [8] (Proposition 1) about the monotonicity results of the ratio \( \frac{B_{2n-1}(t)}{B_{2n+1}(t)} \) in \( t \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1) \) for \( n \in \mathbb{N} \), discussed the monotonicity of three more ratios

\[ \frac{B_{2n}(t)}{B_{2n+1}(t)}, \quad n \in \mathbb{N}_0; \quad \frac{B_{2m}(t)}{B_{2n}(t)}, \quad n > m \in \mathbb{N}_0; \quad \frac{B_{2n}(t)}{B_{2n-1}(t)}, \quad n \in \mathbb{N} \]

and derived some known and new inequalities of the Bernoulli polynomials \( B_n(t) \), the Bernoulli numbers \( B_{2n} \), and their ratios such as \( \frac{B_{2n+2}}{B_{2n}} \).

**6. Conclusions**

In this paper, the authors mainly discussed the following three kinds of properties of the ratios in (17) of the normalized tails \( \sin R_n(x) \) and \( \cos R_n(x) \) of the Maclaurin power series expansions (1) and (2) of \( \sin x \) and \( \cos x \):

1. The limits of the ratios in (17) of the normalized tails \( \sin R_n(x) \) and \( \cos R_n(x) \) by taking \( x \to 0 \) and \( x \to \infty \); see Theorem 1;
2. The monotonicity of the ratios in (17) of the normalized tails \( \sin R_n(x) \) and \( \cos R_n(x) \) in \( x \in (0, \infty) \); see Theorems 2 and 3;
3. The corresponding forms of the above conclusions expressed in terms of the generalized hypergeometric functions \( \frak{F}_2 \); see Section 5.

The novel concept of the normalized tails, also known as the normalized remainders, of the Maclaurin power series expansions of analytic functions was first introduced by Qi implicitly in [3,4] and explicitly in [5,6,8]. The main results in [6,8] and Remark 7 in [7] initially demonstrate that the new notion of normalized tails is significant in mathematics.

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**References**


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