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Hyers–Ulam Stability of Isometries on Bounded Domains–III

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Abstract: The question of whether there is a true isometry that approximates the ε-isometry defined on a bounded set has long interested mathematicians. The first paper on this topic was published by Fickett, whose result was subsequently greatly improved by Alestalo et al., Väisälä and Vestfrid. Recently, the authors published some papers improving the previous results. The main purpose of this paper is to improve all of the abovementioned results by utilizing the properties of the norm and inner product for Euclidean space.

Keywords: Hyers–Ulam stability; isometry; ε-isometry; Euclidean space; bounded domain

MSC: 46C99; 39B82; 39B62; 46B04

1. Introduction

Let $(E, \langle\cdot,\cdot\rangle)$ and $(F, \langle\cdot,\cdot\rangle)$ be real (or complex) Hilbert spaces and let D be a nonempty subset of E. Given an $\varepsilon > 0$, a function $f : D \to F$ is said to be an *ε-isometry* if f satisfies the following inequality

$$|\|f(x) - f(y)\| - \|x - y\|| \leq \varepsilon$$

for all $x, y \in D$. If there exists a positive constant K (independent of f and ε), such that for each ε-isometry $f : D \to F$ there exists an isometry $I : D \to F$ satisfying the inequality $\|f(x) - I(x)\| \leq K\varepsilon$ for every $x \in D$, then the equation $\|f(x) - f(y)\| = \|x - y\|$, which is commonly used to define the isometries, is said to have the *Hyers–Ulam stability*.

Hyers and Ulam [1] were the first mathematicians to begin the practical study of the stability of isometries. Indeed, they were able to prove the Hyers–Ulam stability of surjective isometries defined on the entire space by using properties of the inner product of Hilbert space.

Theorem 1 (Hyers and Ulam). *For any surjective ε-isometry $f : E \to E$ satisfying $f(0) = 0$ there exists a surjective isometry $I : E \to E$ satisfying $\|f(x) - I(x)\| \leq 10\varepsilon$ for all $x \in E$.*

Readers interested in more literature on similar subjects are referred to the papers [2–12] and the references cited therein.

The question of whether there is a true isometry that approximates the ε-isometry defined on the bounded subset of the n-dimensional Euclidean space has long been considered an interesting question. To our knowledge, Fickett [13] was the first mathematician who studied the stability of the isometry whose domain is a bounded subset of $\mathbb{R}^n$.

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Theorem 2 (Fickett). *Given an integer $n \geq 2$, let D be a bounded subset of $\mathbb{R}^n$ and let $\varepsilon > 0$ be given. If a function $f : D \to \mathbb{R}^n$ is an ε-isometry then there is an isometry $I : D \to \mathbb{R}^n$, such that*

$$\|f(x) - I(x)\| \leq 27\varepsilon^{1/2^n} \tag{1}$$

for any $x \in D$.

Comparing Theorem 2 with the definition of Hyers–Ulam stability mentioned above, although Fickett did not prove the Hyers–Ulam stability of isometries in a strict sense, it is obvious that his goal was to prove the Hyers–Ulam stability of isometries on the bounded domain.

For the same kind of inequalities as (1) concerning the stability of isometries, the speed of convergence of the upper bounds of the inequalities seems to be most important when ε approaches zero. From this point of view, an obvious disadvantage of Fickett's theorem is that the upper bound of the inequality (1) decreases relatively slowly to 0 as ε approaches 0. Roughly speaking, the problem is that the speed of convergence is too slow. Fickett's theorem, which has this shortcoming, justifies the purpose of this paper to improve this shortcoming.

Fickett attempted to prove the Hyers–Ulam stability of the isometry whose domain is a bounded subset of $\mathbb{R}^n$. Over the past 40 years, several mathematicians have made steady attempts to improve Fickett's result. For example, see Alestalo et al. [14], Väisälä [15], Vestfrid [16], Jung [17], Choi and Jung [18] and Jung et al. [19].

Assume that D is a bounded subset of $\mathbb{R}^n$, such that $\{0, e_1, e_2, \ldots, e_n\} \subset D \subset B_d(0)$, where $\{e_1, e_2, \ldots, e_n\}$ is the standard basis for $\mathbb{R}^n$ and $B_d(0)$ is the closed ball with radius $d \geq 1$ and center at the origin of $\mathbb{R}^n$. The authors [18] proved that for every ε-isometry $f : D \to \mathbb{R}^n$ there is an isometry $I : D \to \mathbb{R}^n$, such that $\|f(x) - I(x)\| \leq K_n\varepsilon$ for all $x \in D$, where $K_n = 4(d+1)n\sqrt{n}$. In the special cases of $n \in \{2, 3\}$, Jung et al. [19] proved that $K_2 = 8d + 4$ and $K_3 = 16d + 5$, respectively. In fact, paper [19], which specializes in 2-dimensional and 3-dimensional Euclidean spaces, provides some efficient results in the case of these two dimensions.

The purpose of this paper is to extend and improve the results of the existing works by exploiting the properties of the norm and inner product for Euclidean space. More precisely, we prove the Hyers–Ulam stability of isometries defined on a bounded subset of $\mathbb{R}^n$, where $n \geq 4$.

Throughout this paper, to avoid space constraints, we use the symbol $(x_1, x_2, \ldots, x_n)^{tr}$ to represent column vectors of $\mathbb{R}^n$.

2. Preliminaries

First, we present a lemma that will be useful in proving Theorem 3, which is the main result of this paper. We will divide the proof of this lemma into four cases and prove each case separately.

Lemma 1. *If ϱ is a real constant with $0 < \varrho < \sin^{-1}\frac{1}{8}$ then*

$$\frac{\cos\theta_1}{\sin^2\theta_1}\cos\theta_2\cos\theta_3 \geq -\frac{\sin\varrho}{\cos^2\varrho}|\cos\theta_2\cos\theta_3|$$

for all $\frac{\pi}{2} - \varrho < \theta_1, \theta_2, \theta_3 < \frac{\pi}{2} + \varrho$.

Proof. (a) Assume that $\cos\theta_1 \geq 0$ and $\cos\theta_2\cos\theta_3 \geq 0$. Then, we have $\frac{\pi}{2} - \varrho < \theta_1 \leq \frac{\pi}{2}$, and, thus,

$$\frac{\cos\theta_1}{\sin^2\theta_1}\cos\theta_2\cos\theta_3 \geq 0 \geq -\frac{\cos\left(\frac{\pi}{2} - \varrho\right)}{\sin^2\left(\frac{\pi}{2} - \varrho\right)}\cos\theta_2\cos\theta_3 = -\frac{\sin\varrho}{\cos^2\varrho}|\cos\theta_2\cos\theta_3|.$$

(b) If we assume $\cos\theta_1 \geq 0$ and $\cos\theta_2 \cos\theta_3 < 0$ then we have $\frac{\pi}{2} - \varrho < \theta_1 \leq \frac{\pi}{2}$ and

$$\frac{\cos\theta_1}{\sin^2\theta_1}\cos\theta_2\cos\theta_3 > \frac{\cos\left(\frac{\pi}{2}-\varrho\right)}{\sin^2\left(\frac{\pi}{2}-\varrho\right)}\cos\theta_2\cos\theta_3 = -\frac{\sin\varrho}{\cos^2\varrho}|\cos\theta_2\cos\theta_3|.$$

(c) Suppose $\cos\theta_1 < 0$ and $\cos\theta_2\cos\theta_3 \geq 0$. Then, $\frac{\pi}{2} < \theta_1 < \frac{\pi}{2} + \varrho$. Similarly, it follows that

$$\frac{\cos\theta_1}{\sin^2\theta_1}\cos\theta_2\cos\theta_3 \geq \frac{\cos\left(\frac{\pi}{2}+\varrho\right)}{\sin^2\left(\frac{\pi}{2}+\varrho\right)}|\cos\theta_2\cos\theta_3| = -\frac{\sin\varrho}{\cos^2\varrho}|\cos\theta_2\cos\theta_3|.$$

(d) Finally, we assume $\cos\theta_1 < 0$ and $\cos\theta_2\cos\theta_3 < 0$. Then, we have

$$\frac{\cos\theta_1}{\sin^2\theta_1}\cos\theta_2\cos\theta_3 > 0 > -\frac{\sin\varrho}{\cos^2\varrho}|\cos\theta_2\cos\theta_3|,$$

which completes the proof. □

For a fixed integer $n > 1$, let $t_1, t_2, \ldots, t_n$ be any non-negative real numbers. The *arithmetic* and *quadratic means* are defined by

$$A(t_1, t_2, \ldots, t_n) = \frac{1}{n}(t_1 + t_2 + \cdots + t_n),$$
$$Q(t_1, t_2, \ldots, t_n) = \sqrt{\frac{1}{n}(t_1^2 + t_2^2 + \cdots + t_n^2)}.$$

It is known that

$$A(t_1, t_2, \ldots, t_n) \leq Q(t_1, t_2, \ldots, t_n)$$

for any real numbers $t_1, t_2, \ldots, t_n \geq 0$.

3. Hyers–Ulam Stability of Isometry on a Bounded Domain

Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis for the n-dimensional Euclidean space $\mathbb{R}^n$ and let 0 denote the origin of $\mathbb{R}^n$. To denote a closed ball with radius d and center at the origin of $\mathbb{R}^n$ we use the notation $B_d(0)$, *i.e.*, $B_d(0) = \{x \in \mathbb{R}^n : \|x\| \leq d\}$.

The Euclidean inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i = \|x\|\|y\|\cos\theta$$

for all $x = (x_1, x_2, \ldots, x_n)^{tr}, y = (y_1, y_2, \ldots, y_n)^{tr} \in \mathbb{R}^n$, where $\|x\| = \sqrt{\langle x, x \rangle}$ and θ is the angle between vectors x and y.

Even if ε has a very small value, proving the existence of an isometry that satisfies the inequality (2) is not different from studying the Hyers–Ulam stability problem of isometry. Therefore, even if we force ε to be small in the following theorem, it does not violate the concept of Hyers–Ulam stability at all.

Theorem 3. *Given an integer $n \geq 4$, let D be a bounded subset of the n-dimensional Euclidean space $\mathbb{R}^n$, such that $\{0, e_1, e_2, \ldots, e_n\} \subset D \subset B_d(0)$ for some $d \geq 1$. Assume that $f : D \to \mathbb{R}^n$ is a function with $f(0) = 0$. If the function f satisfies the inequality*

$$|\|f(x) - f(y)\| - \|x - y\|| \leq \varepsilon \tag{2}$$

for all $x, y \in D$ and for some $0 < \varepsilon < \frac{1}{8n}$ then there is a linear isometry $I : \mathbb{R}^n \to \mathbb{R}^n$, such that

$$\|f(x) - I(x)\| < \left((10d + 6)n + 8d\sqrt{n}\right)\varepsilon$$

for all $x \in D$.

Proof. (a) For any point $x = (x_1, x_2, \ldots, x_n)^{tr}$ of D, it follows from (2) that

$$|\|f(x)\| - \|x\|| \le \varepsilon \quad \text{and} \quad |\|f(x) - f(e_j)\| - \|x - e_j\|| \le \varepsilon \tag{3}$$

for all $j \in \{1, 2, \ldots, n\}$. Since $\|f(x)\| \le \|x\| + \varepsilon$ and $0 < \varepsilon < \frac{1}{32}$, we can use the first inequality of (3) to obtain

$$\big|\|f(x)\|^2 - \|x\|^2\big| = |\|f(x)\| + \|x\|| \cdot |\|f(x)\| - \|x\|| < \left(2\|x\| + \frac{1}{32}\right)\varepsilon.$$

We note that

$$\|f(x) - f(e_j)\| \le \|f(x)\| + \|f(e_j)\| \le (\|x\| + \varepsilon) + (1 + \varepsilon) < \|x\| + \frac{17}{16}$$

and

$$\|x - e_j\| \le \|x\| + \|e_j\| = \|x\| + 1.$$

Since $0 < \varepsilon < \frac{1}{32}$, it similarly follows from the second inequality of (3) that

$$\big|\|f(x) - f(e_j)\|^2 - \|x - e_j\|^2\big| < \left(2\|x\| + \frac{33}{16}\right)\varepsilon \tag{4}$$

for all $j \in \{1, 2, \ldots, n\}$.

(b) We recall that $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product for $\mathbb{R}^n$ and $\|x\| = \sqrt{\langle x, x \rangle}$. Using (4), we have

$$\big|\|f(x)\|^2 - 2\langle f(x), f(e_j)\rangle + \|f(e_j)\|^2 - \|x - e_j\|^2\big| < \left(2\|x\| + \frac{33}{16}\right)\varepsilon$$

or

$$\begin{aligned} -\left(2\|x\| + \frac{33}{16}\right)\varepsilon + \big(\|f(x)\|^2 + \|f(e_j)\|^2 - \|x - e_j\|^2\big) \\ < 2\langle f(x), f(e_j)\rangle < \\ \left(2\|x\| + \frac{33}{16}\right)\varepsilon + \big(\|f(x)\|^2 + \|f(e_j)\|^2 - \|x - e_j\|^2\big) \end{aligned} \tag{5}$$

for any $j \in \{1, 2, \ldots, n\}$.

Furthermore, using (2) or (3), we obtain

$$\begin{aligned} &\|f(x)\|^2 + \|f(e_j)\|^2 - \|x - e_j\|^2 \\ &\le \big(\|x\| + \varepsilon\big)^2 + \big(\|e_j\| + \varepsilon\big)^2 - \|x - e_j\|^2 \\ &= \big(\|x\|^2 + \|e_j\|^2 - \|x - e_j\|^2\big) + 2\big(\|x\| + \|e_j\| + \varepsilon\big)\varepsilon \\ &= 2\langle x, e_j\rangle + 2\big(\|x\| + \|e_j\| + \varepsilon\big)\varepsilon. \end{aligned} \tag{6}$$

On the other hand, if $\|x\| < \varepsilon$ then we obtain

$$\begin{aligned} &\|f(x)\|^2 + \|f(e_j)\|^2 - \|x - e_j\|^2 \\ &\quad \ge 0^2 + \big(\|e_j\| - \varepsilon\big)^2 - \big(\|x\|^2 - 2\langle x, e_j\rangle + \|e_j\|^2\big) \\ &\quad > -2\varepsilon + 2\langle x, e_j\rangle, \end{aligned} \tag{7}$$

and if $\|x\| \geq \varepsilon$ then

$$\begin{aligned} &\|f(x)\|^2 + \|f(e_j)\|^2 - \|x - e_j\|^2 \\ &\quad \geq \left(\|x\| - \varepsilon\right)^2 + \left(\|e_j\| - \varepsilon\right)^2 - \left(\|x\|^2 - 2\langle x, e_j\rangle + \|e_j\|^2\right) \\ &\quad > -2(\|x\| + 1)\varepsilon + 2\langle x, e_j\rangle. \end{aligned} \tag{8}$$

Hence, it follows from (5)–(8) that

$$|\langle f(x), f(e_j)\rangle - \langle x, e_j\rangle| < \left(2\|x\| + \frac{33}{16}\right)\varepsilon \tag{9}$$

for all $j \in \{1, 2, \ldots, n\}$ and $x \in D$.

On the other hand, we note that

$$2\langle f(e_i), f(e_j)\rangle = \|f(e_i)\|^2 + \|f(e_j)\|^2 - \|f(e_i) - f(e_j)\|^2.$$

Hence, using (3) we have

$$|\langle f(e_i), f(e_j)\rangle - \langle e_i, e_j\rangle| < \left(2 + \sqrt{2} + \frac{1}{64}\right)\varepsilon < \frac{7}{2}\varepsilon \tag{10}$$

for any $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$. We can easily show that the inequality (10) also holds for $i = j$.

(c) We will slightly modify Vestfrid's idea to prove that $f(e_1), f(e_2), \ldots, f(e_n)$ are linearly independent vectors in $\mathbb{R}^n$ (see [16]). On the contrary, assume that there are real numbers $c_1, c_2, \ldots, c_n$, such that

$$c_1 f(e_1) + c_2 f(e_2) + \cdots + c_n f(e_n) = 0,$$

where at least one of $c_1, c_2, \ldots, c_n$ is non-zero. Then, we have

$$c_j\|f(e_j)\|^2 + \sum_{i \neq j} c_i \langle f(e_i), f(e_j)\rangle = \left\langle \sum_{i=1}^{n} c_i f(e_i), f(e_j) \right\rangle = 0 \tag{11}$$

for all $j \in \{1, 2, \ldots, n\}$. Furthermore, it follows from (10) and (11) that

$$(1 - 2\varepsilon)|c_j| \leq |c_j|\|f(e_j)\|^2 = \left|\sum_{i \neq j} c_i \langle f(e_i), f(e_j)\rangle\right| < \frac{7}{2}\sum_{i \neq j} |c_i|\varepsilon$$

for any $j \in \{1, 2, \ldots, n\}$. If we sum the above inequality starting from $j = 1$ to n, we obtain

$$(1 - 2\varepsilon)\sum_{j=1}^{n} |c_j| < \frac{7}{2}\sum_{j=1}^{n}\sum_{i \neq j} |c_i|\varepsilon = \frac{7}{2}(n-1)\sum_{j=1}^{n} |c_j|\varepsilon,$$

which contradicts our assumption that $0 < \varepsilon < \frac{1}{8n}$. That is, $f(e_1), f(e_2), \ldots, f(e_n)$ are linearly independent. Therefore, the set $\{f(e_1), f(e_2), \ldots, f(e_n)\}$ forms a basis for $\mathbb{R}^n$.

(d) Let us define the $n \times n$ matrix $\mathbf{A}$ by

$$\mathbf{A} = \big(f(e_1)\ f(e_2)\ \cdots\ f(e_n)\big).$$

We note that $\mathbf{A}$ is invertible due to (c). Indeed, $\mathbf{A}$ is a transformation matrix for a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, such that $T(x) = \mathbf{A}x$ for each $x \in \mathbb{R}^n$. In particular, $T(e_i) = \mathbf{A}e_i = f(e_i)$ for any $i \in \{1, 2, \ldots, n\}$.

We now quote a well-known statement (see [20] [p. 349] and [16] [Proof of Claim 3]):

If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, such that

$$|\langle T(e_i), T(e_j)\rangle - \langle e_i, e_j\rangle| \leq \delta$$

for all $i, j \in \{1, 2, \ldots, n\}$ and for some $\delta > 0$, then there is a linear isometry $I : \mathbb{R}^n \to \mathbb{R}^n$, such that

$$\|T(x) - I(x)\| \leq n\delta$$

for all $x \in \mathbb{R}^n$.

Therefore, due to (10) and the above statement, there is a linear isometry $I : \mathbb{R}^n \to \mathbb{R}^n$, such that

$$\|\mathbf{A}x - I(x)\| \leq \frac{7}{2}n\varepsilon \tag{12}$$

for every $x \in \mathbb{R}^n$.

(e) According to (10), we have

$$\|f(e_i)\|\|f(e_j)\||\cos\theta_{ij}| < \frac{7}{2}\varepsilon$$

for all $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$, where θ_{ij} denotes the angle between two vectors $f(e_i)$ and $f(e_j)$. In addition, since $0 < \varepsilon < \frac{1}{32}$, it follows from (3) and the previous inequality that

$$|\cos\theta_{ij}| \leq \frac{1}{(1-\varepsilon)^2}\frac{7}{2}\varepsilon < 4\varepsilon \tag{13}$$

for every $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$. In view of (13), we now select a real number $\varrho > 0$, such that

$$\sin\varrho = 4\varepsilon \quad \text{and} \quad \left|\frac{\pi}{2} - \theta_{ij}\right| < \varrho \tag{14}$$

for $i \neq j$. Then, we have

$$\sin\theta_{ij} > \cos\varrho \tag{15}$$

for $i \neq j$.

(f) We now set $\hat{f}(e_i) = \frac{1}{\|f(e_i)\|}f(e_i)$ for $i \in \{1, 2, \ldots, n\}$. We can construct an $n \times n$ orthogonal matrix $\mathbf{V} = (v_1\ v_2\ \cdots\ v_n)$, such that

(i) $v_1 = \hat{f}(e_1)$;

(ii) Let ψ_i stand for the angle between v_i and $\hat{f}(e_i)$. We choose v_2 on the plane determined by $\hat{f}(e_1)$ and $\hat{f}(e_2)$, such that $\psi_2 = |\frac{\pi}{2} - \theta_{12}| < \varrho$ by (14). It then follows from (15) that

$$\langle v_2, \hat{f}(e_2)\rangle = \cos\psi_2 = \sin\theta_{12} > \cos\varrho; \tag{16}$$

(iii) Temporarily, let $v_1 = (1, 0, 0)^{tr}$, $v_2 = (0, 1, 0)^{tr}$ and $v_3 = (0, 0, 1)^{tr}$ for simplicity. Then, by (i) and (ii), we have $\hat{f}(e_1) = (1, 0, 0)^{tr} = v_1$ and $\hat{f}(e_2) = (\cos\theta_{12}, \sin\theta_{12}, 0)^{tr}$, and let $\hat{f}(e_3) = (\alpha, \beta, \gamma)^{tr}$, where θ_{ij} is the angle between $\hat{f}(e_i)$ and $\hat{f}(e_j)$. Hence, we obtain

$$\begin{aligned}
&\langle \hat{f}(e_1), \hat{f}(e_3)\rangle = \alpha = \cos\theta_{13},\\
&\langle \hat{f}(e_2), \hat{f}(e_3)\rangle = \alpha\cos\theta_{12} + \beta\sin\theta_{12} = \cos\theta_{23},\\
&\|\hat{f}(e_3)\|^2 = \alpha^2 + \beta^2 + \gamma^2 = 1.
\end{aligned} \tag{17}$$

Therefore, it follows from (17) that

$$\gamma^2 = 1 - \frac{\cos^2\theta_{13} - 2\cos\theta_{12}\cos\theta_{13}\cos\theta_{23} + \cos^2\theta_{23}}{\sin^2\theta_{12}}.$$

Now we estimate a lower bound for the value of γ^2. Using Lemma 1 and (15), we have

$$\begin{aligned}\gamma^2 &= 1 + 2\frac{\cos\theta_{12}}{\sin^2\theta_{12}}\cos\theta_{13}\cos\theta_{23} - \frac{\cos^2\theta_{13} + \cos^2\theta_{23}}{\sin^2\theta_{12}} \\ &> 1 - 2\frac{\sin\varrho}{\cos^2\varrho}|\cos\theta_{13}\cos\theta_{23}| - \frac{\cos^2\theta_{13} + \cos^2\theta_{23}}{\cos^2\varrho} \\ &> 1 - 2\frac{\sin\varrho}{\cos^2\varrho}\sin^2\varrho - 2\frac{\sin^2\varrho}{\cos^2\varrho},\end{aligned}$$

since $|\cos\theta_{ij}| < \sin\varrho$ by (13) and (14). Due to (14) and the previous inequality, we conclude that

$$1 - \frac{128\varepsilon^3 + 32\varepsilon^2}{1 - 16\varepsilon^2} < \gamma^2 \leq 1.$$

Thus, we have

$$\langle v_3, \hat{f}(e_3)\rangle = \gamma > \sqrt{1 - \frac{128\varepsilon^3 + 32\varepsilon^2}{1 - 16\varepsilon^2}} > \sqrt{1 - 37\varepsilon^2}.$$

If we apply the previous argument to the triplet $\{v_1, v_2, v_i\}$ instead of $\{v_1, v_2, v_3\}$, where $i \in \{4, 5, \ldots, n\}$ then the above argument implies that

$$\langle v_i, \hat{f}(e_i)\rangle > \sqrt{1 - 37\varepsilon^2} \tag{18}$$

for any $i \in \{1, 2, \ldots, n\}$. Indeed, we have $\langle v_1, \hat{f}(e_1)\rangle = \|v_1\|^2 = 1 > \sqrt{1 - 37\varepsilon^2}$ and $\langle v_2, \hat{f}(e_2)\rangle > \cos\varrho = \sqrt{1 - 16\varepsilon^2} > \sqrt{1 - 37\varepsilon^2}$ by (14) and (16).

(g) On account of (18), we obtain

$$\|v_i\|^2 - 2\langle v_i, \hat{f}(e_i)\rangle + \|\hat{f}(e_i)\|^2 < 2 - 2\sqrt{1 - 37\varepsilon^2}.$$

That is,

$$\|v_i - \hat{f}(e_i)\|^2 < 2 - 2\sqrt{1 - 37\varepsilon^2} \tag{19}$$

for every $i \in \{1, 2, \ldots, n\}$. Due to the Cauchy–Schwarz inequality and (19), we can estimate:

$$|\langle f(x), v_i\rangle - \langle f(x), \hat{f}(e_i)\rangle| \leq \|f(x)\|\|v_i - \hat{f}(e_i)\| < \|f(x)\|\sqrt{2 - 2\sqrt{1 - 37\varepsilon^2}} \tag{20}$$

for any $i \in \{1, 2, \ldots, n\}$.

(h) We note that

$$\begin{aligned}\|v_i - f(e_i)\| &= \|v_i - \hat{f}(e_i) + \hat{f}(e_i) - f(e_i)\| \\ &\leq \|v_i - \hat{f}(e_i)\| + \|\hat{f}(e_i) - f(e_i)\| \\ &\leq \|v_i - \hat{f}(e_i)\| + \varepsilon\end{aligned}$$

for all $i \in \{1, 2, \ldots, n\}$. We use (19) to estimate

$$\begin{aligned}\|\mathbf{V}x-\mathbf{A}x\| &= \left\|\sum_{i=1}^{n} x_i\big(v_i-f(e_i)\big)\right\| \le \sum_{i=1}^{n}|x_i|\|v_i-f(e_i)\| \\ &\le \sum_{i=1}^{n}|x_i|\big(\|v_i-\hat{f}(e_i)\|+\varepsilon\big) < \sum_{i=1}^{n}|x_i|\left(\sqrt{2-2\sqrt{1-37\varepsilon^2}}+\varepsilon\right) \\ &\le \left(\sqrt{2-2\sqrt{1-37\varepsilon^2}}+\varepsilon\right) n\sqrt{\frac{1}{n}\sum_{i=1}^{n}|x_i|^2} \\ &= \left(\sqrt{2-2\sqrt{1-37\varepsilon^2}}+\varepsilon\right)\sqrt{n}\,\|x\|,\end{aligned} \tag{21}$$

where we use the fact that the arithmetic mean is not greater than the quadratic mean.

(i) Moreover, using (9) and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}&|\langle f(x),\hat{f}(e_i)\rangle-\langle x,e_i\rangle| \\ &\quad= \big|\langle f(x),f(e_i)\rangle+(1-\|f(e_i)\|)\langle f(x),\hat{f}(e_i)\rangle-\langle x,e_i\rangle\big| \\ &\quad\le |\langle f(x),f(e_i)\rangle-\langle x,e_i\rangle|+|1-\|f(e_i)\||\cdot|\langle f(x),\hat{f}(e_i)\rangle| \\ &\quad< \left(2\|x\|+\frac{33}{16}\right)\varepsilon+\|f(x)\|\|\hat{f}(e_i)\|\varepsilon \\ &\quad< \left(3\|x\|+\frac{67}{32}\right)\varepsilon\end{aligned} \tag{22}$$

for all $i\in\{1,2,\dots,n\}$.

We note that $\{v_1,v_2,\dots,v_n\}$ is an orthonormal basis for $\mathbb{R}^n$. Thus, by (20) and (22), we obtain

$$\begin{aligned}\|f(x)-\mathbf{V}x\| &= \left\|\sum_{i=1}^{n}\langle f(x),v_i\rangle v_i-\sum_{i=1}^{n}x_iv_i\right\| \le \sum_{i=1}^{n}|\langle f(x),v_i\rangle-\langle x,e_i\rangle|\|v_i\| \\ &\le \sum_{i=1}^{n}\big(|\langle f(x),v_i\rangle-\langle f(x),\hat{f}(e_i)\rangle|+|\langle f(x),\hat{f}(e_i)\rangle-\langle x,e_i\rangle|\big) \\ &\le \sum_{i=1}^{n}\left(|\langle f(x),\hat{f}(e_i)\rangle-\langle x,e_i\rangle|+\|f(x)\|\sqrt{2-2\sqrt{1-37\varepsilon^2}}\right) \\ &< \left(3\|x\|+\frac{67}{32}\right)n\varepsilon+n\|f(x)\|\sqrt{2-2\sqrt{1-37\varepsilon^2}}.\end{aligned} \tag{23}$$

(j) Finally, using (12), (21) and (23), we conclude that there exists a linear isometry $I:\mathbb{R}^n\to\mathbb{R}^n$, such that

$$\begin{aligned}\|f(x)-I(x)\| &\le \|f(x)-\mathbf{V}x\|+\|\mathbf{V}x-\mathbf{A}x\|+\|\mathbf{A}x-I(x)\| \\ &< \left(3\|x\|+\frac{179}{32}\right)n\varepsilon+n\|f(x)\|\sqrt{2-2\sqrt{1-37\varepsilon^2}} \\ &\quad+\left(\sqrt{2-2\sqrt{1-37\varepsilon^2}}+\varepsilon\right)\sqrt{n}\,\|x\| \\ &\le \left(\big(3+\sqrt{38}\,\big)\|x\|+\frac{179+\sqrt{38}}{32}\right)n\varepsilon+\big(\sqrt{38}+1\big)\|x\|\sqrt{n}\,\varepsilon \\ &< (10d+6)n\varepsilon+8d\sqrt{n}\,\varepsilon\end{aligned}$$

for all $x\in D$, since $\sqrt{1-37\varepsilon^2}>1-19\varepsilon^2$ and $\sqrt{2-2\sqrt{1-37\varepsilon^2}}<\sqrt{38}\,\varepsilon$ for $0<\varepsilon<\frac{1}{32}$. □

4. Examples

Theorem 3 can be applied when $n \geq 4$. If n is a small positive integer, the result of this paper is inferior to that of [18]. However, while K_n is approximately proportional to n in this paper, K_n is proportional to $n\sqrt{n}$ in [18]. Therefore, if n is a large positive integer, the result of this paper is superior to that of [18]. If n is an integer larger than 8 and $d = 1$, the result of this paper is certainly superior to that of [18].

Derived from Theorem 3, we present examples of cases where n is a small positive integer.

Example 1. *Let $D = \{x \in \mathbb{R}^4 : \|x\| \leq d\}$ for some $d \geq 1$ and let $f : D \to \mathbb{R}^4$ be a function satisfying $f(0) = 0$ and the inequality (2) for all $x, y \in D$ and some constant ε with $0 < \varepsilon < \frac{1}{32}$. According to Theorem 3, there exists an isometry $I : \mathbb{R}^4 \to \mathbb{R}^4$ that satisfies*

$$\|f(x) - I(x)\| < (56d + 24)\varepsilon$$

for all $x \in D$.

Example 2. *Let $D = \{x \in \mathbb{R}^5 : \|x\| \leq d\}$ for some $d \geq 1$ and let $f : D \to \mathbb{R}^5$ be a function satisfying $f(0) = 0$ and the inequality (2) for all $x, y \in D$ and some constant ε with $0 < \varepsilon < \frac{1}{40}$. According to Theorem 3, there exists an isometry $I : \mathbb{R}^5 \to \mathbb{R}^5$ that satisfies*

$$\|f(x) - I(x)\| < (68d + 30)\varepsilon$$

for all $x \in D$.

Example 3. *Let $D = \{x \in \mathbb{R}^9 : \|x\| \leq d\}$ for some $d \geq 1$ and let $f : D \to \mathbb{R}^9$ be a function satisfying $f(0) = 0$ and the inequality (2) for all $x, y \in D$ and some constant ε with $0 < \varepsilon < \frac{1}{72}$. According to Theorem 3, there exists an isometry $I : \mathbb{R}^9 \to \mathbb{R}^9$ that satisfies*

$$\|f(x) - I(x)\| < (114d + 54)\varepsilon$$

for all $x \in D$. When we assume $d = 1$, K_9 in [18] is greater than 189, while in this paper it is less than 168. As mentioned before, if n is an integer larger than 8, the result of this paper is superior to the result of [18].

5. Discussion

Now, to confirm the superiority of the result of this paper, we compare the result of this paper with previously published notable research results and present them in Table 1. More precisely, we compare the coefficients K_n of the upper bounds $K_n\varepsilon$ for the difference between the given function f and the sought isometry I.

Table 1. Comparison of the result of this paper with those of existing papers.

	$\mathbb{R}$	$\mathbb{R}^2$	$\mathbb{R}^3$	$\mathbb{R}^4$	$\mathbb{R}^5$	...
in [15]	4	>79	>799	>7990	>79,900	...
in [16]	27	54	81	108	135	...
in [17]	–	–	<84	128	<179	...
in [18]	–	–	<37	<57	<79	...
in [19]	–	12	21	–	–	–
in this paper	–	–	–	80	98	...

The figures in the first row of Table 1 are obtained by substituting $c = 1$ into the formula presented in the proof of [15] [Theorem 4.1]. The numbers of the third, fourth, fifth and the last rows are due to the formulas with $d = 1$ presented in [17–19] and in this paper, respectively. In the above table, we see that paper [19] provides efficient results in the 2-dimensional and 3-dimensional cases. This is because paper [19] specializes in cases

of these dimensions. It is worth confirming that the result of this paper is superior to that of [16] whenever n is an integer greater than or equal to 4.

The second author was notified recently that Vestfrid had obtained a result similar to that of [17]. In fact, we note that Vestfrid [16] improved the existing results for very large dimensions, and he demonstrated the Hyers–Ulam stability of the isometry defined on the bounded domain. We analyzed the main theorem of his paper for the case $r = R = 1$ and we present the resulting figures in the second row of Table 1. It is also worth noting that the methods used in this paper are more basic and easier for the reader to understand than the methods used by Vestfrid or other authors.

It follows from Theorem 3 that if a function $f : D \to \mathbb{R}^n$ satisfies $f(0) = 0$ and inequality (2) for any $x, y \in D$ and for some small constant $\varepsilon > 0$ then there exists an isometry $I : \mathbb{R}^n \to \mathbb{R}^n$ and a constant $K_n > 0$, such that the inequality $\|f(x) - I(x)\| \leq K_n\varepsilon$ holds true for all $x \in D$. Unfortunately, however, it seems impossible to derive this useful conclusion using Fickett's theorem. From this point of view, we dare to say that we have significantly improved Fickett's theorem.

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