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Differentiation of Solutions of Caputo Boundary Value Problems with Respect to Boundary Data

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Abstract: Under suitable continuity and uniqueness conditions, solutions of an \( \alpha \) order Caputo fractional boundary value problem are differentiated with respect to boundary values and boundary points. This extends well-known results for nth order boundary value problems. The approach used applies a standard algorithm to achieve the result and makes heavy use of recent results for differentiation of solutions of Caputo fractional initial value problems with respect to initial conditions and continuous dependence for Caputo fractional boundary value problems.

Keywords: Caputo fractional differential equation; boundary value problem; continuous dependence; variational equation

MSC: 26A33; 34A08; 34B15

1. Introduction

Let \( n \in \mathbb{N} \) with \( \alpha \in (n-1,n) \) and \( a < t_0 < b \) in \( \mathbb{R} \). Our concern is characterizing partial derivatives with respect to the boundary data for solutions to the Caputo fractional boundary value problem

\[ D^\alpha_{t_0} x(t) = f(t, x(t), x'(t), \ldots, x^{(n-1)}(t)), \quad a < t_0 < t < b, \]  

satisfying conjugate boundary conditions

\[ x(t_i) = x_i \]  

where \( D^\alpha_{t_0} x \) is the Caputo fractional derivative of order \( \alpha \) of the function \( x(t) \) and \( a < t_0 \leq t_1 < t_2 < \ldots < t_n < b \) and \( x_i \in \mathbb{R} \) for \( 1 \leq i \leq n \). These partial derivatives solve the associated Caputo fractional variational equation.

Definition 1. The \( \alpha \) order Caputo fractional variational equation of (1) along a solution \( x(t) \) is the differential equation

\[ D^\alpha_{t_0} z(t) = \sum_{j=0}^{n-1} \frac{\partial f}{\partial x_j}(t, x(t), x'(t), \ldots, x^{(n-1)}(t))z^{(j)}. \]  

In this paper, we impose suitable continuity and uniqueness hypotheses so that given a solution of (1), (2), one may take the derivative with respect to the boundary data. This derivative solves the variational Equation (3) with interesting boundary data where all but one of the boundary values are zero. Colloquially, we refer to this as studying the smoothness of conditions.

The history of initial and boundary data smoothness dates back to Peano and his work on the smoothness of initial conditions as cited by Hartman [1]. Subsequently, Peterson [2], Spencer [3], and Sukup [4] were among the first to shift to studying the smoothness
of boundary conditions. In the following decade, these results were then extended by
Henderson to right-focal boundary conditions [5,6] and conjugate-type boundary condi-
tions [7]. Over the next several decades, results were introduced for nonlocal boundary
conditions [8–10], difference equations [11–13], dynamic equations on time scales [14–16],
and researchers incorporated parameters into the nonlinearity [15,17].

With this work, we broaden the scope even further by analyzing smoothness of solutions
to Caputo fractional boundary value problems. Research into fractional differential
equations has seen an explosion of articles in the past decade that seek to generalize results
for integer order differential equations to fractional order. To name a few, we cite [18–27].
In fact, there also seem to be a limitless number of different ways to define a fractional
derivative. However, two definitions have become the source of focus amongst a broad
range of researchers in the field; namely the Riemann-Liouville and Caputo fractional
derivatives. Brief definitions may be found in Section 2. For expository material on
fractional differential equations, we refer the reader to [28–31].

The theorems and proof in this article are novel as no other research to date has
attempted to extend boundary data smoothness to fractional differential equations. The rea-
son is that the results found in this article rely heavily upon two recent results for Caputo
fractional differential equations. The first establishes differentiation of solutions of Caputo
initial value problems with respect to the initial data [22], and the second establishes the
continuous dependence on boundary conditions for Caputo boundary value problems [32].

The idea behind the proof of our main result is to first assume a unique solution to a
Caputo boundary value problem. Then, we define a difference quotient with respect to the
boundary point or boundary value of interest. We view this difference quotient in terms of
an initial value problem and apply Theorem 3.2 from [22]. This yields that the difference
quotient solves the variational equation. Finally, we take a limit by applying the continuous
dependence result, Theorem 4.2, from [32] which yields the desired result.

The remainder of the paper is organized as follows. In Section 2, one will find brief
definitions of fractional integrals and derivatives. Section 3 is where we establish our
sufficient hypotheses. For Section 4, we present important recent results in continuous
dependence and smoothness of initial conditions. Following this, we have Section 5 that
contains the main result and its proof. Finally, we conclude with a summary of project and
thoughts on future research avenues.

2. Fractional Derivatives

Let \( \alpha > 0 \). The Riemann-Liouville fractional integral of a function \( x \) of order \( \alpha \), denoted
\( I_{t_0}^\alpha x \), is defined as

\[
I_{t_0}^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} x(s) \, ds, \quad t_0 \leq t,
\]

provided the right-hand side exists. Moreover, let \( n \in \mathbb{N} \) denote a positive integer and
assume \( n - 1 < \alpha \leq n \). The Riemann-Liouville fractional derivative of order \( \alpha \) of
the function \( x \), denoted \( D_{t_0}^\alpha x \), is defined as

\[
D_{t_0}^\alpha x(t) = D^n I_{t_0}^{\alpha-n} x(t),
\]

provided the right-hand side exists. If a function \( x \) is such that

\[
D_{t_0}^\alpha \left( x(t) - \sum_{i=0}^{n-1} x^{(i)}(t_0) \frac{(t-t_0)^i}{i!} \right)
\]

exists, then the Caputo fractional derivative of order \( \alpha \) of \( x \) is defined by

\[
D_{t_0}^\alpha x(t) = D_{t_0}^\alpha \left( x(t) - \sum_{i=0}^{n-1} x^{(i)}(t_0) \frac{(t-t_0)^i}{i!} \right).
\]
Remark 1. A sufficient condition to guarantee the existence of the Caputo fractional derivative is the absolute continuity of the \((n - 1)\)st derivative of \(x(t)\). See Theorem 3.1 in [28] and discussion thereafter.

3. Preliminaries

Throughout this work, we make use of the following assumptions which are required to apply the continuous dependence and differentiation results from [22,32]

1. \(f : (a, b) \times \mathbb{R}^n \rightarrow \mathbb{R}\) is continuous;
2. for \(1 \leq i \leq n\), \(\partial f(t, x_1, \ldots, x_n) / \partial x_i : (a, b) \times \mathbb{R}^n \rightarrow \mathbb{R}\) is continuous; and
3. solutions to initial value problems for (1) are unique on \((a, b)\);

Next, we present two more hypotheses which establish a uniqueness condition for (1) and (3), respectively.

4. Given points \(a < t_0 \leq t_1 < t_2 < \ldots < t_n < b\), if \(y\) and \(z\) are solutions of (1) such that for \(1 \leq i \leq n\), \(y(t_i) = z(t_i)\), then \(y(t) = z(t)\) on \([t_0, b]\); and
5. given points \(a < t_0 \leq t_1 < t_2 < \ldots < t_n < b\), if \(u\) is a solution of (3) along (1) such that for \(1 \leq i \leq n\), \(u(t_i) = 0\), then \(u(t) \equiv 0\) on \([t_0, b]\).

Next, we present two crucial results that make this work possible. Let \([c, d] \subset \mathbb{R}\) and for \(x \in C[c, d]\), define

\[
\|x\|_{0,[c,d]} = \max_{t \in [c,d]} |x(t)|.
\]

If \(k \in \mathbb{N}\), for \(x \in C^k[c, d]\), define

\[
\|x\|_{k,[c,d]} = \max\{\|x\|_{0,[c,d]}, \|x'\|_{0,[c,d]}, \ldots, \|x^{(k)}\|_{0,[c,d]}\}.
\]

We seek a boundary value problem result as an analog of the initial value problem result from Eloe et al [22].

Theorem 1. Assume that hypotheses (1)–(3) hold. Let \(y(t) := y(t; t_0, y_0, \ldots, y_{n−1})\) be the unique solution of the initial value problem (1) satisfying

\[
y^{(i)}(t_0) = y_i, \quad i = 0, \ldots, n - 1,
\]

with maximal interval of existence \([t_0, \omega]\). Choose \([c, d] \subset [t_0, \omega]\).

(a) for each \(0 \leq j \leq n − 1\), \(\gamma_j(t) := \partial y(t) / \partial y_j\) exists and is the solution of the variational Equation (3) along \(y(t)\) on \([c, d]\) and hence, \([t_0, \omega]\) satisfying the initial conditions

\[
\gamma^{(i)}_j(t_0) = \delta_{ij}, \quad 0 \leq i \leq n - 1;
\]

(b) if, in addition, \(f\) has a continuous first derivative with respect to \(t\) and

\[
f(t_0, y_0, y_1, \ldots, y_{n−1}) = 0,
\]

then \(\beta(t) := \partial y(t) / \partial t_0\) exists and is the solution of the variational Equation (3) along \(y(t)\) on \([c, d]\) and hence, \([t_0, \omega]\) satisfying the initial conditions

\[
\beta^{(i)}(t_0) = -y^{(i+1)}(t_0), \quad 0 \leq i \leq n - 1; \text{ and}
\]

(c) Under the additional in (b), \(\beta(t) = -\sum_{i=0}^{n-1} y^{(i+1)}(t_0) \gamma_i(t)\).

We also use recent continuous dependence on boundary conditions results for Caputo fractional differential equations [32]. The first one is if the left-most boundary condition is
to the right of the starting point of the Caputo fractional derivative; namely $t_0 < t_1$, and the second is if they are equal; namely $t_0 = t_1$. Note that the second result has an additional condition to establish continuous dependence to the left of $t_0$.

**Theorem 2.** [Case when $t_0 < t_1$] Assume that hypotheses (1), (3), and (4) hold. Let $x(t)$ be a solution of (1) on $[t_0, b]$, $[c, d] \subset [t_0, b]$ with points $t_0 \leq c < t_1 < t_2 < \ldots < t_n < d$, and $\epsilon > 0$. Then, there exists a $\delta(\epsilon, [c, d]) > 0$ such that if for $1 \leq i \leq n$, $|t_i - \tau_i| < \delta$ with $c < \tau_1 < \tau_2 < \tau_3 < \ldots < \tau_n < d$ and $|x(t_i) - y_i| < \delta$ with $y_i \in \mathbb{R}$, then there exists a solution $y(t)$ of (1) satisfying $y(\tau_i) = y_i$. Also,

$$||x(t) - y(t)||_{n-1,[c,d]} < \epsilon.$$ 

**Theorem 3.** [Case when $t_0 = t_1$] Assume that hypotheses (1), (3), and (4) hold. Let $x(t)$ be a solution of (1) on $[t_1, b]$, $[c, d] \subset [t_1, b]$ with points $c = t_1 < t_2 < \ldots < t_n < d$, and $\epsilon > 0$. Then, there exists a $\delta(\epsilon, [c, d]) > 0$ such that if for $2 \leq i \leq n$, $|t_i - \tau_i| < \delta$ with $c < \tau_2 < \tau_3 < \ldots < \tau_n < d$ and for $1 \leq i \leq n$, $|x(t_i) - y_i| < \delta$ with $y_i \in \mathbb{R}$, then there exists a solution $y(t)$ of (1) satisfying $y(t_1) = y_1$ and for $2 \leq i \leq n$, $y(\tau_i) = y_i$. Also,

$$||x(t) - y(t)||_{n-1,[c,d]} < \epsilon.$$ 

Additionally, if $f_k : (a, b) \times \mathbb{R}^n \to \mathbb{R}$ is a sequence of continuous functions that converge uniformly to $f$ on compact subsets of $[c, d] \times \mathbb{R}^n$ and for $k \geq 1$, $t^k_1$ is an increasing sequence such that $t^k_1 \uparrow t_1$ as $k \to \infty$, then there exists a $K$ such that if $k \geq K$, then

$$||x_k(t) - x(t)||_{n-1,[c,d]} \to 0 \quad \text{as} \quad k \to \infty.$$

### 4. Main Results

In this section, we present our boundary value problem analog. First, we state and prove the result when $t_0 < t_1$.

**Theorem 4.** [Case when $t_0 < t_1$] Assume conditions (1)–(5) are satisfied and that $t_0 < t_1$. Let $x(t) := x(t, t_1, \ldots, t_n, x_1, \ldots, x_n)$ be a solution of (1) satisfying $x(t_i) = x_i$ for $1 \leq i \leq n$ on $[t_0, \omega) \subset (a, b)$. Then,

(a) for each $1 \leq j \leq n$, $z_j(t) := \partial x(t)/\partial x_j$ exists and is the solution of the variational Equation (3) along $x(t)$ on $[c, d]$ and hence, $[t_0, \omega)$ satisfying the boundary conditions

$$z_j(t_i) = \delta_{ij}, \quad 1 \leq i \leq n;$$

(b) if $f$ has a continuous first derivative with respect to $t$ and for each $1 \leq j \leq n$,

$$f\left(t, x(t), x'(t), \ldots, x^{(n-1)}(t)\right) = 0,$$

then $w_j(t) := \partial x(t)/\partial t_j$ exists and is the solution of the variational Equation (3) along $x(t)$ on $[c, d]$ and hence, $[t_0, \omega)$ satisfying the boundary conditions

$$w_j(t) = -x'(t)\delta_{ij}, \quad 1 \leq i \leq n; \quad \text{and}$$

(c) Under the conditions of (b), for each $1 \leq j \leq n$, $w_j(t) = -x'(t)z_j(t)$.

**Proof.** We will only prove part (a) as the proof of part (b) is similar. Part (c) is an immediate consequence from parts (a) and (b) when coupled with hypothesis (5).

Let $1 \leq j \leq n$, and consider $\partial x(t)/\partial x_j$. In the interests of conserving space and lessening the tedious notation, we denote $x(t, t_1, \ldots, t_n, x_1, \ldots, x_n)$ by $x(t, x_j)$ as $x_j$ is the boundary value of interest.
Let $\delta > 0$ be as in Theorem 2, $0 < |h| < \delta$ be given, and define the difference quotient with respect to $x_j$ by

$$z_{jh}(t) = \frac{1}{h} [x(t; x_j + h) - x(t; x_j)].$$

Note that for every $h \neq 0$,

$$z_{jh}(t_j) = \frac{1}{h} [x(t_j; x_j + h) - x(t_j; x_j)] = \frac{1}{h} [(x_j + h) - x_j] = \frac{1}{h} |h| = 1.$$

Also, for every $h \neq 0$, $1 \leq k \leq n$ with $k \neq j$,

$$z_{jh}(t_k) = \frac{1}{h} [x(t_k; x_j + h) - x(t_k; x_j)] = \frac{1}{h} [x_k - x_k] = 0.$$

Now that we have established the boundary conditions for $z_{jh}(t)$, we show that $z_{jh}(t)$ solves the variational equation. To that end, for $1 \leq i \leq n - 1$, let

$$v_i = x^{(i)}(t_j; x_j)$$

and

$$e_i = e_i(h) = x^{(i)}(t_j; x_j + h) - v_i.$$

By Theorem 2, for $1 \leq i \leq n - 1$, $e_i = e_i(h) \to 0$ as $h \to 0$. Using the notation of Theorem 1 for solutions of initial value problems for (1), viewing $x(t)$ as the solution of an initial value problem, and denoting the solution $x(t)$ by $y(t; t_j, x_j, v_1, \ldots, v_{n-1})$, we have

$$z_{jh}(t) = \frac{1}{h} [y(t; t_j, x_j + h, v_1 + e_1, \ldots, v_{n-1} + e_{n-1}) - y(t; t_j, x_j, v_1, \ldots, v_{n-1} + e_{n-1})].$$

Then, by utilizing telescoping sums, we have

$$z_{jh}(t) = \frac{1}{h} \left\{ y(t; t_j, x_j + h, v_1 + e_1, \ldots, v_{n-1} + e_{n-1}) - y(t; t_j, x_j, v_1 + e_1, \ldots, v_{n-1} + e_{n-1}) ight\}$$

$$+ \left\{ y(t; t_j, x_j, v_1 + e_1, \ldots, v_{n-1} + e_{n-1}) - y(t; t_j, x_j, v_1, \ldots, v_{n-1} + e_{n-1}) ight\}$$

$$+ \left\{ y(t; t_j, x_j, v_1, \ldots, v_{n-1} + e_{n-1}) - y(t; t_j, x_j, v_1, \ldots, v_{n-1}) ight\} - \cdots$$

$$+ \left\{ y(t; t_j, x_j, v_1, \ldots, v_{n-1}) - y(t; t_j, x_j) \right\} \right\}.$$

By Theorem 1 and the Mean Value Theorem, we obtain
\[ z_{jk}(t) = \frac{1}{h} \left[ \gamma_0(t, y(t; t_j, x_j) + \hat{h}, v_1 + \epsilon_1, \ldots, v_{n-1} + \epsilon_{n-1}) (x_j + h - x_j) \right. \\
+ \gamma_1(t, y(t; t_j, x_j, v_1 + \epsilon_1, \ldots, v_{n-1} + \epsilon_{n-1}) (v_1 + \epsilon_1 - v_1) + \cdots \\
+ \gamma_{n-1}(t, y(t; t_j, x_j, v_1, \ldots, v_{n-1} + \epsilon_{n-1}) (v_{n-1} + \epsilon_{n-1} - v_{n-1}) \right] \\
= \gamma_0(t, y(t; t_j, x_j + \hat{h}, v_1 + \epsilon_1, \ldots, v_{n-1} + \epsilon_{n-1})) \\
+ \frac{\epsilon_1}{h} \gamma_1(t, y(t; t_j, x_j, v_1 + \epsilon_1, \ldots, v_{n-1} + \epsilon_{n-1})) + \cdots \\
+ \frac{\epsilon_{n-1}}{h} \gamma_{n-1}(t, y(t; t_j, x_j, v_1, \ldots, v_{n-1} + \epsilon_{n-1})).
\]

where, for \(0 \leq k \leq n-1\), \(\gamma_k(t, y(\cdot))\) is the solution of the variational Equation (3) along \(y(\cdot)\) satisfying

\[ \gamma_k^{(i)}(t_j) = \delta_{ik}, \quad 0 \leq i \leq n-1. \]

Furthermore, for each \(1 \leq i \leq n-1\), \(v_i + \hat{e}_i\) is between \(v_i\) and \(v_i + e_i\). Thus, to show \(\lim_{h \to 0} z_{jk}(x)\) exists, it suffices to show, for each \(1 \leq i \leq n-1\), \(\lim_{h \to 0} \epsilon_i / h\) exists.

Now, from the construction of \(z_{jk}(t)\), we have

\[ z_{jk}(t_k) = 0, \quad 1 \leq k \leq n \text{ with } k \neq j. \]

Hence, for \(1 \leq k \leq n\) with \(k \neq j\), we have a system of \(n-1\) linear equations with \(n-1\) unknowns:

\[- \gamma_0(t_k, y(t_k; t_j, x_j) + \hat{h}, v_1 + \epsilon_1, \ldots, v_{n-1} + \epsilon_{n-1})) \\
= \frac{\epsilon_1}{h} \gamma_1(t_k, y(t_k; t_j, x_j, v_1 + \epsilon_1, \ldots, v_{n-1} + \epsilon_{n-1})) + \cdots \\
+ \frac{\epsilon_{n-1}}{h} \gamma_{n-1}(t_k, y(t_k; t_j, x_j, v_1, \ldots, v_{n-1} + \epsilon_{n-1})).
\]

In the system of equations above, we notice that \(y(\cdot)\) is not always the same. Therefore, we consider the coefficient matrix \(M\) based on \(y(t)\)

\[
M := \begin{pmatrix}
\gamma_1(t_1, y(t)) & \cdots & \gamma_{n-1}(t_1, y(t)) \\
\gamma_1(t_2, y(t)) & \cdots & \gamma_{n-1}(t_2, y(t)) \\
\vdots & \ddots & \vdots \\
\gamma_1(t_{j-1}, y(t)) & \cdots & \gamma_{n-1}(t_{j-1}, y(t)) \\
\gamma_1(t_{j+1}, y(t)) & \cdots & \gamma_{n-1}(t_{j+1}, y(t)) \\
\vdots & \ddots & \vdots \\
\gamma_1(t_n, y(t)) & \cdots & \gamma_{n-1}(t_n, y(t))
\end{pmatrix}
\]

We claim \(\det(M) \neq 0\). Suppose to the contrary that \(\det(M) = 0\). Then, there exist \(p_i \in \mathbb{R}\) for \(1 \leq i \leq n-1\) not all zero such that

\[
p_1 \begin{pmatrix}
\gamma_1(t_1, y(t)) \\
\gamma_1(t_2, y(t)) \\
\vdots \\
\gamma_1(t_{j-1}, y(t)) \\
\gamma_1(t_{j+1}, y(t)) \\
\vdots \\
\gamma_1(t_n, y(t))
\end{pmatrix} + \cdots + p_{n-1} \begin{pmatrix}
\gamma_{n-1}(t_1, y(t)) \\
\gamma_{n-1}(t_2, y(t)) \\
\vdots \\
\gamma_{n-1}(t_{j-1}, y(t)) \\
\gamma_{n-1}(t_{j+1}, y(t)) \\
\vdots \\
\gamma_{n-1}(t_n, y(t))
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
\vdots \\
0 \\
0 \\
\vdots \\
0
\end{pmatrix}.
\]
Set
\[ w(t, y(t)) := p_1 \gamma_1(t, y(t)) + \cdots + p_{n-1} \gamma_{n-1}(t, y(t)). \]
Then, \( w(t, y(t)) \) is a nontrivial solution of the variational Equation (3). However, \( w(t_j, y(t_j)) = 0 \), and for \( 1 \leq k \leq n-1 \) with \( k \neq j \), \( w(x_k, y(t_j)) = 0 \). By hypothesis (5), \( w(t, y(t)) = 0 \). Thus, \( p_1 = p_2 = \cdots = p_{n-1} = 0 \) which is a contradiction to the choice of the \( p_i \)'s. Hence, \( \det(M) \neq 0 \).

As a result of continuous dependence, for \( h \neq 0 \) and sufficiently small, \( \det(M(h)) \neq 0 \) implying \( M(h) \) has an inverse where \( M(h) \) is the appropriately defined matrix from the system of equations. Therefore, for each \( 1 \leq i \leq n-1 \), we are able to find \( \epsilon_i/h \) using Cramer’s rule.

Note as \( h \to 0 \), \( \det(M(h)) \to \det(M) \), and so for \( 1 \leq i \leq n-1 \), \( \epsilon_i(h)/h \to \det(M_i)/\det(M) := B_i \) as \( h \to 0 \), where \( M_i \) is the \( n-1 \times n-1 \) matrix found by replacing the appropriate column of the matrix defining \( M \) by

\[
\text{col} \left[ -\gamma_0(t_1, x(t)), -\gamma_0(t_2, x(t)), \ldots, -\gamma_0(t_{j-1}, x(t)), -\gamma_0(t_{j+1}, x(t)), \ldots, -\gamma_0(t_k, x(t)) \right].
\]

Now, let \( z_j(t) = \lim_{h \to 0} z_{jh}(t) \), and by construction of \( z_{jh}(t) \),
\[
z_j(t) = \frac{\partial x}{\partial x_j}(t).
\]
Furthermore,
\[
z_j(t) = \lim_{h \to 0} z_{jh}(t) = \gamma_0(t, x(t)) + \sum_{i=1}^{n-1} B_i \gamma_i(t, x(t))
\]
which is a solution of the variational Equation (3) along \( x(t) \). In addition, for \( 1 \leq j \leq n \),
\[
z_j(x_k) = \lim_{h \to 0} z_{jh}(x_k) = \delta_{jk}.
\]
This completes the argument for \( \partial x(t)/\partial x_j \). \( \square \)

Next, with the additional assumption from Theorem 3, the same result is established for \( t_0 = t_1 \) and the proof remains the same. Without this additional assumption, the derivative at \( t_1 \) would only be a right-hand derivative but the result still holds.

**Theorem 5.** [Case when \( t_0 = t_1 \)] Assume conditions (1)–(5) are satisfied and that \( t_0 = t_1 \). Let \( x(t) := x(t_1, \ldots, t_n, x_1, \ldots, x_n) \) be a solution of (1) satisfying \( x(t_i) = x_i \) for \( 1 \leq i \leq n \) on \( [t_0, \omega) \subset (a, b) \). Then,

(a) for each \( 1 \leq j \leq n \), \( z_j(t) := \partial x(t)/\partial x_j \) exists and is the solution of the variational Equation (3) along \( x(t) \) on \( [c, d] \) and hence, \( [t_0, \omega) \) satisfying the boundary conditions
\[
z_j(t_i) = \delta_{ij}, \ 1 \leq i \leq n;
\]
(b) if \( f \) has a continuous first derivative with respect to \( t \),
\[
f \left( t_1, x(t_1), x'(t_1), \ldots, x^{(n-1)}(t_1) \right) = 0,
\]
and additionally, \( f_k : (a, b) \times \mathbb{R}^n \to \mathbb{R} \) is a sequence of continuous functions that converge uniformly to \( f \) on compact subsets of \( [c, d] \times \mathbb{R}^n \) and for \( k \geq 1 \), \( t_k^1 \) is an increasing sequence such that \( t_k^1 \uparrow t_1^1 \) as \( k \to \infty \), then \( w_k(t) := \partial x(t)/\partial t_1 \) exists and is the solution of the variational Equation (3) along \( x(t) \) on \( [c, d] \) and hence, \( [t_0, \omega) \) satisfying the boundary conditions
\[
w_k(t_i) = -x'(t_i) \delta_{i1}, \ 1 \leq i \leq n;
(c) for each \(2 \leq j \leq n\), if \(f\) has a continuous first derivative with respect to \(t\) and
\[
f\left(t_j, x(t_j), x'(t_j), \ldots, x^{(n-1)}(t_j)\right) = 0,
\]
then \(w_j(t) := \frac{\partial x}{\partial t_j}\) exists and is the solution of the variational Equation (3) along \(x(t)\) on \([c, d]\) and hence, \(t_0, \omega\) satisfying the boundary conditions
\[
w_j(t_i) = -x'(t_i)\delta_{ij}, \quad 1 \leq i \leq n; \text{ and}
\]
(d) Under the conditions of (b) and (c), for each \(1 \leq j \leq n\), \(w_j(t) = -x'(t_j)z_j(t)\).

5. Conclusions

In this paper, we showed that under suitable continuity and uniqueness conditions that a solution Caputo fractional conjugate boundary value problem may be differentiated with respect to the boundary points and the boundary values. The resulting function solves the Caputo fractional version of the variational equation. This work only recently became possible as its proof relies extensively upon the differentiation of a solution to a Caputo fractional initial value problem [22] and the continuous dependence of solutions to Caputo fractional boundary value problems with respect to boundary data [32].

The results contained herein are novel and have not been explored or considered previously. We believe this result is foundational in smoothness of solutions for Caputo fractional boundary value problems and the proof sets a template for how to proceed for several future research avenues such as Caputo fractional differential equations with varying types of boundary conditions including parameter dependence, Caputo fractional difference equations, and Caputo fractional dynamic equations.

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When looking at the wide breadth and depth of research conducted for smoothness of solutions for integer order differential, difference, and dynamic equations, it is clear there is a lot of work to be done in this area.

Other future work could entail loosening the hypothesis that the nonlinearity be continuously differentiable. This was posited for future study in [22]. Another avenue would be finding sufficient conditions to guarantee the uniqueness condition in hypothesis (3) in certain contexts.

Funding: This research received no external funding.

Data Availability Statement: No new data were created or analyzed in this study.

Conflicts of Interest: The author declares no conflicts of interest.

References


