Nodal Invulnerability Recovery Considering Power Generation Balance: A Bi-Objective Robust Optimization Framework

Xueyang Zhang *, Shengjun Huang, Qingxia Li, Rui Wang, Tao Zhang, and Bo Guo

College of Systems Engineering, National University of Defense Technology, Changsha 410073, China; huangshengjun@nudt.edu.cn (S.H.); liqingxia22@nudt.edu.cn (Q.L.); ruiwangnudt@gmail.com (R.W.); zhangtao@nudt.edu.cn (T.Z.); guobo@nudt.edu.cn (B.G.)
* Correspondence: zhangxueyang@nudt.edu.cn

Abstract: Nodal invulnerability has broad application prospects because of its emphasis on the differences between buses. Due to their long-term exposure, transmission lines are inevitably susceptible to damage caused by physical attacks or extreme weather. Therefore, restoring nodal invulnerability through a remedial approach or the introduction of mobile generators (MGs) is pivotal for resisting subsequent damage after a system is attacked. However, the research devoted to this field is limited. In order to fill the gap, this study conducts research on the configuration of MGs considering power generation balance to recover nodal invulnerability. First, a defender–attacker–defender (DAD) model is established, corresponding to the bi-objective robust optimization problem. The upper-level model is formulated to obtain the optimal compromise configuration scheme, the uncertainties of the attacked lines are elucidated in the middle level, and the nodal $N-k$ security criterion utilized for measuring nodal invulnerability cooperates in the lower level. Then, a modified column-and-constraint generation (C&CG) algorithm is developed to incorporate fuzzy mathematics into the solution framework. In addition, the nodal invulnerability settings are optimized under limited resources. Numerical experiments are executed on the IEEE 24-bus system to verify the effectiveness and rationality of the proposed method.

Keywords: bi-objective robust optimization; fuzzy mathematics; nodal invulnerability; power generation balance

MSC: 90-10

1. Introduction

Extreme weather and cyber–physical attacks threaten the security and invulnerability of power systems. Furthermore, increasing consumer demand has become a major trend in the development of power grids, and the complexity of the load types has significantly escalated, making systems more fragile than before [1]. At the same time, the large proportion of variable renewable energy (VRE) integration poses great challenges for stable system operation [2,3]. In such cases, the safety stability margin of a power system decreases, increasing the probability of power outage accidents when the system is disturbed [4,5]. Due to the occurrence of high-impact low-probability (HILP) events, researchers’ attention has been drawn to invulnerability assessment.

Invulnerability mainly refers to the fault tolerance and survival ability of a system under extreme conditions, with an important aspect being the continuity of the power supplied to the loads. The $N-k$ security criterion ($k \geq 2$) has been presented to constrain the safe and stable operation of power systems [6–8]. However, traditional $N-k$ security-constrained invulnerability enhancement studies have a fatal disadvantage, which is the excessive system development and serious resource waste with the growth of $k$ and expansion of the system scale. Essentially, to overcome this disadvantage, the invulnerability
requirements of each bus should be determined based on their relative importance instead of being unified. Therefore, the nodal $N - k$ security criterion has emerged.

First, buses can be divided into power supply and load buses based on their net load, which is obtained by subtracting the power generation capacity from the load demand. Power supply buses prioritize meeting their own load demands and are unlikely to lose load. Thus, the nodal $N - k$ security criterion solely constrains load buses, corresponding to buses 4–9 in the example depicted in Figure 1. It declares that if a bus does not shed load when any $k$ components fail due to damage, this bus is considered to meet the nodal $N - k$ security criterion. Compared to the traditional standard, this criterion shifts the core research object from the system as a whole to a single bus, emphasizing the distinctions between buses. Furthermore, it allows buses to be separated into different sets based on their significance, with each set corresponding to an invulnerability level. If the invulnerability requirements of each bus in a system cannot be met, the nodal invulnerability can be upgraded through transmission line expansion planning (TEP), which is related to topology optimization [9–11]. Figure 1 illustrates the upgrade strategy for a nine-bus system, in which all load buses can meet their invulnerability requirements benefit from the invested transmission lines, represented by dashed lines. This paper considers the case in which TEP has already been implemented to upgrade each bus to its corresponding invulnerability level, and explores a method for quickly restoring nodal invulnerability in the event of system damage when short-term repair is not available.

![Schematic diagram of the nodal $N - k$ security criterion and TEP strategy.](image)

Providing mobile or portable emergency generators for endangered areas plays an indispensable role in re-establishing the power supply after a system is devastated [12]. For some remote but crucial loads, the functionality of the power system can be partially restored by means of a suitable configuration of emergency generators providing electrical energy support [13–15]. The authors of [16] combined the long-term planning of newly added underground cables with a short-term configuration of mobile generators (MGs) to enhance system invulnerability, demonstrating better performance compared to simply considering cable investment. However, current research on allocating MGs overlooks the distinctions between buses. MGs are often expensive; therefore, it is more practical to configure them in accordance with nodal invulnerability requirements rather than uniform standards. Additionally, the normal operation of a generator entails a suitable load rate, between 60% and 90%. An excessively low rate will result in an uncontrolled rotor speed, while an particularly high load rate will cause a sharp increase in the unit temperature, resulting in components blowing out easily. Both situations can lead to internal power outages in the absence of external disturbances. Therefore, in this paper, the power generation balance is considered when allocating MGs for nodal invulnerability recovery after transmission line damage provoked by attacks. Furthermore, in order to address the uncertainty of attacks, we adopt a robust optimization (RO) approach [17], leading to a bi-objective RO planning problem.

Generally, there are two kinds of methods for solving optimization problems: analytical and numerical [18]. Analytical methods follow a rigorous mathematical derivation process and enable it to produce precise solutions. Numerical methods can be further divided into two categories: classical and intelligent. The representatives of classical numer-
ical solving methods for multi-objective optimization are linear weighting method [19] and the $\epsilon$-constraint manner [20]. Intelligent optimization methods represented by evolutionary algorithms [21] and swarm intelligence [22], have also received widespread attention in recent years because of their efficient solving performance. Moreover, fuzzy optimization is becoming a rising star, whose basic principle is to seek satisfactory solutions that make each objective as optimal as possible within the fuzzy set of optimal solutions for every single objective [23]. Reference [24] systematically reviewed numerous methods of fuzzy optimization. In contrast to the multi-objective optimization algorithms mentioned previously, fuzzy optimization avoids the problems of subjectivity that may arise when converting multiple objectives into a single objective. Meanwhile, as this study does not concentrate on intelligent optimization methods, fuzzy optimization works harmoniously with the RO solution framework employed in this paper.

The prominent contributions and novelty of the present study are outlined as follows:

1. Oriented by decoupling the invulnerability recovery from the system level to individual nodes, a defender–attacker–defender (DAD) model is established in combination with the nodal $N-k$ security criterion, and the power generation balance is considered when allocating MGs through a bi-objective optimization problem.

2. An improved column-and-constraint generation (C&CG) algorithm is developed, which combines fuzzy mathematics with RO to empower the solution of bi-objective programming problems.

3. The invulnerability requirement of each bus is optimized subject to resource constraints, aiming to maximize benefits at the same or even lower cost as directly restoring to the original nodal invulnerability.

The remainder of this paper is organized as follows: In Section 2, the DAD model based on the uncertainty of attacks on transmission lines is established, in which the bi-objective and nodal $N-k$ security criterion are reflected in the first and third layers, respectively. Then, the modified C&CG algorithm framework and the optimization of the $k$-value settings under limited resources are provided in Section 3. Additionally, extensive numerical experiments are implemented and the findings are discussed in Section 4 to manifest the effectiveness and rationality of the proposed method. We conclude this study and outline promising future research directions in Section 5.

2. Model Formulation

The model proposed in this paper is designed from the perspective of the DAD architecture based on an attack–defense game [25], and the scenario of interest is the restoration of the invulnerability of each bus to a certain level via a suitable configuration of MGs after the system is subjected to physical transmission line attacks. The upper level generates an optimal configuration strategy of MGs, taking both investment cost and power generation balance into account. The uncertainty of attacked lines are depicted in the middle level, and the optimal power flow (OPF) that cooperates with the nodal $N-k$ security criterion is implemented in the lower level.

Compared to AD models that only optimize at the operational level or single-layer optimization models that cannot consider fault scenarios, DAD models have significant advantages. Benefiting from taking another interaction level between the attacker and the defender into account, the tri-level model yields the optimal strategy from a global perspective. It provides not only the optimal configuration scheme of MGs but also an operational condition to maximize the protection of critical loads.

Figure 2 depicts the results of nodal invulnerability recovery, where the red dashed lines represent lines that are attacked and removed from the system topology due to the inability to conduct line maintenance within a short period of time.

In the example presented in Figure 2, there were originally only three generators in the system, located at buses 1–3, i.e., $n_g = 3$. As indicated by the colors of the dashed boxes, bus 8 is configured with a type-1 MG for resisting subsequent $N-1$ fault, and the other two correspond to recovering bus 9 to the nodal $N-3$ security level. The recovery strategy
for the buses is hierarchical, with each round of restoration corresponding to a particular attack intensity. That is, after the first round of recovery, all buses fulfill the $N - 1$ security criterion, and after the second round of restoration, buses 5, 6, and 9 satisfy the nodal $N - 2$ security constraint until bus 9 regains its original invulnerability after the final round.

![Diagram](image)

Figure 2. Nodal invulnerability restoration after attacks. Red dashed lines: damaged lines. Blue dashed box: MG configuration strategy in the first round of restoration. Orange dashed box: Nodal invulnerability restoration after attacks. Red dashed lines: damaged lines.

The model formulated in this section is based on a single round of recovery as part of the whole recovery process, and it is divided into three layers. Equations (1)–(16) represent the upper layer of the DAD model, corresponding to the behavior of the defender:

\[
F = \operatorname{Minimize}\left( f_1, f_2 \right) \quad \text{Minimize} \quad f_1 = \sum_{i \in T} \sum_{j \in N} a_{ij} \bar{G}_t \quad (1)
\]

\[
f_2 = \frac{1}{n_G + 3n - 1} \left\{ \sum_{j \in N_G} (a_{0,j} - v)^2 + \sum_{i \in N, t \in T} (a_{ij} - v)^2 \right\} \quad (2)
\]

\[
s.t. \quad a_{ij}, w_{ij} \in \{0,1\}, \quad \forall i \in N, \forall t \in T \quad (3)
\]

\[
a_{0,j} = a_{0,j} \bar{G}_t, \quad \forall j \in N_G \quad (4)
\]

\[
g^n_{0,j} = a_{ij} \bar{G}_t, \quad \forall j \in N_G \quad (5)
\]

\[
g^n_{ij} = a_{ij} \bar{G}_t, \quad \forall i \in N, \forall t \in T \quad (6)
\]

\[
s = \sum_{j \in N_G} a_{0,j} + \sum_{i \in N, t \in T} a_{ij} \quad (7)
\]

\[
v(n_G + \sum_{i \in N, t \in T} w_{ij}) = s \quad (8)
\]

\[
\bar{a}_{ij} = a_{ij} - w_{ij} + v(1 - w_{ij}), \quad \forall i \in N, \forall t \in T \quad (9)
\]

\[
w_{ij} = c_{ij} + a_{ij}, \quad \forall i \in N, \forall t \in T \quad (10)
\]

\[
0 \leq g^n_{0,j} \leq \bar{G}_t, \quad \forall j \in N_G \quad (11)
\]

\[
0 \leq g^n_{ij} \leq w_{ij} \bar{G}_t, \quad \forall i \in N, \forall t \in T \quad (12)
\]

\[
R_i g^n_{0,j} + \sum_{t \in T} g^n_{ij} + \sum_{l \in L_{s}(t) = i} p^n_l - \sum_{l \in L_{r}(t) = i} p^n_l = d_i, \quad \forall i \in N \quad (13)
\]

\[
p^n_l = B_l (\theta^n_{s(l)} - \theta^n_{r(l)}), \quad \forall l \in L \quad (14)
\]

\[
p^n_l \leq p^n_l \leq \bar{P}_l, \quad \forall l \in L \quad (15)
\]

\[
-2\pi \leq \theta^n_{l} \leq 2\pi, \quad \forall i \in N \quad (16)
\]
where \( a_{j} \) denote load rate on original generator \( j \) and corrected load rate on type-\( t \) MG at bus \( i \) under normal operating conditions, respectively. \( T \) and \( \mathcal{N} \) are sets of MG types and buses. \( f_1 \) and \( f_2 \) are two objective functions of the upper level. The investment cost of MGs is shown in (2); note that the relationship between investment cost and generator capacity is assumed to be linear. \( a_{j} \) is a binary decision variable; it takes 1 if a type-\( t \) MG is integrated at bus \( i \) in this round of recovery, where different MG types are relevant to different power limits.

The balance of generation load rates is measured by the variance in (3), where \( n_g \) and \( n \) are the number of generators and buses in the system. \( n_{g} + 3n \) denotes the number of decision variables related to the calculation of variance, and the denominator is subtracted by one to ensure that the degree of freedom is three less than the total number of data. Moreover, \( v \) is the mean load rate. This method is simple and is capable of effectively capturing fluctuations in data. Excessive variance reflects that the generator load rate is either very low or high, which is apparently unacceptable. However, a low variance does not necessarily indicate that the allocation plan is reasonable because high investment cost may occur, leading to the overbuilding of the system. Therefore, the mutual game between the two objectives determines that the optimal configuration of capacities and load rates are both rational.

Compared with the traditional DAD models, consideration is given to both the investment cost of MGs and the balance of generation load rates, which is unfulfillable by the single objective model. The configuration strategy can not only provide the capacity of MGs located at each bus but also offer a suitable operation power output of generators. It can be obtained from Figure 2 that in the first round, \( a_{8,1} = 1 \), \( a_{0,j}(j \in \{1, 2, 3\}) \) are the load rates of the three original generators and \( \tilde{a}_{8,1} \) denotes the load rate of type-1 MG at bus 8.

Constraint (4) restricts two binary variables, where \( w_{ij} \) is an auxiliary variable, indicating the current configuration status of the type-\( t \) MG at bus \( i \). Constraints (5)–(8) describe the constraints that need to be met for each generator in terms of total, and mean load rate. To facilitate the classification of variables, the generation power and load rates of the original generators and MGs are represented separately. In (5), \( \mathcal{N}_G \) is the set of original generators of the system, and the power generation of original generator \( j \) under normal operating conditions denoted by \( g_{0,j}^n(kW) \) is determined by its load rate and upper limit \( \mathcal{G}_j(kW) \). A constraint for MGs at each bus is given in (6), where \( g_{t,i}^n(kW) \) is the power generation of the type-\( t \) MG at bus \( i \) under normal operating conditions and \( \mathcal{G}_t(kW) \) is the upper limit of type-\( t \) MG. The summation and average value of load rate are shown in (7) and (8).

Due to the lack of MG expansion at some buses, i.e., \( w_{ij} = 0 \), the load rate obtained from (6) can take any value. In response to this situation, constraint (9) corrects \( a_{ij} \) to the mean load rate, this correction ensures that the result of solving the variance will not be affected in any way. Taking the 9-bus system in Figure 2 as an example, in the first round of restoration, no other buses except bus 8 have MGs, so it is necessary to correct \( a_{ij} \) of the other 8 buses to \( v \) so that only the generators actually in the system contribute to the results when calculating the variance.

Constraint (10) implies a rule that if a type-\( t \) MG has already been configured at bus \( i \) before this round of restoration, i.e., \( c_{ij} = 1 \), then the same configuration decision will not be made again; that is, each bus will be allocated at most one MG of each type. In the first round of restoration, \( c_{ij} = 0 \) since no MGs were previously invested. However, as seen from Figure 2, \( c_{8,1} = 1 \) in the subsequent recovery process. Equations (11)–(16) are the constraints on the auxiliary decision variables created to solve for the generation load rates, which correspond to the system operation rules under normal operating conditions. Generation constraints of the original generators are given in (11). Likewise, the power generation of newly added MGs are confined by the investment status and their types, as presented in (12).

The node power balance equation is given in (13), where \( R \) is the generator–node correlation matrix \( (n \times n_g) \), and \( R_i \) is the \( i \)-th row of the matrix. Moreover, \( g_{0,i}^n \) is a column vector with dimensions of \( n_g \times 1 \). Therefore, the former two terms are the power generation of the original generator and MGs located at bus \( i \), respectively. Additionally, net power
injection through transmission lines is expressed by \( \sum_{l \in L \cap r(l) = i} P_l^0 - \sum_{l \in L \cap s(l) = i} P_l^f \), where \( p_l^f \) (kW) is the power flow on transmission line \( l \) under normal operating conditions, and \( r(l) / s(l) \) are the receiving/sending ends of line \( l \), respectively. DC power flow equation is expressed in (14), where \( B_l(S) \) is the susceptance of line \( l \), \( \theta_{s(l)}^i - \theta_{r(l)}^i \) (rad) signifies the power angle difference between the beginning and end of the line, and \( L \) is the set of all transmission lines. The power flow equation reveals the corresponding relationship between power flow of lines and power angle of buses. Constraint (15) restricts the range of power transmission flow, and (16) confines the node power angle to be between \( \pm 2\pi \). It is worth noting that the main decision variables in the upper layer are \( a_{l,t} \), \( a_{0,j} \) and \( \delta_{i,t} \), while the others are auxiliary variables.

The middle layer of the model is formulated as (17)–(19), relating to the actions of the attacker:

\[
\text{Maximum} \quad \delta \quad \text{s.t.} \quad b_l \in \{0, 1\}, \quad \forall l \in L \quad \sum_{l \in L} b_l = k
\]

where (17) indicates the target of the attacker, who seeks to obtain the maximum attack revenue under limited attack resources. \( \delta \) denotes the power imbalance penalty, which is calculated in the third layer. The binary decision variable \( b_l \) in (18) takes 1 if line \( l \) is damaged. The attack mode shown in (19) is a random attack against multiple transmission lines, where \( k \) limits the attack resources.

A detailed description of the defender’s actions in a fault scenario is given below, aiming to reduce the power imbalance penalty to 0 through the adjustment of various electrical decision variables:

\[
\delta = \text{Minimum} \quad \sum_{i \in N_H} a \left( r_i^+ + r_i^- \right) + b \sum_{i \in N \setminus N_H} r_i^+ \quad \text{subject to} \quad R_i g_0 + \sum_{t \in T} g_{i,t} + \sum_{l \in L \cap r(l) = i} p_l - \sum_{l \in L \cap s(l) = i} p_l - r_i^+ + r_i^- = d_i : \lambda_{1_i}, \quad \forall i \in N \quad \sum_{l \in L} b_l = k \quad \text{s.t.} \quad b_l \in \{0, 1\}, \quad \forall l \in L \quad \sum_{l \in L} b_l = k \quad \text{subject to} \quad -M b_l \leq p_l - B_l (\theta_{s(l)} - \theta_{r(l)}) \leq M b_l : \lambda_{2_l}, \lambda_{3_l}, \quad \forall l \in L \quad P_l (1 - b_l) \leq p_l \leq P_l (1 - b_l) : \lambda_{4_l}, \lambda_{5_l}, \quad \forall l \in L \quad 0 \leq g_{0,j} \leq \bar{g}_j : \lambda_{6_j}, \quad \forall j \in \tilde{N}_G \quad r_i^+ \geq 0, \quad \forall i \in N \quad \forall i \in N \quad r_i^- \geq 0 \quad \forall i \in N \quad -2\pi \leq \theta_l \leq 2\pi : \lambda_{7_l}, \lambda_{8_l}, \quad \forall i \in N \quad 0 \leq g_{i,t} \leq \bar{w}_{i,t} : \lambda_{9_{i,t}}, \quad \forall i \in N, \forall t \in T
\]

where the main decision variables in the lower level are \( r_i^+ \) (kW) and \( r_i^- \) (kW), denoting the power surplus and load shedding of bus \( i \), while the remaining electrical quantities are auxiliary variables. The first term in (20) penalizes the power imbalance among the buses in \( N_H \), which is the set of upgraded buses. For the example shown in Figure 2, \( N_H = \{5, 6, 9\} \) in the second round of restoration. Moreover, \( A \) is a penalty factor with a larger value, the purpose of which is to dominate the direction of iteration. The second term is designed to enforce that for a real system in operation, the power surplus on any bus should be 0 to avoid resource waste, here, \( N \setminus N_H \) is the set of unupgraded buses. The value of \( B \) is much smaller than that of \( A \), consequently, the former plays only the role of making the second term of the objective function gradually approach and eventually equal 0 in the iterative process. The objective function is consistent with the nodal \( N - k \) security criterion,
indicating buses that satisfy the invulnerability level of \( k \) are not allowed to shed load. Nevertheless, the same requirement does not apply to buses with lower invulnerability.

Constraint (21) shows the node power balance equation in a fault scenario. The third term denotes the total power injected from transmission lines into bus \( i \), while the fourth term represents the amount of power flowing out from bus \( i \). The system may experience load shedding at some buses, and \( r_j^+ \) is introduced to guarantee that the equation holds at all times. The DC power equation is transformed into the form of inequality in (22), the power on line \( l \) will be 0 in the case of an open circuit fault, and the relaxation factor \( M \) will take effect when \( b_l = 1 \). Constraint (23) associates line power with its state, (25) and (26) stipulates that variables denoted with power imbalance are non-negative values, and the power generation of MGs relates to their configuration situation in (28).

3. Solution Methodology
3.1. Modified C&CG Algorithm

The common solution method for the proposed DAD model is to divide it into two problems, namely a master and a sub-problem. One of the representatives is the column-and-constraint (C&CG) algorithm [26,27], which is widely used in the field of RO on account of its ability to add constraints and variables simultaneously and its high solution efficiency during the iteration process. Because the established model is bi-objective, traditional C&CG algorithm is incompetent. Hence, fuzzy mathematics is combined to enhance the adaptability of the algorithm in the field of bi-objective optimization.

3.1.1. The Sub-Problem

Since the decision variables in the third layer are all continuous, it is natural to apply strong duality theory to transform the min-problem into a max-problem. On this basis, the third- and second-layer models can be combined to construct the corresponding sub-problem. The dual variables denoted by \( \lambda_1^+, \lambda_8+\), are given in (21)–(28), where \( n_t \) is the number of MG types. The converted model for the \( m \)-th iteration is elaborated below:

\[
\begin{align*}
\text{Maximize} \quad f_s &= \sum_{i \in N} \sum_{t \in T} c_{1w}^{(m)} \lambda_{8+i,t} + \sum_{i \in N} d_i \lambda_{1i} + 2\pi (\lambda_{7i} + \lambda_{8i}) + \sum_{l \in L} M b_l (\lambda_{2l} + \lambda_{3l}) + \sum_{i \in N} c_{l} \lambda_{6i} \\
\text{s.t.} \quad & (18) - (19) \\
& (R^T \lambda_1)_j + \lambda_{6j} \leq 0, \quad \forall j \in N_C \\
& \lambda_{1i} + \lambda_{8+i,t} \leq 0, \quad \forall i \in N, \forall t \in T \\
& \lambda_{i} - \lambda_{8+i,t} = 0, \quad \forall i \in N, \forall t \in T \\
& -\sum_{\forall l \in \mathcal{L}|i \in l} b_l (\lambda_{2l} - \lambda_{3l}) + \sum_{\forall l \in \mathcal{L}(|i \in l)} b_l (\lambda_{2l} - \lambda_{3l}) + \lambda_{7i} - \lambda_{8i} = 0 \quad \forall i \in N \\
& \begin{cases} 
-\lambda_{1i} \leq A, & \text{if } i \in N_H, \\
-\lambda_{1i} \leq B, & \text{if } i \in N \setminus N_H. 
\end{cases} \\
& \begin{cases} 
\lambda_{1i} \leq A, & \text{if } i \in N_H, \\
\lambda_{1i} \leq 0, & \text{if } i \in N \setminus N_H. 
\end{cases}
\end{align*}
\]

The objective function of the sub-problem is shown in (29)–(30), where the decision variables are composed of the dual variables and the attack scheme, with the goal of identifying the worst attack scenario. \( w_{i}^{(m)} (t \in T) \) is the optimal current configuration scheme calculated from the master-problem in the \( m \)-th iteration. (32)–(38) are the dual constraints associated with the power generation of all kinds of generators, where \( R^T \lambda_1 \) is a column
vector with dimensions of \( n_g \times 1 \). The constraint on the power flow in the transmission lines is described in (34), where \( \lambda^r_j/\lambda^s_j \) are the dual variables for the receiving/sending ends of line \( l \). The dual constraint on the voltage power angle is expressed in (35). It is necessary to divide buses into two categories when constructing constraints for power imbalance because they have already been separated in (20). Constraint (38) restricts the range of values for each dual variable except \( \lambda_1 \), as (21) is an equality constraint.

Significantly, the objective function is nonlinear, including products of binary and continuous variables, that is, \( b_1(\lambda_{21} + \lambda_{31}) \) and \( b_1(\lambda_{41} + \lambda_{51}) \). Thus, the original bi-level mixed integer linear programming (MILP) problem evolves into a single-level mixed integer nonlinear programming (MINLP) problem.

Regarding the nonlinear terms appearing in (29), the product terms can be replaced by introducing auxiliary variables and constrained by inequalities to achieve linearization \([1]\). For example, for \( c = ab \), where \( a/b \) are continuous/binary variables. It can be found that \( -D(1-b) \leq c - a \leq D(1-b) \) and \( a_{\min}b \leq c \leq a_{\max}b \) are valid, where \( D \) is a parameter used for limitation, and \( a_{\min} / a_{\max} \) are the lower/upper bounds on \( a \). Oriented by this method, all nonlinear terms are transformed and the complete expression of the sub-problem is as follows:

\[
\text{(SP)} \quad \text{Maximize} \quad f_s \quad \begin{equation}
\begin{aligned}
&\sum_{l \in L} \sum_{i \in T} c_1 w_{i,l}^{(m)} \lambda_{8+ij} + \sum_{l \in L} d_i \lambda_{1l} + 2\pi(\lambda_{7l} + \lambda_{8l}) + \sum_{l \in L} M\gamma_{1l} + P_l(\lambda_{4l} + \lambda_{5l} - \gamma_{2l}) + \sum_{j \in N_c} g_j \lambda_{6j} \\
&\quad \text{s.t.} \quad \begin{cases} 
\gamma_{1l} = b_l(\lambda_{2l} + \lambda_{3l}), \\
0 \leq \gamma_{1l} - (\lambda_{2l} + \lambda_{3l}) \leq \eta_{1l}(1-b_l), \quad \forall l \in L \\
\gamma_{2l} = b_l(\lambda_{4l} + \lambda_{5l}), \\
0 \leq \gamma_{2l} - (\lambda_{4l} + \lambda_{5l}) \leq \eta_{2l}(1-b_l), \quad \forall l \in L \\
\epsilon_{2l}b_l \leq \gamma_{2l} \leq 0
\end{cases}
\end{aligned}
\end{equation}
\]

(31) − (38)

where \( \gamma_{1l}/\gamma_{2l} \) are auxiliary variables introduced to replace the nonlinear products in (40), \( \eta_{1l}/\eta_{2l} \) are parameters of the big-M method for linearization, and \( \epsilon_{1l}/\epsilon_{2l} \) limit the lower bounds on dual variables in (41) and (42). One of the boundary values of each inequality is 0, which is determined by the limitations of the dual variables themselves.

3.1.2. The Master Problem

In this section, some modifications are made to the process of solving the master problem based on fuzzy mathematics, enabling the solution of the bi-objective programming problem. First, two single-objective optimization problems are solved. For convenience, these two problems are referred to as \( MP_1 \) and \( MP_2 \). The formulation of \( MP_1 \) is expressed below:

\[
\text{(MP\_1)} \quad \text{Minimize} \quad f_1 + \beta \quad \begin{equation}
\begin{aligned}
&\text{s.t.} \quad \beta \geq A \sum_{l \in N_H} (r^+_{i,l}(m) + r^-_{i,l}(m)) + B \sum_{i \in \mathcal{N} \setminus N_H} r^+_{i,l}(m), \quad m = 1, 2, \ldots, q \\
&R_{i,s_0}^{(m)} + \sum_{l \in L \setminus \{l\}} S_{i,l}^{(m)} + \sum_{l \in L \setminus \{l\}} p_{l,i}^{(m)} - \sum_{l \in L \setminus \{l\}} P_{l,i}^{(m)} - r^+_{i,l}(m) + r^-_{i,l}(m) = d_i, \quad \forall i \in \mathcal{N}, m = 1, 2, \ldots, q \\
&-Mb_{l,i}^s \leq p_{l,i}^{(m)} - B_l(\varphi_{l,i}^{(m)} - \theta_{l,i}^{(m)}) \leq Mb_{l,i}^s, \quad \forall l \in L, m = 1, 2, \ldots, q \\
P_l(1 - b_{l,i}^r) \leq p_{l,i}^{(m)} \leq \overline{P}_l(1 - b_{l,i}^r), \quad \forall l \in L, m = 1, 2, \ldots, q
\end{aligned}
\end{equation}
\]

(44) – (47)
where $\beta$ represents a cutting plane constrained by (45), and variables marked by $(m)$ under fault scenario in (45)–(53) are the created recourse decision variables after the calculation of $m$-th SP, and $q$ is the total iterations. Furthermore, $b_i^*$ denotes the optimal attack scheme of the $m$-th iteration. Additionally, (54) represents constraints of the upper model.

The objective function of $MP_2$ is shown in Equation (55):

$$ (MP_2) \quad \text{Minimize} \quad f_2 + \beta $$

The corresponding constraints are the same as $MP_1$. More details of the original C&CG algorithm is included in [28].

Second, the values of $f_1$ and $f_2$ in solving $MP_1$ are obtained, which are denoted by $z_1$ and $z_2$, as well as the objective function values $z_3$ and $z_4$ corresponding to $MP_2$. Third, the scaling factors are calculated and the membership functions are established as follows:

$$ e_1 = z_3 - z_1, \quad e_2 = z_2 - z_4 $$

$$ H_1 = \begin{cases} 
1, & \text{if } f_1 < z_1 \\
1 - \frac{f_1 - z_1}{e_1}, & \text{if } z_1 \leq f_1 \leq z_3 \\
0, & \text{if } f_1 > z_3 
\end{cases} $$

$$ H_2 = \begin{cases} 
1, & \text{if } f_2 < z_4 \\
1 - \frac{f_2 - z_4}{e_2}, & \text{if } z_4 \leq f_2 \leq z_2 \\
0, & \text{if } f_2 > z_2 
\end{cases} $$

Then, to maximize the membership functions, a new decision variable named $\psi$ is created, the objective function of $MP_3$ is constructed below:

$$ (MP_3) \quad \text{Maximize} \quad \psi $$

Furthermore, constraints of $MP_3$ are two more than those in $MP_1/MP_2$, which are given below:

$$ e_1 \psi + f_1 \leq z_3, \quad e_2 \psi + f_2 \leq z_2 $$

where (60) is obtained from $H_1/H_2 \geq \psi$. Finally, the decision variables determined by solving this problem are taken as the solution to the master problem, and the objective function values of $f_1$ and $f_2$ are denoted by $z_5$ and $z_6$, respectively.

Since the calculation of the lower and upper bounds serves for iterative convergence, $LB^{(m)} = z_5 + z_6 + \beta^{(m)}$ and $UB^{(m)} = \min\{z_5 + z_6 + f_3^{(m)}, UB^{(m-1)}\}$ are defined for the $m$-th iteration. Note that $\beta^{(m)}$ is the cutting plane obtained from the $m$-th master problem, and $f_3^{(m)}$ is the optimal value of the $m$-th SP. Moreover, the process of adding variables and constraints based on the most severe attack scenario accomplished by the sub-problem of the last iteration is the same as in the traditional algorithm, and thus will not be detailed in this paper. The flow chart of the modified C&CG algorithm is depicted in Figure 3.
where $k$ is the resource restriction of attacked lines and $k^{\text{max}}$ denotes the highest nodal invulnerability requirement in the system.

3.2. Optimization of the $k$-Value Settings under Limited Resources

When the invulnerability of each bus is given, the total power needed for the entire configuration scheme can be easily accessed by the process described above. For instance, under the assumption that the generation capacities of type-1 and type-2 MGs are 50 MW and 100 MW, respectively, the total power investment for restoring buses to their original invulnerability requirements is 200 MW. However, it is worth asking whether there may be better $k$-value settings available for the same power investment. This concept is illustrated in Figure 4, where the $k_i$ values are arranged in ascending order of bus number. Accordingly, the original settings are $\{1,2,1,1,3\}$, and the optimized settings are $\{2,3,1,3,1,2\}$. Essentially, the transformation of the $k$-value settings is equivalent to a shift in the nodal invulnerability
requirements, with the aim of enhancing the overall capability of the system. Driven by this motivation, a method based on greedy rules is provided to solve the problem.

![Diagram](image)

Figure 4. Schematic diagrams of different \( k \)-value settings. (a) Original nodal invulnerability settings. (b) Optimal nodal invulnerability settings under limited resources.

First, the current invulnerability level of each bus after the system is attacked is ascertained, denoted by \( k_{0,i} \). According to the rules explained before, the \( k \) values of power supply buses are not differentiated because they always satisfy their own load demands. Therefore, only the values of the load buses need to be solved for. On the premise that no MGs have been introduced and no faulty lines have gone out of operation, the load shedding at each bus is calculated under different scenarios in accordance with the paradigm of \( (20)\)–\( (28) \). If bus \( i \) violates the nodal \( N - k \) constraint, it is considered that the invulnerability level of this bus drops to \( k - 1 \). The minimum invulnerability standard for each load bus is

\[
 k_{\text{min}}^i = \max(1, k_{0,i}), \quad \forall i \in \mathcal{N}_C
\]

where \( \mathcal{N}_C \) is the set of load buses.

Additionally, it is decisive to establish an indicator for measuring the quality of the \( k \)-value settings. A weighted power indicator is considered a suitable choice, where the weight factor of each bus is \( k_i \), and the net load demand is used as the power, denoted by \( d_{0,i} \). Moreover, this indicator also reflects the overall capability of the system. The mathematical formulation of the evaluation indicator is given below:

\[
 I = \sum_{i \in \mathcal{N}_C} k_i d_{0,i}
\]

Then, the load buses are arranged in descending order in accordance with their net loads. A flowchart of the optimization strategy for \( k \)-value settings is shown in Figure 5. The outer loop begins at the bus with the largest net load denoted as \( i = 1 \), whereas the inner loop starts from the highest invulnerability level signified by \( q = k_{\text{max}} \). Moreover, \( k_{1} \) is the new setting. Buses for which the corresponding calculations have not been performed maintain \( k_{\text{min}}^i \), and the necessary power investment is computed under these settings. If the power limit is not violated, \( k_{\text{min}}^i \) is updated to the current invulnerability level of the inner loop represented by \( q \), and the weighted power indicator \( I \) is further calculated to verify the capability of the system. In case the current \( k \)-value settings offer a higher system capability, the settings are recorded as a candidate for the optimal solution, and \( k_{1,i} \) is fixed as \( q \) to proceed to the calculation for the next bus. If \( I \) does not meet the requirement, the optimal solution \( k_{1}^* \) will not be updated. Otherwise, if the power investment does not meet the standard, the invulnerability level of bus \( i \) will be reduced, and the calculation will be repeated until the inner loop reaches \( q = k_{\text{min}}^i \). The process is not complete until all buses belonging to \( \mathcal{N}_C \) have been traversed.
Note that the fine-tuning of \( k \)-value settings is optional after the direct restoration based on their original invulnerability requirements. This proposal gives more flexibility to the decision makers.

- Arrange load buses in descending order according to \( d_{o,i} \).
- Verify the actual invulnerability of each bus under the current attack denoted by \( k_{0,i} \).

![Flowchart of the optimization strategy for \( k \)-value settings.](image)

**Figure 5.** Flowchart of the optimization strategy for \( k \)-value settings.

### 4. Case Studies

In order to confirm the validity of the proposed method, substantial experiments were conducted on the IEEE 24-bus system. The data are contained in the MATPOWER
toolbox [29]. The proposed method is programmed with MATLAB 2021b (The MathWorks Inc., Natick, MA, USA) and solved by Gurobi 10.0.1 (Gurobi Optimization LLC., Houston, TX, USA) on a laptop with an Intel i9-12900H CPU and 16 GB of memory.

The IEEE 24-bus system commonly used for reliability tests is composed of 33 generators and 38 transmission lines. A total of 11 out of the 17 buses with load demands belong to \(N_C\), making them the primary research subjects for these experiments. Others are classified as power supply buses, which are not within the scope of the invulnerability recovery discussed below. The generation capacity and load demand of each bus are given in Figure 6, where the former is expressed as a negative number for the convenience of display.

![Figure 6. Generation capacities and load demands in the IEEE 24-bus system.](image)

The invulnerability levels of the 11 load buses and the corresponding line reinforcement scheme based on the primary topology are shown in Table 1. The transmission capability of each line is set to 500 MW and \(T = \{1, 2, 3\}\).

<table>
<thead>
<tr>
<th>Installed Lines</th>
<th>Invulnerability Level</th>
<th>Corresponding Load Buses</th>
</tr>
</thead>
<tbody>
<tr>
<td>1–19, 14–17</td>
<td>(N - 1)</td>
<td>4, 5, 6, 15, 20</td>
</tr>
<tr>
<td></td>
<td>(N - 2)</td>
<td>3, 8, 14, 19</td>
</tr>
<tr>
<td></td>
<td>(N - 3)</td>
<td>9, 10</td>
</tr>
</tbody>
</table>

4.1. Restoration of Nodal Invulnerability through the Configuration of MGs

When the system is subjected to physical attacks targeting transmission lines, the invulnerability level of each bus will decrease to varying degrees, implying that the buses are not capable of achieving the nodal \(N - k\) security constraints. In military operations or other situations where remedial measures such as line repairs cannot be carried out immediately, a suitable configuration of MGs becomes an excellent choice for maintaining and restoring power supply capabilities. The upper generation limits for the three types of MGs are 50 MW, 100 MW, and 200 MW. In conjunction with it, the values of the factors \(A\) and \(B\) in the third layer of the model are 500 and 100, respectively. In the following, these parameters remain unchanged if there are no special explanations.

Single-objective optimization oriented by \(f_1\) is implemented to demonstrate the effectiveness of mobile power investment for nodal invulnerability recovery. The attacked lines in this case are 9–12, 10–12, 11–14, and 20–23, which are removed from the system topology. Therefore, the number of binary variables is 250, while the number of continuous variables is 374. On this basis, the most severe load-shedding scenarios under each attack intensity before and after MG allocation are utilized to measure the overall capability of the system.
According to the results illustrated in Table 2 (where, for example, 4:74 represents that bus 4 loses 74 MW of load), if the system is attacked without the support of a mobile power supply, even the basic $N - 1$ security criterion cannot be met. In addition, buses 10 and 14 violate their respective invulnerability criterion and the overall load shedding situation is relatively woeful. However, the above matter does not exist after the investment of MGs. Therefore, the invulnerability of each bus has been properly restored, and—as seen in Table 2—no buses violate the constraints. In addition, the overall load shedding is much lower than that before the allocation of MGs.

**Table 2.** Comparison of the load shedding under most severe attack scenarios.

<table>
<thead>
<tr>
<th>$N - k$</th>
<th>Before MG Configuration</th>
<th>After MG Configuration</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Total Load Shedding (MW)</td>
<td>Buses That Violate Nodal $N - k$ Security Criteria</td>
</tr>
<tr>
<td>$N - 1$</td>
<td>4:74, 5:71, 6:136, 10:150</td>
<td>4, 5, 6, 10</td>
</tr>
</tbody>
</table>

### 4.2. Nodal Invulnerability Restoration Considering Power Generation Balance

Although the effectiveness of restoring nodal invulnerability through configuring MGs has been proven via the above experiment, if adequate measures are not taken, radically undesirable situations are likely to occur, as shown in Figure 7. For the convenience of the display, all generation at a single bus has been integrated, and buses 1, 14, 20, and 21 are equipped with MGs. On the one hand, the generators at buses 1, 2 and 7 are completely idle, leading to a waste of resources in the system. On the other hand, the components of the heavily loaded generators located on buses 16, 18, and 20–23 will age rapidly and be prone to other failures due to temperature rise. Both of these circumstances are extreme operating conditions that are not conducive to the long-term stable operation of the system. The above experimental results have exposed the shortcomings of single-objective optimization for traditional generator configuration problems, and it is necessary to consider the load balancing of power generation.

![Figure 7. Load rates on generators before generation balance is considered.](image_url)

Thus, based on the single-objective solution of the configuration scheme, further optimization is carried out with the goal of $f_2$. The results are depicted in Figure 8.
Although the actual power outputs in both Figures 7 and 8 are feasible from the perspective of nodal power balance equation (i.e., Equation (13)), the optimized power generation load rates of Figure 8 are within the range of 60–80%, and extreme operating conditions are eliminated, proving the rationality of considering power generation balance.

Moreover, to clarify the superiority of the proposed bi-objective integrated optimization framework, the stratified sequencing method is introduced for comparison. Both approaches are used to restore nodal invulnerability after damage to lines 9–12, 10–12, 11–14, and 20–23. The comparison is listed in Table 3, where Case 1 concerns optimizing the load rates of generators based on the configuration scheme obtained from single-objective optimization, which only considers investment cost. Case 2 corresponds to the bi-objective optimization. The round number represents the round of recovery process in which the invulnerability level is restored to \( k = 1, 2 \) or 3. For instance, the second round signifies recovering the nodal invulnerability of relevant buses from \( k = 1 \) to \( k = 2 \). The explanation of bus 20(I, III) is to allocate 1 type-1 MG and 1 type-3 MG at bus 20. Moreover, – denotes that no additional MG is invested in this round.

![Figure 8. Load rates on generators after considering generation balance.](image)

**Table 3.** Comparison between separate and integrated optimization of the configuration scheme and power generation load balancing.

<table>
<thead>
<tr>
<th>Case No.</th>
<th>Configuration Scheme</th>
<th>Round No.</th>
<th>( f_1 ) (MW)</th>
<th>( f_2 )</th>
<th>CPU Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>bus 20(I, III), bus 21(III)</td>
<td>1</td>
<td>450</td>
<td>7.2593</td>
<td>16.30</td>
</tr>
<tr>
<td></td>
<td>bus 20(II), bus 14(III)</td>
<td>2</td>
<td>300</td>
<td>2.3574</td>
<td>17.04</td>
</tr>
<tr>
<td></td>
<td>bus 1(I)</td>
<td>3</td>
<td>50</td>
<td>1.8587</td>
<td>12.79</td>
</tr>
<tr>
<td>Case 2</td>
<td>bus 20(I, II), bus 4(III), bus 21(III), bus 24(III)</td>
<td>1</td>
<td>750</td>
<td>2.4709</td>
<td>627.98</td>
</tr>
<tr>
<td></td>
<td>bus 14(III), bus 19(III)</td>
<td>2</td>
<td>400</td>
<td>0.0813</td>
<td>33.44</td>
</tr>
<tr>
<td></td>
<td>–</td>
<td>3</td>
<td>0</td>
<td>0.0813</td>
<td>8.33</td>
</tr>
</tbody>
</table>

According to the results, although the power investment is 800 MW in Case 1 versus 1150 MW in Case 2, bi-objective optimization further alleviates the data fluctuations. Taking the final round for each of the two cases as an example, the variance differs between them by a factor of approximately 22. Under the condition that both cases have 250 binary decision variables, the number of continuous variables in the single-objective optimization problem considering only \( MP_1 \) is 374, while the number is 553 in the bi-objective optimization problem. Thus, 179 continuous variables related to load rates are solved in the second step of Case 1. 553 continuous variables are directly optimized in Case 2, but due to the increase in the size of the master problem from one to three, its solving process is more complex than Case 1. Although the solving time of Case 2 is longer than Case 1, the time difference is acceptable during the planning stage. In addition, the results obtained
from bi-objective optimization are more conducive to system operation, and the practical significance is demonstrated.

The final load rates on the generators calculated in Case 2 are investigated in Figure 9. Contrasting to the results given in Figure 8, although the configuration scheme is different, bi-objective optimization results in more balanced load rates, and the overall value is also appropriate. It should be noted that the schemes in Figures 8 and 9 are both feasible, and decision makers can choose between them based on the actual situation or their own preferences.

Notably, it is unrealistic to obtain a configuration strategy solely based on $f_2$. For instance, in the first round of recovery, such an approach would indicate that the numbers of the three types of MGs that should be allocated are 18, 11, and 14, which is obviously unacceptable. Although balanced power generation load rates would be achieved, the invested power would be extremely high, causing serious resource waste. Instead, the solution in Case 2 offers a compromise between the two objectives, making it the final solution.

Figure 9. Load rates on generators when the problem is solved via bi-objective optimization.

4.3. Optimal k-Value Settings under Limited Resources

The total power investment required to restore the invulnerability level of each bus to the original standard under the attack scheme of 9–12, 10–12, 11–14 and 20–23 is 800 MW. In order to reset the settings for $k$, it is essential to first calculate the nodal invulnerability after attack, which is denoted by $k_{0,i}$. Then, the optimal settings of $k$ can be obtained under the current power resources, and the results are given in Table 4.

Table 4. Invulnerability of the load buses under different conditions of the IEEE 24-bus system.

<table>
<thead>
<tr>
<th>Bus ID</th>
<th>$k_i^T$</th>
<th>$k_i^{R}$</th>
<th>$k_{0,i}$</th>
<th>$k_{1,i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>10</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>14</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>15</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>19</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>20</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>

Evaluation indicator (i.e., Equation (62)) - 3523 - 3625
Expectation of load shedding (MW) - 689.5658 - 631.6355

Bold values represent these buses are assigned to a higher invulnerability level after optimization.
The results indicate that the damage to these 4 lines posed a significant threat to the invulnerability of most buses, only the security levels of buses 3, 9 and 15 did not decrease. Further examination of Table 4 reveals that the access of MGs enables buses 14 and 20 to achieve and even exceed their original invulnerability, where $k^T_i$ denotes the theoretical value of $k_i$ and $k^R_i$ represents the real value after nodal invulnerability restoration via MG configuration, indicating that we have traded lower costs for higher returns than expected. Therefore, when evaluating the $k$-value settings, the real values rather than the theoretical values should be considered.

Interestingly, buses 5, 14, 15, and 20 are all assigned to a higher level based on the comparison between $k_{1i}$ and $k^T_i$. Furthermore, there may also be situations in which the invulnerability requirements of some buses are reduced after optimization, while the values of another part of buses are increased. It seems clear that the expected load shedding value under the security analysis performed after changing the $k$-value settings has been effectively decreased. Moreover, the system capability as evaluated in terms of the indicator $I$ has become larger, confirming the rationality of the estimation mechanism.

According to this key discovery, the overall system capability can be further enhanced by fine-tuning the requirements for nodal invulnerability under the premise of allowable conditions or special needs.

5. Conclusions

In this paper, nodal invulnerability in a power system is appropriately restored through the configuration of MGs after multiple lines are attacked. First, a DAD model that comprehensively considers investment costs and power generation balance is established, combined with the nodal $N-k$ security criterion. For compatibility with the classical solution framework of RO, an improved C&CG algorithm associated with fuzzy mathematics for solving bi-objective optimization problems is provided. Moreover, to seek better settings for nodal invulnerability requirements, an optimization strategy illuminated by greedy rules is proposed.

As seen from the numerical experiments conducted on the IEEE 24-bus system, restoring nodal invulnerability reduces the number of buses that violate constraints in the next round of attacks from 6 to 0. The optimization of the power generation load rates effectively avoids the occurrence of extreme operating conditions. Furthermore, the integrated optimization of the configuration scheme and load balancing yields better performance, with a variance that differs between them by a factor of approximately 22 compared to separate optimization. In addition, the optimization of nodal invulnerability requirements not only improves the weighted power indicator from 3523 to 3625, but also effectively cuts down the expected load shedding of nearly 60 MW.

Future work will focus on investigating the impact of interval load fluctuations on nodal invulnerability enhancement and further exploring schemes for reinforcing nodal invulnerability from the perspective of reactive power regulation.

Author Contributions: X.Z.: conceptualization, writing—original draft, methodology, software, validation. S.H.: conceptualization, supervision, writing—reviewing and editing. Q.L.: writing—reviewing and validation. R.W.: supervision. T.Z.: supervision. B.G.: supervision. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the National Natural Science Foundation of China (Grant No. 61973310), the Natural Science Foundation for Excellent Young Scholars of Hunan Province (Grant No. 2023JJ20054) and the Science and Technology Innovation Program of Hunan Province (Grant No. 2023RC1002).

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.


**Disclaimer/Publisher’s Note:** The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.