mathematics

MDPI

Article

On a Mixed Transient–Asymptotic Result for the Sequential Interval Reliability for Semi-Markov Chains

Guglielmo D'Amico [1,*,†] and Thomas Gkelsinis [2,†]

[1] Department of Economics, University "G. d'Annunzio" of Chieti-Pescara, 65127 Pescara, Italy
[2] Laboratory of Mathematics Raphaël Salem, University of Rouen-Normandy, UMR 6085, 76801 Saint-Étienne-du-Rouvray, France; gkelsinis@univ-rouen.fr
* Correspondence: g.damico@unich.it
† These authors contributed equally to this work.

Abstract: In this paper, we are concerned with the study of sequential interval reliability, a measure recently introduced in the literature. This measure represents the probability of the system working during a sequence of nonoverlapping time intervals. In the cited work, the authors proposed a recurrent-type formula for computing this indicator in the transient case and investigated the asymptotic behavior as all the time intervals go to infinity. The purpose of the present work is to further explore the asymptotic behavior when only some of the time intervals are allowed to go to infinity while the remaining ones are not. In this way, we provide a unique indicator that is able to describe the process evolution in the transient and asymptotic cases as well. It is important to mention that this is not a straightforward result since, in order to achieve it, we need to develop several mathematical ingredients that generalize the classical renewal and Markov renewal frameworks. A numerical example illustrates our theoretical results.

Keywords: sequential measures; convolution product; semi-Markov processes; asymptotic results

MSC: 60K15; 60K20; 90B25

check for updates

Citation: D'Amico, G.; Gkelsinis, T. On a Mixed Transient–Asymptotic Result for the Sequential Interval Reliability for Semi-Markov Chains. *Mathematics* **2024**, *12*, 1842. https://doi.org/10.3390/math12121842

Academic Editors: Panagiotis-Christos Vassiliou and Andreas C. Georgiou

Received: 30 April 2024
Revised: 6 June 2024
Accepted: 12 June 2024
Published: 13 June 2024

Copyright: © 2024 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction

In this paper, we study a specific reliability indicator of discrete-time semi-Markov systems called sequential interval reliability (*SIR*) which has been recently introduced in [1].

The choice for semi-Markov models has a twofold meaning: on the practical side, it is important to consider flexible models able to describe real problems; on the theoretical side, it is essential to provide mathematical results for general systems that encompass interesting particular cases already studied or worthy to be investigated.

In this respect, it is well known that semi-Markov processes are among the most important modeling techniques for real-world issues in diverse applied fields, like reliability, financial mathematics, earthquake studies, bioinformatics, etc. (see, e.g., [2–5]). Furthermore, there are valuable theoretical reasons supporting the semi-Markov choice. Indeed, they generalize the Markov chain framework by taking into account the duration of stay in the states. Hence, any result established in the semi-Markov case has a corresponding particular result in the Markovian setting. The latter is recovered whenever the sojourn time in a state of the process is modeled through a memoryless distribution, the exponential one in continuous time or the geometric one in discrete time.

Several researchers have investigated the dependability metrics of semi-Markov systems.

In the literature, there is a distinction based on the choice of discrete-time or continuous-time models.

The reliability methodology for continuous-time semi-Markov processes and the corresponding statistical inference can be found in [2,6–10].

The complexities that arise from the solution of semi-Markov models in continuous time led to the development of numerical approximations based on the discrete counterpart [11–14]. Also, for this last requirement, a study has recently emerged on the theory of reliability for discrete-time semi-Markov systems and on their estimation problem (see, e.g., [15–17]).

Numerical aspects related to a corresponding `R` package (https://cran.r-project.org/web/packages/smmR/index.html and https://cran.r-project.org/web/packages/SMM, accessed on 11 March 2024) can be found in [18].

Employing interval-based reliability indicators is a promising approach that allows us to examine the dependency of reliability in response to a particular specification of the time interval of interest expressed in terms of a starting point and a length. This was the original idea that brought to light the notion of interval reliability for continuous-time semi-Markov models [19,20]. The analogous discrete-time models were studied in [21,22] and recently extended to include duration-dependent versions in [23]. In short, interval reliability is the probability that the system will work at any time within a fixed time interval $[t, t+p]$. This measure includes, in particular, the availability function at time t whenever $p = 0$ and the reliability function at p whenever $t = 0$.

The present work is an extension of the work developed in [1], where we proposed a new reliability measure called sequential interval reliability (*SIR*).

The *SIR* generalizes the notion of interval reliability as representing the probability that a system is in a working state during a sequence of nonoverlapping intervals that are not necessarily equi-spaced. This performance indicator is of importance in several application domains where system performance is important only for specific temporal intervals. As an example, we can think of the reliability of an air-conditioning system in an office; it is recommended that the reliability be high on working days from 8 a.m. to 5 p.m. Unreliability during the night or at the weekend is not a serious issue, and the engineer can disregard it using the SIR indicator, which has been designed according to this scope. Following the line in [24], we are also interested in taking into account the dependence on the initial and/or final backward.

In [1], we proposed a recurrent-type formula for computing this indicator in the transient case, and we investigated the asymptotic behavior as the first time point goes to infinity, and hence, all the successive time points diverge to infinity as well. The purpose of the present work is to further explore the asymptotic behavior as some other time points are allowed to go to infinity. This means that we will consider a number of time intervals over which we assess the reliability in the transient regime and the remaining ones, which diverge to infinity, over which we measure the asymptotic sequential interval reliability. This leads to a result of mixed type, simultaneously considering transient and asymptotic behaviors in a unique formula.

It is very important to stress that this is not a straightforward work; in order to achieve this, we needed to develop several mathematical ingredients, like proposing a specific operator between two sets of functions (cf. Equation (11)), introducing a new matrix convolution product (in Definition 2), investigating the relationship with the classical one (in Proposition 2), and applying renewal-type arguments (like the key renewal theorem and Markov renewal equation techniques) to the generalized framework formalized by this new matrix convolution product (cf. the proof of Theorem 2).

The rest of this article is structured as follows: in the next section, we introduce the semi-Markov framework and sequential interval reliability by recalling some previous results. The main contribution of this article, presented in Section 3, is the investigation of the asymptotic behavior of the sequential interval reliability as a time of interest goes to infinity. A numerical example is provided in Section 4, illustrating some aspects of our theoretical work.

2. Mathematical Model and System Performance Metrics

This section introduces the mathematical framework by presenting a short description of semi-Markov chains and known performability measures used in the reliability field.

2.1. Semi-Markov Chains

A semi-Markov chain is a random process that is frequently used in several applied problems. It exhibits a particular type of time dependence between events where the Markovian property holds not at every time point but only at moments when the system changes its states. Consider a generic random system taking values in the finite state space $E = \{1, \ldots, s\}$. The system evolves in time, and its behavior can be described by two sequences of random variables defined over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

The first sequence of random variables is $J = (J_n)_{n\in\mathbb{N}}$, with $J_n : \Omega \to E$. It denotes the states successively visited by the system. The successive points when the random system changes its states are denoted by $T = (T_n)_{n\in\mathbb{N}}$, with $T_n : \Omega \to \mathbb{N}$. Let us further introduce $N(t) := \max\{n \in \mathbb{N} \mid T_n \leq t\}$ as the discrete-time counting process denoting the number of transitions within time t. Then, the process $(Z_t)_{t\in\mathbb{N}}$, defined by $Z_t = J_{N(t)}$, is called a semi-Markov chain.

If we assume that

$$\mathbb{P}(J_{n+1} = j, T_{n+1} - T_n = k | J_n, \ldots, J_0, T_n, \ldots, T_0) = \mathbb{P}(J_{n+1} = j, T_{n+1} - T_n = k | J_n), \quad (1)$$

then the process $Z = (Z_k, k \in \mathbb{N})$ is a semi-Markov chain. Property (1) allows the joint process $(J, T) = (J_n, T_n)_n$ to be a Markov renewal Chain. Whenever the probability $\mathbb{P}(J_{n+1} = j, T_{n+1} - T_n = k | J_n = i)$ is independent of the number of transitions n, the semi-Markov chain is time-homogeneous. When we relax this assumption, we deal with time-inhomogeneous semi-Markov processes; see, e.g., [25,26]. This class of stochastic processes has demonstrated high potential for describing real-life problems, among which are credit risk and financial applications; see, e.g., [27–29].

A semi-Markov chain is uniquely determined (almost surely) through an initial distribution $(\mu_i)_{i\in E}$ with $\mu_i = \mathbb{P}(Z_0 = i) = \mathbb{P}(J_0 = i)$ and a matrix of function $\mathbf{q}(t) = (q_{ij}(t))_{i,j\in E, t\in\mathbb{N}}$ called the semi-Markov kernel. The latter collects the probabilities

$$q_{ij}(t) = \mathbb{P}(J_{n+1} = j, T_{n+1} - T_n = t | J_n = i).$$

Simple probabilistic computations allow us to represent the semi-Markov kernel as

$$q_{ij}(k) = p_{ij} f_{ij}(k),$$

where $p_{ij} = \mathbb{P}(J_{n+1} = j | J_n = i)$, and $f_{ij}(k) = \mathbb{P}(T_{n+1} - T_n = k | J_n = i, J_{n+1} = j)$.

Hence, the semi-Markov kernel can be identified by providing a transition probability matrix $\mathbf{P} = (p_{ij})_{i,j\in E}$ for the embedded Markov chain J_n and a probability distribution function $f_{ij}(\cdot)$ for each couple of states i, j. The element p_{ij} represents the conditional probability of transitioning from state i to state j independently in time. The function $f_{ij}(\cdot)$ identifies the probability distribution of the sojourn time length in state i before making the next transition in state j.

The theory of semi-Markov chains is well developed, and the interested reader can find several results described in [4].

2.2. Reliability Metrics

To assess the reliability behavior of a semi-Markov system, usually the state space E is split into two subsets, U for the working states and D for the failure states; hence, $E = U \cup D$ and $E = U \cap D = \varnothing$.

The literature abounds with various measures of performance. One of the most interesting is the sequential interval reliability introduced in [1]. The main interest of this measure resides in its high generality, as it encompasses availability, reliability, and interval

reliability at the same time. Here, we report some definitions and results given in [1] for improving readability and understanding the new result that we are going to show in the next section.

Let us consider two time sequences of nonnegative times $\underline{t} := (t_i)_{i=1,\dots,N}$ and $\underline{p} := (p_i)_{i=1,\dots,N}$ such that the following apply:

1. $t_i < t_{i+1}$ for all $i = 1, \dots, N-1$;
2. $t_i + p_i < t_{i+1}$ for all $i = 1, \dots, N-1$.

Previous conditions guarantee that $\forall i, j = 1, \dots, N$ with $i \neq j$,

$$[t_i, t_i + p_i] \bigcap [t_j, t_j + p_j] = \emptyset.$$

Hence, they form a sequence of nonoverlapping time intervals.

Sequences $\underline{t}$ and $\underline{p}$ can be considered row vectors. Hereinafter, vectors will always be intended as rows unless otherwise specified.

According to [1], for $v \in \mathbb{N}$ and $k \in E$, we can define the conditional sequential interval reliability, $SIR_k^{(N)}(v; \underline{t}, \underline{p})$, as the conditional probability that the system is in the up-states U during the time intervals $\{[t_i, t_i + p_i]\}_{i=1,\dots,N}$, given the information set $(k, v) := \{J_{N(0)} = k, T_{N(0)} = -v\}$, i.e.,

$$\begin{aligned} SIR_k^{(N)}(v; \underline{t}, \underline{p}) &:= \mathbb{P}(Z_l \in U, \text{ for all } l \in [t_i, t_i + p_i], i = 1, \dots, N \mid J_{N(0)} = k, T_{N(0)} = -v) \\ &= \mathbb{P}_{(k,v)}(Z_l \in U, \text{ for all } l \in [t_i, t_i + p_i], i = 1, \dots, N). \end{aligned} \tag{2}$$

It is important to note that the SIR function ignores the behavior of the system within two subsequent time intervals. In other words, for any discrete time point in the set $[t_i + p_i + 1, t_{i+1} - 1]$, the system is free to occupy up-states or down-states without altering the indicator's value.

A particular case of the former definition is obtained when $v = 0$; thus, we will set

$$SIR_k^{(N)}(\underline{t}, \underline{p}) := SIR_k^{(N)}(0; \underline{t}, \underline{p}) = \mathbb{P}(Z_l \in U, \text{ for all } l \in [t_i, t_i + p_i], i = 1, \dots, N \mid J_{N(0)} = k).$$

A recursive formula for computing the sequential interval reliability of a discrete-time semi-Markov chain is known.

Proposition 1 ([1]). *Let $\underline{t} := (t_i)_{i=1,\dots,N}$ and $\underline{p} := (p_i)_{i=1,\dots,N}$, $N \in \mathbb{N}^*$ be two time sequences such that $\{[t_i, t_i + p_i]\}_{i=1,\dots,N}$ is a sequence of nonoverlapping real intervals. For any $v \in \mathbb{N}$ and $k \in E$, the indicator $SIR_k^{(N)}(v; \underline{t}, \underline{p})$ is given by the following:*

$$SIR_k^{(N)}(v; \underline{t}, \underline{p}) = g_k^{(N)}(v; \underline{t}, \underline{p}) + \sum_{r \in E} \sum_{\theta=1}^{t_1} \frac{q_{kr}(v + \theta)}{\overline{H}_k(v)} SIR_r^{(N)}(0; \underline{t} - \theta \mathbf{1}_{1:N}, \underline{p}). \tag{3}$$

Here, $\overline{H}_k(v) = \sum_{s=v+1}^{\infty} \sum_{j \in E} f_{kj}(s)$, $\mathbf{1}_{1:N}$ is a vector of 1s of length N, and $g_k^{(N)}(v; \underline{t}, \underline{p})$ is given by

$$g_k^{(N)}(v;\underline{t},\underline{p}) := 1_{\{k\in U\}}\Bigg[\frac{\overline{H}_k(t_N+p_N+v)}{\overline{H}_k(v)}$$
$$+\sum_{\theta=t_1+1}^{t_1+p_1}\sum_{r\in E}\sum_{m\in U}\sum_{v'=0}^{t_1+p_1-\theta}\frac{q_{kr}(v+\theta)}{\overline{H}_k(v)}R_{rm}^{b}(v';t_1+p_1-\theta)SIR_m^{(N-1)}(v';t_{2:N}-\mathbf{1}_{2:N}(t_1+p_1),p_{2:N})$$
$$+\sum_{j=2}^{N}\sum_{\theta=t_j}^{t_j+p_j}\sum_{r\in E}\frac{q_{kr}(v+\theta)}{\overline{H}_k(v)}SIR_r^{(N-j+1)}\big(0;(\theta,t_{j+1:N}-\mathbf{1}_{j+1:N}\theta),(t_j+p_j-\theta,p_{j+1:N})\big)$$
$$+\sum_{j=1}^{N-1}\sum_{\theta=t_j+p_j+1}^{t_{j+1}-1}\sum_{r\in E}\frac{q_{kr}(v+\theta)}{\overline{H}_k(v)}SIR_r^{(N-j)}(0;t_{j+1:N}-\mathbf{1}_{j+1:N}\theta,p_{j+1:N})\Bigg], \tag{4}$$

where $1_{\{k\in U\}}$ *is the indicator function of the event* $\{k\in U\}$*, and* $R_{ij}^{b}(v;t)$ *is the reliability with the final backward defined by*

$$R_{ij}^{b}(v;t) := \mathbb{P}(Z_s\in U,\ \textit{for all } s\in\{0,\dots,t-v\}, Z_t=j, B_t=v \mid Z_0=i, T_{N(0)}=0). \tag{5}$$

Remark 1. *Equation (3) is a recurrent-type formula. The unknown function* $SIR_k^{(N)}(v;\underline{t},\underline{p})$ *depends on itself but is evaluated at different values of the variables. First, we observe that the value of the backward recurrence time process is* v *on the left-hand side of the formula and is reset to zero on the right-hand side. This is due to the Markovian property of the semi-Markov process at transition times. The vector* $\underline{t}$ *of the initial points of the* N *intervals is shifted according to the time* θ *of the first transition when occurring before the initial time* t_1 *of the first interval. This transition occurs with probability* $\frac{q_{kr}(v+\theta)}{\overline{H}_k(v)}$ *in the state* r*. The summations over all possible states* $r\in E$ *and times* $\theta\in\{1,2,\dots,t_1\}$ *consider all the possible cases.*

The function $g_k^{(N)}(v;\underline{t},\underline{p})$ *considers the remaining possibilities, which consist of transitions inside one of the intervals* $[t_j,t_j+p_j]$ *or between two intervals, i.e.,* $[t_j+p_j+1,t_{j+1}-1]$ *or exceeding the considered time horizon* t_N+p_N*. Then, a further recursion shows the dependence of* $SIR_k^{(N)}$ *on* $SIR_k^{(M)}$ *for every positive integer* $M<N$*.*

If $v=0$, Equation (3) collapses into

$$SIR_k^{(N)}(\underline{t},\underline{p}) \;=\; g_k^{(N)}(\underline{t},\underline{p})+\sum_{r\in E}\sum_{\theta=1}^{t_1}q_{kr}(\theta)SIR_r^{(N)}(\underline{t}-\theta\mathbf{1}_{1:N},\underline{p}), \tag{6}$$

where we have set $g_k^{(N)}(\underline{t},\underline{p}) := g_k^{(N)}(0;\underline{t},\underline{p})$.

In [1], the asymptotic analysis of the sequential interval reliability $SIR_k^{(N)}(v;t_{1:N},b_{1:N})$ was also studied, allowing the time point t_1 to go to infinity. As a result of the sequence t_i rising, we see that all these time points also diverge to infinity. The next result given in Theorem 1 answers this question.

Theorem 1 ([1])**.** *For an ergodic semi-Markov chain, under the previous notations,*

$$\lim_{t_1\to\infty}SIR_k^{(N)}(v;\underline{t},\underline{p})=\lim_{t_1\to\infty}SIR_k^{(N)}(\underline{t},\underline{p})=\frac{1}{\sum_{i\in E}\nu(i)m_i}\sum_{j\in U}\nu(j)\sum_{t_1\geq 0}g_j^{(N)}(\underline{t},\underline{p}), \tag{7}$$

where $(\nu(i))_{i\in E}$ *represents the stationary distribution of the embedded Markov chain* $(J_n)_{n\in\mathbb{N}}$*, and* m_i *is the mean sojourn time in state* $i\in E$*, so it is the mean time of the distribution* $(h_i(k))_{k\in\mathbb{N}}, i\in E$*. The form of the function* $g_j^{(N)}(\underline{t},\underline{p})$ *is given in Equation (4).*

3. A Mixed Transient–Asymptotic Result for the Sequential Interval Reliability

It is possible to generalize and combine Proposition 1 with Theorem 1, allowing t_h, $h \geq 2$ to go to infinity at any time. Hence, the main question we want to answer is about the value of

$$\lim_{t_h \to \infty} SIR_k^{(N)}(\underline{t}, \underline{p}).$$

We observe that a positive answer to this problem will provide a mixed result because the part of the probability $SIR^{(N)}(\underline{t}, \underline{p})$ corresponding to the time intervals $\{[t_i, t_i + p_i]\}_{i=1,\ldots,h-1}$ is devoted to the description of the behavior of the system in the transient case, while the part containing the remaining intervals $\{[t_i, t_i + p_i]\}_{i=h,\ldots,N}$, having considered $t_h \to \infty$, is responsible for the description of the asymptotic behavior of the system.

First, we present some definitions related to matrix convolution products and some new operations necessary for the proof of our main result. Let $\mathcal{M}_E$ be the set of real matrices of dimension E, and let $\mathcal{M}_E(\mathbb{N})$ be the set of matrix-valued functions defined on $\mathbb{N}$, with values in $\mathcal{M}_E$. A matrix of functions $\mathbf{A} \in \mathcal{M}_E(\mathbb{N})$ can be interpreted in two different ways. First, if we fix any time $t \in \mathbb{N}$, we obtain $\mathbf{A}(t) \in \mathcal{M}_E$, which is a real-valued matrix. The second possibility consists of fixing a couple of states $i, j \in E$, and then, the element $A_{ij}(\cdot)$ is a function of the time. A special element of $\mathcal{M}_E(\mathbb{N})$ is the null matrix $\mathbf{0} := (\mathbf{0}(t);\ t \in \mathbb{N})$, with $\mathbf{0}(t) := \mathbf{0}$ for any $t \in \mathbb{N}$.

Definition 1. *Let* $\mathbf{A}, \mathbf{B} \in \mathcal{M}_E(\mathbb{N})$*, where their matrix convolution product* $\mathbf{A} * \mathbf{B}$ *is defined as the matrix-valued function* $\mathbf{C} \in \mathcal{M}_E(\mathbb{N})$ *such that*

$$C_{ij}(t) := \sum_{r \in E} \sum_{s=0}^{t} A_{ir}(t-s)\, B_{rj}(s), \quad i, j \in E, \quad t \in \mathbb{N}. \tag{8}$$

In matrix form,

$$\mathbf{C}(t) := \sum_{s=0}^{t} \mathbf{A}(t-s)\, \mathbf{B}(s), \quad k \in \mathbb{N}.$$

Define a matrix-valued function $\boldsymbol{\delta I} = (d_{ij}(t);\ i, j \in E) \in \mathcal{M}_E(\mathbb{N})$ such that $d_{ii}(0) = 1$, $\forall i \in E$, and $d_{ij}(t) = 0$ otherwise. It is simple to realize that $\boldsymbol{\delta I}$ is the identity element for the matrix convolution product. Hence, iterating the convolution operation, we can easily obtain the n-fold convolution for any element $\mathbf{A} \in \mathcal{M}_E(\mathbb{N})$. Indeed, $\mathbf{A}^{(n)}$ can be obtained recursively as follows:

$$\mathbf{A}^{(0)} := \boldsymbol{\delta} I,\ \mathbf{A}^{(1)} := \mathbf{A} \ \text{ and } \mathbf{A}^{(n)} := \mathbf{A} * \mathbf{A}^{(n-1)}, \quad n \geq 2.$$

Second, let us introduce two sets of functions that will be useful in our proof. So, for $h, N \in \mathbb{N}$, first, let us define

$$\mathcal{A}(h; N) := \left\{ f : D_{\mathcal{A}(h;N)} \to \mathbb{R} \right\}, \tag{9}$$

where the domain $D_{\mathcal{A}}(h; N)$ is defined by

$$\begin{aligned} D_{\mathcal{A}}(h; N) \ := \ & \Big\{ (\underline{x}, \underline{z}, \underline{w}) \in \mathbb{N}^{h-1} \times \mathbb{N}^{N-h} \times \mathbb{N}^{N} \mid \underline{x} = (x_1, \ldots, x_{h-1}), x_i \leq x_{i+1}, \\ & \forall i = 1, \ldots, h-2; \underline{z} = (z_{h+1}, \ldots, z_N), z_i \geq 0, \forall i = h+1, \ldots, N-1; \\ & \underline{w} = (w_1, \ldots, w_N), 0 \leq w_i \leq x_{i+1} - x_i, \forall i = 1, \ldots, h-1, \\ & 0 \leq w_i \leq z_{i+1}, \forall i = h, \ldots, N, \\ & \text{having set } z_{N+1} := \infty \}. \end{aligned}$$

Similarly, for $h, N \in \mathbb{N}$, let us define a set of one-variable functions indexed by three vectors of parameters

$$\begin{aligned}\mathcal{B}_{\underline{s},\underline{l},\underline{p}}(h;N) := &\left\{\widetilde{f} : \mathbb{N} \to \mathbb{R} \mid y = \widetilde{f}(t_1;\underline{s},\underline{l},\underline{p})\right\}, \text{ such that}\\ &t_1 \in \mathbb{N},\\ &\underline{s} \in \mathbb{N}^{h-1},\ \underline{s} = (s_1,\ldots,s_{h-1}), 0 \le s_i\ \forall i = 1,\ldots,h-1,\\ &\underline{l} \in \mathbb{N}^{N-h},\ \underline{l} = (l_{h+1},\ldots,l_N),\ 0 \le l_i,\ \forall i = h+1,\ldots,N,\\ &\underline{p} \in \mathbb{N}^N,\ 0 \le p_i \le s_i,\ \forall i = 1,\ldots,h-1\\ &0 \le p_i \le l_{i+1},\ \forall i = h,\ldots,N, \text{ having set } l_{N+1} := \infty.\end{aligned} \tag{10}$$

Let us consider an operator acting on these two sets, $\Phi : \mathcal{A}(h;N) \to \mathcal{B}_{\underline{s},\underline{l},\underline{p}}(h;N)$ such that $\forall y = f(\underline{x},\underline{z},\underline{w}) \in \mathcal{A}(h;N)$, we have that

$$\Phi(f(\underline{x},\underline{z},\underline{w})) := \widetilde{f}(t_1;\underline{s},\underline{l},\underline{p}), \tag{11}$$

where

$$\begin{aligned}t_1 = x_1;\ s_i = x_{i+1} - x_i =: \Delta x_i\ \forall i = 1,\ldots,h-1,\\ l_i = z_i\ \forall i = h+1,\ldots,N,\\ p_i = w_i\ \forall i = h+1,\ldots,N,\end{aligned}$$

and

$$\widetilde{f}(x_1;\Delta\underline{x},\underline{z},\underline{w}) = f(\underline{x},\underline{z},\underline{w}).$$

The operator Φ maps any function $f(\underline{x},\underline{z},\underline{w}) \in \mathcal{A}(h;N)$ to an element of the set of functions $\mathcal{B}_{\underline{s},\underline{l},\underline{p}}(h;N)$ in such a way as to preserve the values of the images. Hence, to a vectorial function, we associate a scalar function, namely of the variable t_1, with the parameter set $\Theta = \{s_{1:h-1}, l_{h+1:N}, w_{1:N}\}$.

It is simple to realize that operator (11) is bijective and satisfies the following two properties:

- the image of a product is equal to the product of the images:
  $\forall f(\underline{x}^1,\underline{z}^1,\underline{w}^1),\ h(\underline{x}^2,\underline{z}^2,\underline{w}^2) \in \mathcal{A}(h;N)$, it results that

  $$\Phi(f(\underline{x}^1,\underline{z}^1,\underline{w}^1)\cdot h(\underline{x}^2,\underline{z}^2,\underline{w}^2)) = \Phi(f(\underline{x}^1,\underline{z}^1,\underline{w}^1))\cdot\Phi(h(\underline{x}^2,\underline{z}^2,\underline{w}^2)),$$

  This property is sometimes referred to as Cauchy's multiplicative functional equation.
- The image of a sum is equal to the sum of the images:
  $\forall f(\underline{x}^1,\underline{z}^1,\underline{w}^1),\ h(\underline{x}^2,\underline{z}^2,\underline{w}^2) \in \mathcal{A}(h;N)$, it results that

  $$\Phi(f(\underline{x}^1,\underline{z}^1,\underline{w}^1) + h(\underline{x}^2,\underline{z}^2,\underline{w}^2)) = \Phi(f(\underline{x}^1,\underline{z}^1,\underline{w}^1)) + \Phi(h(\underline{x}^2,\underline{z}^2,\underline{w}^2))$$

  This property is sometimes referred to as Cauchy's additive functional equation.

The last point before presenting our main result about the mixed transient–asymptotic behavior of the SIR function is to introduce a new matrix convolution product, which is important for our framework, and to see the relationship between the classical matrix convolution product and the one defined in [1].

Definition 2. *Let* $\mathbf{A} \in \mathcal{M}_E(\mathbb{N})$ *be a matrix-valued function, and let* $\mathbf{b} = (b_1,\ldots,b_s)^T$ *be a vector-valued function such that every component* $b_r \in \mathcal{A}(h;N), r \in E$. *Let* $\underline{t} := (t_i)_{i=1,\ldots,N}$ *and* $\underline{p} := (p_i)_{i=1,\ldots,N}$, $N \in \mathbb{N}^*$ *be two time sequences such that* $\{[t_i, t_i + p_i]\}_{i=1,\ldots,N}$ *is a sequence of nonoverlapping real intervals, and set* $l_{h+1:N} = (l_{h+1},\ldots,l_N)$ *with* $l_i := t_i - t_{i-1}$, $\forall i = h+1,\ldots,N$ *and* $x_{1:h-1} = (x_1,\ldots,x_{h-1})$ *with* $x_i := t_i$, $\forall i = 1,\ldots,h-1$.

The matrix convolution product $*_h$ *is defined by*

$$(\mathbf{A}*_h\mathbf{b})_k(x_{1:h-1}, l_{h+1:N}, p_{1:N}) := \sum_{r\in E}\sum_{\theta=1}^{t_1} A_{kr}(\theta)b_r(x_{1:h-1} - \theta\mathbf{1}_{1:h-1}, l_{h+1:N}, p_{1:N}),$$

or, in matrix form,

$$(\mathbf{A}*_h\mathbf{b})(x_{1:h-1}, l_{h+1:N}, p_{1:N}) := \sum_{\theta=1}^{t_1} \mathbf{A}(\theta)\mathbf{b}(x_{1:h-1} - \theta\mathbf{1}_{1:h-1}, l_{h+1:N}, p_{1:N}).$$

The next result presents a relationship between the new matrix convolution product introduced above and the classical one defined in (8).

Proposition 2. *Let* $\mathbf{q} \in \mathcal{M}_E(\mathbb{N})$ *be a semi-Markov kernel, and let* $\mathbf{f} = (f_1, \ldots, f_s)^T$ *be a vector-valued function such that every component* $f_r \in \mathcal{A}(h; N), r \in E$. *Then,* $\forall(\underline{x}, \underline{z}, \underline{w}) \in \mathbb{N}^{h-1} \times \mathbb{N}^{N-h} \times \mathbb{N}^N$, *it results that*

$$\Phi\big((\mathbf{q}*_h\mathbf{f})_k(\underline{x}, \underline{z}, \underline{w})\big) = \big(\mathbf{q} * \widetilde{\mathbf{f}}\big)_k(x_1; s_{1:h-1}, l_{h+1:N}, \underline{w}),$$

where $s_i = x_{i+1} - x_i, \forall i = 1, \ldots, h-1$, *and* $l_{i+1} = x_{i+1} - x_i, \forall i = h, \ldots, N-1$.

Proof. By the definitions of the operator Φ and of the $*_h$ convolution product, we have

$$\begin{aligned}\Phi\big((\mathbf{q}*_h\mathbf{f})_k(\underline{x}, \underline{z}, \underline{w})\big) &= \Phi\Big(\sum_{r\in E}\sum_{\theta=1}^{x_1} q_{kr}(\theta)\cdot f_r(x_{1:h-1} - \theta\mathbf{1}_{1:h-1}, l_{h+1:N}, w_{1:N})\Big)\\ &= \sum_{r\in E}\sum_{\theta=1}^{t_1}\Phi\big(q_{kr}(\theta)\cdot f_r(x_{1:h-1} - \theta\mathbf{1}_{1:h-1}, l_{h+1:N}, w_{1:N})\big), \qquad (12)\end{aligned}$$

where the last equality is due to the additivity property of the Φ-operator. Next, we observe that any function $i : \{1, 2, \ldots, x_1\} \to \mathbb{R}$ can be seen as an element of the set of function $\mathcal{A}(h; N)$ by simply observing that $\forall\theta \in \{1, 2, \ldots, x_1\}$, we can write $i(\theta) = i(\theta, \underline{0}, \underline{\tilde{0}})$, where $\underline{\theta} = (\theta, \ldots, \theta) \in \mathbb{N}^{h-1}$, $\underline{0} = (0, \ldots, 0) \in \mathbb{N}^{N-h}$, and $\underline{\tilde{0}} = (0, \ldots, 0) \in \mathbb{N}^N$. Thus, in also using the second property of the Φ-operator (multiplicative property), Equation (12) becomes

$$\begin{aligned}&= \sum_{r\in E}\sum_{\theta=1}^{t_1}\Phi\big(q_{kr}(\underline{\theta}, \underline{0}, \underline{\tilde{0}})\cdot f_r(x_{1:h-1} - \theta\mathbf{1}_{1:h-1}, l_{h+1:N}, w_{1:N})\big)\\ &= \sum_{r\in E}\sum_{\theta=1}^{t_1}\Phi\big(q_{kr}(\underline{\theta}, \underline{0}, \underline{\tilde{0}})\big)\cdot\Phi\big(f_r(x_{1:h-1} - \theta\mathbf{1}_{1:h-1}, l_{h+1:N}, w_{1:N})\big)\\ &= \sum_{r\in E}\sum_{\theta=1}^{t_1} q_{kr}(\theta)\cdot\tilde{f}_r(x_1 - \theta; s_{1:h-1}, l_{h+1:N}, w_{1:N})), \qquad (13)\end{aligned}$$

where $s_i = x_{i+1} - \theta - (x_i - \theta) = x_{i+1} - x_i$.

The proof is complete once we observe that Equation (13) coincides with the ordinary convolution product $(q * \tilde{f})_k(x_1; s_{1:h-1}, l_{h+1:N}, w_{1:N})$. □

Now, we are in the position of formulating the main result of this study, but first, we introduce a notation that we will also use in the proof of our main result.

Assume for the moment that the $\lim_{t_h\to\infty} SIR_k^{(N)}(\underline{t}, \underline{p})$ exists, and denote it by the following notation:

$$\lim_{t_h\to\infty} SIR_k^{(N)}(\underline{t}, \underline{p}) = \begin{cases}\mathcal{L}_k^{(N)}(t_{1:h-1}, l_{h+1:N}, p_{1:N}) & \text{if } \ 2 \le h \le N,\\ \mathcal{L}^{(N)}(l_{2:N}, p_{1:N}) & \text{if } \ h = 1 \wedge N > 1\\ \mathcal{L}^{(1)}(p) & \text{if } \ h = 1 \wedge N = 1\,.\end{cases}$$

Theorem 2. *Assume that the semi-Markov chain is ergodic. Let* $\underline{t} := (t_i)_{i=1,\dots,N}$ *and* $\underline{p} := (p_i)_{i=1,\dots,N}$, $N \in \mathbb{N}^*$ *be two time sequences such that* $\{[t_i, t_i + p_i]\}_{i=1,\dots,N}$ *is a sequence of nonoverlapping real intervals. Then,* $\forall 1 \leq h \leq N$, *it results that*

$$\begin{aligned}\mathcal{L}_k^{(N)}(t_{1:h-1}, l_{h+1:N}, p_{1:N}) &:= \lim_{t_h \to \infty} SIR_k^{(N)}(\underline{t}, \underline{p}) \\ &= \sum_{r \in E} \sum_{\theta=1}^{t_1} \psi_{kr}(\theta) \mathcal{G}_r^{(N)}(t_{1:h-1} - \theta \mathbf{1}_{1:h-1}, l_{h+1:N}, p_{1:N}),\end{aligned} \tag{14}$$

where the function $\mathcal{G}_r^{(N)}(t_{1:h-1} - \theta \mathbf{1}_{1:h-1}, l_{h+1:N}, p_{1:N})$ *is fully determined in Equation (31), and the convention* $x_{1:0} = \emptyset$ *is used.*

Proof of Theorem 2. First, we observe that the theorem is true for $h = 1$, which was proved in Theorem 1 in [1]. Indeed, the authors proved that

$$\begin{aligned}\lim_{t_1 \to \infty} SIR^{(N)}(\underline{t}, \underline{p}) &= \lim_{t_1 \to \infty} \mu \psi * \widetilde{g}(t_1; l_{2:N}, p_{1:N}) =: \mathcal{L}^{(N)}(l_{2:N}, p_{1:N}) \\ = \quad \sum_{i \in E} \mu_i \sum_{j \in U} \frac{1}{\mu_{jj}} \sum_{t_1 \geq 0} g_j^{(N)}(\underline{t}, \underline{p}) &= \frac{1}{\sum_{i \in E} \nu(i) m_i} \sum_{j \in U} \nu(j) \sum_{t_1 \geq 0} g_j^{(N)}(\underline{t}, \underline{p}),\end{aligned}$$

where μ_{jj} is the mean recurrence time to state j for the semi-Markov chain.

Now, we assume that the statement in the theorem is true $\forall n \leq N - 1$, and we proceed to verify its overall validity using the mathematical induction principle.

Based on Equation (3), and taking the limit for t_h going to infinity, we have

$$\lim_{t_h \to +\infty} SIR_k^{(N)}(\underline{t}, \underline{p}) \quad = \quad \lim_{t_h \to +\infty} g_k^{(N)}(\underline{t}, \underline{p}) + \sum_{r \in E} \sum_{\theta=1}^{t_1} q_{kr}(\theta) \lim_{t_h \to +\infty} SIR_r^{(N)}(\underline{t} - \theta \mathbf{1}_{1:N}, \underline{p}). \tag{15}$$

Now, we begin to evaluate $\lim_{t_h \to +\infty} g_k^{(N)}(\underline{t}, \underline{p})$.

According to Equation (4), we have

$$\begin{aligned}\lim_{t_h \to +\infty} g_k^{(N)}(\underline{t}, \underline{p}) &:= \lim_{t_h \to +\infty} \Big\{ 1_{\{k \in U\}} \big[\overline{H}_k(t_N + p_N) \\ &+ \sum_{\theta=t_1+1}^{t_1+p_1} \sum_{r \in E} \sum_{m \in U} \sum_{v'=0}^{t_1+p_1-\theta} q_{kr}(\theta) R_{rm}^b(v'; t_1 + p_1 - \theta) SIR_m^{(N-1)}(v'; t_{2:N} - \mathbf{1}_{2:N} \cdot (t_1 + p_1), p_{2:N}) \\ &+ \sum_{j=2}^{h-1} \sum_{\theta=t_j}^{t_j+p_j} \sum_{r \in E} q_{kr}(\theta) SIR_r^{(N-j+1)}\big(0; ((\theta, t_{j+1:N}) - \mathbf{1}_{j:N} \cdot \theta), (t_j + p_j - \theta, p_{j+1:N})\big) \\ &+ \sum_{j=h}^{N} \sum_{\theta=t_j}^{t_j+p_j} \sum_{r \in E} q_{kr}(\theta) SIR_r^{(N-j+1)}\big(0; ((\theta, t_{j+1:N}) - \mathbf{1}_{j:N} \cdot \theta), (t_j + p_j - \theta, p_{j+1:N})\big) \big] \\ &+ \sum_{j=1}^{h} \sum_{\theta=t_j+p_j+1}^{t_{j+1}-1} \sum_{r \in E} q_{kr}(\theta) SIR_r^{(N-j)}(0, t_{j+1:N} - \mathbf{1}_{j+1:N} \cdot \theta, p_{j+1:N}) \\ &+ \sum_{j=h}^{N-1} \sum_{\theta=t_j+p_j+1}^{t_{j+1}-1} \sum_{r \in E} q_{kr}(\theta) SIR_r^{(N-j)}(0, t_{j+1:N} - \mathbf{1}_{j+1:N} \cdot \theta, p_{j+1:N}) \Big\},\end{aligned} \tag{16}$$

The next step is the computation of the previous limits for each of its components that we are going to enumerate for the sake of clarity:

$$(1) \quad \lim_{t_h \to +\infty} \overline{H}_k(t_N + p_N) = \lim_{t_h \to +\infty} \overline{H}_k(t_h + \sum_{s=h+1}^{N} l_s + p_N) = \overline{H}_k(\infty) = 0. \tag{17}$$

$$(2)\ \lim_{t_h\to+\infty}\sum_{\theta=t_1+1}^{t_1+p_1}\sum_{r\in E}\sum_{m\in U}\sum_{v'=0}^{t_1+p_1-\theta} q_{kr}(\theta)R^b_{rm}(v';t_1+p_1-\theta)$$
$$\cdot SIR_m^{(N-1)}(v';t_{2:N}-\mathbf{1}_{2:N}\cdot(t_1+p_1),p_{2:N}) \tag{18}$$
$$=\sum_{\theta=t_1+1}^{t_1+p_1}\sum_{r\in E}\sum_{m\in U}\sum_{v'=0}^{t_1+p_1-\theta} q_{kr}(\theta)R^b_{rm}(v';t_1+p_1-\theta)$$
$$\cdot\lim_{t_h\to+\infty} SIR_m^{(N-1)}(t_{2:N}-\mathbf{1}_{2:N}\cdot(t_1+p_1),p_{2:N}), \tag{19}$$

Here, in the last equality, we used the fact that $\forall v'\in\mathbb{N}$ for an ergodic semi-Markov process,

$$\lim_{t_h\to+\infty} SIR_m^{(N-1)}(v';t_{2:N}-\mathbf{1}_{2:N}\cdot(t_1+p_1),p_{2:N})=\lim_{t_h\to+\infty} SIR_m^{(N-1)}(t_{2:N}-\mathbf{1}_{2:N}\cdot(t_1+p_1),p_{2:N}).$$

Moreover, using the inductive hypothesis, we obtain

$$\lim_{t_h\to+\infty} SIR_m^{(N-1)}(t_{2:N}-\mathbf{1}_{2:N}\cdot(t_1+p_1),p_{2:N})$$
$$=\lim_{t^*_{h-1}\to+\infty} SIR_m^{(N-1)}(t^*_{1:N-1},p^*_{1:N-1})=\mathcal{L}_m^{(N-1)}(t^*_{1:h-2},l^*_{h:N-1},p^*_{1:N-1}), \tag{20}$$

where

$$t^*_j=t_{j+1}-(t_1+p_1)\quad\forall j=1,\dots,N-1,$$
$$l^*_j=t^*_j-t^*_{j-1}\quad\forall j=2,\dots,N-1,$$
$$p^*_j=p_{j+1}\quad\forall j=1,\dots,N-1.$$

A substitution of (20) in (18) produces

$$=\sum_{\theta=t_1+1}^{t_1+p_1}\sum_{r\in E}\sum_{m\in U}\sum_{v'=0}^{t_1+p_1-\theta} q_{kr}(\theta)R^b_{rm}(v';t_1+p_1-\theta)\mathcal{L}_m^{(N-1)}(t^*_{1:h-2},l^*_{h:N-1},p^*_{1:N-1})$$
$$=\sum_{\theta=t_1+1}^{t_1+p_1}\sum_{r\in E}\sum_{m\in U} q_{kr}(\theta)R^b_{rm}(t_1+p_1-\theta)\mathcal{L}_m^{(N-1)}(t^*_{1:h-2},l^*_{h:N-1},p^*_{1:N-1}),$$

having observed that $\sum_{v'=0}^{t_1+p_1-\theta} R^b_{rm}(v';t_1+p_1-\theta)=R^b_{rm}(t_1+p_1-\theta)$.

$$(3)\ \lim_{t_h\to+\infty}\sum_{j=2}^{h-1}\sum_{\theta=t_j}^{t_j+p_j}\sum_{r\in E} q_{kr}(\theta)SIR_r^{(N-j+1)}((\theta,t_{j+1:N})-\mathbf{1}_{j:N}\cdot\theta),(t_j+p_j-\theta,p_{j+1:N})$$
$$=\sum_{j=2}^{h-1}\sum_{\theta=t_j}^{t_j+p_j}\sum_{r\in E} q_{kr}(\theta)\lim_{t_h\to+\infty} SIR_r^{(N-j+1)}((\theta,t_{j+1:N})-\mathbf{1}_{j:N}\cdot\theta),(t_j+p_j-\theta,p_{j+1:N}). \tag{21}$$

Now, through a change in variables

$$\dot{t}_s=\begin{cases}0 & \text{if } s=1,\\ t_{j+s-1}-\theta & \text{for } s=2,\dots,N-j+1,\end{cases}$$

$$\dot{p}_s=\begin{cases}t_j+p_j-\theta & \text{if } s=1,\\ p_{j+s-1} & \text{for } s=2,\dots,N-j+1,\end{cases}$$

we obtain that

$$\lim_{t_h\to+\infty} SIR_r^{(N-j+1)}((\theta, t_{j+1:N}) - \mathbf{1}_{j:N}\cdot\theta), (t_j + p_j - \theta, p_{j+1:N})$$
$$= \lim_{\dot{t}_{h-j+1}\to+\infty} SIR_r^{(N-j+1)}(\dot{t}_{1:N-j+1}, \dot{p}_{1:N-j+1}) = \mathcal{L}_r^{(N-j+1)}(\dot{t}_{1:h-j}, \dot{l}_{h-j+2:N-j+1}, \dot{p}_{1:N-j+1}), \quad (22)$$

where the last equality is due to the inductive hypothesis. A substitution of (22) in (21) produces

$$\sum_{j=2}^{h-1}\sum_{\theta=t_j}^{t_j+p_j}\sum_{r\in E} q_{kr}(\theta)\mathcal{L}_r^{(N-j+1)}(\dot{t}_{1:h-j}, \dot{l}_{h-j+2:N-j+1}, \dot{p}_{1:N-j+1}). \quad (23)$$

$$(4)\ \sum_{j=h}^{N}\sum_{\theta=t_j}^{t_j+p_j}\sum_{r\in E} q_{kr}(\theta) SIR_r^{(N-j)}(0, t_{j+1:N} - \mathbf{1}_{j+1:N}\cdot\theta, p_{j+1:N})$$
$$\leq \sum_{j=h}^{N}\sum_{\theta=t_j}^{t_j+p_j}\sum_{r\in E} q_{kr}(\theta) = \sum_{j=h}^{N}(H_k(t_j + p_j) - H_k(t_j)). \quad (24)$$

Now, observe that if $t_h \to +\infty$, all times t_s with $s \geq h$ go to infinity as well; hence, we have

$$\lim_{t_h\to+\infty}\sum_{j=h}^{N}\sum_{\theta=t_j}^{t_j+p_j}\sum_{r\in E} q_{kr}(\theta) SIR_r^{(N-j)}(0, t_{j+1:N} - \mathbf{1}_{j+1:N}\cdot\theta, p_{j+1:N})$$
$$\leq \lim_{t_h\to+\infty}\sum_{j=h}^{N}(H_k(t_j + p_j) - H_k(t_j)) = \sum_{j=h}^{N}\lim_{t_j\to+\infty}(H_k(t_j + p_j) - H_k(t_j)) = 0. \quad (25)$$

$$(5)\ \lim_{t_h\to+\infty}\sum_{j=1}^{h-1}\sum_{\theta=t_j+p_j+1}^{t_{j+1}-1}\sum_{r\in E} q_{kr}(\theta) SIR_r^{(N-j)}(t_{j+1:N} - \mathbf{1}_{j+1:N}\cdot\theta, p_{j+1:N})$$
$$\lim_{t_h\to+\infty}\left(\sum_{j=1}^{h-2}\sum_{\theta=t_j+p_j+1}^{t_{j+1}-1}\sum_{r\in E} q_{kr}(\theta) SIR_r^{(N-j)}(t_{j+1:N} - \mathbf{1}_{j+1:N}\cdot\theta, p_{j+1:N})\right.$$
$$\left. + \sum_{\theta=t_{h-1}+p_{h-1}+1}^{t_h-1}\sum_{r\in E} q_{kr}(\theta) SIR_r^{(N-h+1)}(t_{h:N} - \mathbf{1}_{h:N}\cdot\theta, p_{h:N})\right). \quad (26)$$

Let us consider the above:

$$\lim_{t_h\to+\infty}\sum_{\theta=t_{h-1}+p_{h-1}+1}^{t_h-1}\sum_{r\in E} q_{kr}(\theta) SIR_r^{(N-h+1)}(t_{h:N} - \mathbf{1}_{h:N}\cdot\theta, p_{h:N}).$$

To this end, we observe that

$$\lim_{t_h\to+\infty} SIR_r^{(N-h+1)}(t_{h:N} - \mathbf{1}_{h:N}\cdot\theta, p_{h:N})$$
$$= \lim_{\tilde{t}_1\to+\infty} SIR_r^{(N-h+1)}(\tilde{t}_{1:N-h+1}, \tilde{p}_{1:N-h+1}) = \mathcal{L}^{(N-h+1)}(\tilde{l}_{2:N-h+1}, \tilde{p}_{1:N-h+1}), \quad (27)$$

where the last equality is a consequence of the inductive hypothesis and

$$\tilde{t}_j = t_{h-1+j} - \theta \ \ \forall j = 1, \ldots, N-h+1\,,$$

$$\tilde{p}_j = p_{h-1+j} \ \ \forall j = 1, \ldots, N-h+1\,,$$

$$\tilde{l}_s = \begin{cases} \tilde{t}_s - \tilde{t}_{s-1} & \text{for } s = 2, \dots, N-h+1, \\ 0 & \text{for } s = 1, \end{cases}$$

Moreover, we observe that

$$\sum_{\theta=t_{h-1}+p_{h-1}+1}^{t_h-1} \sum_{r\in E} q_{kr}(\theta) = \sum_{\theta=t_{h-1}+p_{h-1}+1}^{t_h-1} h_k(\theta) = \overline{H}_k(t_{h-1}+p_{h-1}),$$

Then, from the key renewal theorem (see, e.g., [4]), we obtain that

$$\begin{aligned} &\lim_{t_h\to+\infty} \sum_{\theta=t_{h-1}+p_{h-1}+1}^{t_h-1} \sum_{r\in E} q_{kr}(\theta) SIR_r^{(N-h+1)}(t_{h:N} - \mathbf{1}_{h:N}\cdot\theta, p_{h:N}) \\ &= \overline{H}_k(t_{h-1}+p_{h-1}) \mathcal{L}^{(N-h+1)}(\tilde{l}_{2:N-h+1}, \tilde{p}_{1:N-h+1}). \end{aligned}$$

It remains to compute

$$\begin{aligned} &\lim_{t_h\to+\infty} \sum_{j=1}^{h-2} \sum_{\theta=t_j+p_j+1}^{t_{j+1}-1} \sum_{r\in E} q_{kr}(\theta) SIR_r^{(N-j)}(t_{j+1:N} - \mathbf{1}_{j+1:N}\cdot\theta, p_{j+1:N}) \\ &= \sum_{j=1}^{h-2} \sum_{\theta=t_j+p_j+1}^{t_{j+1}-1} \sum_{r\in E} q_{kr}(\theta) \lim_{t_h\to+\infty} SIR_r^{(N-j)}(t_{j+1:N} - \mathbf{1}_{j+1:N}\cdot\theta, p_{j+1:N}). \end{aligned}$$

Now, set

$$\begin{aligned} \hat{t}_s &= t_{j+s} - \theta \ \ \forall s = 1, \dots, N-j, \\ \hat{p}_s &= p_{j+s} - \theta \ \ \forall s = 1, \dots, N-j, \end{aligned}$$

and observe that

$$\begin{aligned} &\lim_{t_h\to+\infty} SIR_r^{(N-j)}(t_{j+1:N} - \mathbf{1}_{j+1:N}\cdot\theta, p_{j+1:N}) \\ &= \lim_{\hat{t}_{h-j}\to+\infty} SIR_r^{(N-j)}(\hat{t}_{1:N-j}, \hat{p}_{1:N-j}) = \mathcal{L}_r^{(N-j)}(\hat{t}_{1:h-j-1}, \hat{l}_{h-j+1:N-j}, \hat{p}_{1:N-j}), \end{aligned}$$

where in the last equality, we used the inductive hypothesis. Therefore, we may conclude that limit (26) is equal to

$$\begin{aligned} &\overline{H}_k(t_{h-1}+p_{h-1}) \mathcal{L}^{(N-h+1)}(\tilde{l}_{2:N-h+1}, \tilde{p}_{1:N-h+1}) \\ &+ \sum_{j=1}^{h-2} \sum_{\theta=t_j+p_j+1}^{t_{j+1}-1} \sum_{r\in E} q_{kr}(\theta) \mathcal{L}_r^{(N-j)}(\hat{t}_{1:h-j-1}, \hat{l}_{h-j+1:N-j}, \hat{p}_{1:N-j}). \end{aligned} \tag{28}$$

It remains to compute

$$(6) \quad \lim_{t_h\to+\infty} \sum_{j=h}^{N-1} \sum_{\theta=t_j+p_j+1}^{t_{j+1}-1} \sum_{r\in E} q_{kr}(\theta) SIR_r^{(N-j)}(t_{j+1:N} - \mathbf{1}_{j+1:N}\cdot\theta, p_{j+1:N}).$$

Clearly, this limit is upper bounded by

$$\lim_{t_h\to+\infty} \sum_{j=h}^{N-1} \sum_{\theta=t_j+p_j+1}^{t_{j+1}-1} \sum_{r\in E} q_{kr}(\theta). \tag{29}$$

Now, set $a = \theta - (t_j + p_j + 1)$ to obtain the equality of (29) with

$$\lim_{t_h \to +\infty} \sum_{j=h}^{N-1} \sum_{a=0}^{(t_{j+1}-1)-(t_j+p_j+1)} \sum_{r \in E} q_{kr}(a + t_j + p_j + 1) = \sum_{j=h}^{N-1} \sum_{a=0}^{t_{j+1}-t_j-p_j-2} \sum_{r \in E} q_{kr}(\infty) = 0 \tag{30}$$

The limits from (1) to (6) computed before provide the following result:

$$\begin{aligned}
&\lim_{t_h \to +\infty} g_k^{(N)}(\underline{t}, \underline{p}) = 1_{k \in U} \Bigg\{ \sum_{\theta=t_1+1}^{t_1+p_1} \sum_{r \in E} \sum_{m \in U} q_{kr}(\theta) R_{rm}^b(t_1 + p_1 - \theta) \mathcal{L}_m^{(N-1)}(t_{1:h-2}^*, l_{h:N-1}^*, p_{1:N-1}^*) \\
&+ \sum_{j=2}^{h-1} \sum_{\theta=t_j}^{t_j+p_j} \sum_{r \in E} q_{kr}(\theta) \mathcal{L}_r^{(N-j+1)}(\dot{t}_{1:h-j}, \dot{l}_{h-j+2:N-j+1}, \dot{p}_{1:N-j+1}) \Bigg\} \\
&+ \overline{H}_k(t_{h-1} + p_{h-1}) \mathcal{L}^{(N-h+1)}(\tilde{l}_{2:N-h+1}, \tilde{p}_{1:N-h+1}) \\
&+ \sum_{j=1}^{h-2} \sum_{\theta=t_j+p_j+1}^{t_{j+1}-1} \sum_{r \in E} q_{kr}(\theta) \mathcal{L}_r^{(N-j)}(\hat{t}_{1:h-j-1}, \hat{l}_{h-j+1:N-j}, \hat{p}_{1:N-j}) =: \mathcal{G}_k^{(N)}(t_{1:h-1}, l_{h+1:N}, p_{1:N}).
\end{aligned} \tag{31}$$

According to previous computations, Equation (15) can be expressed as

$$\begin{aligned}
&\mathcal{L}_k^{(N)}(t_{1:h-1}, l_{h+1:N}, p_{1:N}) = \mathcal{G}_k^{(N)}(t_{1:h-1}, l_{h+1:N}, p_{1:N}) \\
&\sum_{r \in E} \sum_{\theta=1}^{t_1} q_{kr}(\theta) \mathcal{L}_r^{(N)}(t_{1:h-1} - \mathbf{1}_{1:h-1} \cdot \theta, l_{h+1:N}, p_{1:N}).
\end{aligned} \tag{32}$$

By applying the Φ-operator, we transform Equation (32) into an ordinary Markov renewal equation:

$$\begin{aligned}
&\tilde{\mathcal{L}}_k^{(N)}(t_1; s_{1:h-1}, l_{h+1:N}, p_{1:N}) = \tilde{\mathcal{G}}_k^{(N)}(t_1; s_{1:h-1}, l_{h+1:N}, p_{1:N}) \\
&\sum_{r \in E} \sum_{\theta=1}^{t_1} q_{kr}(\theta) \tilde{\mathcal{L}}_r^{(N)}(t_1 - \theta; s_{1:h-1}, l_{h+1:N}, p_{1:N}).
\end{aligned} \tag{33}$$

Equation (33) can be expressed in a more compact form denoted by $\Theta = \{s_{1:h-1}, l_{h+1:N}, p_{1:N}\}$, the set of parameters of the transformed function, and using a matrix-form representation:

$$\tilde{\mathcal{L}}_k^{(N)}(t_1; \Theta) = \tilde{\mathcal{G}}_k^{(N)}(t_1; \Theta) + (\mathbf{q} * \tilde{\mathcal{L}}^{(N)})_k(t_1; \Theta).$$

The solution of this Markov renewal equation is well known (cf. [4]), and it is given by

$$\tilde{\mathcal{L}}^{(N)}(t_1; \Theta) = (\boldsymbol{\psi} * \tilde{\mathcal{G}}^{(N)})(t_1; \Theta),$$

or element-wise,

$$\begin{aligned}
\tilde{\mathcal{L}}_k^{(N)}(t_1; \Theta) &= \sum_{r \in E} \sum_{\theta=1}^{t_1} \psi_{kr}(\theta) \tilde{\mathcal{G}}_r^{(N)}(t_1 - \theta; \Theta) \\
&= \sum_{r \in E} \sum_{\theta=1}^{t_1} \psi_{kr}(\theta) \mathcal{G}_r^{(N)}(t_{1:h-1} - \mathbf{1}_{1:h-1} \cdot \theta, l_{h+1:N}, p_{1:N}) \\
&= \mathcal{L}_k^{(N)}(t_{1:h-1}, l_{h+1:N}, p_{1:N}) = \lim_{t_h \to +\infty} SIR_k^{(N)}(\underline{t}, \underline{p}),
\end{aligned}$$

which concludes the proof. □

Remark 2. *Observe that the function $\mathcal{G}^{(N)}$ depends on the $\mathcal{L}^{(\diamond)}$ at the number of intervals $\diamond < N$; hence, it should be evaluated recursively in the number of intervals.*

Example 1. *As an application of Theorem 2, we can obtain an explicit representation of the mixed transient–asymptotic result of the sequential interval reliability for $N = 2$:*

$$\lim_{t_2\to\infty} SIR_k^{(2)}(t_{1:2}, p_{1:2})$$
$$= \sum_{r\in E}\sum_{\theta=1}^{t_1} \psi_{kr}(\theta)\cdot\left[1_{\{r\in U\}}\sum_{x=t_1-\theta+1}^{t_1-\theta+p_1}\sum_{v\in E} q_{rv}(x)R_v(t_1-\theta+p_1-x) + \overline{H}_r(t_1+p_1)\right]\cdot IR(\infty, p_2),$$

where $IR(\infty, p_2)$ is the asymptotic value of the interval reliability function, which can be recovered from Theorem 1 once we observe that if $t_i + p_i = t_{i+1} - 1$ for all $i = 1, \ldots, N-1$ and $v = 0$, then $SIR_k^{(N)}(0; \underline{t}, \underline{p}) = IR(t_1, t_N + p_N - t_1)$.

4. A Numerical Example

In this section, we will present a numerical example considering three semi-Markov models that govern three different repairable systems. The difference among these systems is located in the difficulty of repairing them through a repairability index (transition probability). In order to make it clear, we fully present the setting of the experiment. The state space of the systems consists of three possible states, $E = \{1, 2, 3\}$, where the operational states are the first two, $U = \{1, 2\}$, and the non-working state is the last one, $D = \{3\}$. The first state is considered to be a fully operational state, while the second one is thought to be barely operational.

The transitions of the repairable semi-Markov models are shown in the following flowgraph, Figure 1.

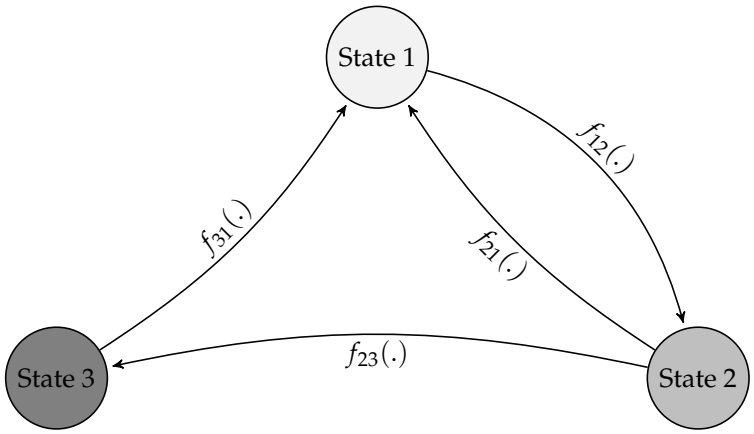


Figure 1. Semi-Markov model.

The transition matrix $\boldsymbol{P}$ of the embedded Markov chain J_n along with the initial distribution $\boldsymbol{\mu}$ are given by

$$\boldsymbol{P} = \begin{pmatrix} 0 & 1 & 0 \\ p_{21} & 0 & 1-p_{21} \\ 1 & 0 & 0 \end{pmatrix}, \ \boldsymbol{\mu} = (1, 0, 0)\,.$$

Now, let X_{ij} be the conditional sojourn time of the SMCs, and $\boldsymbol{Z}$ is state i, given that the next state is j ($j \neq i$). The conditional sojourn times are given as follows:

$$
\begin{aligned}
X_{12} &\sim \text{Geometric}(0.3)\,,\\
X_{21} &\sim \text{discrete Weibull}(0.9, 1.1)\,,\\
X_{23} &\sim \text{discrete Weibull}(0.4, 1.3)\,,\\
X_{31} &\sim \text{discrete Weibull}(0.6, 1.8)\,.
\end{aligned}
$$

The difficulty of repairing the system is located in the transition probability p_{21} of going from the barely operated state, 2, back to the fully operational state, 1. The three models are classified according to the probability p_{21} (repairability index) as easily repairable for $p_{21} = 0.9$ (Model 1), repairable for $p_{21} = 0.5$ (Model 2), and difficult to repair for $p_{21} = 0.1$ (Model 3).

Figures 2–4 illustrate each model's conditional sequential interval reliability for two time-varying intervals as the time t_2 is moving forward through time. This is the probability that the system will be operational in the time intervals $[1, 2]$ and $[k+2, k+3]$ for $k \in \{15, \ldots, 30\}$. It can be easily identified that, as time t_2 becomes large enough, each system's sequential interval reliability tends to converge to the asymptotic analogous $\lim_{t_2 \to \infty} SIR_k^{(N)}(v; \underline{t}, \underline{p})$. On the other hand, Figures 5–7 depict each model's conditional sequential interval reliability for two nonoverlapping, contiguous time intervals as the time t_1 is moving forward through time. Also, in this case, the sequential interval reliability function exhibits a tendency to converge asymptotically, as expected from our theoretical result.

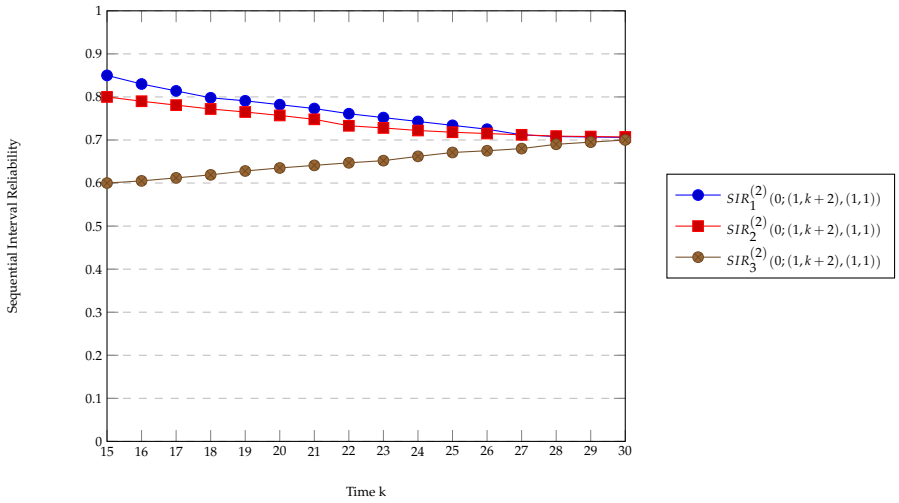


Figure 2. Model 1. Sequential interval reliability plot with two separate time-varying intervals.

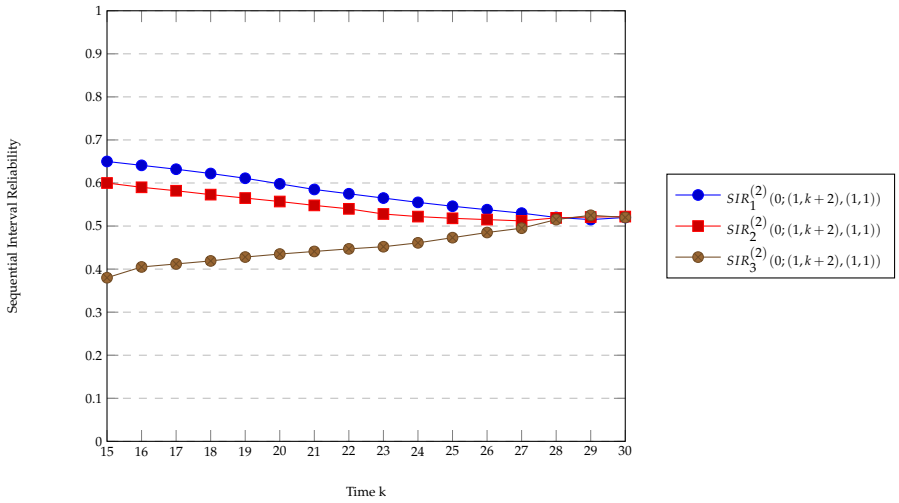


Figure 3. Model 2. Sequential interval reliability plot with two separate time-varying intervals.

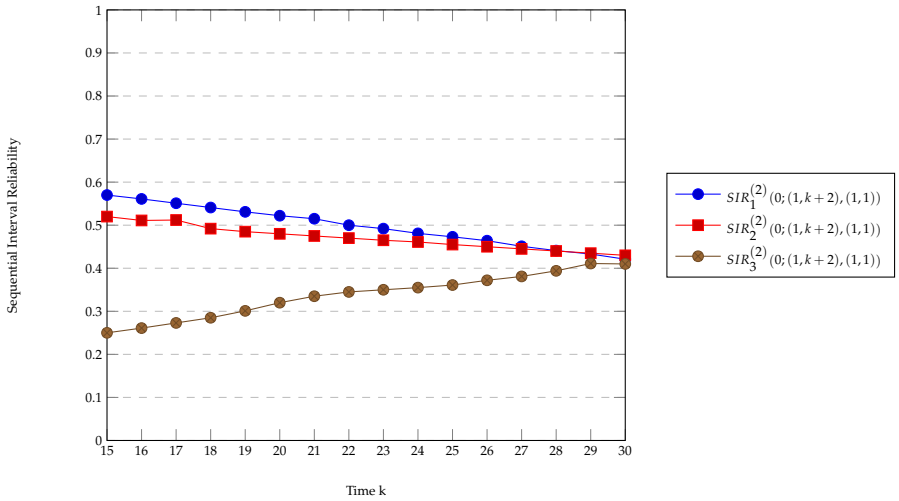


Figure 4. Model 3. Sequential interval reliability plot with two separate time-varying intervals.

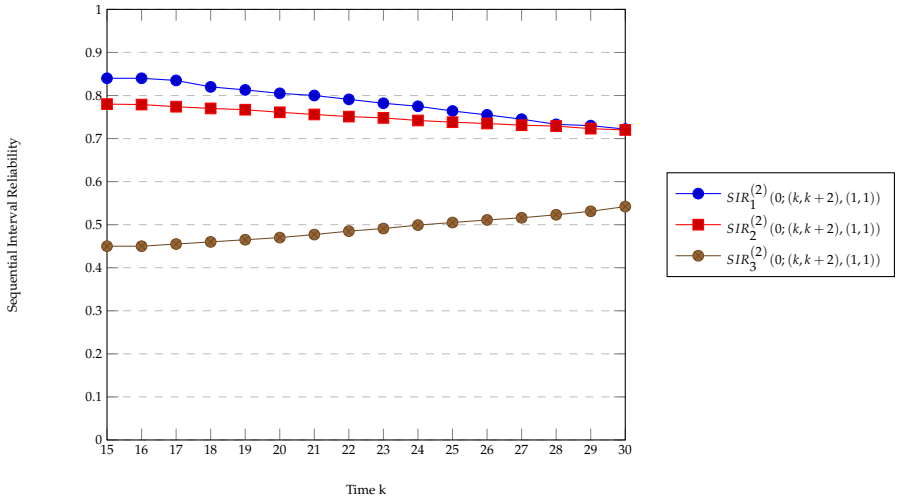


Figure 5. Model 1. Sequential interval reliability plot with two nonoverlapping, contiguous time intervals.

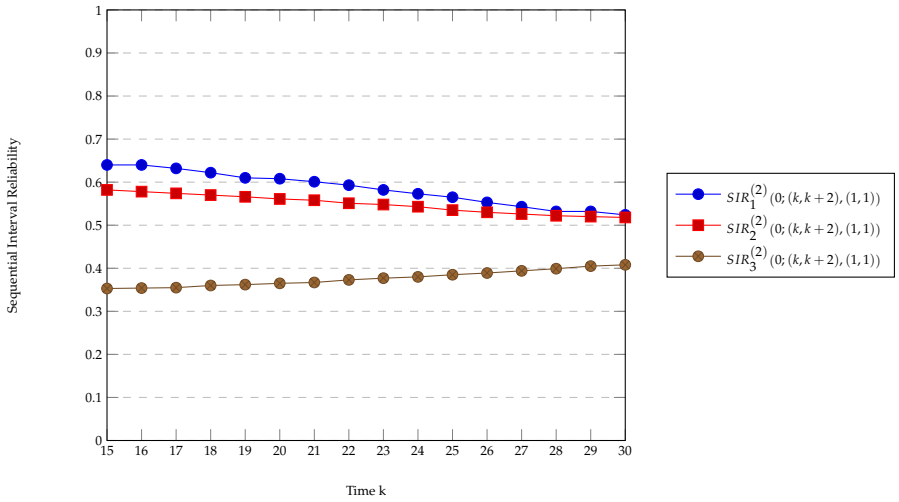


Figure 6. Model 2. Sequential interval reliability plot with two nonoverlapping, contiguous time intervals.

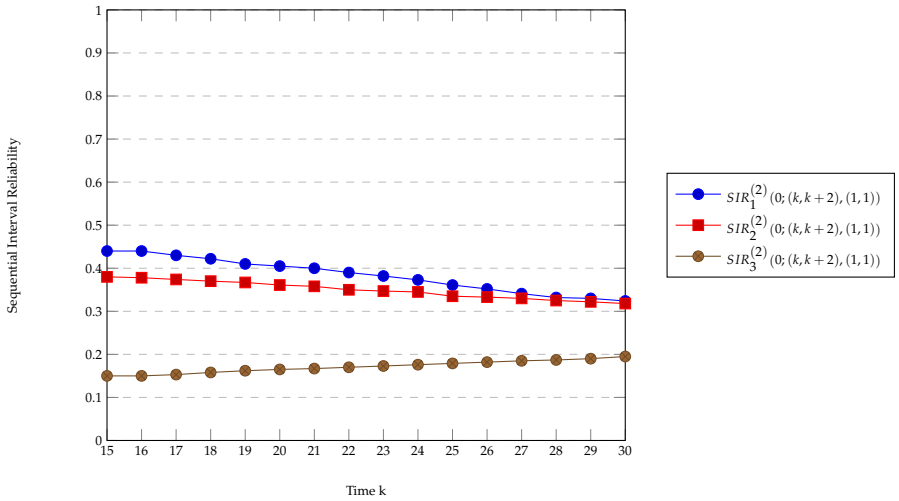


Figure 7. Model 3. Sequential interval reliability plot with two nonoverlapping, contiguous time intervals.

5. Conclusions

This paper presents a new result for the sequential interval reliability (SIR), an indicator recently introduced in the literature. The new result provides a mixed transient and asymptotic description for the system, merging these two aspects into a unique measure. The achievement of this result needs dedicated mathematical development, which includes several innovative aspects. Indeed, this paper introduces a specific operator acting between two sets of functions, a new matrix convolution product, and the use of classical results in probability theory such as the key renewal theorem and Markov renewal equation techniques in the generalized framework. Future work may include an evaluation of this indicator and related transient–asymptotic results in different applied problems involving the use of real data.

Author Contributions: Conceptualization, G.D.; methodology, G.D.; software, T.G.; validation, G.D. and T.G.; formal analysis, G.D.; investigation, G.D. and T.G.; resources, G.D. and T.G.; data curation, T.G.; writing—original draft preparation, G.D. and T.G.; writing—review and editing, G.D. and T.G.; visualization, T.G.; supervision, G.D.; project administration, G.D.; funding acquisition, G.D. All authors have read and agreed to the published version of the manuscript.

Funding: Guglielmo D'Amico acknowledges the financial support from the program MUR PRIN 2022 n. 2022ETEHRM "Stochastic models and techniques for the management of wind farms and power systems". Guglielmo D'Amico is a member of the Gruppo Nazionale Calcolo Scientifico-Istituto Nazionale di Alta Matematica (GNCS-INdAM).

Data Availability Statement: Data are contained within the article.

Acknowledgments: The authors would like to express their gratitude to Panagiotis- Christos Vassiliou and Andreas C. Georgiou for the opportunity to submit the present manuscript to the Special Issue *Stochastic Processes and Their Applications: In Honor of Prof. Sally McClean* for possible publication.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Barbu, V.S.; D'Amico, G.; Gkelsinis, T. Sequential interval reliability for discrete-time homogeneous semi-Markov repairable systems. *Mathematics* **2021**, *9*, 1997. [CrossRef]
2. Limnios, N.; Oprisan, G. *Semi-Markov Processes and Reliability*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2001.
3. Janssen, J.; Manca, R. *Applied Semi-Markov Processes*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2006.
4. Barbu, V.S.; Limnios, N. *Semi-Markov Chains and Hidden Semi-Markov Models toward Applications: Their Use in Reliability and DNA Analysis*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2009; Volume 191.
5. Vassiliou, P.C.; Georgiou, A.C. Markov and Semi-Markov Chains, processes, systems, and emerging related fields, 2021.
6. Limnios, N. Dependability analysis of semi-Markov systems. *Reliab. Eng. Syst. Saf.* **1997**, *55*, 203–207. [CrossRef]

7. Ouhbi, B.; Limnios, N. Nonparametric reliability estimation of semi-Markov processes. *J. Stat. Plan. Inference* **2003**, *109*, 155–165. [CrossRef]
8. D'Amico, G. Age-usage semi-Markov models. *Appl. Math. Model.* **2011**, *35*, 4354–4366. [CrossRef]
9. Votsi, I.; Gayraud, G.; Barbu, V.; Limnios, N. Hypotheses testing and posterior concentration rates for semi-Markov processes. *Stat. Inference Stoch. Process.* **2021**, *24*, 707–732. [CrossRef]
10. D'Amico, G.; Petroni, F. ROCOF of higher order for semi-Markov processes. *Appl. Math. Comput.* **2023**, *441*, 127719. [CrossRef]
11. das Chagas Moura, M.; Droguett, E.L. Mathematical formulation and numerical treatment based on transition frequency densities and quadrature methods for non-homogeneous semi-Markov processes. *Reliab. Eng. Syst. Saf.* **2009**, *94*, 342–349. [CrossRef]
12. Blasi, A.; Janssen, J.; Manca, R. Numerical treatment of homogeneous and non-homogeneous semi-Markov reliability models. *Commun. Stat. Theory Methods* **2004**, *33*, 697–714. [CrossRef]
13. Corradi, G.; Janssen, J.; Manca, R. Numerical treatment of homogeneous semi-Markov processes in transient case—A straightforward approach. *Methodol. Comput. Appl. Probab.* **2004**, *6*, 233–246. [CrossRef]
14. Wu, B.; Maya, B.I.G.; Limnios, N. Using semi-Markov chains to solve semi-Markov processes. *Methodol. Comput. Appl. Probab.* **2021**, *23*, 1419–1431. [CrossRef]
15. Barbu, V.; Boussemart, M.; Limnios, N. Discrete-time semi-Markov model for reliability and survival analysis. *Commun. Stat.-Theory Methods* **2004**, *33*, 2833–2868. [CrossRef]
16. Trevezas, S.; Limnios, N. Exact MLE and asymptotic properties for nonparametric semi-Markov models. *J. Nonparametric Stat.* **2011**, *23*, 719–739. [CrossRef]
17. D'Amico, G. Single-use reliability computation of a semi-Markovian system. *Appl. Math.* **2014**, *59*, 571–588. [CrossRef]
18. Barbu, V.S.; Berard, C.; Cellier, D.; Sautreuil, M.; Vergne, N. Sequential interval reliability for discrete-time homogeneous semi-Markov repairable systems. *R J.* **2018**, *10*, 226–247. [CrossRef]
19. Csenki, A. On the interval reliability of systems modelled by finite semi-Markov processes. *Microelectron. Reliab.* **1994**, *34*, 1319–1335. [CrossRef]
20. Csenki, A. An integral equation approach to the interval reliability of systems modelled by finite semi-Markov processes. *Reliab. Eng. Syst. Saf.* **1995**, *47*, 37–45. [CrossRef]
21. Georgiadis, S.; Limnios, N. Interval reliability for semi-Markov systems in discrete time. *J. Soc. Fr. Stat.* **2014**, *155*, 152–166.
22. Georgiadis, S.; Limnios, N. Nonparametric estimation of interval reliability for discrete-time semi-Markov systems. *J. Stat. Theory Pract.* **2016**, *10*, 20–39. [CrossRef]
23. D'Amico, G.; Manca, R.; Petroni, F.; Selvamuthu, D. On the computation of some interval reliability indicators for semi-Markov systems. *Mathematics* **2021**, *9*, 575. [CrossRef]
24. D'Amico, G.; Janssen, J.; Manca, R. Initial and final backward and forward discrete time non-homogeneous semi-Markov credit risk models. *Methodol. Comput. Appl. Probab.* **2010**, *12*, 215–225. [CrossRef]
25. De Dominicis, R.; Manca, R. An algorithmic approach to non-homogeneous semi-Markov processes. *Commun. Stat.-Simul. Comput.* **1984**, *13*, 823–838. [CrossRef]
26. Vassiliou, P.C.; Papadopoulou, A. Non-homogeneous semi-Markov systems and maintainability of the state sizes. *J. Appl. Probab.* **1992**, *29*, 519–534. [CrossRef]
27. Vasileiou, A.; Vassiliou, P.C. An inhomogeneous semi-Markov model for the term structure of credit risk spreads. *Adv. Appl. Probab.* **2006**, *38*, 171–198. [CrossRef]
28. D'Amico, G.; Janssen, J.; Manca, R. Downward migration credit risk problem: A non-homogeneous backward semi-Markov reliability approach. *J. Oper. Res. Soc.* **2016**, *67*, 393–401. [CrossRef]
29. Vassiliou, P.C. Non-homogeneous semi-Markov and Markov renewal processes and change of measure in credit risk. *Mathematics* **2020**, *9*, 55. [CrossRef]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.