

Article

Extended Efficient Multistep Solvers for Solving Equations in Banach Spaces

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Abstract: In this paper, we investigate the local and semilocal convergence of an iterative method for solving nonlinear systems of equations. We first establish the conditions under which these methods converge locally to the solution. Then, we extend our analysis to examine the semilocal convergence of these methods, considering their behavior when starting from initial guesses that are not necessarily close to the solution. Iterative approaches for solving nonlinear systems of equations must take into account the radius of convergence, computable upper error bounds, and the uniqueness of solutions. These points have not been addressed in earlier studies. Moreover, we provide numerical examples to demonstrate the theoretical findings and compare the performance of these methods under different circumstances. Finally, we conclude that our examination offers a significant understanding of the convergence characteristics of previous iterative techniques for solving nonlinear equation systems.

Keywords: multi-step solver; nonlinear systems; Banach space; convergence order; Lipschitz condition

MSC: 47H17; 65H10; 65G99



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1. Introduction

This study deals with the task of finding a solution x_* for the equation

$$F(x) = 0, \quad (1)$$

which is locally unique, where $\mathbb{Z}_1, \mathbb{Z}_2$ are the Banach spaces [1] and $\Omega \subset \mathbb{Z}_1$ is the convex set, $F : \Omega \rightarrow \mathbb{Z}_2$ is a differentiable operator in the Fréchet sense. Solving a system of nonlinear equations of the form (1) is one of the most challenging and complicated tasks, as there is no general method that works for all problems. Thus, the iterative solvers [2] are the only option to obtain the solution of (1). Numerous scientific and engineering domains, such as steering issues, Fisher problems, and BVPs, can be converted into higher-order systems of nonlinear equations. Thus, such problems can be solved by iterative approaches without losing generality. Therefore, these methods provide powerful tools for solving complex problems and are essential for many practical applications [3,4].

The Newton iterative solver [2,5,6] is particularly useful for solving systems of nonlinear equations, which is defined below:

$$x^{(m+1)} = x^{(m)} - F'(x^{(m)})^{-1}F(x^{(m)}), \quad m = 0, 1, 2, 3, \dots$$

It involves updating a vector of estimates for the solution by iteratively solving a linear system of equations derived from the Jacobian matrix $F'(x^{(m)})$ of the system. The importance

of the Newton iterative solver for systems of nonlinear equations lies in its ability to provide accurate solutions and handle complex systems with multiple variables and equations.

In particular, we present the local convergence analysis of a multistep solver, described as follows:

$$\begin{aligned} y_1^{(m)} &= x^{(m)} - F'(x^{(m)})^{-1}F(x^{(m)}), \\ y_2^{(m)} &= x^{(m)} - 2\left(F'(x^{(m)}) + F'(y_1^{(m)})^{-1}F(x^{(m)})\right), \\ y_3^{(m)} &= y_2^{(m)} - A_m F'(x^{(m)})^{-1}F(y_2^{(m)}), \\ &\vdots \\ x_{n+1}^{(m+1)} &= y_n^{(m+1)} = y_{n-1}^{(m)} - A_m F'(x^{(m)})^{-1}F(y_{n-1}^{(m)}), \end{aligned} \quad (2)$$

where n is a given natural number, initial guess $x_0 \in \Omega$, and

$$A_m = \frac{7}{2}I - 4F'(x^{(m)})^{-1}F'(y_1^{(m)}) + \frac{3}{2}\left(F'(x^{(m)})^{-1}F'(y_1^{(m)})\right)^2, \quad m = 0, 1, 2, 3, 4, \dots$$

The convergence order $3(n-1)$ ($n \geq 3$) was shown in ref. [7] for $\mathbb{Z}_1 = \mathbb{Z}_2 = \mathbb{R}^j$. The solver (2) uses $(n-1)$ operator evaluations and two frozen derivatives per iteration. The order of convergence was derived in an earlier study [7] by using derivatives of the involved operator of order at least eight (which do not appear in solver (2)) and Taylor expansions.

As a motivating experiment, let us assume Ω as any interval containing 0 to 1. Then, define the following function F on Ω by

$$F(t) = \begin{cases} t^4 \log t + t^7 - t^6 & , t \neq 0 \\ 0 & , t = 0. \end{cases}$$

Notice that for the required solution $x_* = 1$, we have $F(x_*) = 0$, but $F^{(5)}(t)$ does not exist on Ω , even though the solver converges to $x_* = 1$. But, the result in [7], which requires the existence of at least the eighth derivative, cannot be applied. This is true as [7]'s requirements are sufficient but also not necessary. In addition, the radius of convergence is a critical aspect of iterative solvers. It determines the convergence properties of the solver and indicates the maximum range of the initial values that leads to a convergent solution. A small radius of convergence implies a slow convergence rate, while a large radius of convergence ensures faster and more robust convergence. The accurate computation of the radius of convergence is essential for optimizing the performance and reliability of iterative solvers in practical applications. Further, the computable upper error bounds on $\|y_n^{(j)} - x_*\|$ and results on the uniqueness of x_* were not considered in a previous study [7] or in similar methods using Taylor series to determine the order of convergence [2,8–14].

In this study, we cover these factors, which are important for determining the effectiveness and reliability of every iterative solver. By considering the local, semi-local, radius of convergence, computable error bounds, and uniqueness of the root, we can assert the granted convergence of the iterative method for a particular problem. We demonstrate the application in steering problems, Fisher problems, BVPs, and higher-order systems of nonlinear equations, as well as suggest the radii of convergence.

2. Analysis: Local

Let $S = (-\infty, +\infty)$. Consider the function $\aleph_0 : S \rightarrow S$ to be continuous and nondecreasing (CN).

Assume:

- (i) The equation

$$\aleph_0(t) - 1 = 0, \quad (3)$$

has a minimal positive solution (mps) κ_0 . Set $S_0 = [0, \kappa_0]$.

Consider the function $\aleph : S_0 \rightarrow S$ be continuous and nondecreasing (CN). Define functions g_1, h_1, g_2, h_2 on the interval, so by

$$g_1(t) = \frac{\int_0^1 \aleph((1-\theta)t) d\theta}{1 - \aleph_0(t)} \text{ and } h_1(t) = g_1(t) - 1.$$

(ii) Equation $h_1(t) = 0$ has an mps in $(0, \kappa_0)$, denoted by r_1 .

(iii) The equation

$$p(t) - 1 = 0, \quad (4)$$

has an mps, denoted by κ_1 , where

$$p(t) = \frac{1}{2} \left(\aleph_0(t) + \aleph_0(g_1(t)t) \right).$$

Set $\kappa = \min\{\kappa_0, \kappa_1\}$ and $S_1 = [0, \kappa]$.

Consider the function $v : S_1 \rightarrow S$ to be CN. Define functions g_2 and h_2 on the interval S_1 by

$$g_2(t) = \frac{\int_0^1 \aleph((1-\theta)t) d\theta}{1 - \aleph_0(t)} + \frac{\left(\aleph_0(t) + \aleph_0(g_1(t)t) \right) \int_0^1 \aleph_1(\theta t) d\theta}{2(1 - \aleph_0(t))(1 - p(t))}$$

and $h_2(t) = g_2(t) - 1$.

(iv) Equation $h_2(t) = 0$ has an mps in S_1 , denoted by r_2 . Define functions g_n, h_n ($n \geq 3$) on the interval S_1 by

$$\begin{aligned} g_n(t) = & \left[\frac{\int_0^1 \aleph((1-\theta)g_{n-1}(t)t) d\theta}{1 - \aleph_0(g_{n-1}(t)t)} + \frac{\left(\aleph_0(t) + \aleph_0(g_{n-1}(t)t) \right) \int_0^1 \aleph_1(\theta g_{n-1}(t)t) d\theta}{\left(1 - \aleph_0(g_{n-1}(t)t) \right) \left(1 - \aleph_0(t) \right)} \right. \\ & + \frac{1}{2} \left\{ \left(\frac{\aleph_0(t) + \aleph_0(g_{n-2}(t)t)}{1 - \aleph_0(t)} \right)^2 + 4 \left(\frac{\aleph_0(t) + \aleph_0(g_{n-2}(t)t)}{1 - \aleph_0(t)} \right) \right\} \\ & \left. \times \frac{\int_0^1 \aleph_1(\theta g_{n-1}(t)t) d\theta}{1 - \aleph_0(t)} \right] g_{n-1}(t) \end{aligned}$$

and

$$h_n(t) = g_n(t) - 1.$$

(v) Equations $h_n(t) = 0$ have an mps, denoted by r_n in S_1 .

Define a radius of convergence r by

$$r = \min\{r_i\}, \quad i = 1, 2, 3, \dots, n. \quad (5)$$

These definitions imply that

$$0 \leq \aleph_0(t) < 1, \quad (6)$$

$$0 \leq p(t) < 1 \quad (7)$$

and

$$0 \leq g_i(t) < 1, \quad (8)$$

for all $t \in [0, r]$.

We denote the open and closed balls in $\mathbb{Z}_1$ by $\mathbb{E}(z, \gamma)$, $\mathbb{E}[z, \gamma]$, respectively with a center of $z \in \mathbb{Z}_1$ and radius of $\gamma > 0$.

The convergence conditions (A) will be used.

Assume:

(A₁) $F : \Omega \rightarrow \mathbb{Z}_2$ is differentiable; $x_* \in \Omega$ exists, and $F'(x_*) = \Delta \in \ell(\mathbb{Z}_1, \mathbb{Z}_2)$, such that

$$F(x_*) = 0 \quad \text{and} \quad \Delta^{-1} \in \ell(\mathbb{Z}_2, \mathbb{Z}_1).$$

(A₂) For each $x \in \Omega$

$$\left\| \Delta^{-1} (F'(x) - \Delta) \right\| \leq \aleph_0(\|x - x_*\|).$$

Set $\Omega_0 = \Omega \cap \mathbb{E}(x_*, \kappa_0)$.

(A₃) For each $x, y \in S_0$

$$\left\| \Delta^{-1} (F'(x) - F'(y)) \right\| \leq \aleph(\|x - y\|).$$

Set $\Omega_1 = \Omega \cap \mathbb{E}(x_*, \kappa_0)$.

(A₄) For each $x \in \Omega_1$

$$\left\| \Delta^{-1} F'(x) \right\| \leq \aleph_1(\|x - x_*\|).$$

(A₅) $\mathbb{E}(x_*, r) \subset \Omega$

and

(A₆) There exists $r_* \geq r$, such that

$$\int_0^1 \aleph_0(\theta r_*) d\theta < 1.$$

Set $\Omega_2 = \Omega \cap \mathbb{E}[x_*, r_*]$.

Based on the preceding notations and conditions (A), we arrive at the main local convergence result for solver (2)

Theorem 1. Assume conditions (A) hold. Then, starting from $x_0 \in \mathbb{E}(x_*, r) - \{x_*\}$ sequence $\{y_n^{(m)}\}$ generated by solver (2) is defined, stays in $\mathbb{E}(x_*, r)$, and converges to x_* . Moreover, solution x_* is unique in the set Ω_2 given in (A₆).

Proof. We use mathematical induction. Then, by the condition (A₂), we have that for each $x \in \mathbb{E}(x_*, r) - \{x_*\}$

$$\left\| \Delta^{-1} (F'(x) - \Delta) \right\| \leq \aleph_0(\|x - x_*\|) \leq \aleph_0(r) < 1. \quad (9)$$

It follows from (9) and the celebrated Lemma due to Banach on invertible operators [1,6,15] that $F'(x)^{-1} \in (\mathbb{Z}_2, \mathbb{Z}_1)$, which stands for the space of bounded linear operators from $\mathbb{Z}_2$ to $\mathbb{Z}_1$, and

$$\left\| F'(x)^{-1} \Delta \right\| \leq \frac{1}{1 - \aleph_0(\|x - x_*\|)}. \quad (10)$$

Thus, for $x = x^{(m)}$, we have from (5), (8) (for $i = 1$), (A₃) and (10)

$$\begin{aligned} \|y_1^m - x_*\| &= \left\| \rho^{(m)} - F'(x^{(m)})^{-1} F(x^{(m)}) \right\| \\ &\leq \left\| F'(x^{(m)})^{-1} \Delta \right\| \left\| \int_0^1 \Delta^{-1} (F'(x_* + \theta(\rho^{(m)})) - F'(x^{(m)})) d\theta \rho^{(m)} \right\| \\ &\leq \frac{\int_0^1 \aleph((1 - \theta)\|\rho^{(m)}\|) d\theta}{1 - \aleph_0(\|\rho^{(m)}\|)} \|\rho^{(m)}\| \\ &\leq \|\rho^{(m)}\| < r, \end{aligned} \quad (11)$$

where $x^{(m)} - x_* = \rho^{(m)}$. Hence, the iterate $y_1^{(m)} \in \mathbb{E}(x_*, r)$. We need the estimate

$$\begin{aligned} & \left\| (2\Delta)^{-1} \left(F'(x^{(m)}) + F'(y_1^{(m)}) - 2\Delta \right) \right\| \\ & \leq \frac{l}{2} \left[\left\| \Delta^{-1} \left(F'(x^{(m)}) - \Delta \right) \right\| + \left\| \Delta^{-1} \left(F'(y_1^{(m)}) - \Delta \right) \right\| \right] \\ & \leq \frac{l}{2} \left(\aleph_0(\|\rho^{(m)}\|) + \aleph_0(\|y_1^{(m)} - x_*\|) \right) \\ & \leq \frac{l}{2} \left[\aleph_0(\|\rho^{(m)}\|) + \aleph_0(g_1(\|\rho^{(m)}\|)\|\rho^{(m)}\|) \right] \\ & = p(\|\rho^{(m)}\|) \leq p(r) < 1. \end{aligned}$$

Thus, we have

$$\left\| \left(F'(x^{(m)}) + F'(y_1^{(m)}) \right)^{-1} \Delta \right\| \leq \frac{l}{2(1 - p(\|\rho^{(m)}\|))}. \quad (12)$$

Hence, the iterate $y_2^{(m)}$ is also well defined. Thus, similar to the second sub step of solver, we have the (2)

$$\begin{aligned} \|y_2^{(m)} - x_*\| &= \left\| \left(\rho^{(m)} - F'(x^{(m)})^{-1} F(x^{(m)}) \right) + \left[-2 \left(F'(x^{(m)}) + F'(y_1^{(m)}) \right)^{-1} \right. \right. \\ & \quad \left. \left. + F'(x^{(m)})^{-1} \right] F(x^{(m)}) \right\| \\ &= \left\| \left(\rho^{(m)} - F'(x^{(m)})^{-1} F(x^{(m)}) \right) + F'(x^{(m)})^{-1} \left(F'(y_1^{(m)}) - F'(x^{(m)}) \right) \right. \\ & \quad \left. \times \left(F'(x^{(m)}) + F'(y_1^{(m)}) \right)^{-1} F(x^{(m)}) \right\| \\ &\leq \left[\frac{\int_0^1 \aleph((1-\theta)\|\rho^{(m)}\|) d\theta}{1 - \aleph_0(\|\rho^{(m)}\|)} \right. \\ & \quad \left. + \frac{(\aleph_0(\|\rho^{(m)}\|) + \aleph_0(\|y_1^{(m)} - x_*\|)) \int_0^1 \aleph_1(\theta\|\rho^{(m)}\|) d\theta}{(1 - \aleph_0(\|\rho^{(m)}\|))(1 - p(\|\rho^{(m)}\|))} \right] \|\rho^{(m)}\| \\ &\leq g_2(\|\rho^{(m)}\|)\|\rho^{(m)}\| \leq \|\rho^{(m)}\|. \end{aligned} \quad (13)$$

Therefore, the iterate $y_2^{(m)} \in \mathbb{E}(x_*, r)$.

Next, for $i \geq 3$, we similarly obtain,

$$\begin{aligned} \|y_i^{(m)} - x_*\| &= \left\| \left(y_{i-1}^{(m)} - x_* - F'(y_{i-1}^{(m)})^{-1} F(y_{i-1}^{(m)}) \right) + \left(F'(y_{i-1}^{(m)})^{-1} - F'(x^{(m)})^{-1} \right) F(y_{i-1}^{(m)}) \right. \\ & \quad \left. + \left(\frac{5}{2} I - 4F'(x^{(m)})^{-1} F'(y_{i-2}^{(m)}) + \frac{3}{2} \left(F'(x^{(m)})^{-1} F'(y_{i-2}^{(m)}) \right)^2 \right) F'(x^{(m)})^{-1} F(y_{i-1}^{(m)}) \right\| \\ &\leq \left[\frac{\int_0^1 \aleph((1-\theta)\|y_{i-1}^{(m)} - x_*\|) d\theta}{1 - \aleph_0(\|y_{i-1}^{(m)} - x_*\|)} \right. \\ & \quad \left. + \frac{(\aleph_0\|y_{i-1}^{(m)} - x_*\| + \aleph_0(\|\rho^{(m)}\|)) \int_0^1 \aleph_1(\theta\|y_{i-1}^{(m)} - x_*\|) d\theta}{(1 - \aleph_0(\|y_{i-1}^{(m)} - x_*\|))(1 - \aleph_0(\|\rho^{(m)}\|))} \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{l}{2} \left\{ \left(\frac{\aleph_0(\|\rho^{(m)}\|) + \aleph_0(\|y_{i-2}^{(m)} - x_*\|)}{1 - \aleph_0(\|\rho^{(m)}\|)} \right)^2 + 4 \left(\frac{\aleph_0(\|\rho^{(m)}\|) + \aleph_0(\|y_{i-2}^{(m)} - x_*\|)}{1 - \aleph_0(\|\rho^{(m)}\|)} \right) \right\} \\
& \times \frac{\int_0^1 \aleph_1(\theta \|y_{i-1}^{(m)} - x_*\|) d\theta}{1 - \aleph_0(\|\rho^{(m)}\|)} \Big] \|y_{i-1}^{(m)} - x_*\| \\
& \leq g_i(\|y_{i-1}^{(m)} - x_*\|) \|y_{i-1}^{(m)} - x_*\| \leq \|y_{i-1}^{(m)} - x_*\| < r.
\end{aligned}$$

Thus, the iterate $y_i^{(m)} \in \mathbb{E}(x_*, r)$. Thus, using the estimate $\|y_{n+1}^{(m)} - x_*\| \leq c \|y_n^{(m)} - x_*\| < r$, where $c = g_n(\|x_0 - x_*\|) \in [0, 1]$, we deduce that $y_{n+1}^{(m)} \in \mathbb{E}(x_*, r)$ and $\lim_{m \rightarrow \infty} y_{n+1}^{(m)} = x_*$. Moreover, the uniqueness of the solution part follows if we consider $y_* \in \Omega_2$ with $F(y_*) = 0$. Then, using (A_2) and (A_5) , for $T = \int_0^1 F'(x_* + \theta(y_* - x_*)) d\theta$, we get

$$\|\Delta^{-1}(T - \Delta)\| \leq \int_0^1 \aleph_0(\theta \|y_* - x_*\|) d\theta \leq \int_0^1 \aleph_0(\theta r_*) d\theta < 1.$$

Hence, $T^{-1} \in \ell(\mathbb{Z}_2, \mathbb{Z}_1)$. Furthermore, we use the approximation $0 = F(y_*) - F(x_*) = T(y_* - x_*)$ to conclude that $x_* = y_*$. $\square$

Remark 1. A usual pick for $\Delta = F'(x_*)$, which makes x_* a simple solution. Note that this hypothesis is not made in Theorem 2. Consequently, method (2) is applicable for compute solutions of multiplicity greater than 1, if $\Delta \neq F'(x_*)$. Moreover, the pick $\Delta = F'(x_*)$, is not necessarily the most flexible.

3. Semi-Local Convergence

The proof of the semi-local convergence requires the utilization of majorizing sequences [1,3,4,6].

Assume that a CN function $w_0 : [0, +\infty) \rightarrow \mathbb{R}$ exists, such that equation $w_0(t) - 1 = 0$ has an mps, denoted by α . Let $w : [0, \alpha) \rightarrow \mathbb{R}$ be a CN.

Define the scalar sequence $\{a_m\}$ for $a_0 = a_0^{(0)} = 0$, $a_0^{(1)} \geq 0$, n is a fixed natural number and each $a_{m+1} = a_n^{(m)}$, $m = 0, 1, 2, \dots$ by

$$\begin{aligned}
\bar{w}_m &= \begin{cases} w(a_m^{(1)} - a_0^{(m)}) \\ \text{or} \\ w_0(a_0^{(m)}) + w_0(a_m^{(1)}), \end{cases} \\
a_2^{(m)} &= a_1^{(m)} + \frac{\bar{w}_m(a_m^{(1)} - a_m^{(0)})}{2[1 - 0.5(w_0(a_0^{(m)}) + w_0(a_1^{(m)}))]}, \\
b_2^{(m)} &= \left(1 + \int_0^1 w_0(a_0^{(m)} + \theta(a_2^{(m)} - a_0^{(m)})) d\theta\right) (a_m^2 - a_0^{(m)}) \\
&\quad + (1 + w_0(a_0^{(m)})) (a_1^{(m)} - a_0^{(m)}), \\
p_m &= 6 \left(\frac{\bar{w}_n}{1 - w_0(a_0^{(m)})} \right)^2 + 4 \left(\frac{\bar{w}_n}{1 - w_0(a_0^{(m)})} \right) + 5, \\
a_3^{(m)} &= a_2^{(m)} + \frac{p_m b_2^{(m)}}{1 - w_0(a_0^{(m)})},
\end{aligned}$$

$$\begin{aligned}
b_{k-1}^{(m)} &= \left(1 + \int_0^1 w_0(a_0^{(m)} + \theta(a_{k-1}^{(m)} - a_0^{(m)})) d\theta\right) (a_{k-1}^{(m)} - a_0^{(m)}) \\
&\quad + \left(1 + w_0(a_0^{(m)})\right) (a_1^{(m)} - a_0^{(m)}), \quad k = 2, \dots, (n-1) \\
a_k^{(m)} &= a_{k-1}^{(m)} + \frac{p_m b_{k-1}^{(m)}}{1 - w_0(a_0^{(m)})}, \\
a_{m+1} &= a_n^{m+1} = a_{n-1}^{(m)} + \frac{p_m b_{n-1}^{(m)}}{1 - w_0(a_0^{(m)})}, \\
\lambda_{m+1} &= \int_0^1 w((1-\theta)(a_{m+1} - a_m)) d\theta (a_{m+1} - a_m) + (1 + w_0 a_m) (a_{m+1} - a_1^{(m)}),
\end{aligned}$$

and

$$a_{m+1}^{(1)} = a_{m+1} + \frac{\lambda_{m+1}}{1 - w_0(a_{m+1})}. \quad (14)$$

General convergence conditions follow for the sequence $\{a_m\}$.

Lemma 1. Assume

$$w_0(a_0^{(m)}) + w_0(a_1^{(m)}) < 2, \quad w_0(a_0^{(m)}) < 1, \quad \text{and} \quad a_m \leq \alpha. \quad (15)$$

Then, the following assertions hold

$$a_m \leq a_{m+1} \leq \alpha, \quad (16)$$

and $\alpha_* \in [0, \alpha]$ exists, such that $\lim_{m \rightarrow \infty} a_m = \alpha_*$.

Proof. The condition (2) and the definition of the sequence $\{a_m\}$ given by formula (14) imply the assertion (16). Hence, the limit point α_* exists. $\square$

Note that α_* is the unique least upper bound of sequence $\{a_m\}$. Next, functions w_0 , w and the limit point α_* are connected to the operators on the solver (2).

Assume:

(H₁) There exists $x_0 \in \Omega$, $\Lambda \in \ell(z_1, z_2)$ and a parameter $a_0^{(1)} \geq 0$, such that linear operator Λ is invertible and $\|\Lambda^{-1}F(x_0)\| \leq a_0^{(1)}$.

(H₂) $\|\Lambda^{-1}(F'(v) - \Lambda)\| \leq w_0(\|v - x_0\|)$ for each $v_1 \in \Omega$.

Set $\Omega_2 = \Omega \cap \mathbb{E}(x_0, \alpha)$. The choice of α , and the conditions (H₂) for $v_1 = x_0$, imply

$$\|\Lambda^{-1}(F'(v_1) - \Lambda)\| \leq w_0(\|v_1 - x_0\|) = w_0(0) < 1.$$

Thus, Λ^{-1} , and consequently, we can pick $\|\Lambda^{-1}F(x_0)\| \leq \alpha_0^1$.

(H₃) $\|\Lambda^{-1}(F'(v_2) - F'(v_1))\| \leq w(\|v_2 - v_1\|)$ for each $v_1, v_2 \in \Omega_2$.

(H₄) Condition (2) holds

and

(H₅) $\mathbb{E}[x_0, \alpha_*] \subset \Omega$.

The preceding notation and the conditions (H) allow us to present the semi-local convergence result for the solver (2).

Theorem 2. Assume that the conditions (H) hold. Thus, the sequence $\{x^{(m)}\}$ generated by solver (2) exists in $\mathbb{E}(x_0, \alpha_*)$ for each $m = 0, 1, 2, 3, \dots$ and is convergent to some $x_* \in \mathbb{E}[x_0, \alpha_*]$. Moreover, the following assertions hold

$$\begin{aligned}
\|y_1^{(m)} - x^{(m)}\| &\leq a_1^{(m)} - a_0^{(m)}, \\
\|y_2^{(m)} - y_1^{(m)}\| &\leq a_2^{(m)} - a_1^{(m)}, \\
&\vdots \\
\|y_k^{(m)} - y_{k-1}^{(m)}\| &\leq a_k^{(m)} - a_{k-1}^{(m)}, \quad k = 2, \dots, (n-1), \\
&\text{and} \\
\|x^{(m+1)} - y_{n-1}^{(m)}\| &= \|y_n^{(m+1)} - y_{n-1}^{(m)}\| \leq a_{m+1} - a_{n-1}^{(m)}.
\end{aligned}$$

Proof. Mathematical induction is employed to prove these assertions. The first one holds for $m = 0$, as by (H_1) and (14)

$$\|y_1^{(0)} - x_0\| = \left\| -\Lambda^{-1}F(x_0) \right\| \leq a_1^{(0)} = a_1^{(0)} - a_0^{(0)} < a_*$$

and the iterate $y_1^{(0)} \in \mathbb{E}(x_0, a_*)$.

We need the following estimate

$$\begin{aligned}
&\left\| (2\Lambda)^{-1} \left(F'(x^{(m)}) + F'(y_1^{(m)}) - 2\Lambda \right) \right\| \\
&\leq \frac{1}{2} \left[\left\| \Lambda^{-1} \left(F'(x^{(m)}) - \Lambda \right) \right\| + \left\| \Lambda^{-1} \left(F'(y_1^{(m)}) - \Lambda \right) \right\| \right] \\
&\leq \frac{1}{2} \left[w_0(\|x^{(m)} - x_0\|) + w_0(\|y_1^{(m)} - x_0\|) \right] \\
&\leq \frac{1}{2} \left[w_0(a_0^{(m)}) + w_0(a_1^{(m)}) \right] < 1,
\end{aligned}$$

so,

$$\left\| \left(F'(x^{(m)}) + F'(y_1^{(m)})^{-1} \right) \Lambda \right\| \leq \frac{1}{2 \left[1 - \left(w_0(a_0^{(m)}) + w_0(a_1^{(m)}) \right) \right]}. \quad (17)$$

Thus, we can rewrite the first two substeps of the solver (2) in the following way:

$$\begin{aligned}
(y_2^{(m)} - y_1^{(m)}) &= \left(F'(x^{(m)}) \right)^{-1} - 2 \left(F'(x^{(m)}) + F'(y_1^{(m)}) \right)^{-1} F(x^{(m)}) \\
&= - \left[2 \left(F'(x^{(m)}) + F'(y_1^{(m)}) \right)^{-1} - F'(x^{(m)})^{-1} \right] F(x^{(m)}) \\
&= - \left(F'(x^{(m)}) + F'(y_1^{(m)}) \right)^{-1} \left(2F'(x^{(m)}) - F'(x^{(m)}) - F'(y_1^{(m)}) \right) F(x^{(m)}) \\
&= \left(F'(x^{(m)}) + F'(y_1^{(m)}) \right)^{-1} \left(F'(x^{(m)}) - F'(y_1^{(m)}) \right) (y_1^{(m)} - x^{(m)}).
\end{aligned}$$

This further yield

$$\|y_2^{(m)} - y_1^{(m)}\| = \frac{w_m \|y_1^{(m)} - x^{(m)}\|}{2 \left[1 - \left(w_0(a_0^{(m)}) + w_0(a_1^{(m)}) \right) \right]} \leq a_2^{(m)} - a_1^{(m)}$$

and

$$\begin{aligned}
\|y_2^{(m)} - x_0\| &\leq \|y_2^{(m)} - y_1^{(m)}\| + \|y_1^{(m)} - x_0\| \\
&\leq a_2^{(m)} - a_1^{(m)} + a_1^{(m)} - a_0^{(0)} = a_2^{(m)} < a_*,
\end{aligned}$$

where we also use the following estimates

$$\begin{aligned}\left\|\Lambda^{-1}\left(F'(x^{(m)}) - F'(y_1^{(m)})\right)\right\| &\leq w(\|x^{(m)} - y_1^{(m)}\|) \\ &\leq w(a_1^{(m)} - a_0^{(m)})\bar{w}_m\end{aligned}$$

and

$$\begin{aligned}\left\|\Lambda^{-1}\left(F'(x^{(m)}) - F'(y_1^{(m)})\right)\right\| &\leq \left\|\Lambda^{-1}\left(F'(x^{(m)}) - \Lambda\right)\right\| + \left\|\Lambda^{-1}\left(F'(y_1^{(m)}) - \Lambda\right)\right\| \\ &\leq w_0(\|x^{(m)} - x_0\|) + w_0(\|y_1^{(m)} - x_0\|) \\ &\leq w_0(a_0^{(m)}) + w_0(a_1^{(m)}) \\ &\leq w_m.\end{aligned}$$

Hence, the iterate $y_2^{(m)} \in \mathbb{E}(x_0, a_*)$ and the second assertion holds. An upper bound on $\|A_m\|$ is needed. As in the local case

$$A_m = 6\left(I - F'(x^{(m)})^{-1}F'(y_1^{(m)})^2 - 4\left(I - F'(x^{(m)})^{-1}F'(y_1^{(m)})\right) + 5I\right),$$

thus,

$$\|A_m\| \leq 6\left(\frac{\bar{w}_m}{1 - w_0(a_0^{(m)})}\right)^2 + 4\left(\frac{\bar{w}_m}{1 - w_0(a_0^{(m)})}\right) + 5 = p_m,$$

where

$$\begin{aligned}\left\|I - F'(x^{(m)})^{-1}F'(y_1^{(m)})\right\| &\leq \left\|F'(x^{(m)})^{-1}\Lambda\right\| \left\|F(x_0)^{-1}\left(F'(y_1^{(m)}) - \Lambda\right)\right\| \\ &\leq \frac{\bar{w}_m}{1 - w_0(a_0^{(m)})}.\end{aligned}$$

Moreover, we can write that

$$\begin{aligned}F(y_2^{(m)}) &= F(y_2^{(m)}) - F(x^{(m)}) + F(x^{(m)}) \\ &= F(y_2^{(m)}) - F(x^{(m)}) - F'(x^{(m)})(y_1^{(m)} - x^{(m)}) \\ &= \int_0^1 F'(x^{(m)} + \theta(y_2^{(m)} - x^{(m)}))d\theta(y_2^{(m)} - x^{(m)}) - F'(x^{(m)})(y_1^{(m)} - x^{(m)}).\end{aligned}$$

Hence, we have

$$\begin{aligned}\left\|\Lambda^{-1}F(y_2^{(m)})\right\| &\leq \left[1 + \int_0^1 w_0(\|x^{(m)} - x_0\| + \theta\|y_2^{(m)} - x^{(m)}\|)d\theta\|y_2^{(m)} - x^{(m)}\| \right. \\ &\quad \left. + \left(1 + w_0(\|x^{(m)} - x_0\|)\right)\right]\|y_1^{(m)} - x^{(m)}\| \\ &\leq \left[1 + \int_0^1 w_0(a_0^{(m)} + \theta\|a_2^{(m)} - a_0^{(m)}\|)d\theta\|a_2^{(m)} - a_0^{(m)}\| + \left(1 + w_0(a^{(m)})\right)\right] \\ &\quad \times (a_1^{(m)} - a_0^{(m)}) \\ &\leq b_2^{(m)}.\end{aligned}$$

Consequently, by the third substep of the solver (2), we get

$$\begin{aligned}\|y_3^{(m)} - y_2^{(m)}\| &\leq \|A_m\| \|F'(x^{(m)})^{-1} \Lambda\| \|\Lambda^{-1} F(y_2^{(m)})\| \\ &\leq \frac{p_m b_2^{(m)}}{1 - w_0(a_0^{(m)})} \\ &= a_3^{(m)} - a_2^{(m)}\end{aligned}$$

and

$$\begin{aligned}\|y_3^m - x^0\| &\leq \|y_3^m - y_2^m\| + \|y_2^m - x^0\| \\ &\leq a_3^m - a_2^m + a_2^m - a^0 = a_3^m < a_*.\end{aligned}$$

Hence, the iterate $y_3^m \in \mathbb{E}(x_0, a_*)$ and the third assertion holds. The rest of the assertions are validated, simply by replacing 2 and 3 by $(k-1)$ and (k) , respectively, in the last calculations. Thus, we can write by the first substep of the solver

$$\begin{aligned}F(x^{(m+1)}) &= F(x^{(m+1)}) - F(x^{(m)}) + F(x^{(m)}) \\ &= F(x^{(m+1)}) - F(x^{(m)} - F'(x^{(m)})) (x^{(m+1)} - x^{(m)}) \\ &\quad + F'(x^{(m)})(x^{(m+1)} - x^{(m)}) - F'(x^{(m)})(y_1^{(m)} - x^{(m)}).\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\|\Lambda^{-1} F(x^{(m+1)})\| &\leq \int_0^1 w((1-\theta)\|x^{(m+1)} - x^{(m)}\|) d\theta \|x^{(m+1)} - x^{(m)}\| \\ &\quad + (1 + w_0(\|x^{(m)} - x_0\|)) \|y_1^{(m)} - x^{(m)}\| \\ &\leq \int_0^1 w((1-\theta)(a_{m+1} - a_m) + (1 + w_0(a_m)) d\theta (a_1^{(m)} - a_m) = \lambda_{m+1}.\end{aligned}\tag{18}$$

Therefore, we have

$$\begin{aligned}\|y_1^{(m+1)} - x^{(m+1)}\| &\leq \|F'(x^{(m+1)})^{-1} F''(x_0)\| \|\Lambda^{-1} F(x^{(m+1)})\| \\ &\leq \frac{\lambda_{m+1}}{1 - w_0(a_0^{(m+1)})} = a_1^{(m+1)} - a_0^{(m+1)},\end{aligned}$$

and

$$\begin{aligned}\|y_1^{(m+1)} - x_0\| &\leq \|y_1^{(m+1)} - x^{(m+1)}\| + \|x^{(m+1)} - x_0\| \\ &\leq a_1^{(m+1)} - a_0^{(m+1)} + a_0^{(m+1)} - a_0^{(0)} = a_1^{m+1} < a_*.\end{aligned}$$

Thus, the iterate $y_1^{(m+1)} \in \mathbb{E}(x_0, a_*)$ and the induction for the assertions is terminated. Then, sequence $\{a_m\}$ is shown to be majorizing for $\{x^m\}$. Moreover, the latter sequence is also complete in the Banach space $\mathbb{Z}_1$. Hence, $x_* \in \mathbb{E}[x_0, a_*]$ exists, such that $\lim_{m \rightarrow \infty} x_m = x_*$. Notice that the continuity of the operator F together with the estimate (18) lead to $F(x_*) = 0$, provided that $m \rightarrow \infty$.

A uniqueness domain of a solution of the equation $F(x) = 0$ can be specified. $\square$

Proposition 1. Suppose there exists a solution $y_* \in \mathbb{E}(x_0, d_0)$ of the equation $F(x) = 0$ for some $d_0 > 0$; the condition (H_2) holds in the ball $\mathbb{E}(x_0, d_0)$ and there exists $d \geq d_0$, such that

$$\int_0^1 w_0((1-\theta)d_0 + \theta d) d\theta < 1.\tag{19}$$

Set $\Omega_3 = \Omega \cap \mathbb{E}[x_0, d]$.

Thus the equation $F(x) = 0$ is uniquely solvable by y_* in the domain Ω_3 .

Proof. Define the linear operator

$$M = \int_0^1 F'(y_* + \theta(z_* - y_*))d\theta, \text{ for } z_* \in \Omega_3,$$

with $F(z_*) = 0$.

Thus, it follows by (H_2) and (19) that

$$\begin{aligned} \|\Lambda^{-1}(M - \Lambda)\| &\leq \int_0^1 w_0((1 - \theta)\|y_* - x_0\| + \theta\|z_* - x_0\|)d\theta \\ &\leq \int_0^1 w_0((1 - \theta)d_0 + \theta d)d\theta < 1. \end{aligned}$$

Therefore, we deduce that z_* is equal to y_* . $\square$

Remark 2. (1) A usual pick (but not necessarily the most appropriate) is $\Lambda = F'(x_0)$.

(2) The limit point a_* can be exchanged by α (see Lemma 1) in the condition (H_5) .

(3) If all the conditions (H_1) – (H_5) hold in the Proposition 1, then we can set $y_* = x_*$ and $d_0 = a_*$. $(H_3)'$ Assume that

$$\|\Lambda^{-1}F'(x)\| \leq w_1(\|x - x_0\|),$$

for each $x \in \Omega_2$, where $w_1 : [0, \alpha] \rightarrow \mathbb{R}$ is a CN function.

Notice that

$$\begin{aligned} \|\Lambda^{-1}F'(x)\| &= \|\Lambda^{-1}(F'(x) - \Lambda) + \Lambda\| \\ &\leq 1 + \|\Lambda^{-1}(F'(x) - \Lambda)\| \\ &\leq 1 + w_0(\|x - x_0\|) \end{aligned}$$

Hence, the estimate $1 + w_0(t)$ can be replaced by w_1^t in the local as well as in the semi-local case. In this case, the condition $(H_3)'$ must be added in the conditions (H_1) – (H_5) . The same modification is used for the local case. This is important to do, as it is possible that $w_1(t) < 1 + w_0(t)$. As an example, let us assume that $F(x) = \sin x$. Then, $x_* = 0$, $w_1(t) = t$, $w_0(t) = 1 + t$ and $w_1(t) < w_0(t)$. Clearly, if this is not true, then we do not add the condition $(H_3)'$.

4. Numerical Applications

In this numerical segment, we emphasize the significance of verifying and validating both local and semilocal convergence across six numerical problems. For this purpose, we choose three cases from the iterative solver (2) for $n = 3$, $n = 4$, and $n = 5$, denoted by (Case-1), (Case-2), and (Case-3), respectively. The first two problems, referred to as Examples 1 and 2, are analyzed in order to assess the local convergence. Later on, we will focus on the semilocal convergence. For computational findings, we select the following problems: the steering problem (usually three-dimensional), the Fisher problem (using a system of dimensions 121×121), and the boundary value problem (BVP), involving a system of dimensions 60×60 . In the sixth example, we study a large system of 600×600 -dimensional nonlinear equations.

We initially determine the radii of convergence for the iterative solver (2) in order to tackle the given problem. We execute the iterative procedure after choosing a suitable initial approximation and determining the computational order of convergence. This makes it

easier for us to see how fast the iterative solver was getting close to the required answer. The computation order of convergence (COC) [9] is calculated using the following formulas:

$$\kappa = \frac{\ln \frac{\|x_{\rho+1} - x_*\|}{\|x_{\rho} - x_*\|}}{\ln \frac{\|x_{\rho} - x_*\|}{\|x_{\rho-1} - x_*\|}}, \quad \text{for } \rho = 1, 2, \dots \quad (20)$$

or approximated computational order of convergence ACOC [11,16] by:

$$\kappa^* = \frac{\ln \frac{\|x_{\rho+1} - x_{\rho}\|}{\|x_{\rho} - x_{\rho-1}\|}}{\ln \frac{\|x_{\rho} - x_{\rho-1}\|}{\|x_{\rho-1} - x_{\rho-2}\|}}, \quad \text{for } \rho = 2, 3, \dots \quad (21)$$

In order to ensure the effectiveness of the iterative solver, we additionally recorded the CPU timing during the computation. The residual error and the number of iterations required to reach the desired precision were ultimately established.

Stopping criteria are important in iterative solvers to determine when to stop the iteration process and consider the current approximation as the solution. Depending on the type of problem being solved and the algorithm being used, several stopping conditions might be applied. We choose the following stopping criterion:

- (i) $\|x_{k+1} - x_k\| < \epsilon$, and
- (ii) $\|F(x_k)\| < \epsilon$,

where $\epsilon = 10^{-300}$ is the error tolerance. The multiple precision arithmetic operations are performed using Mathematica-v.11 software.

Example 1. Let $\mathbb{Z}_1 = \mathbb{Z}_2 = \mathbb{R}^3$ and $\Omega = \mathbb{E}[0, 1]$. Consider the mapping for $\vartheta = \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \\ \vartheta_3 \end{bmatrix}$ on the ball Ω by

$$F(\vartheta) = \begin{bmatrix} \frac{e-1}{2} \vartheta_1^2 + \vartheta_1 \\ e^{\vartheta_2} - 1 \\ \vartheta_3 \end{bmatrix}. \quad (22)$$

Then, the derivative F' is found to be:

$$F'(\vartheta) = \begin{bmatrix} (e-1)\vartheta_1 + 1 & 0 & 0 \\ 0 & e^{\vartheta_2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Set $\Delta = F'(\vartheta_*)$. We obtain $F(\vartheta_*) = 0$ for $\vartheta_* = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Then, $\Delta = I$, and the conditions (H_1) – (H_4) are verified for

$$\aleph_0(t) = (e-1)t, \quad \aleph(t) = e^{\frac{1}{e-1}t}, \quad \aleph_t(t) = e^{\frac{1}{e-1}t}.$$

The radii of convergence and computational results are depicted in Tables 1 and Table 2, respectively.

Table 1. Radii of convergence for Example 1.

Cases of (2)	κ_0	r_1	κ_1	r_2	r_3	r_4	r_5	r
Case-1	0.58198	0.3827	0.4415	0.1966	0.1215	-	-	0.1215
Case-2	0.58198	0.3827	0.4415	0.1966	0.1215	0.1118	-	0.1118
Case-3	0.58198	0.3827	0.4415	0.1966	0.1215	0.1118	0.0923	0.0923

Table 2. Numerical results of solver (2) for Example 1.

Cases of (2)	x_0	$\ F(x^3)\ $	$\ x^{(4)} - x^{(3)}\ $	m	ρ	CPU Timing
Case-1	$(0.11, 0.11, 0.11)^T$	1.2×10^{-174}	1.2×10^{-174}	4	5.0054	0.258927
Case-2	$(0.1, 0.1, 0.1)^T$	2.6×10^{-568}	2.6×10^{-568}	3	7.0023	0.157492
Case-3	$(0.08, 0.08, 0.08)^T$	8.6×10^{-2058}	2.9×10^{-2058}	3	9.0025	0.463888

Example 2. Let $\Omega = \mathbb{E}[0, 1]$ and $\mathbb{Z}_1 = \mathbb{Z}_2 = C[0, 1]$. Consider the nonlinear integral equation of the first kind Hammerstein operator F be defined by

$$F(Y)(x) = Y(x) - 3 \int_0^1 x \zeta Y(\zeta)^3 d\zeta.$$

The calculation for the derivative gives

$$F'(Y(q))(x) = q(x) - 9 \int_0^1 x \zeta Y(\zeta)^2 q(\zeta) d\zeta,$$

for $q \in C[0, 1]$. For this value of the operator F' , the conditions $(H_1) - (H_4)$, for $\Delta = F'(x_*) = I$ and $x_* = 0$ are verified, so we choose

$$\aleph_0(t) = 4.5t, \quad \aleph(t) = 9t, \quad \aleph(t) = 1 + 4.5t.$$

By adopting these functions, we calculate the radii for compositions (2) of the Example 2 in Table 3.

Table 3. Radii of solver (2) of Example 2.

Cases	κ_0	r_1	κ_1	r_2	r_3	r_4	r_5	r
Case-1	0.2222	0.1111	0.1481	0.07127	0.05056	-	-	0.05056
Case-2	0.2222	0.1111	0.1481	0.07127	0.05056	0.04729	-	0.04729
Case-3	0.2222	0.1111	0.1481	0.07127	0.05056	0.04729	0.04522	0.04522

Example 3. We use a well-known partial differential equation (PDE), which was proposed by Fisher in [17]. This PDE describes the spatiotemporal dynamics of a population in a homogeneous environment, which is also known as Fisher's equation. The mathematical expression for Fisher's equation is defined by the following:

$$\begin{aligned}
 \theta_t &= D\theta_{xx} + \theta(1 - \theta) = 0, \\
 &\text{with homogeneous Neumann's boundary conditions} \\
 \theta(x, 0) &= 1.5 + 0.5\cos(\pi x), \quad 0 \leq x \leq 1, \\
 \theta_x(0, t) &= 0, \quad \forall \quad t \geq 0, \\
 \theta_x(1, t) &= 0, \quad \forall \quad t \geq 0.
 \end{aligned} \tag{23}$$

The differential Equation (23) is discretized using a finite difference method to create a system of nonlinear equations. In order to do this, we denote $w_{i,j}$ as $\theta(x_i, t_j)$, which stands for the desired solution at the mesh's grid points. Here, x and t denote the number of steps in the directions of M and N , respectively. Further, h and k correspond to step sizes of M and N , respectively. By adopting forward, backward, and central differences, we obtain, in turn, that

$$\begin{aligned}
 \theta_{xx}(x_i, t_j) &= (w_{i+1,j} - 2w_{i,j} + w_{i-1,j})/h^2, \\
 \theta_t(x_i, t_j) &= (w_{i,j} - w_{i,j-1})/k, \\
 &\text{and} \\
 \theta_x(x_i, t_j) &= (w_{i+1,j} - w_{i,j})/(h), \quad t \in [0, 1],
 \end{aligned}$$

leading to

$$\frac{w_{1,j} - w_{i,j-1}}{k} - w_{i,j}(1 - w_{i,j}) - D \frac{w_{i+1,j} - 2w_{i,j} + w_{i-1,j}}{h^2}, \quad (24)$$

where $i = 1, 2, 3, \dots, M, j = 1, 2, 3, \dots, N, h = \frac{1}{M}$ and $k = \frac{1}{N}$. For specific values of $M = 11, N = 11, h = \frac{1}{11}, k = \frac{1}{11}$, and $D = 1$, this results in a large nonlinear system of size 121×121 . Convergence leads us to the following column vector solution $u(x_i, t_j) = x_*$ (not a matrix), which is given below:

$$x_* = \begin{pmatrix} 1.645 \dots, 1.473 \dots, 1.375 \dots, 1.312 \dots, 1.269 \dots, 1.236 \dots, 1.210 \dots, 1.188 \dots, \\ 1.169 \dots, 1.153 \dots, 1.138 \dots, 1.623 \dots, 1.464 \dots, 1.370 \dots, 1.310 \dots, 1.268 \dots, \\ 1.236 \dots, 1.210 \dots, 1.188 \dots, 1.169 \dots, 1.153 \dots, 1.138 \dots, 1.583 \dots, 1.445 \dots, \\ 1.361 \dots, 1.306 \dots, 1.266 \dots, 1.235 \dots, 1.209 \dots, 1.188 \dots, 1.169 \dots, 1.153 \dots, \\ 1.138 \dots, 1.528 \dots, 1.419 \dots, 1.349 \dots, 1.300 \dots, 1.263 \dots, 1.233 \dots, 1.208 \dots, \\ 1.187 \dots, 1.169 \dots, 1.153 \dots, 1.138 \dots, 1.463 \dots, 1.388 \dots, 1.334 \dots, 1.292 \dots, \\ 1.259 \dots, 1.231 \dots, 1.208 \dots, 1.187 \dots, 1.169 \dots, 1.153 \dots, 1.138 \dots, 1.395 \dots, \\ 1.355 \dots, 1.317 \dots, 1.284 \dots, 1.255 \dots, 1.229 \dots, 1.207 \dots, 1.186 \dots, 1.168 \dots, \\ 1.152 \dots, 1.138 \dots, 1.328 \dots, 1.322 \dots, 1.301 \dots, 1.276 \dots, 1.251 \dots, 1.227 \dots, \\ 1.206 \dots, 1.186 \dots, 1.168 \dots, 1.152 \dots, 1.138 \dots, 1.268 \dots, 1.292 \dots, 1.286 \dots, \\ 1.269 \dots, 1.248 \dots, 1.226 \dots, 1.205 \dots, 1.186 \dots, 1.168 \dots, 1.152 \dots, 1.138 \dots, \\ 1.219 \dots, 1.267 \dots, 1.274 \dots, 1.263 \dots, 1.245 \dots, 1.224 \dots, 1.204 \dots, 1.185 \dots, \\ 1.168 \dots, 1.152 \dots, 1.138 \dots, 1.185 \dots, 1.250 \dots, 1.265 \dots, 1.258 \dots, 1.242 \dots, \\ 1.223 \dots, 1.204 \dots, 1.185 \dots, 1.168 \dots, 1.152 \dots, 1.138 \dots, 1.168 \dots, 1.240 \dots, \\ 1.261 \dots, 1.256 \dots, 1.241 \dots, 1.223 \dots, 1.203 \dots, 1.185 \dots, 1.168 \dots, 1.152 \dots, \\ 1.138 \dots \end{pmatrix}^{tr}.$$

We depicted the numerical outcome in Table 4 based on the initial guess

$$x^{(0)} = \begin{pmatrix} 1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 1.1, 1.1, 1.1, 1.1, 1.1, 1.1, 1.1, 1.1, 1.1, 1.1, 1.1, 1.1, \\ 1.2, 1.2, 1.2, 1.2, 1.2, 1.2, 1.2, 1.2, 1.2, 1.2, 1.2, 1.2, 1.3, 1.3, 1.3, 1.3, 1.3, 1.3, \\ 1.3, 1.3, 1.3, 1.3, 1.3, 1.3, 1.4, 1.4, 1.4, 1.4, 1.4, 1.4, 1.4, 1.4, 1.4, 1.4, 1.4, 1.4, \\ 1.5, 1.5, 1.5, 1.5, 1.5, 1.5, 1.5, 1.5, 1.5, 1.5, 1.5, 1.5, 1.6, 1.6, 1.6, 1.6, 1.6, 1.6, \\ 1.6, 1.6, 1.6, 1.6, 1.6, 1.6, 1.7, 1.7, 1.7, 1.7, 1.7, 1.7, 1.7, 1.7, 1.7, 1.7, 1.7, 1.7, \\ 1.8, 1.8, 1.8, 1.8, 1.8, 1.8, 1.8, 1.8, 1.8, 1.8, 1.8, 1.8, 1.9, 1.9, 1.9, 1.9, 1.9, 1.9, \\ 1.9, 1.9, 1.9, 1.9, 1.9, 1.9, 2.0, 2.0, 2.0, 2.0, 2.0, 2.0, 2.0, 2.0, 2.0, 2.0, 2.0, 2.0 \end{pmatrix}^{tr}.$$

Table 4. Numerical results of solver (2) for Example 3.

Cases of (2)	$\ F(x^{(3)})\ $	$\ x^{(4)} - x^{(3)}\ $	m	ρ	CPU Timing
Case-1	7.0×10^{-113}	5.5×10^{-114}	4	5.0066	93.2924
Case-2	5.7×10^{-306}	2.6×10^{-307}	3	7.0602	30.7952
Case-3	2.0×10^{-646}	7.1×10^{-648}	3	9.0338	62.7814

Example 4. The motion issue for steering involves determining the optimal course for a vehicle to take while taking into consideration its dynamic constraints, such as the maximum turning radius and acceleration. Solving this issue is essential for establishing effective and safe navigation in practical settings, especially when it comes to driverless cars. The steering problem's mathematical

equation of motion is defined by Tsoulos & Stavrakoudis and Awawdeh [18,19], which is provided below:

$$\left[\beta_1 \left(\beta_2 \sin(\tau_\theta) - \beta_3 \right) \left(\beta_2 \cos(\varphi_\theta) + 1 \right) - \beta_1 \left(\beta_2 \cos(\tau_\theta) - \beta_3 \right) \left(\beta_2 \sin(\varphi_\theta) - \beta_3 \right) \right]^2 - \left[G_\theta \left(\beta_2 \cos(\varphi_\theta) + 1 \right) - G_\theta \left(\beta_2 \cos(\tau_\theta) - 1 \right) \right]^2 - \left[E_\theta \left(\beta_2 \sin(\tau_\theta) - \beta_3 \right) - G_\theta \left(\beta_2 \sin(\varphi_\theta) - \beta_3 \right) \right]^2 = 0, \theta = 1, 2, 3,$$

where

$$G_\theta = \beta_2(\beta_3 - \beta_1) \sin(\tau_0) + \beta_2 \cos(\tau_0) - \beta_3 \beta_2 \sin(\tau_\theta) - \beta_2 \cos(\tau_\theta) + \beta_1 \beta_3, \\ E_\theta = \beta_1 \left(-\beta_2 \sin(\varphi_\theta) + \beta_3 \right) + \beta_2 \left(\cos(\varphi_\theta) - \cos(\varphi_0) \right) - \beta_3 \beta_2 \left(\sin(\varphi_\theta) - \sin(\varphi_0) \right).$$

The values of τ_i, φ_i (in radians) are provided below:

$$\tau_0 = 1.3954, \tau_1 = 1.7444, \tau_2 = 2.0656, \tau_3 = 2.4601, \\ \varphi_0 = 1.7461, \varphi_1 = 2.0364, \varphi_2 = 2.2391, \varphi_3 = 2.4601.$$

The number of iterations, residual error, error difference between two consecutive iterations, initial guess, convergence order, and CPU timing of Example 4 are shown in Table 5. These values correspond to the initial approximation $(0.88, 0.68, 0.64)^{tr}$.

Table 5. Numerical results of solver (2) for Example 4.

Cases of (2)	$\ F(x^{(3)})\ $	$\ x^{(4)} - x^{(3)}\ $	m	ρ	CPU Timing
Case-1	2.9×10^{-75}	2.3×10^{-73}	4	4.9945	0.401338
Case-2	7.9×10^{-173}	7.8×10^{-171}	4	6.9966	0.924285
Case-3	7.6×10^{-331}	8.6×10^{-329}	2	9.0008	0.358374

The solver (2) converges to $x_* = \begin{bmatrix} 0.9051 \dots \\ 0.6977 \dots \\ 0.6508 \dots \end{bmatrix}$.

Example 5. A mathematics problem known as a boundary value problem (BVP) aims to solve a differential equation under certain boundary conditions. The values of the solution at the boundary of the domain where the problem is being solved are specified by the boundary conditions. One of the main areas of inquiry in Physics and Mathematics is solving BVPs. Therefore, we select the following boundary value problem (see [6]).

$$u'' + a^2 u'^2 + 1 = 0 \quad (25)$$

with $u(0) = u(1) = 0$. The interval $[0, 1]$ is divided into k parts, which yields

$$\gamma_0 = 0 < \gamma_1 < \gamma_2 < \dots < \gamma_{k-1} < \gamma_k, \quad \gamma_{k+1} = \gamma_k + h, h = \frac{1}{k}.$$

Then, we can choose $u_0 = u(\gamma_0) = 0, u_1 = u(\gamma_1), \dots, u_{k-1} = u(\gamma_{k-1}), u_k = u(\gamma_k) = 1$. We have

$$u'_\theta = \frac{u_{\theta+1} - u_{\theta-1}}{2h}, \quad u''_\theta = \frac{u_{\theta-1} - 2u_\theta + u_{\theta+1}}{h^2}, \quad \theta = 1, 2, 3, \dots, \ell - 1.$$

This is achieved using a discretization technique. The following nonlinear system of equations for $(\theta - 1) \times (\theta - 1)$ is thus obtained.

$$u_{\theta-1} - 2u_{\theta} + u_{\theta+1} + \frac{\mu^2}{4}(u_{\theta+1} - u_{\theta-1})^2 + h^2 = 0.$$

For $\ell = 61$ and $\mu = \frac{1}{2}$, we have a nonlinear 60×60 system of equations. In Table 6, we present iterations and the CO of Example 5.

Table 6. Numerical results of solver (2) for Example 5.

Cases of (2)	$\ F(x^{(3)})\ $	$\ x^{(4)} - x^{(3)}\ $	m	ρ	CPU Timing
Case-1	1.3×10^{-139}	1.5×10^{-137}	4	5.0680	969.673
Case-2	5.4×10^{-384}	1.2×10^{-381}	3	7.0599	941.422
Case-3	3.8×10^{-782}	7.8×10^{-781}	3	9.1715	1197.97

The solver (2) converges to the approximated root.

$$x_* = \begin{pmatrix} 0.02756 \dots, 0.05467 \dots, 0.08133 \dots, 0.1075 \dots, 0.1333 \dots, 0.1586 \dots, 0.1836 \dots, 0.2081 \dots, \\ 0.2321 \dots, 0.2558 \dots, 0.2791 \dots, 0.3019 \dots, 0.3244 \dots, 0.3465 \dots, 0.3682 \dots, 0.3895 \dots, \\ 0.4104 \dots, 0.4309769, 0.4511 \dots, 0.4709 \dots, 0.4903 \dots, 0.5094 \dots, 0.5281 \dots, 0.5465 \dots, \\ 0.5645 \dots, 0.5822 \dots, 0.5995 \dots, 0.6165 \dots, 0.6331 \dots, 0.6494 \dots, 0.6654 \dots, 0.6810 \dots, \\ 0.6963 \dots, 0.7113 \dots, 0.7260 \dots, 0.7403 \dots, 0.7543 \dots, 0.7680 \dots, 0.7814 \dots, 0.7945 \dots, \\ 0.8072 \dots, 0.8197 \dots, 0.8319 \dots, 0.8437 \dots, 0.8552 \dots, 0.8665 \dots, 0.8774 \dots, 0.8881 \dots, \\ 0.8984 \dots, 0.9085 \dots, 0.9182 \dots, 0.9277 \dots, 0.9369 \dots, 0.9457 \dots, 0.9543 \dots, 0.9626 \dots, \\ 0.9707 \dots, 0.9784 \dots, 0.9859 \dots, 0.9931 \dots \end{pmatrix}^{tr}.$$

Example 6. Solving large systems of nonlinear equations is a sophisticated problem that arises in many scientific and engineering applications. To illustrate, we choose to tackle the complexity presented by the ensuing system of nonlinear equations, characterized by an order of 600×600 :

$$F(x) = \begin{cases} x_j^2 x_{j+1} - 1 = 0, & 1 \leq j \leq 600, \\ x_j^2 x_1 - 1 = 0. \end{cases} \quad (26)$$

In the aforementioned system, (26), $x_* = (1, 1, 1, \dots, (600\text{times}))^T$ is the convergence point. Table 7 presents the numerical results based on the initial approximation $(1.1, 1.1, \dots, 1.1)^{tr}$.

Table 7. Numerical results of solver (2) for Example 6.

Cases of (2)	$\ F(x^{(m)})\ $	$\ x^{(m+1)} - x^{(m)}\ $	m	ρ	CPU Timing
Case-1	9.9×10^{-275}	3.3×10^{-275}	3	6.1883	5065.5
Case-2	1.4×10^{-886}	4.8×10^{-887}	3	9.1288	6889.65
Case-3	9.5×10^{-2057}	3.2×10^{-2057}	3	12.098	8694.25

5. Conclusions

In this research article, we established the local and semilocal convergence of method (2), along with the corresponding radius of convergence. In addition, we derived computable upper error bounds that allowed us to estimate the accuracy of the approximate solutions obtained using the method. Further, we also demonstrated the uniqueness of solutions

produced by the method (2) under certain conditions. Finally, we conclude that our research contributes to the development of more efficient and robust numerical methods for solving nonlinear equations in a variety of scientific and engineering applications. Similar methodology is applicable to other methods [2,8–14]. Thus, this can be the chosen course of action for upcoming projects.

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