

Article

Constructing a New Multi-Scroll Chaotic System and Its Circuit Design

Yinfang Ye ¹ and Jianbin He ^{2,*} ¹ School of Business Administration, Jimei University, Xiamen 361021, China; yeyinfang@jmu.edu.cn² School of Mathematics and Statistics, Minnan Normal University, Zhangzhou 363000, China

* Correspondence: hejianbin@mnnu.edu.cn

Abstract: Multi-scroll chaotic systems have complex dynamic behaviors, and the multi-scroll chaotic system design and analysis of their dynamic characteristics is an open research issue. This study explores a new multi-scroll chaotic system derived from an asymptotically stable linear system and designed with a uniformly bounded controller. The main contributions of this paper are given as follows: (1) The controlled system can cause chaotic behavior with an appropriate control position and parameters values, and a new multi-scroll chaotic system is proposed using a bounded sine function controller. Meanwhile, the dynamical characteristics of the controlled system are analyzed through the stability of the equilibrium point, a bifurcation diagram, and Lyapunov exponent spectrum. (2) According to the Poincaré section, the existence of a topological horseshoe is proven using the rigorous computer-aided proof in the controlled system. (3) Numerical results of the multi-scroll chaotic system are shown using Matlab R2020b, and the circuit design is also given to verify the multi-scroll chaotic attractors.

Keywords: chaos anti-control; multi-scroll chaotic attractor; topological horseshoe; circuit design

MSC: 37M05



Citation: Ye, Y.; He, J. Constructing a New Multi-Scroll Chaotic System and Its Circuit Design. *Mathematics* **2024**, *12*, 1931. <https://doi.org/10.3390/math12131931>

Academic Editor: Daniel-Ioan Curiac

Received: 13 May 2024

Revised: 14 June 2024

Accepted: 19 June 2024

Published: 21 June 2024



Copyright: © 2024 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

Over the past six decades, chaotic systems have received much attention. As a special phenomenon in nonlinear science, chaos theory has been widely applied in many fields, such as chaos-based encryption and liquid mixing. In 1963, the Lorenz system showed butterfly-shaped attractors with two wings. Thereafter, generalized Lorenz family chaotic systems with multiple scrolls have been proposed [1–3]. Many new methods have been designed for chaotic systems with multi-scroll attractors, such as the switching piecewise-linear control approach [4–7], threshold approach [8], sawtooth or triangular wave series [9,10], hyperbolic functions [11,12], sine function [13], and Josephson junctions [14].

The systematic saturated function series methodology can create multi-scroll chaotic attractors from a three-dimensional linear autonomous system with a simple saturated function series controller, including one-directional n -scroll, two-directional $n \times m$ -grid scroll, and 3-D $n \times m \times l$ -grid scroll chaotic attractors [15]. An overview of the subject of multi-scroll chaotic attractor generation, including some fundamental theories, design methodologies, circuit implementations, and practical applications was given in [16]. The generation of $n \times m$ -wing Lorenz-type attractors from a modified Shimizu–Morioka system was explored by introducing a multi-segment quadratic function and a stair function into the 2-D state-space of the system [17]. A general multi-scroll and multi-wing attractor system combining Julia’s process with a Lorenz’s attractor was proposed in [18]. A modified Sprott-A system without an equilibrium point but with perpetual points was presented using a sine function, and it had the conservative property of zero-sum Lyapunov exponents and thus could generate a chaotic sea rather than an attractor [19]. A quadratic system with two stable node-foci can generate a double-wing chaotic hidden attractor. A

multi-fold cover for a quadratic system was constructed using rotation symmetry [20]. An eight-wing chaotic attractor, replacing a constant parameter with a switch function in a Qi four-wing 3-D chaotic system, was presented, and the physical existence of an eight-wing chaotic attractor was verified using an electronic circuit FPGA [21]. A new chaotic system model was designed, to generate multi-direction multi-scroll chaotic attractors, and the chaotic sequences generated by the proposed system were used to encrypt images [22]. A systematic scheme for synthesizing a Chua diode with multi-segment piecewise-linearity was investigated, and different numbers of scrolls could be generated by changing the number of passive nonlinear resistor cells or adjusting two coupling parameters [23]. The generation of $n \cdot m$ -scroll attractors based on a two-port network was presented according to the RCL circuit suggested in the conventional Chua circuit [24]. A systematic methodology for generating various grid multi-wing hyperchaotic attractors with switching control and for constructing super-heteroclinic loops from the piecewise linear hyperchaotic Lorenz system family was investigated, and a module-based circuit design approach was developed to realize the designed piecewise linear grid multi-wing hyperchaotic Lorenz and Chen attractors [25]. In reference [26], the FPGA realization of two multi-scroll chaotic oscillator was investigated by applying forward Euler and Runge–Kutta numerical methods.

In recent years, many multi-scroll chaotic systems have also been investigated. A multi-wing chaotic system inserting a nonlinear feedback controller was generated based on a novel 3-D Lorenz-like chaotic system [27]. A brief review of lower-dimensional chaotic systems with an unusual complex was presented in [28]. Lower-dimensional chaotic systems can preserve basic chaotic properties, and some special features are beneficial for real application characteristics. A new controlled multi-double-scroll chaotic system and corresponding implementation on a digital platform FPGA were designed, and it could generate $2N + 1$ and $2N + 2$ numbers of double-scroll attractors [29]. A hyperchaotic system with multi-scroll coexistence attractors using a piecework function was proposed based on a Lorenz system [30], and a chaos-based cryptosystem for the digital image was designed. A method for generating multi-scroll attractors was introduced using a single parametric control nonlinearity function, and the proposed multi-scroll chaotic system evolved several fluctuating attractors by varying a single parameter instead of replacing the nonlinear function [31]. An expanded multi-scroll chaotic system consisting of eight terms with one nonlinearity was studied using a simple multi-segment function [32].

Meanwhile, certain design methods of anti-control were investigated based on a linear system and bounded controller. In reference [33], a simple and systematic control design method was proposed for making a continuous-time Takagi–Sugeno fuzzy system chaotic, and the generated chaos was of the Li–Yorke type. A universal and practical anti-control approach to design a general continuous-time autonomous chaotic system via Lyapunov exponent placement was researched, it was proven that the new approach yielded a heteroclinic orbit in a three-dimensional autonomous system, therefore the resulting system was indeed chaotic in the sense of Shilnikov [34]. In reference [35], a new methodology for designing a dissipative hyperchaotic system with the desired number of positive Lyapunov exponents was investigated, and a general design principle and the corresponding implementation steps were then developed. A simple model for dissipative hyperchaotic systems was constructed, and the controlled system could generate any desired number of positive Lyapunov exponents, as long as the dimension of the system was sufficiently high [36]. In reference [37], a unified chaotification framework for generating desired higher-dimensional dissipative hyperchaotic systems using a single-parameter controller was presented. The impulse control was shown to generate chaos from a non-chaotic system based on a Chen system, and the existence of topological horseshoe was demonstrated through a rigorous computer-aided proof [38].

Furthermore, the application of chaotic systems has been widely investigated. Based on an enhanced logistic chaotic system, an efficient tweakable voice encryption algorithm was proposed, to protect the security of digital voice transmission [39]. A new chaotic cryptosystem for the encryption of very high-resolution digital images based on the design

of a digital chaos generator using arbitrary precision arithmetic was presented, and the security analysis confirmed that the proposed chaotic cryptosystem was secure and robust against several known attacks [40]. A new method for obtaining pseudo-random numbers based on a discrete chaotic map was demonstrated, and the randomness of the pseudo-random sequences was verified using NIST 800-22 test suite and TestU01 [41].

Multi-scroll chaos has complex nonlinear dynamic system behaviors, and the time series generated from chaotic systems are highly sensitive to the initial values, which can be applied to the design of information encryption algorithms, such as for video, image, and speech. Therefore, the design of multi-scroll chaotic systems is an important research topic, and the existence of chaos in dynamical systems needs to be proven through rigorously mathematical methods, such as the topological horseshoe method. In addition, a software or hardware implementation is also an important practical means to verify the existence of chaos. This paper aimed to solve the design problem of multi-scroll chaotic systems and to rigorously prove the existence of chaos, as well as verifying the theoretical results through software and hardware methods.

Multi-scroll chaotic attractors have been widely studied using the above-mentioned methods, and this paper proposes a new method for multi-scroll chaotic systems through chaos anti-control of a simple linear system, while the existence of chaos is rigorously proven by the methods of topological horseshoe and numerical simulation. The main contributions of this article are given as follows: (1) A new method of constructing a chaotic system is proposed based on an asymptotically stable linear system and a simple controller; (2) The controlled system with multi-scroll chaotic attractor is generated by the controller of a sine function, and the dynamic characteristics are discussed using the Lyapunov exponent spectrum and a bifurcation diagram; (3) The topological horseshoe is given to show the chaos existence in the controlled system. Meanwhile, a circuit diagram was designed based on software simulation, and the experimental results verified the chaotic phenomena of the multi-scroll chaotic attractors in the controlled system.

The rest of this paper is arranged as follows: Section 2 gives a new method for designing multi-scroll chaotic attractors. The dynamic characteristics of the controlled system are analyzed with a bifurcation diagram and Lyapunov exponent spectra in Section 3. Based on the topological horseshoe theory, a topological horseshoe is found in the Poincaré section of controlled system in Section 4. In Section 5, an analog circuit based on Multisim 14 software was designed to verify the existence of multi-scroll chaotic attractors in the controlled system. Section 6 gives the conclusions.

2. A New Design Method for Multi-Scroll Chaotic Attractors

2.1. Chaos Anti-Control for Linear Dynamical Systems

According to the chaos anti-control method of dynamical systems, a continuous linear system is given as follows:

$$\dot{x} = Ax \quad (1)$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

$a_{12} \cdot a_{21} < 0$, and $a_{jj} < 0$ ($j = 1, 2, 3$), then system (1) is asymptotically stable. A uniformly bounded continuous controller u and a similar transformation matrix P are designed, and then the controlled system is given by [35,37,42]

$$\dot{x} = PAP^{-1}x + Cu \quad (2)$$

where C is a control matrix, it is used to design the position of the controller u , so the controlled system (2) can generate chaotic attractors. One considers a continuous sine function to design the controller, and the controller is given as follows:

$$u = [\varepsilon_1 \sin(\sigma_1 x_1), \varepsilon_2 \sin(\sigma_2 x_2), \varepsilon_3 \sin(\sigma_3 x_3)]^T$$

where ε_i and σ_i ($i = 1, 2, 3$) are control parameters, and they represent the amplitude and frequency of the controlled system, respectively. If the matrices A , C and P are given by

$$A = \begin{bmatrix} -0.3 & -9 & 0 \\ 6 & -0.51 & 0 \\ -9 & 6 & -0.1 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, P = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

then the controlled system is obtained as

$$\begin{cases} \dot{x}_1 = 4.195x_1 - 4.295x_2 + 1.295x_3 \\ \dot{x}_2 = 3.1x_1 - 3.2x_2 - 6.1x_3 \\ \dot{x}_3 = -7.605x_1 + 7.605x_2 - 1.905x_3 + \varepsilon_2 \sin(\sigma_2 x_2) \end{cases} \quad (3)$$

When the amplitude $\varepsilon_2 = 3.3$ and frequency $\sigma_2 = 4.7$, the Lyapunov exponents of controlled system (3) are obtained as [43–45]

$$LE_1 = 1.2294, LE_2 = 0.0000, LE_3 = -2.1394.$$

Obviously, there exists a positive Lyapunov exponent, so the controlled system (3) is in a state of chaos. The initial values of the controlled system (3) are $(x_1(0), x_2(0), x_3(0)) = (0.2, 0.6, 0.3)$, and corresponding phase diagrams of the chaotic attractor are shown in Figure 1.

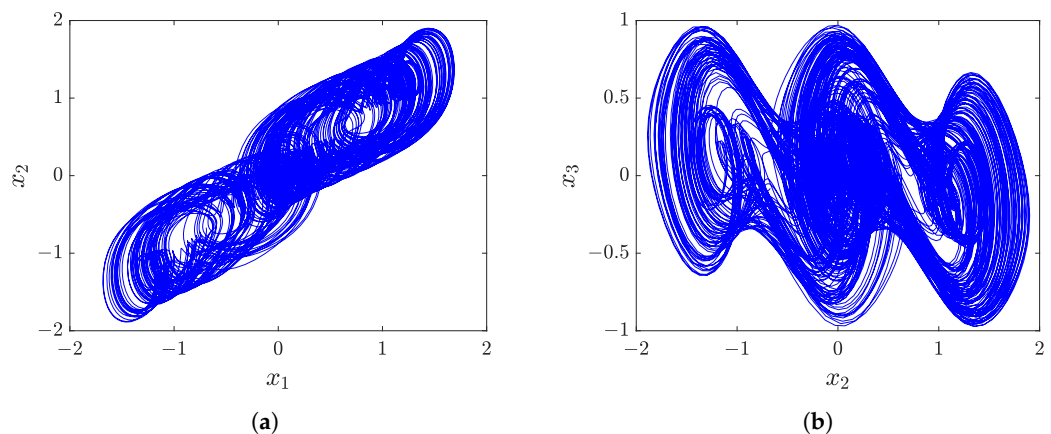


Figure 1. Chaotic attractor of controlled system (3). (a) Phase diagram x_1 vs. x_2 . (b) Phase diagram x_2 vs. x_3 .

2.2. Analysis of Equilibrium Point

Obviously, one of the equilibrium points of the controlled system is $(0, 0, 0)$. In addition, this satisfies $Dx = B$, where

$$D = \begin{bmatrix} 4.195 & -4.295 & 1.295 \\ 3.1 & -3.2 & -6.1 \\ -7.605 & 7.605 & -1.905 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ \varepsilon_2 \sin(\sigma_2 x_2) \end{bmatrix}$$

Then, the equilibrium points are given by

$$\begin{cases} x_1 = \frac{|D_1|}{|D|} = \frac{30.3435 \cdot \varepsilon_2 \sin(\sigma_2 x_2)}{-5.4153}, \\ x_2 = \frac{|D_2|}{|D|} = \frac{29.6040 \cdot \varepsilon_2 \sin(\sigma_2 x_2)}{-5.4153}, \\ x_3 = \frac{|D_3|}{|D|} = \frac{-0.1095 \cdot \varepsilon_2 \sin(\sigma_2 x_2)}{-5.4153}, \end{cases}$$

where D_i ($i = 1, 2, 3$) denotes a new matrix where the i -th column of matrix D is replaced by vector B , and $|\bullet|$ is the determinant of a matrix. According to the second equation

$$x_2 = \frac{29.6040 - 5.4153}{\varepsilon_2} \sin(\sigma_2 x_2),$$

if one lets

$$y_1 = x_2, y_2 = \frac{29.6040}{-5.4153} \varepsilon_2 \sin(\sigma_2 x_2),$$

and they are shown in Figure 2, then there are multiple non-zero equilibrium points besides the origin, such as equilibrium points $(0.6929, 0.6760, -0.0025)$ and $(1.3540, 1.3210, -0.0049)$.

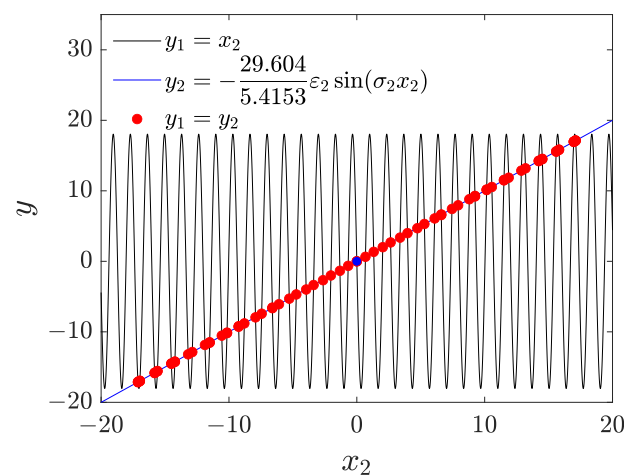


Figure 2. The equilibrium points of the controlled system (3).

The Jacobian matrix of the controlled system (3) is given by

$$J = \begin{bmatrix} 4.195 & -4.295 & 1.295 \\ 3.1 & -3.2 & -6.1 \\ -7.605 & 7.605 + \varepsilon_2 \sigma_2 \cos(\sigma_2 x_2) & -1.905 \end{bmatrix},$$

then, the Jacobian matrix at equilibrium point $(0, 0, 0)$ is obtained as

$$J_0 = \begin{bmatrix} 4.195 & -4.295 & 1.295 \\ 3.1 & -3.2 & -6.1 \\ -7.605 & 7.605 + \varepsilon_2 \sigma_2 & -1.905 \end{bmatrix},$$

and the corresponding eigenvalues are

$$\lambda_1 = 2.8444, \lambda_{2,3} = -1.8772 \pm 12.4900i.$$

There exists a positive eigenvalue, so the equilibrium point $(0, 0, 0)$ is unstable.

Similarly, the eigenvalues of equilibrium points $(0.6929, 0.6760, -0.0025)$ and $(1.3540, 1.3210, -0.0049)$ are given by

$$\begin{cases} \lambda_{1,2} = 4.4571 \pm 5.2338i, \lambda_3 = -9.8243, \\ \lambda_1 = 2.8414, \lambda_{2,3} = -1.8757 \pm 12.4789i. \end{cases}$$

Obviously, the eigenvalues of two equilibrium points have positive real parts, so they are unstable.

The controlled system (3) has multiple unstable equilibrium points, and the trajectories switch back and forth near these equilibrium points. Through the design of a controller u , the controlled system can behave as a multi-scroll chaotic attractor.

3. Dynamic Analysis of the Controlled System

The characteristics of dynamic systems are closely related to the control parameters, and different dynamic behaviors can be observed through the design of control parameters. In particular, chaotic systems are highly sensitive to parameters and initial conditions. The bifurcation characteristics and the Lyapunov exponent spectra of controlled systems with different control amplitudes and frequencies are discussed.

3.1. Bifurcation Analysis with Amplitude of Dynamic System

When the other parameters of the controlled system (3) were fixed, the dynamical behavior was investigated with different amplitudes ε_2 of the controller. Through software simulation testing, the cross-section of $x_1 = x_2$ was taken to obtain a trajectory bifurcation diagram, and the dynamic behavior of the system with different parameters was obtained in Figure 3a. According to the bifurcation diagram and Lyapunov exponent spectrum in Figure 3b, the parameter ε_2 belongs to the intervals $[-1.7, -0.5] \cup [0.5, 1.7]$, the trajectory of the controlled system is periodic, and the controlled system approaches the equilibrium origin when the parameter $\varepsilon_2 \in [-1.7, -0.5] \cup [0.5, 1.7]$. In addition, the trajectory of the controlled system shows chaotic behavior when the parameter $\varepsilon_2 < -1.7$ or $\varepsilon_2 > 1.7$.

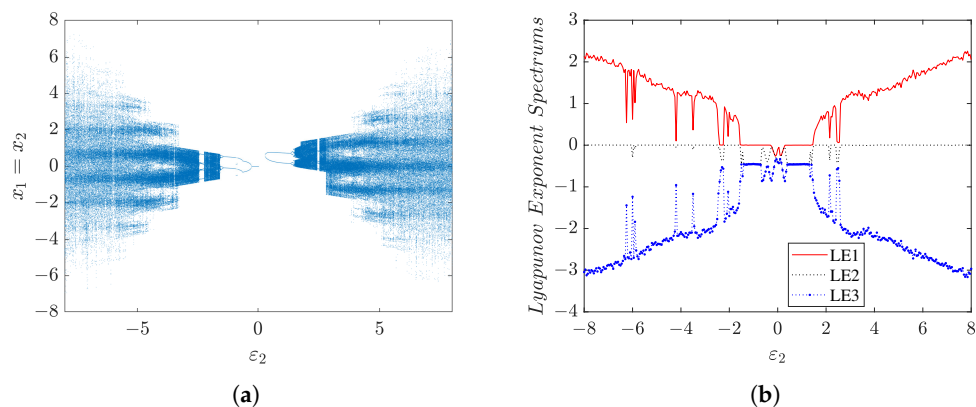


Figure 3. The bifurcation diagram and Lyapunov exponent of the controlled system (3) with respect to parameter ε_2 . (a) Bifurcation diagram with control parameter ε_2 . (b) Lyapunov exponent spectrum with control parameter ε_2 .

3.2. Bifurcation Analysis with Frequency of Controlled System

Similarly, when the other parameters of the controlled system (3) were fixed, the dynamical behavior was investigated with different frequencies σ_2 . Based on the software simulation, the cross-section of $x_1 = x_2$ was selected to obtain the bifurcation of the controlled system, and this is shown in Figure 4a. Meanwhile, the Lyapunov exponent spectrum is given to shown the states of the controlled system (3) with different frequencies σ_2 . Obviously, the system (3) is in a state of period when $\sigma_2 \in (-2, -0.3) \cup (0.3, 2)$, and chaos exists when $\sigma_2 \in (-8, -2) \cup (2, 8)$.

In the past few decades, many methods of designing multi-scroll chaotic systems have been proposed, such as the piecewise-linear function approach, nonlinear modulating function method, including a sine function, adjustable sawtooth wave function, adjustable triangular wave function, etc. [16,31,46]. In order to analyze the characteristics of different methods more intuitively, a brief comparison of some different methods for designing multi-scroll chaotic systems is shown in Table 1.

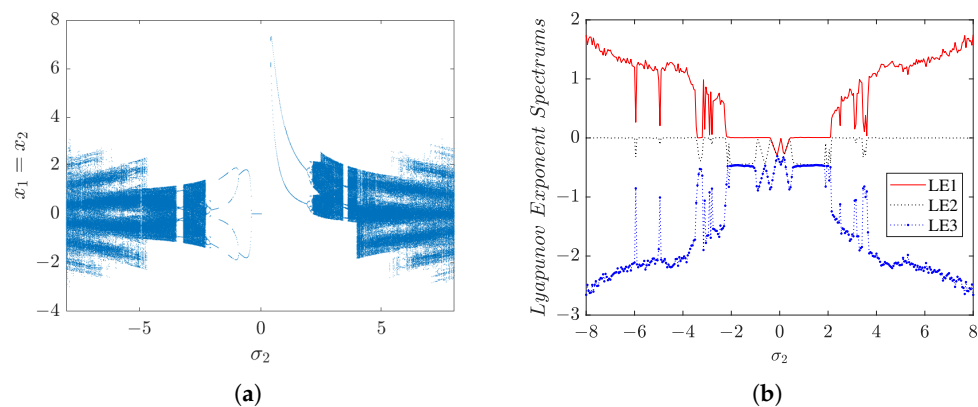


Figure 4. The bifurcation diagram and Lyapunov exponent of the controlled system (3) with respect to the parameter σ_2 . (a) Bifurcation diagram with control parameter σ_2 . (b) Lyapunov exponent spectrum with control parameter σ_2 .

As is well known, it is extremely difficult to construct a multi-scroll chaotic system without any theoretical guidance or criteria. For some multi-scroll chaotic systems based on chaos anti-control of a linear system, the selection and adjustment of their parameters is within a certain interval or by taking a fixed value. However, the selection of parameters is very difficult for the chaos anti-control of a nonlinear system, and this requires many tests by trial-and-error methods.

Table 1. Different methods of generating multi-scroll chaotic systems.

System	Equation	Controller
Generalized Chua's circuit [47]	$\begin{cases} \dot{x} = \alpha(y - h(x)) \\ \dot{y} = x - y + z \\ \dot{z} = -\beta y - \gamma z \end{cases}$	Piecewise-linear function $h(x) = m_{2n-1}x + \sum_{k=1}^{2n-1} \frac{m_{k-1} - m_k}{2} (x + b_k - x - b_k)$
Ref. [48]	$\begin{cases} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = -ax - by - cz + d_1 f(x; q) \end{cases}$	Irregular saturated function $f(x; q) = \begin{cases} k, & \text{if } x > q(2), \\ \frac{k}{q}, & \text{if } q(2) > x > q(1), \\ -k, & \text{if } x < q(1) \end{cases}$
Ref. [49]	$\begin{cases} \dot{x} = y \\ \dot{y} = -x + ayz + byf(x) \\ \dot{z} = 1 - y^2 \end{cases}$	Sine function $f(z) = \sin(z)$
Ref. [50]	$\begin{cases} \dot{x} = g(y - ax + f(x)) - c \\ \dot{y} = x - y + z \\ \dot{z} = -dy \end{cases}$	Sign function $f(x) = \operatorname{sgn}(x) + \operatorname{sgn}(x + b) + \operatorname{sgn}(x - b) + \operatorname{sgn}(x + 2b) + \operatorname{sgn}(x - 2b)$
Proposed system in Equation (3)	$\dot{x} = PAP^{-1}x + Cu$	Sine function $u = \varepsilon \sin(\sigma x)$

In the proposed method, there are many parameters for matrices C , P , and A , and one can generate chaotic systems with more scrolls by adjusting the control parameters ε_i and σ_i , ($i = 1, 2, \dots, n$). Moreover, the parameters of matrices P and A only need to satisfy that the linear system is asymptotically stable, i.e., the eigenvalues of matrix A are negative

and the P is non-singular. Therefore, one can easily find the parameters in Equation (3) for designing multi-scroll chaotic system and successfully construct a multi-scroll chaotic system. The proposed method provides some criteria for constructing a multi-scroll chaotic system, which can reduce the excessive parameter testing by trial-and-error methods.

4. Topological Horseshoe of Multi-Scroll Chaotic Systems

4.1. Topological Horseshoes

A topological horseshoe provides a powerful tool in the rigorous study of chaos and dynamical systems, and one can use it to obtain the topological entropy and verify the existence of chaos. There are some definitions and theories of topological horseshoes, and they are given as follows:

Let D be the compact subset of metric space X , and D_i ($i = 1, 2, \dots, m$) are mutually disjoint compact subsets of D . Meanwhile, the map $f : D \rightarrow X$ is continuous in the subsets D_i ($i = 1, 2, \dots, m$).

Definition 1. Let D_i^1, D_i^2 ($i = 1, 2, \dots, m$) be two fixed disjoint compact subsets of D_i , if there exists a connected compact set $\gamma_i \subset \gamma \subset D$ such that $\gamma_i \subset D_i$, $\gamma_i \cap D_i^1 \neq \emptyset$, $\gamma_i \cap D_i^2 \neq \emptyset$, then γ_i is said to connect D_i^1 and D_i^2 of D_i , we denote this by $D_i^1 \xleftarrow{\gamma_i} D_i^2$.

Theorem 1 ([51,52]). Suppose that the map $f : D \rightarrow X$ satisfies following assumptions:

- (a) There exists m mutually disjoint compact subsets $D_1, D_2, \dots, D_m$ of D , $f|_{D_i}$ is continuous;
- (b) Then, the relation $f(D_i) \mapsto D_j$ holds for every pair with i, j taken from $1 \leq i, j \leq m$.

Then, there exists a invariant set $K \subset D$, such that $f|_K$ is semi-conjugate to full m -shift dynamics $\sigma|_{\Sigma_m}$, and entropy $\text{ent}(f) \geq \log m$.

Corollary 1 ([53]). If $f^p(D_1) \mapsto D_1, f^p(D_1) \mapsto D_2, f^q(D_2) \mapsto D_2, f^q(D_2) \mapsto D_1$ ($p, q \in \mathbf{Z}^+$), then there exists a invariant set $K \subset D$, such that $f^{p+q}|_K$ is semi-conjugate to full 2-shift dynamics, and $\text{ent}(f) \geq \frac{1}{p+q} \log 2$.

4.2. Topological Horseshoes in the Controlled System

In order to find the topological horseshoes in the controlled chaotic system (3), the Poincaré section is first defined by $\Gamma = \{x | -2 \leq x_1 \leq 2, -1 \leq x_3 \leq 1, x_2 = 0 \text{ and } \dot{x}_2 < 0\}$. The Poincaré map $H : \Gamma \rightarrow \Gamma$ is chosen as follows: For each $x \in \Gamma$, $H(x)$ is taken to be the first return point in Γ under the flow of (3) with the initial condition x [53]. Therefore, the Poincaré section Γ of the controlled system (3) is shown in Figure 5, and then two disjoint compact subsets D_1 and D_2 can be chosen from the Poincaré section Γ . Furthermore, the Poincaré section Γ is a dense set of points, so the controlled system (3) is also shown in a state of chaos when the amplitude $\varepsilon_2 = 3.3$ and frequency $\sigma_2 = 4.7$.

The polygon subsets D_1 and D_2 in Γ with four vertices are given by

$$\begin{aligned} &[-0.4661, 0.0000, 0.3679], [-0.4636, 0.0000, 0.3584], \\ &[-0.4586, 0.0000, 0.3633], [-0.4613, 0.0000, 0.3724], \end{aligned}$$

and

$$\begin{aligned} &[-0.3929, 0.0000, 0.4368], [-0.3902, 0.0000, 0.4268], \\ &[-0.3781, 0.0000, 0.4370], [-0.3809, 0.0000, 0.4472]. \end{aligned}$$

The images of polygons D_1 and D_2 under the map H are shown in Figure 6. The D_1^1 (green) and D_1^2 (pink) denote the compact subsets on the left-side and right-side of D_1 , respectively. Based on the numerical simulation of the Poincaré map in Figure 6a, the image $H(x)$ fully crosses D_1 and D_2 for all $x \in D_1$. Similarly, the image $H(x)$ fully crosses D_1 and D_2 for all $x \in D_2$ in Figure 6b. According to Corollary 1, there exists a compact invariant set $K \subset D$, such that $f^2|_K$ is semi-conjugate to 2-shift dynamics, and $\text{ent}(f) \geq \frac{1}{2} \log 2$.

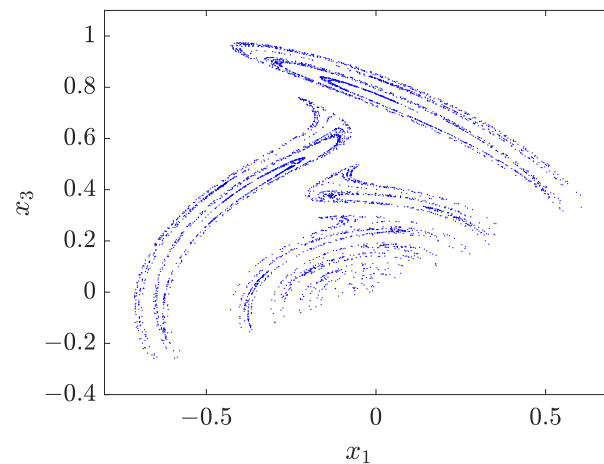


Figure 5. The Poincaré section of the controlled system (3) when $x_2 = 0$ and $\dot{x}_2 < 0$.

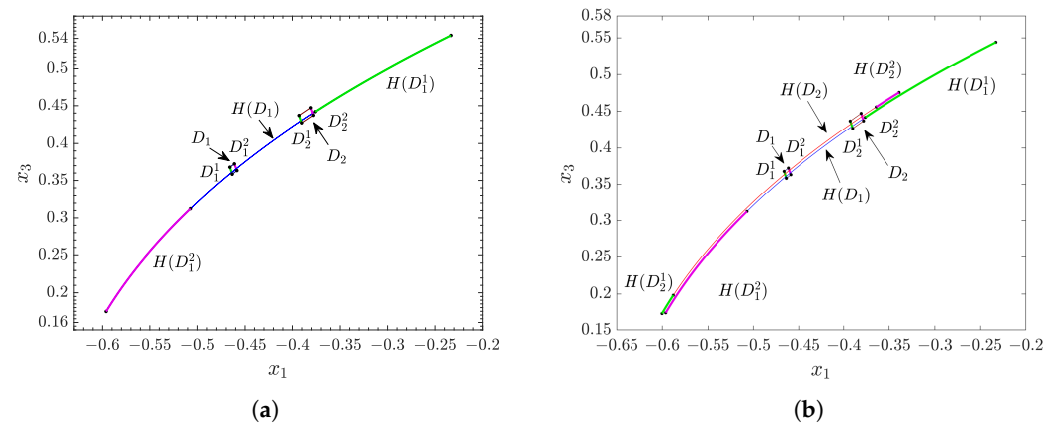


Figure 6. A new horseshoe of the controlled system (3). (a) D_1 is suitably across itself under H . (b) A chosen D_2 that can be suitably across D_1 under H .

5. Circuit Design

Using the software Multisim 14, the resistance, capacitance, and other device values of the designed circuit are shown in Figure 7. As the variable values of the controlled system are within the allowable range of the device, linear transformation compression is not required. Based on the variables x_1 , x_2 and x_3 of oscilloscope and the phase diagrams x_1 vs. x_2 and x_2 vs. x_3 , the values are consistent with the numerical simulation results in Figure 1, so the existence of chaotic attractors with three scrolls in the controlled system (3) is also shown in Figure 8b,c.

According to the bifurcation analysis in Section 3, the controlled system (3) was in a state of chaos when the controller parameters were increased, and the system variables could increase and move back and forth among multiple equilibrium points, so the controlled system behaved as chaotic attractors with scrolls. For example, when the controller parameter $\varepsilon_2 = 4.3$ and other parameters remained unchanged, chaotic attractors with five scrolls could be generated. The results from the oscilloscope based on the circuit are shown in Figure 9a. When the controller parameter $\varepsilon_2 = 4.9$ and other parameters remained unchanged, there were chaotic attractors with six scrolls, as shown in Figure 9b. When the controller parameter $\varepsilon_2 = 5.6$ and other parameters remained unchanged, there were chaotic attractors with seven scrolls, as shown in Figure 9c. Through experimental observation, the controlled system (3) could generate chaotic attractors with more scrolls as the parameter ε_2 was increased, i.e., the larger the control parameters ε_2 , the more scrolls the controlled system had.

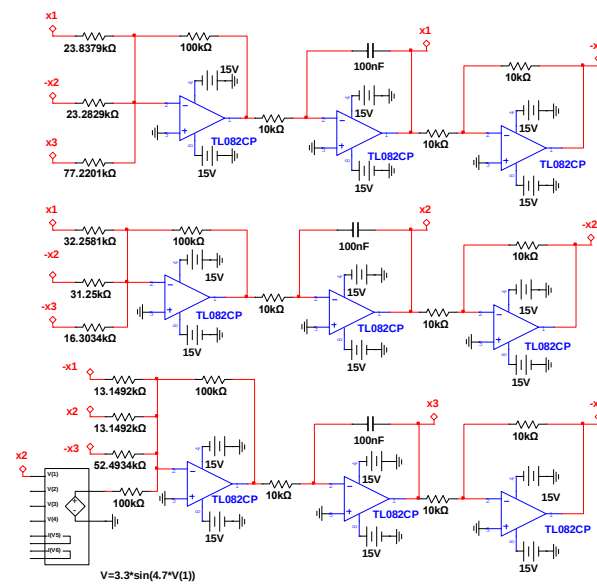


Figure 7. Circuit diagram of controlled system (3).

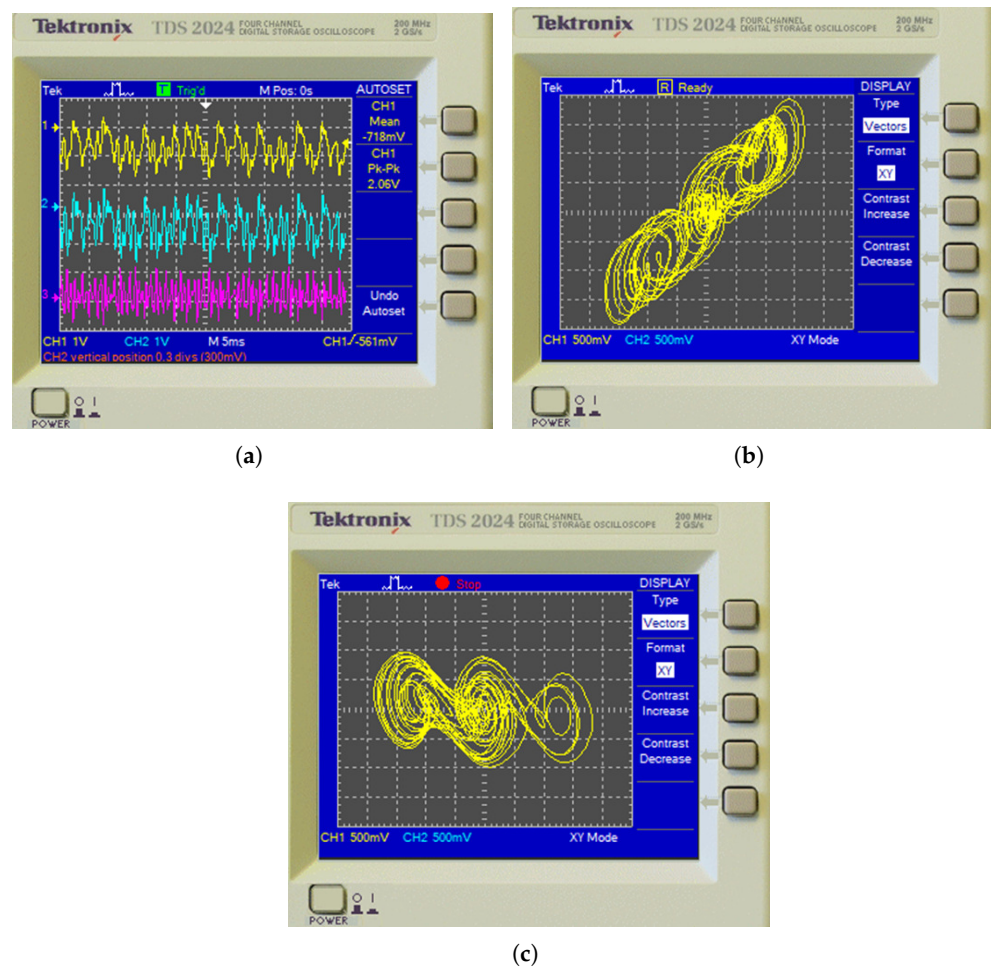


Figure 8. The results from the circuit oscilloscope. (a) The time series of variables x_1 (yellow), x_2 (blue) and x_3 (purple). (b) Phase diagram x_1 vs. x_2 . (c) Phase diagram x_2 vs. x_3 .

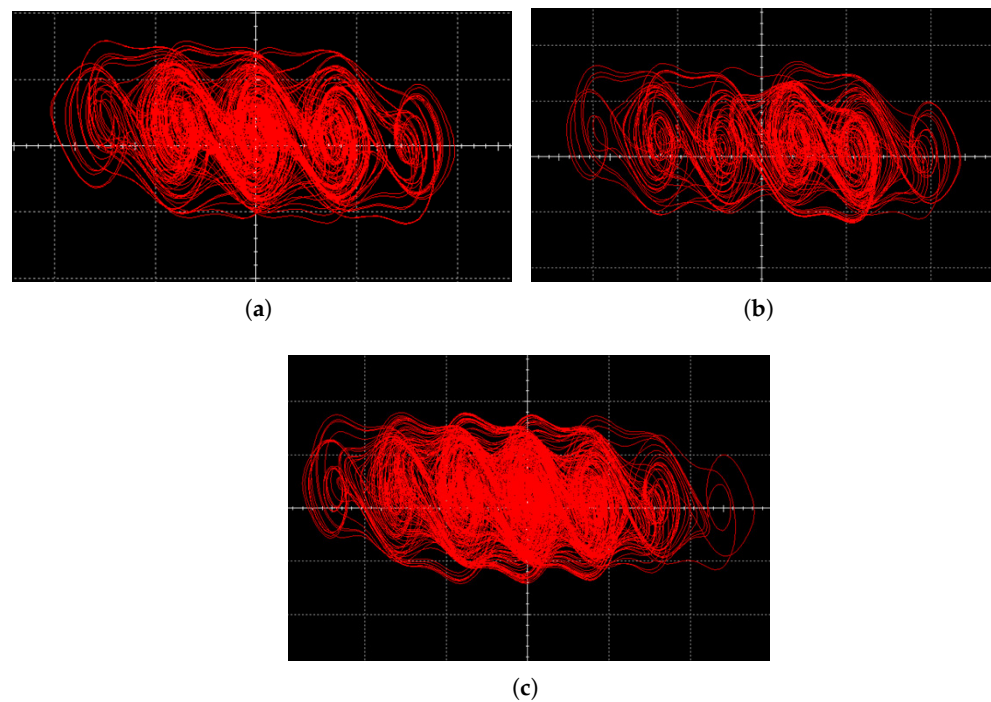


Figure 9. The multi-scroll chaotic system with different values of the parameter ε_2 . (a) The chaotic attractors with five scrolls when $\varepsilon_2 = 4.3$; (b) The chaotic attractors with six scrolls when $\varepsilon_2 = 4.9$; (c) The chaotic attractors with seven scrolls when $\varepsilon_2 = 5.6$.

Similarly, if the frequency σ_2 was changed, then chaotic attractors with different numbers of scrolls could also be obtained. When $\sigma_2 = 6.3$, there were chaotic attractors with five scrolls for the controlled system, (3) based on the oscilloscope image shown in Figure 10a. When $\sigma_2 = 9.3$, the chaotic attractor had seven scrolls, as shown in Figure 10b. Therefore, the number of scrolls in the controlled system (3) increased with the control frequency σ_2 , but the control effect of frequency σ_2 was not as significant as that of the amplitude ε_2 .

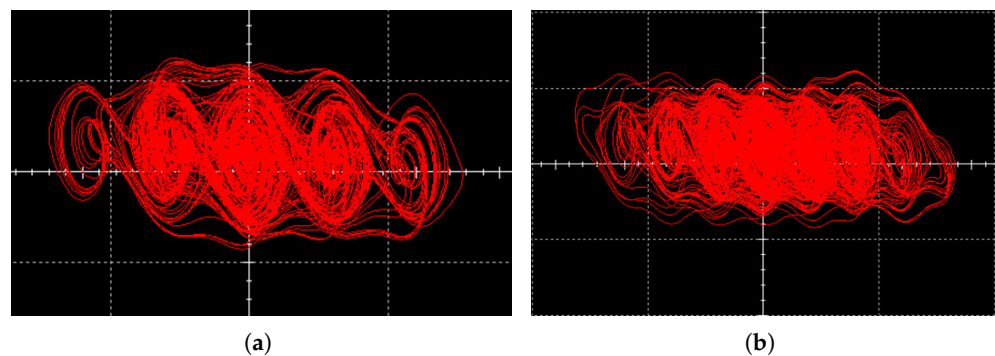


Figure 10. The multi-scroll chaotic system with different values of parameter σ_2 . (a) The chaotic attractors with five scrolls when $\sigma_2 = 6.3$; (b) The chaotic attractors with seven scrolls when $\sigma_2 = 9.3$.

6. Conclusions

In this paper, a new method for constructing multi-scroll chaotic attractors was explored using a class of asymptotically stable linear systems and a bounded sine function controller. There are many unstable equilibrium points in the controlled system, so it can behave as a multi-scroll chaotic attractor with different values of amplitude and frequency. The controlled system can be changed between a periodic and chaotic state using the control parameters based on the bifurcation diagram and Lyapunov exponent spectrum. Moreover,

a topological horseshoe was found in the Poincaré section of the controlled system, and then the existence of chaos was proven through a rigorous computer-aided proof. Finally, the existence of multi-scroll chaotic attractors was experimentally verified using a circuit design based on the software Multisim 14. Multi-scroll chaotic systems have complex dynamic characteristics and important applications in information encryption of multimedia, and they will be applied to the design of encryption algorithms in the future.

Author Contributions: Writing—original draft preparation, Y.Y. and J.H.; writing—review and editing, Y.Y. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the Natural Science Foundation of Fujian Province (No. 2022J01895), the Digital Fujian Meteorological Big Data Research Institute, and the Key Laboratory of Data Science and Statistics.

Data Availability Statement: The data in this study are available from the corresponding author upon request.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Lü, J.; Han, F.; Yu, X.; Chen, G. Generating 3-D multi-scroll chaotic attractors: A hysteresis series switching method. *Automatica* **2004**, *40*, 1677–1687. [\[CrossRef\]](#)
2. Li, F.; Yao, C. The infinite-scroll attractor and energy transition in chaotic circuit. *Nonlinear Dyn.* **2016**, *84*, 2305–2315. [\[CrossRef\]](#)
3. Luo, X.; Wang, C.; Wan, Z. Grid multi-wing butterfly chaotic attractors generated from a new 3-D quadratic autonomous system. *Nonlinear Anal. Model. Control* **2014**, *19*, 272–285. [\[CrossRef\]](#)
4. Chua, L.; Komuro, M.; Matsumoto, T. The double scroll family. *IEEE Trans. Circuits Syst.* **1986**, *33*, 1072–1118. [\[CrossRef\]](#)
5. Suykens, J.; Vandewalle, J. Quasilinear approach to nonlinear systems and the design of n-double scroll ($n = 1, 2, 3, 4, \dots$). *IEE Proc. G (Circuits Devices Syst.)* **1991**, *138*, 595–603. [\[CrossRef\]](#)
6. Lü, J.; Zhou, T.; Chen, G.; Yang, X. Generating chaos with a switching piecewise-linear controller. *Chaos Interdiscip. J. Nonlinear Sci.* **2002**, *12*, 344–349. [\[CrossRef\]](#) [\[PubMed\]](#)
7. Zhang, C.; Yu, S. Generation of grid multi-scroll chaotic attractors via switching piecewise linear controller. *Phys. Lett. A* **2010**, *374*, 3029–3037. [\[CrossRef\]](#)
8. Lü, J.; Murali, K.; Sinha, S.; Leung, H.; Aziz-Alaoui, M. Generating multi-scroll chaotic attractors by thresholding. *Phys. Lett. A* **2008**, *372*, 3234–3239. [\[CrossRef\]](#)
9. Wang, F.Q.; Liu, C.X. Generation of multi-scroll chaotic attractors via the saw-tooth function. *Int. J. Mod. Phys. B* **2008**, *22*, 2399–2405. [\[CrossRef\]](#)
10. Joshi, M.; Ranjan, A. Dual feedback IRC ring for chaotic waveform generation. *IET Circuits Devices Syst.* **2021**, *15*, 595–601. [\[CrossRef\]](#)
11. Xu, F.; Yu, P. Chaos control and chaos synchronization for multi-scroll chaotic attractors generated using hyperbolic functions. *J. Math. Anal. Appl.* **2010**, *362*, 252–274. [\[CrossRef\]](#)
12. Joshi, M.; Ranjan, A. An autonomous simple chaotic jerk system with stable and unstable equilibria using reverse sine hyperbolic functions. *Int. J. Bifurc. Chaos* **2020**, *30*, 2050070. [\[CrossRef\]](#)
13. Tang, W.K.; Zhong, G.; Chen, G.; Man, K. Generation of n -scroll attractors via sine function. *IEEE Trans. Circuits Syst. I Fundam. Theory Appl.* **2001**, *48*, 1369–1372. [\[CrossRef\]](#)
14. Yalçın, M.E. Multi-scroll and hypercube attractors from a general jerk circuit using Josephson junctions. *Chaos Solitons Fractals* **2007**, *34*, 1659–1666. [\[CrossRef\]](#)
15. Lü, J.; Chen, G.; Yu, X.; Leung, H. Design and analysis of multiscroll chaotic attractors from saturated function series. *IEEE Trans. Circuits Syst. I Regul. Pap.* **2004**, *51*, 2476–2490. [\[CrossRef\]](#)
16. Lü, J.; Chen, G. Generating multiscroll chaotic attractors: Theories, methods and applications. *Int. J. Bifurc. Chaos* **2006**, *16*, 775–858. [\[CrossRef\]](#)
17. Yu, S.; Tang, W.K.; Lü, J.; Chen, G. Generation of $n \times m$ -Wing Lorenz-Like Attractors From a Modified Shimizu–Morioka Model. *IEEE Trans. Circuits Syst. II Express Briefs* **2008**, *55*, 1168–1172. [\[CrossRef\]](#)
18. Bouallegue, K.; Chaari, A.; Toumi, A. Multi-scroll and multi-wing chaotic attractor generated with Julia process fractal. *Chaos Solitons Fractals* **2011**, *44*, 79–85. [\[CrossRef\]](#)
19. Wang, N.; Zhang, G.; Kuznetsov, N.V.; Li, H. Generating grid chaotic sea from system without equilibrium point. *Commun. Nonlinear Sci. Numer. Simul.* **2022**, *107*, 106194. [\[CrossRef\]](#)
20. Yang, Y.; Huang, L.; Xiang, J.; Bao, H.; Li, H. Design of multi-wing 3D chaotic systems with only stable equilibria or no equilibrium point using rotation symmetry. *AEU-Int. J. Electron. Commun.* **2021**, *135*, 153710. [\[CrossRef\]](#)
21. Qi, G.; Wang, Z.; Guo, Y. Generation of an eight-wing chaotic attractor from Qi 3-D four-wing chaotic system. *Int. J. Bifurc. Chaos* **2012**, *22*, 1250287. [\[CrossRef\]](#)

22. Liu, S.; Wei, Y.; Liu, J.; Chen, S.; Zhang, G. Multi-scroll chaotic system model and its cryptographic application. *Int. J. Bifurc. Chaos* **2020**, *30*, 2050186. [\[CrossRef\]](#)
23. Wang, N.; Li, C.; Bao, H.; Chen, M.; Bao, B. Generating multi-scroll Chua's attractors via simplified piecewise-linear Chua's diode. *IEEE Trans. Circuits Syst. I Regul. Pap.* **2019**, *66*, 4767–4779. [\[CrossRef\]](#)
24. Yu, S.; Tang, W.K. Generation of $n \times m$ -scroll attractors in a two-port RCL network with hysteresis circuits. *Chaos Solitons Fractals* **2009**, *39*, 821–830. [\[CrossRef\]](#)
25. Yu, S.; Lü, J.; Yu, X.; Chen, G. Design and implementation of grid multiwing hyperchaotic Lorenz system family via switching control and constructing super-heteroclinic loops. *IEEE Trans. Circuits Syst. I Regul. Pap.* **2012**, *59*, 1015–1028. [\[CrossRef\]](#)
26. Tlelo-Cuautle, E.; Rangel-Magdaleno, J.; Pano-Azucena, A.; Obeso-Rodelo, P.; Nunez-Perez, J. FPGA realization of multi-scroll chaotic oscillators. *Commun. Nonlinear Sci. Numer. Simul.* **2015**, *27*, 66–80. [\[CrossRef\]](#)
27. Liu, S.; An, X.; Wang, Y.; Shi, Q. Design of a new multi-wing chaotic system and its application in color image encryption. *Optik* **2023**, *290*, 171334. [\[CrossRef\]](#)
28. Wang, Z.; Ahmadi, A.; Tian, H.; Jafari, S.; Chen, G. Lower-dimensional simple chaotic systems with spectacular features. *Chaos Solitons Fractals* **2023**, *169*, 113299. [\[CrossRef\]](#)
29. Peng, H.; Ji'e, M.; Du, X.; Duan, S.; Wang, L. Design of pseudorandom number generator based on a controllable multi-double-scroll chaotic system. *Chaos Solitons Fractals* **2023**, *174*, 113803. [\[CrossRef\]](#)
30. Yan, M.; Xu, H. The multi-scroll hyper-chaotic coexistence attractors and its application. *Signal Process. Image Commun.* **2021**, *95*, 116210. [\[CrossRef\]](#)
31. Joshi, M.; Bhatt, V.; Ranjan, A. A single parametrically controlled megastable multiscroll attractor with an unstable node. *Eur. Phys. J. B* **2023**, *96*, 62. [\[CrossRef\]](#)
32. Qin, M.; Lai, Q. Expanded multi-scroll attractor system analysis and application for remote sensing image encryption. *Appl. Math. Model.* **2024**, *125*, 125–146. [\[CrossRef\]](#)
33. Kim, T.R.; Joo, Y.H.; Park, J.B.; Chen, G. Anticontrol of chaos for a continuous-time Takagi–Sugeno fuzzy system via local time-delay feedback. *Int. J. Bifurc. Chaos* **2005**, *15*, 3883–3894. [\[CrossRef\]](#)
34. Yu, S.; Chen, G. Anti-control of continuous-time dynamical systems. *Commun. Nonlinear Sci. Numer. Simul.* **2012**, *17*, 2617–2627. [\[CrossRef\]](#)
35. Shen, C.; Yu, S.; Lu, J.; Chen, G. A Systematic Methodology for Constructing Hyperchaotic Systems With Multiple Positive Lyapunov Exponents and Circuit Implementation. *IEEE Trans. Circuits Syst. I Regul. Pap.* **2017**, *61*, 854–864. [\[CrossRef\]](#)
36. Shen, C.; Yu, S.; Lü, J.; Chen, G. Designing hyperchaotic systems with any desired number of positive Lyapunov exponents via a simple model. *IEEE Trans. Circuits Syst. I Regul. Pap.* **2014**, *61*, 2380–2389. [\[CrossRef\]](#)
37. Shen, C.; Yu, S.; Lü, J.; Chen, G. Constructing hyperchaotic systems at will. *Int. J. Circuit Theory Appl.* **2015**, *43*, 2039–2056. [\[CrossRef\]](#)
38. Tian, K.; Grebogi, C.; Ren, H.P. Chaos generation with impulse control: Application to non-chaotic systems and circuit design. *IEEE Trans. Circuits Syst. I Regul. Pap.* **2021**, *68*, 3012–3022. [\[CrossRef\]](#)
39. Herbadji, D.; Herbadji, A.; Haddad, I.; Kahia, H.; Belmeguenai, A.; Derouiche, N. An enhanced logistic chaotic map based tweakable speech encryption algorithm. *Integration* **2024**, *97*, 102192. [\[CrossRef\]](#)
40. Flores-Vergara, A.; García-Guerrero, E.E.; Inzunza-González, E.; López-Bonilla, O.R.; Rodríguez-Orozco, E.; Cárdenas-Valdez, J.R.; Tlelo-Cuautle, E. Implementing a chaotic cryptosystem in a 64-bit embedded system by using multiple-precision arithmetic. *Nonlinear Dyn.* **2019**, *96*, 497–516. [\[CrossRef\]](#)
41. Lambić, D.; Nikolić, M. Pseudo-random number generator based on discrete-space chaotic map. *Nonlinear Dyn.* **2017**, *90*, 223–232. [\[CrossRef\]](#)
42. He, J.; Yu, S. Construction of Higher-Dimensional Hyperchaotic Systems with a Maximum Number of Positive Lyapunov Exponents under Average Eigenvalue Criteria. *J. Circuits Syst. Comput.* **2019**, *28*, 1950151. [\[CrossRef\]](#)
43. Wolf, A.; Swift, J.B.; Swinney, H.L.; Vastano, J.A. Determining Lyapunov exponents from a time series. *Phys. D Nonlinear Phenom.* **1985**, *16*, 285–317. [\[CrossRef\]](#)
44. Eckmann, J.P.; Ruelle, D. Ergodic theory of chaos and strange attractors. *Rev. Mod. Phys.* **1985**, *57*, 617. [\[CrossRef\]](#)
45. Dieci, L.; Russell, R.D.; Van Vleck, E.S. On the computation of Lyapunov exponents for continuous dynamical systems. *SIAM J. Numer. Anal.* **1997**, *34*, 402–423. [\[CrossRef\]](#)
46. Yan, S.; Li, L.; Gu, B.; Cui, Y.; Wang, J.; Song, J. Design of hyperchaotic system based on multi-scroll and its encryption algorithm in color image. *Integration* **2023**, *88*, 203–221. [\[CrossRef\]](#)
47. Arena, P.; Baglio, S.; Fortuna, L.; Manganaro, G. Generation of n -double scrolls via cellular neural networks. *Int. J. Circuit Theory Appl.* **1996**, *24*, 241–252. [\[CrossRef\]](#)
48. Muñoz-Pacheco, J.M.; Guevara-Flores, D.K.; Félix-Beltrán, O.G.; Tlelo-Cuautle, E.; Barradas-Guevara, J.E.; Volos, C.K. Experimental verification of optimized multiscroll chaotic oscillators based on irregular saturated functions. *Complexity* **2018**, *2018*, 3151840. [\[CrossRef\]](#)
49. Jafari, S.; Pham, V.T.; Kapitaniak, T. Multiscroll chaotic sea obtained from a simple 3D system without equilibrium. *Int. J. Bifurc. Chaos* **2016**, *26*, 1650031. [\[CrossRef\]](#)

50. Rajagopal, K.; Çiçek, S.; Naseradinmousavi, P.; Khalaf, A.J.M.; Jafari, S.; Karthikeyan, A. A novel parametrically controlled multi-scroll chaotic attractor along with electronic circuit design. *Eur. Phys. J. Plus* **2018**, *133*, 354. [[CrossRef](#)]
51. Yang, X.S. Topological horseshoes and computer assisted verification of chaotic dynamics. *Int. J. Bifurc. Chaos* **2009**, *19*, 1127–1145. [[CrossRef](#)]
52. Li, Q. A topological horseshoe in the hyperchaotic Rössler attractor. *Phys. Lett. A* **2008**, *372*, 2989–2994. [[CrossRef](#)]
53. Li, Q.; Yang, X.S. A simple method for finding topological horseshoes. *Int. J. Bifurc. Chaos* **2010**, *20*, 467–478. [[CrossRef](#)]

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.