Article

Construction of $S^{(3)}(2,3)$-Designs of Any Index

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Abstract: Let $H^{(3)}$ be a uniform hypergraph of rank 3. A hyperstar $S^{(3)}(2,3)$ of centre $C = \{x, y\}$ is a 3-uniform hypergraph with three hyperedges, all having the centre $C = \{x, y\}$ in common, with $x$ and $y$ of degree 3 and the remaining vertices of degree 1. In this paper, we determine the spectrum of $S^{(3)}(2,3)$-designs for any index $\lambda$.

Keywords: G-designs; uniform hypergraphs

MSC: 05C51

1. Introduction

Let $K_v^{(h)} = (X, \mathcal{E})$ be the complete hypergraph, uniform of rank $h$, defined on the vertex set $X = \{x_1, x_2, \ldots, x_v\}$. This means that $\mathcal{E}$ is the collection of all the subsets of $X$ whose cardinality is $h$; we will call a set of cardinality $h$ an $h$-subset.

Let $H^{(h)}$ be a subhypergraph of $K_v^{(h)}$. An $H^{(h)}$-design, or also a design of type $H^{(h)}$ or a system of $H^{(h)}$, with order $v$ and index $\lambda$, is a pair $\Sigma = (X, B)$, where $X$ is a finite set of cardinality $v$, whose elements are called vertices, and $B$ is a collection of hypergraphs over $X$, called blocks, all isomorphic to $H^{(h)}$, under the condition that every $h$-subset of $X$ is a hyperedge of exactly $\lambda$ hypergraphs of the collection $B$. An $H^{(h)}$-design, of order $v$ and index $\lambda$, is also called an $H^{(h)}$-decomposition of $\lambda K_v^{(h)}$ (see, for example, [1–5]).

It is important to note that in the definition of index $\lambda$, we do not require that the blocks be distinct. That is, in a given example, a block may be repeated as many as $\lambda$ times.

In what follows, we will indicate by Spectrum($H^{(h)}$) the spectrum of the corresponding $H^{(h)}$-designs, i.e., the set of all positive integers $\nu$ such that there exist $H^{(h)}$-designs of order $\nu$ with blocks isomorphic to $H^{(h)}$. Furthermore, we shall call:

- *Hyperstar* $S^{(h)}(r, s)$ the $h$-uniform hypergraph with $s$ hyperedges and order $(h - r)s + r$, such that all the edges have in common exactly the same $r$ vertices, which form its centre, and all the vertices of the centre have degree $s$;
- *Hyperpath* $P^{(h)}(r, v)$ the $h$-uniform hypergraph, with $v$ vertices and $m$ hyperedges $E_1, E_2, \ldots, E_m$ which can be ordered in such a way that $|E_i \cap E_{i+1}| = r$ and $E_i \cap E_j = \emptyset$ if $j \geq i + 2$.

Following this notation, the symbols $P^{(h)}(h - 1, h + 1)$ and $S^{(h)}(h - 1, 2)$ define the same class of isomorphism of hypergraphs. We refer to [6] for details. Some results on $P^{(h)}(r, v)$-designs have been proven in refs. [7,8].

In the literature, we can find many results on graph decompositions for simple graphs. The results on hypergraph decompositions are limited to small hypergraphs of small uniformity or to a limited class of hypergraphs (see [9]).

In refs. [10,11], the authors introduced the notion of edge-balanced designs and completely determined the spectrum of edge-balanced $S^{(3)}(2,3)$-designs. Here, we drop the hypothesis of balanced edges. In refs. [1], the authors solved the problem of finding a $H$-decomposition of $\lambda K_v^{(h)}$ for $h = 3$ and $v = 4$ and gave general methods which could be...
used for other values of \( h \) and other hypergraphs \( H \) in the case that \( \lambda \) is always equal to 1 and \( H = H_1 = \{ \{ a, b, c \}, \{ a, b, d \} \} \) and \( H = H_2 = \{ \{ a, b, c \}, \{ a, b, d \}, \{ a, c, d \} \} \). The methods and tools used are standard, such as methods of differences and composition methods.

Here, we study another \( H \)-decomposition of \( \lambda K_S^{(3)} \), where \( H = \{ \{ a, b, c \}, \{ a, b, d \}, \{ a, b, e \} \} \) and \( \lambda = 1, 2, 3 \).

The study of hypergraph designs has become an important research area of combinatorial design. In this field, the focus has always been on construction techniques. In general, in the literature, \( B \) is a multiset (see [12]). In this paper, we provide an exhaustive result about the spectrum of \( S^{(3)}(2,3) \)-designs for any index \( \lambda \) with the further condition that in the system, all the blocks are distinct; i.e., we find the cardinality of \( X \), where \( (X, B) \) is a \( \lambda S^{(3)}(2,3) \)-design and \( B \) is a set and not a multiset for \( \lambda = 1, 2, 3 \). For \( \lambda > 3 \), we deal with \( B \) as a multiset. We end of the paper by posing a question regarding the general cases for further investigation.

2. Preliminary Results

In the following, we will use hypergraph terminology and will indicate the \( n \)-star graph, i.e., the connected graph with \( n \) vertices and a centre vertex of degree \( n - 1 \), by \( S^{(2)}(1, n - 1) \).

The following is well known, see [13].

**Theorem 1.** There exists an \( S^{(2)}(1,k) \)-design of order \( v \) if and only if

\[ v(v - 1) \equiv 0 \mod 2k \quad \text{and} \quad v \geq 2k \]

From this, the following result.

**Corollary 1.** There exists an \( S^{(2)}(1,2) \)-design of order \( v \) if and only if

\[ v \equiv 0 \mod 4 \quad \text{or} \quad v \equiv 1 \mod 4 \quad \text{and} \quad v \geq 4 \]

**Corollary 2.** There exists an \( S^{(2)}(1,3) \)-design of order \( v \) if and only if

\[ v \equiv 0 \mod 3 \quad \text{or} \quad v \equiv 1 \mod 3 \quad \text{and} \quad v \geq 6 \]

Regarding systems of \( P^{(3)}(r,s) \), i.e., \( P^{(3)}(r,s) \)-designs, we have the following result.

**Theorem 2.** There exist \( P^{(3)}(1,5) \)-designs of order \( v \) if and only if

\[ v \equiv 0 \mod 2 \quad \text{and} \quad v \geq 4 \quad \text{or} \quad v \equiv 1 \mod 4 \quad \text{and} \quad v \geq 5 \]

In what follows, we will denote the rank 3 uniform hypergraph \( H = (X, E) \), where, e.g., \( X = \{ a, b, c, d \} \) and \( E = \{ \{ a, b, c \}, \{ a, b, d \} \} \), by \( [(a, b), c, d] \).

**Theorem 3.** There exist \( P^{(3)}(2,4) \)-designs of order \( v \) if and only if \( v \equiv 0 \), or 1, or 2 \( \mod 4 \) and \( v \geq 4 \).

The previous results can be found in lrefs. [8,14].

3. The Spectrum of \( S^{(3)}(2,3) \)-Designs of Index 1

In this section, we determine the spectrum of \( S^{(3)}(2,3) \)-designs with index \( \lambda = 1 \); that is, the designs considered will always be of index 1. Let \( X = \{ x, y, z_1, z_2, z_3 \} \) be a set of vertices, \( \{ x, y \} \) be the centre and \( E = \{ \{ x, y, z_i \} : i = 1, 2, 3 \} \) be the set of hyperedges. Such a hypergraph will be denoted by \( [(x,y), z_1, z_2, z_3] \).

First of all, we prove the following results:

**Theorem 4.** If \( \Sigma = (X, B) \) is an \( S^{(3)}(2,3) \)-design of order \( v \) and index \( \lambda \), then:
1. \(|B| = \lambda \cdot \frac{v(v-1)(v-2)}{3} - 1\).

2. If \(\lambda = 1\), then \(v \equiv 0\) or \(v \equiv 1\) or \(v \equiv 2 \mod 9\), and \(v \geq 9\).

**Proof.** Let \(\Sigma = (X, B)\) be an \(S(3)^{(2,3)}\)-design of order \(v\) and index \(\lambda\).

1. Since the blocks of \(\Sigma = (X, B)\) contain, among their hyperedges, all the triples of \(X\) with multiplicity \(\lambda\), and each block contains three of them, it follows that:

\[|B| = \lambda \cdot \left(\frac{v}{3}\right) \cdot \frac{1}{3} = \lambda \cdot \frac{v(v-1)(v-2)}{3 \cdot 2 \cdot 1} = \lambda \cdot \frac{v(v-1)(v-2)}{18}\]

2. As necessary, the only factor in the numerator which is a multiple of 3 must be a multiple of 9.

We recall the following definitions.

**Definition 1.** If \(\Sigma = (X, B)\) is an \(H\)-design and \(x \in X\), we call degree of the vertex \(x\) the number \(d(x)\) of blocks of \(B\) containing \(x\); for any \(x \neq y \in X\), we call degree of edge \((x, y)\) the number \(d(x, y)\) of blocks of \(B\) containing edge \((x, y)\).

**Definition 2.** An \(H^{(3)}\)-design is said to be balanced if the degree \(d(x)\) of each vertex \(x \in X\) is a constant.

**Definition 3.** An \(H^{(3)}\)-design is called edge-balanced if, for any \(x, y \in X, x \neq y\), the degree \(d(x, y)\) is constant.

See [6,10,14] for more details.

**Definition 4 ([15]).** A Steiner quadruple system is a pair \((X, B)\), where \(X\) is a finite set and \(B\) is a collection of four-subsets of \(X\) (called blocks) such that any three-subset of \(X\) belongs to exactly one block of \(B\). The number \(|X| = v\) is called the order of the quadruple system and it is denoted by \(SQS(v)\).

**Theorem 5.** If \(\Sigma = (X, B)\) is an \(SQS(v)\) and

\[C = \{[(y,z), x, t] : [(x, t), y, z] \in B\},\]

then \(\Gamma = (X, C)\) is a \(P^{(3)}(2,4)\)-design of order \(v\).

**Proof.** See ref. [8].

**Theorem 6.** There exist \(S(3)^{(2,3)}\)-designs of order \(v = 9\).

**Proof.** Let us denote by \(X\) the set \([0,1,2,\ldots,8]\). Using Theorem 5, one can verify that \(\Gamma = (X \setminus \{0\}, C)\) is a \(P^{(3)}(2,4)\)-design of order 8, where \(C\) is the following set:

\[
\begin{align*}
\{(1,2,3,4), (1,3,5,7), (1,4,5,8), (1,5,2,6), \\
(1,6,3,8), (1,7,4,6), (1,8,2,7), (2,3,6,7), \\
(2,4,6,8), (2,5,4,7), (2,6,1,5), (2,7,1,8), \\
(2,8,3,5), (3,4,1,2), (3,5,2,8), (3,6,4,5), \\
(3,7,4,8), (3,8,1,6), (4,5,3,6), (4,6,1,7), \\
(4,7,2,5), (4,8,3,7), (5,6,7,8), (5,7,1,3), \\
(5,8,1,4), (6,7,2,3), (6,8,2,4), (7,8,5,6)\}.
\end{align*}
\]
We also observe that in every block, each pair is repeated one time, that is, \( \Gamma = (X \setminus \{0\}, C) \) is edge-balanced. From \( \Gamma \), we construct the family \( B \) of hypergraph \( S^{(3)}(2,3) \) by appending the vertex 0 to the blocks \( C \in C \) in the following way:

\[
C = [(x,y), z, t] \mapsto B = [(x, y), z, t, 0]
\]

One can verify that \( \Sigma = (X, B) \) is an \( S^{(3)}(2,3) \)-design of order \( v = 9 \). \( \square \)

In the following lemmas, we prove that given an \( S^{(3)}(2,3) \)-design of order \( v = 9h \), there exists an \( S^{(3)}(2,3) \)-design of order \( v' \), where \( v' \in \{v+1, v+2, v+9\} \).

**Lemma 1.** If \( v = 9h \in \text{Spectrum}(S^{(3)}(2,3)) \), then \( v' = 9h + 1 \) belongs to \( \text{Spectrum}(S^{(3)}(2,3)) \).

**Proof.** **Construction from \( v = 9h \) to \( v = 9h + 1 \).**

Let \( \Sigma = (X, B) \) be an \( S^{(3)}(2,3) \)-design of order \( v = 9h, h \geq 1 \). Let \( * \not\in X, X' = X \cup \{*\} \).

Since \( v = 9h \in \text{Spectrum}(S^{(2)}(1,2)) \) by Corollary 1, let \( \Delta = (X, D) \) be an \( S^{(2)}(1,2) \)-design of order \( v = 9h \) defined over \( X \).

Denote by \( [(x), y, z, t] \) the element of \( D \) with vertices \( \{x, y, z, t\} \) and centre \( \{x\} \), we construct the family \( D' \) of blocks isomorphic to \( S^{(3)}(2,3) \) by adding \( * \) to the centre of \( [(x), y, z, t] \), i.e.,

\[
D' = \{\{(*, x), y, z, t\} : [(x), y, z, t] \in D\}
\]

We show that the system \( \Sigma' = (X', B \cup D') \), where \( X' = X \cup \{*\} \), is an \( S^{(3)}(2,3) \)-design of order \( v' = 9h + 1 \).

Given any triple \( \{x, y, z\} \subseteq X' \), one can verify that there exists at least one block of \( B \cup D' \) containing it. Indeed, if \( \{x, y, z\} \) is a triple of \( X \), then there exists a block of \( B \) containing it as a hyperedge. If \( \{x, y, *\} \) is a triple of \( X' \), then there exists a block of \( D \) containing the pair \( \{x, y\} \subseteq X \) and, therefore, there exists a block of \( D' \) containing the triple \( \{x, y, *\} \).

Finally, since \( \lambda = 1 \), we get

\[
|B \cup D'| = |B| + |D'| = \frac{(9h+1)(9h-1)}{18}
\]

which is the exact number of blocks in any \( S^{(3)}(2,3) \)-design of order \( v' = 9h + 1 \). \( \square \)

**Lemma 2.** If \( v = 9h + 1 \in \text{Spectrum}(S^{(3)}(2,3)) \), then \( v' = 9h + 2 \) belongs to \( \text{Spectrum}(S^{(3)}(2,3)) \).

**Proof.** **Construction from \( v = 9h + 1 \) to \( v = 9h + 2 \).** Since \( v = 9h + 1 \in \text{Spectrum}(S^{(2)}(1,3)) \) by Corollary 2, we can follow the same procedure as in Lemma 1. Finally, since \( \lambda = 1 \), we have that:

\[
|B'| = |B| + |D'| = \frac{(9h+1)(9h-1)}{18} + \frac{(9h+1)9h}{6} = \frac{(9h+1)9h}{6} - \frac{9h - 1}{3} + 1 = \frac{(9h+2)(9h+1)9h}{18}
\]

\( \square \)

**Lemma 3.** If \( v = 9h \in \text{Spectrum}(S^{(3)}(2,3)) \), then \( v' = 9h + 9 \) belongs to \( \text{Spectrum}(S^{(3)}(2,3)) \).

**Proof.** **Construction from \( v = 9h \) to \( v = 9h + 9 \).**

Let \( \Sigma_1 = (X_1, B_1) \) be an \( S^{(3)}(2,3) \)-design of order \( v_1 = 9h \) and \( \Sigma_2 = (X_2, B_2) \) be an \( S^{(3)}(2,3) \)-design of order \( v_2 = 9 \) such that \( X_1 \cap X_2 = \emptyset \).
Since \( v_1 = 9h \in \text{Spectrum}(S^{(2)}(1,3)) \), let \( \Delta_1 = (X, D_1) \) be an \( S^{(2)}(1,3) \)-design of order \( v = 9h \) defined on \( X_1 \), and define the following family of blocks isomorphic to \( S^{(3)}(2,3) \):

\[
C_2 = \{ [(a, x), y, z, t] : a \in X_2 \text{ and } [(x), y, z, t] \in D_1 \}.
\]

Similarly, since \( v_2 = 9 \in \text{Spectrum}(S^{(2)}(1,3)) \), if \( \Delta_2 = (X, D_2) \) is an \( S^{(2)}(1,3) \)-design of order \( v_2 = 9 \) defined on \( X_2 \), we can consider the following family of blocks isomorphic to \( S^{(3)}(2,3) \):

\[
C_1 = \{ [(x, a), b, c, d] : x \in X_1 \text{ and } [(a), b, c, d] \in D_2 \}.
\]

The system \( (X, B') = (X_1 \cup X_2, B_1 \cup B_2 \cup C_1 \cup C_2) \) is an \( S^{(3)}(2,3) \)-design of order \( v = 9h + 9 \).

Indeed, it is easy to see that for every triple contained in \( X_1 \cup X_2 \), there exists at least a block of \( B' \) containing it.

Since \( \Sigma_1 \) and \( \Sigma_2 \) are \( S^{(3)}(2,3) \)-designs with \( \lambda = 1 \), we have that:

\[
|B'| = |B_1| + |B_2| + |C_1| + |C_2| = \frac{(9h + 9)(9h + 8)(9h + 7)}{18}
\]

which is the exact number of blocks in any \( S^{(3)}(2,3) \)-design of order \( v = v_1 + v_2 = 9h + 9 \). □

We summarize all the previous results in the main theorem of this section.

**Theorem 7.** There exist \( S^{(3)}(2,3) \)-designs of order \( v \) if and only if

\[
v \equiv 0 \text{ or } v \equiv 1 \text{ or } v \equiv 2 \mod 9 \text{ and } v \geq 9
\]

4. The Spectrum of \( S^{(3)}(2,3) \)-Designs of Index 2

In this section, we investigate the spectrum of \( S^{(3)}(2,3) \)-designs with index \( \lambda = 2 \) (\( 2S^{(3)}(2,3) \)-designs for short) with the further condition that in the system, all the blocks are distinct; i.e., we find the cardinality of \( X \), where \( (X, B) \) is a \( 2S^{(3)}(2,3) \)-design and \( B \) is a set and not a multiset. We prove that \( v \) belongs to \( \text{Spectrum}(2S^{(3)}(2,3)) \) iff \( v \equiv 0, 1, 2 \mod 9 \) and \( v \geq 5 \).

In what follows, we will denote by \( \mathbb{Z}_n \) the additive cyclic group of order \( n \) whose elements are listed from 1 to \( n \), and by \( [(x), y_1, \ldots, y_n] \) the star graph with centre \( x \) and edges \( xy_i \), i.e., \( S^{(2)}(1, n) \). We recall some known results that we will use in the following.

**Lemma 4 ([15]).** For every \( v \equiv 2, 4 \mod 6 \), \( v > 4 \), there exists at least two disjoint Steiner quadruple systems (SQSs for short) \( \Sigma_1 = (X, B_1) \) and \( \Sigma_2 = (X, B_2) \) defined on the same set \( X \).

**Lemma 5.** There exists a \( P^{(3)}(2,4) \)-design of order \( v \) and index 2 such that all the blocks are pairwise distinct.

**Proof.** Let \( \Sigma_1 = (X, B_1) \) and \( \Sigma_2 = (X, B_2) \) denote two disjoint SQS(\( v \)) as in Lemma 4, and \( \Gamma_1 = (X, C_1) \) and \( \Gamma_2 = (X, C_2) \) are two disjoint \( P^{(3)}(2,4) \)-designs obtained from \( \Sigma_1 \) and \( \Sigma_2 \), respectively, as in Theorem 5. One can verify that the system \( \Gamma = (X, C_1 \cup C_2) \) is a \( P^{(3)}(2,4) \)-design of order \( v \) and index 2 such that all the blocks are pairwise distinct. □

Thus, we get the following.

**Theorem 8.** For every \( v \) such that \( v \equiv 2, 4 \mod 6 \) and \( v > 4 \), there exists at least one \( P^{(3)}(2,4) \)-design of order \( v \) and index 2 such that all the blocks are pairwise distinct.

In order to prove the result, we still need other preliminary lemmas.

**Lemma 6.** There exist \( 2S^{(2)}(1,3) \)-designs of order 4.
Proof. The number of blocks of $2S^{(2)}(1,3)$-designs of order 4 must be 4; hence, there is a
unique design whose blocks are the four stars which one can exhibit. □

Lemma 7. There exist $2S^{(2)}(1,3)$-designs of order 6.

Proof. The number of blocks of a $2S^{(2)}(1,3)$-design of order 6 is 10. Let us consider the set
of vertices $X$ as $X = \mathbb{Z}_9 \cup \{\ast\}$ with $\mathbb{Z}_9$ equipped with the structure of a cyclic additive group.
We can consider two blocks of the form $[(1), 2, 3, \ast]$ and $[(1), 3, 5, \ast]$ and their translations
by $1 \in \mathbb{Z}_9$, leaving $\ast$ fixed. □

Lemma 8. There exist $2S^{(2)}(1,3)$-designs of order 9.

Proof. The number of blocks of a $2S^{(2)}(1,3)$-design of order 9 is equal to 24, as seen in the
following set:

\[
\begin{align*}
\{ (0,4), 1,8], & \quad [(1,5), 0, 2], & \quad [(2,6), 3, 1], & \quad [(3,7), 4, 2], \\
[4,7], 5, 3, & \quad [(5,8), 6, 4], & \quad [(6,1), 7, 5], & \quad [(7,1), 8, 6], \\
[8,4], 3, 7, & \quad [(0,3), 2, 7], & \quad [(1,4), 3, 8], & \quad [(2,5), 5, 0], \\
[3,6], 5, 1, & \quad [(4,7), 6, 8], & \quad [(5,8), 7, 3], & \quad [(6,0), 8, 4], \\
[7,1], 0, 5, & \quad [(8,2), 1, 6], & \quad [(0,3), 4, 5], & \quad [(1,4), 5, 6], \\
[2,5], 6, 7, & \quad [(3,6), 7, 8], & \quad [(2,7), 4, 8], & \quad [(0,6), 5, 8].
\end{align*}
\]

□

Lemma 9. Given a $2S^{(2)}(1,3)$-design of order $v = 3h$, there exists a $2S^{(2)}(1,3)$-design of order
$v' = v + 1 = 3h + 1$.

Proof. Let $\Sigma = (\mathbb{Z}_{3h}, B)$ be a $2S^{(2)}(1,3)$-design over $\mathbb{Z}_{3h}$ and denote by $C$ the set of blocks
of the form $[[\ast], j, j + 1, j + 2]$ and $[(\ast), j + 1, j + 2, j + 3]$ with $j = 1 + 3a \in \mathbb{Z}_{3h}$ for $0 \leq a \leq h - 1$. It is straightforward that $(\{\ast\} \cup \mathbb{Z}_{3h}, B \cup C)$ is a $2S^{(2)}(1,3)$-design of order $3h + 1$. □

Lemma 10. Given a $2S^{(2)}(1,3)$-design of order $v = 3h$, there exists a $2S^{(2)}(1,3)$-design of order
$v' = v + 6 = 3h + 6$.

Proof. Let us denote by $\Sigma_1 = (X_1, B_1)$ and $\Sigma_2 = (X_2, B_2)$ two block designs of type
$2S^{(2)}(1,3)$ of order $3h$ and 6, respectively. By Lemma 9, given $a \in X_1$, one can construct a
$2S^{(2)}(1,3)$-design over $\{a\} \cup X_2$ and let $C_a$ denote the set of blocks over $a \cup X_2$. Analogously,
for each $\beta \in X_2$, one can construct a $2S^{(2)}(1,3)$-design over $X_1 \cup \{\beta\}$ and denote by $C_{\beta}$ the
set of blocks over $X_1 \cup \{\beta\}$. One can check that $B_1 \cup B_2 \cup_{\alpha \in X_1} C_{\alpha} \cup_{\beta \in X_2} C_{\beta}$ are the blocks
of a $2S^{(2)}(1,3)$-design over $X_1 \cup X_2$. □

Corollary 3. There exist $2S^{(2)}(1,3)$-designs of order $3h$ if $h \geq 2$.

Proof. By Lemmas 7, 8 and 10, the conclusion is immediate. □

Corollary 4. There exist $2S^{(2)}(1,3)$-designs of order $v = 3h + 1$.

In order to prove the main result of this section, we also need the following lemmas.

Lemma 11. There exists a $2S^{(3)}(2,3)$-design of order 9.

Proof. Take two disjoint families $(X, C_1)$ and $(X, C_2)$ of $P^{(3)}(2,4)$-designs over a set $X$ of
cardinality 8, as in Theorem 6, that are edge-balanced and consider $(X, C_1 \cup C_2)$ that is
There exist 3S

Theorem 9.  

\begin{proof}

The proof proceeds by induction, with a base case which holds true by Lemma 11. Let \( \Sigma_1 = (X_1, B_1) \) and \( \Sigma_2 = (X_2, B_2) \) be two 2S\((3,2,3)\)-designs of order \( v_1 = 9h \) and \( v_2 = 9 \). Let us denote by \( (X_1, C_1) \) a 2S\((2,1,3)\)-design over \( X_1 \) and for each block \( \gamma = [(x), y, z, \ell] \in C_1 \) and \( a \in X_2 \), consider \( \gamma_a = [(a, x), y, z, \ell] \) and let \( F_1 \) denote the set of blocks \( \gamma_a \), i.e., \( F_1 = \{ \gamma_a : \gamma \in C_1, a \in X_2 \} \). Analogously, consider a 2S\((2,1,3)\)-design \( (X_2, C_2) \) and \( F_2 = \{ \gamma_b : \gamma \in C_2, b \in X_1 \} \). The reader can easily check that \( (X_1 \cup X_2, B_1 \cup B_2 \cup F_1 \cup F_2) \) is a 2S\((3,2,3)\)-design of order \( v' = v + 9 = 9h + 9 \). 

\end{proof}

Lemma 12.  

There exist 2S\((3,2,3)\)-designs of order \( v \) if \( v \equiv 0 \mod 9 \) and \( v \geq 9 \).

\begin{proof}

By Lemma 12, there exist 2S\((3,2,3)\)-designs of order \( 9h \) which we will denote by \( \Sigma = (X, B) \); furthermore, by Lemma 12, there exists an \((X, C)\) with blocks isomorphic to \( S\((2,1,3)\) \) and index \( 2 \). For each \( \gamma \in C \), consider \( \gamma_* = [(\ast, x), y, z, \ell] \) and let \( F = \{ \gamma_* : \gamma \in C \} \); it is easy to verify that \( (X \cup \{ \ast \}, B \cup F) \) is a 2S\((3,2,3)\)-design of order \( v = 9h + 1 \).

\end{proof}

Lemma 13.  

There exist 2S\((3,2,3)\)-designs of order \( v = 9h + 1 \).

\begin{proof}

By the preceding lemma, there exists a 2S\((3,2,3)\)-design of order \( 9h + 1 \), and by Corollary 3, there exists a 2S\((2,1,3)\)-design over \( X \) which we denote by \( (X, C) \). Define \( F = \{ \gamma_* : \gamma \in C \} \); it is easy to check that \( (X \cup \{ \ast \}, B \cup F) \) is a 2S\((3,2,3)\)-design of order \( 9h + 2 \).

Collecting together all the previous results, the following results.

\begin{theorem}

There exist 2S\((3,2,3)\)-designs of order \( v \) if and only if \( v \equiv 0 \) or \( v \equiv 1 \) or \( v \equiv 2 \mod 9 \) and \( v \geq 9 \).

\end{theorem}

5. The Spectrum of \( S\((3,2,3)\)\)-Designs of Index 3

In this section, we determine the spectrum of \( S\((3,2,3)\)\)-designs with index 3 (for short \( 3S\((3,2,3)\)\)-designs) with the further condition that in the system, all the blocks are distinct. We prove that \( v \) belongs to \( \text{Spectrum}(3S\((3,2,3)\)) \) if and only if \( v \geq 5 \). We need some preliminary lemmas.

Lemma 15.  

There exist 3S\((2,1,3)\)-designs of order \( v \) iff \( v \geq 4 \).

\begin{proof}

It is immediately clear that there exists a (unique) 3S\((2,1,3)\)-design of order 4 and there are not any for \( v \leq 3 \). We now proceed by induction on the number of vertices \( v \); given \( v \), consider the table \( A \) with three rows and \( v \) columns whose elements are \( a_{ij} = i + j - 1 \) for \( i \in \{1, 2, 3\}, j \in \mathbb{Z}_v = \{1, \ldots, v\} \) and the sum \( i + j - 1 \) is performed modulo \( v \). It is evident that the columns of \( A \) are all distinct and each element of \( \mathbb{Z}_v \) appears three times. Denote by \( a_{*j} \) the \( j \)-th column of \( a \) and consider the graph \( \{\ast\}a_{*j} \), which is isomorphic to \( S\((2,1,3)\) \). Denote by \( F \) the set of these graphs.

By induction, there exists a 3S\((2,1,3)\)-design \( (X, B) \) of order \( v \); it is easy to check that \( (X \cup \{\ast\}, B \cup F) \) is a 3S\((2,1,3)\)-design of order \( v + 1 \).

Using the previous results, we get the following.

\begin{theorem}

There exist 3S\((3,2,3)\)-designs of order \( v \) iff \( v \geq 5 \).

\end{theorem}
Proof. We can proceed by induction with the base case \( v = 5 \), where \((X, \mathcal{D})\) is an \( S^{(3)}(2, 3) \)-design of order \( v = 5 \) with \( X = \mathbb{Z}_6 \), and \( \mathcal{D} \) is the following set:

\[
\{ [(1, 2), 3, 4, 5], [(2, 4), 1, 3, 5], [(2, 3), 4, 5, 1], [(3, 5), 2, 4, 1], [(3, 4), 5, 1, 2],
(4, 1), 3, 5, 2], [(4, 5), 1, 2, 3], [(5, 2), 4, 1, 3], [(5, 1), 2, 3, 4], [(1, 3), 5, 2, 4] \}.
\]

By the induction hypothesis, there exists a \( 3S^{(3)}(2, 3) \)-design \((X, B)\) of order \( v \), and by Lemma 15, there exists a \( 3S^{(2)}(1, 3) \)-design \((X, C)\). For each \( \gamma = [(x), y, z, t] \in C \), consider \( \gamma_+ = [(x, y, z, t)] \) and \( \mathcal{F} = \{ \gamma_+ : \gamma \in C \} \). It is easy to check that \((X \cup \{\star\}, B \cup \mathcal{F})\) is a \( 3S^{(2)}(2, 3) \)-design. \( \square \)

6. The Spectrum of \( S^{(3)}(2, 3) \)-Designs of Index \( \lambda > 3 \)

For the sake of completeness, in this section, we deal with the problem of existence of \( \lambda S^{(3)}(2, 3) \)-designs for any index \( \lambda \), where the blocks are repeated with multiplicity \( \lambda \).

To be more precise, we prove that if \( \lambda \equiv 0, 1 \mod 3 \), then there exist \( \lambda S^{(3)}(2, 3) \)-designs of order \( v \) if and only if \( v \equiv 0, 1, 2 \mod 9 \), while if \( \lambda \equiv 0 \mod 3 \), then there exist \( \lambda S^{(3)}(2, 3) \)-designs for every \( v \geq 5 \).

It is an immediate consequence of Theorem 4 that if \( \lambda \equiv 1, 2 \mod 3 \) and a \( \lambda S^{(3)}(2, 3) \)-design exists, then \( v \equiv 0, 1, 2 \mod 9 \).

In Section 3, we have already proven that if \( v \equiv 0, 1, 2 \mod 9 \), then there exists a \( S^{(3)}(2, 3) \)-design \((X, B)\) of order \( v \) and index \( \lambda = 1 \). If repetitions of blocks are allowed, to construct a \( \lambda S^{(3)}(2, 3) \)-design of order \( v \), it is sufficient to consider the design \((X, \lambda B)\), where \( \lambda B \) is the uniform multiset with underlying set \( B \) and multiplicity \( \lambda \).

Similar arguments apply to the case \( \lambda \equiv 0 \mod 3 \) and \( v \geq 5 \): in Section 5, we proved that there exists a \( 3S^{(3)}(2, 3) \)-design \((X, B)\) of order \( v \geq 5 \); in order to construct a \( \lambda S^{(3)}(2, 3) \)-design, it is sufficient to consider the design \((X, \frac{\lambda}{3} B)\), where \( \frac{\lambda}{3} B \) is the uniform multiset with underlying set \( \frac{\lambda}{3} B \) and multiplicity \( \frac{\lambda}{3} \).

To conclude our paper, we suggest the following open problem:

Determining the spectrum of \( S^{(3)}(2, n) \) for any positive integer \( n \).

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