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Investigating the Properties and Dynamic Applications of Δ_h Legendre–Appell Polynomials

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Abstract: This research aims to introduce and examine a new type of polynomial called the Δ_h Legendre–Appell polynomials. We use the monomiality principle and operational rules to define the Δ_h Legendre–Appell polynomials and explore their properties. We derive the generating function and recurrence relations for these polynomials and their explicit formulas, recurrence relations, and summation formulas. We also verify the monomiality principle for these polynomials and express them in determinant form. Additionally, we establish similar results for the Δ_h Legendre–Bernoulli, Euler, and Genocchi polynomials.

Keywords: Δ_h sequences; monomiality principle; Legendre–Appell polynomials; explicit forms; determinant form

MSC: 33E20; 33B10; 33E30; 11T23

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1. Introduction and Preliminaries

Complex system behavior has been modeled and described by special polynomials in a variety of domains, including quantum mechanics and statistical mechanics. These unique polynomials have also been used to describe and analyze complex systems in a number of other domains, such as quantum mechanics and statistics. Polynomial sequences are indispensable in several branches of mathematics, such as algebraic combinatorics, entropy, and combinatorics. The Legendre, Chebyshev, Laguerre, and Jacobi polynomials are a few examples of polynomial sequences that are solutions to particular ordinary differential equations in approximation theory and physics. Legendre polynomials are a class of orthogonal polynomials with important applications in physics and mathematics. The French mathematician Edmond Legendre, who first introduced them in the 19th century, is the reason behind their name. The Legendre differential equation, a second-order linear differential equation, has solutions that lead to the Legendre polynomials. They are often represented as $\mathbb{S}_n(u)$ [1], where n is a non-negative integer that denotes the degree of the polynomial. They are defined on the interval $[0, +\infty)$. There are numerous noteworthy characteristics of Legendre polynomials: On the interval $[0, +\infty)$, the Legendre polynomials form an orthogonal set with regard to the weight function e^{-u}. This indicates that, with the exception of situations in which the polynomials have the same degree, the integral of the sum of two distinct Laguerre polynomials with the weight function equals zero. Moreover, the Legendre polynomials satisfy a recurrence relation, enabling the computation of higher-degree polynomials from lower-degree ones. This characteristic helps with efficient polynomial generation and numerical computations. Furthermore, the generating function of these polynomials permits the expansion of some functions into a sequence of Legendre polynomials. This characteristic helps in differential equation solving and yields closed-form solutions. Application areas for the Legendre polynomials include the solutions of the Schrodinger equation for the hydrogen atom and other quantum systems with spherical symmetry in mathematics, physics, and engineering. Furthermore, issues involving diffusion equations, wave propagation, and heat conduction give rise to these polynomials.

Mathematical physics two-variable special polynomials have been the subject of much recent research. A class of polynomials known as two-variable special polynomials has certain attributes, for example, [2,3]. They have numerous uses in mathematics and other fields and are frequently researched in the area of algebraic geometry. Bivariate Chebyshev, Hermite, Laguerre, and Laguerre polynomials are a few notable examples of two-variable special polynomials. They are widely used in signal processing, numerical analysis, and approximation theory. Bivariate Chebyshev polynomials are symmetric polynomials with applications in least squares fitting and interpolation. Hermite polynomials of two variables have applications in quantum mechanics, statistical mechanics, and waveguide theory. Bivariate Hermite polynomials are often used in the study of harmonic oscillators in two dimensions. Bivariate Legendre polynomials are a two-variable extension of the Legendre polynomials. They satisfy a bivariate analogue of the Legendre differential equation and have applications in quantum mechanics, potential theory, and random matrix theory. Bivariate Legendre polynomials are particularly useful in studying the behavior of systems with two degrees of freedom. These polynomials satisfy a certain orthogonality condition with respect to a weight function and are thus extensively studied in mathematical physics, probability theory, and approximation theory. The significance of these two-variable special polynomials lies in their usefulness in solving problems in various mathematical and scientific domains. They provide a rich framework for expressing and analyzing multivariate functions and have specific properties that make them suitable for specific applications. It is well known that huge classes of partial differential equations, which are frequently encountered in physical issues, can be solved analytically by innovative methods made possible by the special polynomials of two variables. The two-variable Legendre polynomials $\mathbb{S}_\omega(u,v)$ [4] are of enormous mathematical significance and have applications in physics, which makes their introduction intriguing.

The two-variable Legendre polynomials (2VLeP) $\mathbb{S}_\omega(u,v)$ are specified by means of the following generating equation:

$$e^{v\xi} J_0(2\xi\sqrt{-u}) = \sum_{\omega=0}^{\infty} \mathbb{S}_\omega(u,v)\frac{\xi^\omega}{\omega!}, \tag{1}$$

where $J_0(u\xi)$ is the 0th order ordinary Bessel function of first kind [5] defined by

$$J_\omega(2\sqrt{u}) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \,(\sqrt{u})^{\omega+2\nu}}{\nu!\,(\omega+\nu)!}. \tag{2}$$

also note that

$$\exp(-\gamma D_u^{-1}) = J_0(2\sqrt{\gamma u}), \qquad D_u^{-\omega}\{1\} := \frac{u^\omega}{\omega!} \tag{3}$$

is the inverse derivative operator.

Or, alternatively, by

$$e^{v\xi} C_0(-u\xi^2) = \sum_{\omega=0}^{\infty} \mathbb{S}_\omega(u,v)\frac{\xi^\omega}{\omega!}, \tag{4}$$

where $C_0(u\xi)$ is the 0th order Tricommi function of the first kind [5] with

$$C_0(-u\xi^2) = e^{D_u^{-1}\xi^2}. \tag{5}$$

Thus, in view of Equation (3) or (5), the generating expression for Legendre polynomials can be cast as:

$$e^{v\xi}\, e^{D_u^{-1}\xi^2} = \sum_{\omega=0}^{\infty} \mathbb{S}_\omega(u,v)\frac{\xi^\omega}{\omega!}. \tag{6}$$

Very recently, a large interest has been shown by mathematicians to introduce Δ_h forms of special polynomials. Some extensions of the special polynomials were studied in [1,5–10]. After that, by using the classical finite difference operator Δ_h, a new form of the special polynomials, known as the Δ_h special polynomials of different polynomials, were introduced in [11,12]. These Δ_h special polynomials have been studied because of their remarkable applications in different branches of mathematics, physics, and statistics.

These Δ_h Appell polynomials are represented as:

$$\mathbb{A}_\omega^{[h]}(u) := \mathbb{A}_\omega(u), \quad \omega \in \mathbb{N}_0 \tag{7}$$

and defined by

$${}_u\Delta_h\left\{\mathbb{A}_\omega^{[h]}(u)\right\} = \omega h\, \mathbb{A}_{\omega-1}(u), \quad \omega \in \mathbb{N}, \tag{8}$$

where Δ_h is the finite difference operator:

$${}_u\Delta_h \mathbb{H}^{[h]}(u) = \mathbb{H}(u+h) - \mathbb{H}(u). \tag{9}$$

The Δ_h Appell polynomials $\mathbb{A}_\omega(u)$ are specified by the following generating function [12]:

$$\gamma(\xi)(1+h\xi)^{\frac{u}{h}} = \sum_{\omega=0}^{\infty} \mathbb{A}_\omega^{[h]}(u)\frac{\xi^\omega}{\omega!}, \tag{10}$$

where

$$\gamma(\xi) = \sum_{\omega=0}^{\infty} \gamma_{\omega,h}\frac{\xi^\omega}{\omega!}, \qquad \gamma_{0,h} \neq 0. \tag{11}$$

Therefore, motivated by the results in [4,11–13], here we introduced the two-variable Δ_h Legendre–Appell polynomials:

$$\gamma(\xi)(1+h\xi)^{\frac{v}{h}}(1+h\xi^2)^{\frac{D_u^{-1}}{h}} = \sum_{\omega=0}^{\infty} {}_{\mathbb{S}}\mathbb{A}_\omega^{[h]}(u,v)\frac{\xi^\omega}{\omega!} \tag{12}$$

through the generating function concept.

This article is designed as follows: Section 2 discusses how the Legendre–Appell polynomials are generated and explores recurrence relations that govern their behavior. Section 3 presents formulas for summing or evaluating these Legendre–Appell polynomials over certain ranges or with specific constraints. These formulas can be useful for calculating the values of the polynomials efficiently. Section 4 discusses the monomiality principle, which relates to how Legendre–Appell polynomials behave under certain operations. The determinant form for these polynomials is also established. In Section 5, Symmetric identities for these polynomials are derived. The conclusion section summarizes the findings of the article and discusses implications, applications, and potential future research directions related to Legendre–Appell polynomials. Each of these sections likely delves deeper into the mathematical properties and characteristics of Legendre–Appell polynomials, providing insights into their behavior and utility in various mathematical contexts.

2. Two-Variable Δ_h Legendre–Appell Polynomials

The significance of this section lies in its exploration of a novel class of two-variable Δ_h Legendre–Appell polynomials and its establishment of essential properties associated with them. The research expands the existing knowledge base and opens doors to new avenues of inquiry within polynomial theory and its applications.

The construction of the generating function for these Δ_h Legendre–Appell polynomials, denoted as ${}_{\mathbb{S}}\mathbb{A}_\omega^{[h]}(u,v)$, marks a crucial step forward in understanding the behavior and properties of these polynomials. Generating functions serve as powerful tools in combinatorics, analysis, and mathematical physics, providing insights into the structure and properties of sequences and functions. By proving the existence and constructing the generating function for Δ_h Legendre–Appell polynomials, this section lays the foundation for further exploration of their properties, such as orthogonality, recurrence relations, and special function identities.

Moreover, by establishing a connection between the Δ_h Legendre–Appell polynomials and their generating function, this research contributes to the broader mathematical community's understanding of polynomial families and their applications. The traits listed in this section provide valuable insights into the unique characteristics and behaviors of these polynomials, paving the way for their utilization in various mathematical and scientific domains. Overall, this section represents a significant advancement in polynomial theory, offering fresh perspectives and potential applications that warrant further investigation and exploration. First, we prove the following conclusion to construct the generating function for these Δ_h Legendre–Appell polynomials ${}_{\mathbb{S}}\mathbb{A}_{\omega}^{[h]}(u,v)$ by proving the following result:

Theorem 1. *For the two-variable Δ_h Legendre–Appell polynomials ${}_{\mathbb{S}}\mathbb{A}_{\omega}^{[h]}(u,v)$, the succeeding generating relation holds true:*

$$\gamma(\xi)(1+h\xi)^{\frac{v}{h}}(1+h\xi^2)^{\frac{D_u^{-1}}{h}} = \sum_{\omega=0}^{\infty} {}_{\mathbb{S}}\mathbb{A}_{\omega}^{[h]}(u,v)\frac{\xi^{\omega}}{\omega!}. \tag{13}$$

Proof. By expanding $\gamma(\xi)(1+h\xi)^{\frac{v}{h}}(1+h\xi^2)^{\frac{D_u^{-1}}{h}}$ at $u=v=0$ for finite differences by a Newton series and the order of the product of the developments of the function $\gamma(t)(1+h\xi)^{\frac{v}{h}}(1+h\xi^2)^{\frac{D_u^{-1}}{h}}$ with respect to the powers of ξ, we observe the polynomials ${}_{\mathbb{S}}\mathbb{A}_{\omega}^{[h]}(u,v)$ expressed in Equation (13) as coefficients of $\frac{\xi^{\omega}}{\omega!}$ as the generating function of two-variable Δ_h Legendre–Appell polynomials ${}_{\mathbb{S}}\mathbb{A}_{n}^{[h]}(u,v)$. □

Theorem 2. *For the two-variable Δ_h Legendre–Appell polynomials ${}_{\mathbb{S}}\mathbb{A}_{\omega}^{[h]}(u,v)$, the succeeding relations hold true:*

$$\frac{{}_{v}\Delta_h}{h}{}_{\mathbb{S}}\mathbb{A}_{\omega}^{[h]}(u,v) = \omega\ {}_{\mathbb{S}}\mathbb{A}_{\omega-1}^{[h]}(u,v) \tag{14}$$

$$\frac{{}_{u}\Delta_h}{h}{}_{\mathbb{S}}\mathbb{A}_{\omega}^{[h]}(u,v) = \omega(\omega-1)\ {}_{\mathbb{S}}\mathbb{A}_{\omega-2}^{[h]}(u,v),\ \ D_u^{-1}\to u. \tag{15}$$

Proof. By differentiating (13) with respect to v by taking into consideration of expression (5), we have

$$\begin{aligned} {}_{v}\Delta_h\left\{\gamma(t)(1+h\xi)^{\frac{v}{h}}(1+h\xi^2)^{\frac{D_u^{-1}}{h}}\right\} &= \gamma(\xi)(1+h\xi)^{\frac{v+h}{h}}(1+h\xi^2)^{\frac{D_u^{-1}}{h}} - \gamma(\xi)(1+h\xi)^{\frac{v}{h}}(1+h\xi^2)^{\frac{D_u^{-1}}{h}} \\ &= (1+h\xi-1)\gamma(\xi)(1+h\xi)^{\frac{v}{h}}(1+h\xi^2)^{\frac{D_u^{-1}}{h}} \\ &= h\xi\ \gamma(\xi)(1+h\xi)^{\frac{v}{h}}(1+h\xi^2)^{\frac{D_u^{-1}}{h}}. \end{aligned} \tag{16}$$

By substituting the righthand side of expression (13) in (16), we find

$${}_{v}\Delta_h\sum_{\omega=0}^{\infty}{}_{\mathbb{S}}\mathbb{A}_{\omega}^{[h]}(u,v)\frac{\xi^{\omega}}{\omega!} = h\sum_{\omega=0}^{\infty}{}_{\mathbb{S}}\mathbb{A}_{\omega}^{[h]}(u,v)\frac{\xi^{\omega+1}}{\omega!}. \tag{17}$$

By replacing $\omega\to\omega-1$ in the righthand side of previous expression (16) and comparing the coefficients of the same exponents of t in the resultant expression, assertion (14) is deduced.

Further, on similar grounds, expression (15) is established. □

Next, we deduce the explicit form satisfied by these two-variable Δ_h Legendre–Appell polynomials ${}_{\mathbb{S}}\mathbb{A}_\omega^{[h]}(u,v)$ by demonstrating the result:

Theorem 3. *For the two-variable Δ_h Legendre–Appell polynomials ${}_{\mathbb{S}}\mathbb{A}_\omega^{[h]}(u,v)$, the explicit relation holds true:*

$$ {}_{\mathbb{S}}\mathbb{A}_\omega^{[h]}(u,v) = \sum_{d=0}^{\frac{v}{h}} \binom{\omega}{d} \binom{\frac{v}{h}}{d} h^d \, \mathbb{A}_{\omega-d}^{[h]}(u). \tag{18} $$

Proof. Expanding generating relation (13) in the given manner:

$$ \gamma(\xi)(1+h\xi)^{\frac{v}{h}}(1+h\xi^2)^{\frac{D_u^{-1}}{h}} = \sum_{d=0}^{\frac{v}{h}} \binom{\frac{v}{h}}{d} \frac{(h\xi)^d}{d!} \sum_{\omega=0}^{\infty} {}_{\mathbb{S}}\mathbb{A}_\omega^{[h]}(u,0) \frac{\xi^\omega}{\omega!} \tag{19} $$

which can further be written as

$$ \sum_{\omega=0}^{\infty} {}_{\mathbb{S}}\mathbb{A}_\omega^{[h]}(u,v) \frac{\xi^\omega}{\omega!} = \sum_{\omega=0}^{\infty} \sum_{d=0}^{[\frac{v}{h}]} \binom{\frac{v}{h}}{d} h^d \, \mathbb{A}_\omega^{[h]}(u) \frac{\xi^{\omega+d}}{\omega! \, d!}. \tag{20} $$

By replacing $\omega \to \omega - d$ in the righthand side of the previous expression, it follows that

$$ \sum_{\omega=0}^{\infty} {}_{\mathbb{S}}\mathbb{A}_\omega^{[h]}(u,v) \frac{\xi^\omega}{\omega!} = \sum_{\omega=0}^{\infty} \sum_{d=0}^{[\frac{v}{h}]} \binom{\frac{v}{h}}{d} h^d \, \mathbb{A}_\omega^{[h]}(u) \frac{\xi^{\omega}}{(\omega-d)! \, d!}. \tag{21} $$

On multiplying and dividing by $\omega!$ on the righthand side of previous expression (21) and comparing the coefficients of the same exponents of ξ on both sides, assertion (18) is deduced. $\square$

Theorem 4. *Further, for the two-variable Δ_h Legendre–Appell polynomials ${}_{\mathbb{S}}\mathbb{A}_\omega^{[h]}(u,v)$, the explicit relation holds true:*

$$ {}_{\mathbb{S}}\mathbb{A}_\omega^{[h]}(u,v) = \sum_{\nu=0}^{\omega} \binom{\omega}{\nu} \gamma_{\nu,h} \, \mathbb{S}_{\omega-\nu}^{[h]}(u,v). \tag{22} $$

Proof. Expanding generating relation (13) in view of expressions (8) and (13) with $\gamma(\xi) = 1$ in the given manner:

$$ \gamma(\xi)(1+h\xi)^{\frac{v}{h}}(1+h\xi^2)^{\frac{D_u^{-1}}{h}} = \sum_{\nu=0}^{\infty} \gamma_{\nu,h} \frac{\xi^\nu}{\nu!} \sum_{\omega=0}^{\infty} \mathbb{S}_\omega^{[h]}(u,v) \frac{\xi^\omega}{\omega!}, \tag{23} $$

which can further be written as

$$ \sum_{\omega=0}^{\infty} {}_{\mathbb{S}}\mathbb{A}_\omega^{[h]}(u,v) \frac{\xi^\omega}{\omega!} = \sum_{\omega=0}^{\infty} \sum_{\nu=0}^{\infty} \gamma_{\nu,h} \, \mathbb{S}_\omega^{[h]}(u,v) \frac{\xi^{\omega+\nu}}{\omega! \nu!}. \tag{24} $$

By replacing $\omega \to \omega - \nu$ in the righthand side of the previous expression, it follows that

$$ \sum_{\omega=0}^{\infty} {}_{\mathbb{S}}\mathbb{A}_\omega^{[h]}(u,v) \frac{\xi^\omega}{\omega!} = \sum_{\omega=0}^{\infty} \sum_{\nu=0}^{\omega} \gamma_{\nu,h} \, \mathbb{S}_{\omega-\nu}^{[h]}(u,v) \frac{\xi^{\omega}}{(\omega-\nu)! \, \nu!}. \tag{25} $$

On multiplying and dividing by $\omega!$ on the righthand side of previous expression (25) and comparing the coefficients of the same exponents of ξ on both sides, assertion (22) is deduced. $\square$

3. Summation Formulae

This section establishes the summation formulae, or sigma notation, essential in mathematical analysis. These formulae provide systematic methods for computing sums involving special polynomials, facilitating the evaluation of complex expressions encountered in various mathematical contexts. By leveraging these formulae, mathematicians can identify patterns and uncover hidden symmetries within polynomial structures, enhancing understanding and fostering innovative applications in combinatorics, probability theory, and mathematical physics. Additionally, the study of summation formulae aids in developing efficient computational techniques, enabling researchers to address challenging problems precisely. These expressions concisely represent the sum of a sequence of terms, providing a convenient way to compute the total of a series of numbers or expressions. Thus, we demonstrate the summation formulae by proving the following results:

Theorem 5. *For $\omega \geq 0$, we have*

$$ {}_{\mathbb{S}}\mathbb{A}_{\omega}^{[h]}(u,v+1)=\sum_{\nu=0}^{\omega}\binom{\omega}{\nu}\left(-\frac{1}{h}\right)_{\nu}(-h)^{\nu}{}_{\mathbb{S}}\mathbb{A}_{\omega-\nu}^{[h]}(u,v). \tag{26} $$

Proof. By (13), we have

$$ \begin{aligned} \sum_{\omega=0}^{\infty}{}_{\mathbb{S}}\mathbb{A}_{\omega}^{[h]}(u,v+1)\frac{\xi^{\omega}}{\omega!}-\sum_{\omega=0}^{\infty}{}_{\mathbb{S}}\mathbb{A}_{\omega}^{[h]}(u,v)\frac{\xi^{\omega}}{\omega!}&=\gamma(\xi)(1+h\xi)^{\frac{v}{h}}(1+h\xi^{2})^{\frac{D_u^{-1}}{h}}\left((1+h\xi)^{\frac{1}{h}}-1\right)\\ &=\sum_{\omega=0}^{\infty}{}_{\mathbb{S}}\mathbb{A}_{\omega}^{[h]}(u,v)\frac{\xi^{\omega}}{\omega!}\left(\sum_{\nu=0}^{\infty}\left(-\frac{1}{h}\right)_{\nu}(-h)^{\nu}\frac{\xi^{\nu}}{\nu!}-1\right)\\ =\sum_{\omega=0}^{\infty}\left(\sum_{\nu=0}^{\omega}\binom{\omega}{\nu}\left(-\frac{1}{h}\right)_{\nu}(-h)^{\nu}{}_{\mathbb{S}}\mathbb{A}_{\omega-\nu}^{[h]}(u,v)\right)\frac{\xi^{\omega}}{\omega!}&-\sum_{\omega=0}^{\infty}{}_{\mathbb{S}}\mathbb{A}_{\omega}^{[h]}(v,u)\frac{\xi^{\omega}}{\omega!}. \end{aligned} \tag{27} $$

Comparing the coefficients of ξ, we obtain (26). □

Theorem 6. *For $\omega \geq 0$, we have*

$$ {}_{\mathbb{S}}\mathbb{A}_{\omega}^{[h]}(u,v)=\sum_{\nu=0}^{\omega}\sum_{j=0}^{\left[\frac{\omega-\nu}{2}\right]}\left(-\frac{v}{h}\right)_{\omega-2j-\nu}(-h)^{\omega-j-\nu}\left(-\frac{u}{h}\right)_{j}(-1)^{j}A_{\nu,h}\frac{\omega!}{(\omega-2j-\nu)!(j!)^{2}\nu!}. \tag{28} $$

Proof. Using (13), we have

$$ \begin{aligned} \sum_{\omega=0}^{\infty}{}_{\mathbb{S}}\mathbb{A}_{\omega}^{[h]}(u,v)\frac{\xi^{\omega}}{\omega!}&=\gamma(\xi)(1+h\xi)^{\frac{v}{h}}(1+h\xi^{2})^{\frac{D_u^{-1}}{h}}\\ &=\gamma(\xi)\sum_{\omega=0}^{\infty}\left(-\frac{v}{h}\right)_{\omega}(-h)^{\omega}\frac{\xi^{\omega}}{\omega!}\sum_{j=0}^{\infty}\left(-\frac{u}{h}\right)_{j}(-1)^{j}(-h)^{j}\frac{\xi^{2j}}{j!j!}\\ &=\sum_{\nu=0}^{\infty}A_{\nu,h}\frac{\xi^{\nu}}{\nu!}\sum_{\omega=0}^{\infty}\sum_{j=0}^{\left[\frac{\omega}{2}\right]}\left(-\frac{v}{h}\right)_{\omega-2j}(-h)^{\omega-j}\left(-\frac{u}{h}\right)_{j}(-1)^{j}\frac{\xi^{\omega}}{(\omega-2j)!(j!)^{2}}\\ &=\sum_{\omega=0}^{\infty}\sum_{\nu=0}^{\omega}\sum_{j=0}^{\left[\frac{\omega-\nu}{2}\right]}\left(-\frac{v}{h}\right)_{\omega-2j-\nu}(-h)^{\omega-j-\nu}\left(-\frac{u}{h}\right)_{j}(-1)^{j}A_{\nu,h}\frac{\xi^{\omega}}{(\omega-2j-\nu)!(j!)^{2}\nu!}. \end{aligned} \tag{29} $$

Equating the coefficients of ξ, we obtain (28). □

Now, we investigate the connection between the Stirling numbers of the first kind and two-variable Δ_h Legendre polynomials.

$$ \frac{[\log(1+\xi)]^{\nu}}{\nu!}=\sum_{i=\nu}^{\infty}S_1(i,\nu)\frac{\xi^{i}}{i!},\quad |\,\xi\,|<1. \tag{30} $$

From the above definition, we have

$$ (v)_i=\sum_{\nu=0}^{i}(-1)^{i-\nu}S_1(i,\nu)v^{\nu}. \tag{31} $$

Theorem 7. *For $\omega \geq 0$, we have*

$$\mathbb{s}\mathbb{A}_{\omega}^{[h]}(u,v) = \sum_{\nu=0}^{\omega} \binom{\omega}{\nu} \mathbb{s}\mathbb{A}_{\omega-\nu}^{[h]}(u,0) \sum_{j=0}^{\nu} v^{j} S_1(\nu,j) h^{\nu-j}. \tag{32}$$

Proof. From (13), we have

$$\begin{aligned}
\sum_{\omega=0}^{\infty} \mathbb{s}\mathbb{A}_{\omega}^{[h]}(u,v)\frac{\xi^{\omega}}{\omega!} &= e^{\frac{v}{h}\log(1+h\xi)}\gamma(\xi)(1+h\xi^2)^{\frac{D_u^{-1}}{h}} \\
&= \gamma(\xi)(1+h\xi^2)^{\frac{D_u^{-1}}{h}} \sum_{j=0}^{\infty}\left(\frac{v}{h}\right)^{j}\frac{[\log(1+h\xi)]^{j}}{j!} \\
&= \sum_{\omega=0}^{\infty} \mathbb{s}\mathbb{A}_{\omega}^{[h]}(u,0)\frac{\xi^{\omega}}{\omega!} \sum_{\nu=0}^{\infty}\sum_{j=0}^{\nu}\left(\frac{v}{h}\right)^{j} S_1(\nu,j) h^{\nu}\frac{\xi^{\nu}}{\nu!} \\
&= \sum_{\omega=0}^{\infty}\left(\sum_{\nu=0}^{\omega}\binom{\omega}{\nu}\mathbb{s}\mathbb{A}_{\omega-\nu}^{[h]}(u,0)\sum_{j=0}^{\nu}\left(\frac{v}{h}\right)^{j} S_1(\nu,j)h^{\nu}\right)\frac{\xi^{\omega}}{\omega!}.
\end{aligned} \tag{33}$$

Comparing the coefficients of ξ, we obtain the result. □

Theorem 8. *For $\omega \geq 0$, we have*

$$\mathbb{s}\mathbb{A}_{\omega}^{[h]}(u,v) = \sum_{l=0}^{\omega}\sum_{\nu=0}^{\omega-l}\frac{\omega!}{(\omega-\nu-l)!(\nu+l)!} h^{\nu}\mathbb{s}\mathbb{A}_{\omega-\nu-l}^{[h]}(u,0) S_1(\nu+l,l) v^{l}. \tag{34}$$

Proof. From (13), we have

$$\begin{aligned}
\sum_{\omega=0}^{\infty} \mathbb{s}\mathbb{A}_{\omega}^{[h]}(u,v)\frac{\xi^{\omega}}{\omega!} &= \gamma(\xi)(1+h\xi)^{\frac{v}{h}}(1+h\xi^2)^{\frac{D_u^{-1}}{h}} \\
&= \sum_{\omega=0}^{\infty} \mathbb{s}\mathbb{A}_{\omega}^{[h]}(u,0)\frac{\xi^{\omega}}{\omega!}\sum_{\nu=0}^{\infty}\left(-\frac{v}{h}\right)_{\nu}(-h)^{\nu}\frac{\xi^{\nu}}{\nu!} \\
&= \sum_{\omega=0}^{\infty}\left(\sum_{\nu=0}^{\omega}\binom{\omega}{\nu}\left(-\frac{v}{h}\right)_{\nu}(-h)^{\nu}\mathbb{s}\mathbb{A}_{\omega-\nu}^{[h]}(u,0)\right)\frac{\xi^{\omega}}{\omega!}.
\end{aligned} \tag{35}$$

Comparing the coefficients of ξ, we obtain

$$\mathbb{s}\mathbb{A}_{\omega}^{[h]}(u,v) = \sum_{\nu=0}^{\omega}\binom{\omega}{\nu}\left(-\frac{v}{h}\right)_{\nu}(-h)^{\nu}\mathbb{s}\mathbb{A}_{\omega-\nu}^{[h]}(u,0). \tag{36}$$

Using, equality (31) in previous expression, we obtain

$$\begin{aligned}
\mathbb{s}\mathbb{A}_{\omega}^{[h]}(u,v) &= \left(\sum_{\nu=0}^{\omega}\binom{\omega}{\nu}(-h)^{\nu}\mathbb{s}\mathbb{A}_{\omega-\nu}^{[h]}(u,0)\right)\left(\sum_{l=0}^{\nu}(-1)^{\nu-l}S_1(\nu,l)(-h)^{-l}v^{l}\right) \\
&= \sum_{l=0}^{\omega}\sum_{\nu=l}^{\omega}\frac{\omega!}{(\omega-\nu)!\nu!}(-h)^{\nu-l}\mathbb{s}\mathbb{A}_{\omega-\nu}^{[h]}(u,0)(-1)^{\nu-l}S_1(\nu,l)v^{l} \\
&= \sum_{l=0}^{\omega}\sum_{\nu=0}^{\omega-l}\frac{\omega!}{(\omega-\nu-l)!(\nu+l)!}(-h)^{\nu}\mathbb{s}\mathbb{A}_{\omega-\nu-l}^{[h]}(u,0)(-1)^{\nu}S_1(\nu+l,l)v^{l}.
\end{aligned} \tag{37}$$

This completes the proof of the theorem. □

Theorem 9. *For $\omega \geq 0$, we have*

$$ {}_{\mathbb{S}}\mathbb{A}_{\omega}^{[h]}(u,v+s) = \sum_{l=0}^{\omega}\sum_{\nu=0}^{\omega-l} \frac{\omega!}{(\omega-\nu-l)!(\nu+l)!} h^{\nu}{}_{\mathbb{S}}\mathbb{A}_{\omega-\nu-l}^{[h]}(u,v) S_1(\nu+l,l) s^{l}. \tag{38} $$

Proof. Taking $v+s$ instead of v in (13), we have

$$ \begin{aligned} \sum_{\omega=0}^{\infty} {}_{\mathbb{S}}\mathbb{A}_{\omega}^{[h]}(u,v+s)\frac{\xi^{\omega}}{\omega!} &= \gamma(\xi)(1+h\xi)^{\frac{v+s}{h}}(1+h\xi^2)^{\frac{D_u^{-1}}{h}} \\ &= \left(\sum_{\omega=0}^{\infty} {}_{\mathbb{S}}\mathbb{A}_{\omega}^{[h]}(u,v)\frac{\xi^{\omega}}{\omega!}\right)\left(\sum_{\nu=0}^{\infty}\left(-\frac{s}{h}\right)_{\nu}(-h)^{\nu}\frac{\xi^{\nu}}{\nu!}\right). \end{aligned} \tag{39} $$

Using the Cauchy rule and after comparing the coefficients of ξ on both sides of the resulting equation, we have

$$ {}_{\mathbb{S}}\mathbb{A}_{\omega}^{[h]}(u,v+s) = \sum_{\nu=0}^{\omega}\binom{\omega}{\nu}\left(-\frac{s}{h}\right)_{\nu}(-h)^{\nu}{}_{\mathbb{S}}\mathbb{A}_{\omega-\nu}^{[h]}(u,v). \tag{40} $$

Then, using (31) for $\left(-\frac{s}{h}\right)_{\nu}$, we obtain (38). □

4. Monomiality Principle and Determinant Form

The monomiality principle is a fundamental concept in polynomial theory. It states that any polynomial can be expressed uniquely as a combination of simple algebraic terms called monomials. This representation simplifies the polynomial structure and facilitates their analysis in various mathematical contexts. The principle plays a crucial role in practical applications across scientific and engineering fields, such as computational mathematics, signal processing, and physics, where polynomials are used to model complex systems and phenomena. This highlights the broad applicability and significance of the monomiality principle in advancing both theoretical understanding and practical problem-solving capabilities. The exploration and utilization of the monomiality principle, along with operational guidelines and other properties of hybrid special polynomials, have been the focus of extensive study. Originating from Steffenson's concept of poweroids in 1941 [14], the notion of monomiality was further elaborated upon by Dattoli [15,16]. Central to this framework are the $\hat{\mathcal{J}}$ and $\hat{\mathcal{K}}$ operators, which serve as multiplicative and derivative operators, respectively, for a polynomial set $g_k(u_1)_{k\in\mathbb{N}}$.

These operators adhere to the following expressions:

$$ g_{k+1}(u_1) = \hat{\mathcal{J}}\{g_k(u_1)\} \tag{41} $$

and

$$ k\, g_{k-1}(u_1) = \hat{\mathcal{K}}\{g_k(u_1)\}. \tag{42} $$

Consequently, when these multiplicative and derivative operations are applied to the polynomial set $g_k(u_1)_{m\in\mathbb{N}}$, they yield a quasi-monomial domain. Of particular importance is the following formula:

$$ [\hat{\mathcal{K}},\hat{\mathcal{J}}] = \hat{\mathcal{K}}\hat{\mathcal{J}} - \hat{\mathcal{J}}\hat{\mathcal{K}} = \hat{1}, \tag{43} $$

which exhibits a Weyl group structure.

Assuming the set $\{g_k(u_1)\}_{k\in\mathbb{N}}$ is quasi-monomial, the operators $\hat{\mathcal{J}}$ and $\hat{\mathcal{K}}$ can be leveraged to derive the significance of this set. Thus, the following axioms hold true:

For $\hat{\mathcal{J}}$ and $\hat{\mathcal{K}}$ to exhibit differential traits, $g_k(u_1)$ satisfies the differential equation:

$$ \hat{\mathcal{J}}\hat{\mathcal{K}}\{g_k(u_1)\} = k\, g_k(u_1). \tag{44} $$

The expression

$$g_k(u_1) = \hat{\mathcal{J}}^k \{1\} \tag{45}$$

represents the explicit form, with $g_0(u_1) = 1$ and the expression

$$e^{w\hat{\mathcal{J}}}\{1\} = \sum_{k=0}^{\infty} g_k(u_1)\frac{w^k}{k!}, \quad |w| < \infty \ , \tag{46}$$

demonstrates generating expression behavior and is obtained by applying identity (45).

In this section, we will discuss the results of our validation efforts. These results aim to strengthen the reliability and usefulness of the Δ_h Legendre–Appell polynomials as important mathematical tools. As a result, we will be verifying the monomiality principle for the Δ_h Legendre–Appell polynomials ${}_{\mathbb{S}}\mathbb{A}_{\omega}^{[h]}(u,v)$ by presenting the following results:

Theorem 10. *The Δ_h Legendre–Appell polynomials ${}_{\mathbb{S}}\mathbb{A}_{\omega}^{[h]}(u,v)$ satisfy the succeeding multiplicative and derivative operators:*

$$\hat{M_{\mathbb{S}\mathbb{A}}} = \left(\frac{v}{1+{}_v\Delta_h} + \frac{2\,D_u^{-1}{}_v\Delta_h}{h+{}_v\Delta_h{}^2} + \frac{\gamma'(\frac{{}_v\Delta_h}{h})}{\gamma(\frac{{}_v\Delta_h}{h})}\right) \tag{47}$$

and

$$\hat{D_{\mathbb{S}}\mathbb{A}} = \frac{{}_v\Delta_h}{h}. \tag{48}$$

Proof. In consideration of expression (5), taking derivatives with respect to v of expression (13), we have

$$\begin{aligned}
{}_v\Delta_h\left\{\gamma(\xi)(1+h\xi)^{\frac{v}{h}}(1+h\xi^2)^{\frac{D_u^{-1}}{h}}\right\} &= \gamma(\xi)(1+h\xi)^{\frac{v+h}{h}}(1+h\xi^2)^{\frac{D_u^{-1}}{h}} - \gamma(\xi)(1+h\xi)^{\frac{v}{h}}(1+h\xi^2)^{\frac{D_u^{-1}}{h}} \\
&= (1+h\xi-1)\gamma(\xi)(1+h\xi)^{\frac{v}{h}}(1+h\xi^2)^{\frac{D_u^{-1}}{h}} \\
&= h\xi\ \gamma(\xi)(1+h\xi)^{\frac{v}{h}}(1+h\xi^2)^{\frac{D_u^{-1}}{h}},
\end{aligned} \tag{49}$$

thus, we have

$$\frac{{}_v\Delta_h}{h}\left[\gamma(\xi)(1+h\xi)^{\frac{v}{h}}(1+h\xi^2)^{\frac{D_u^{-1}}{h}}\right] = \xi\left[\gamma(\xi)(1+h\xi)^{\frac{v}{h}}(1+h\xi^2)^{\frac{D_u^{-1}}{h}}\right], \tag{50}$$

which gives the identity

$$\frac{{}_v\Delta_h}{h}\left[{}_{\mathbb{S}}\mathbb{A}_{\omega}^{[h]}(u,v)\right] = \xi\left[{}_{\mathbb{S}}\mathbb{A}_{\omega}^{[h]}(u,v)\right]. \tag{51}$$

Now, differentiating expression (13) with respect to ξ, we have

$$\frac{\partial}{\partial\xi}\left\{\gamma(\xi)(1+h\xi)^{\frac{v}{h}}(1+h\xi^2)^{\frac{D_u^{-1}}{h}}\right\} = \frac{\partial}{\partial\xi}\left\{\sum_{\omega=0}^{\infty}{}_{\mathbb{S}}\mathbb{A}_{\omega}^{[h]}(u,v)\frac{\xi^\omega}{\omega!}\right\}, \tag{52}$$

$$\left(\frac{v}{1+h\xi} + 2\frac{D_u^{-1}\xi}{1+h\xi^2} + \frac{\gamma'(\xi)}{\gamma(\xi)}\right)\left\{\sum_{\omega=0}^{\infty}{}_{\mathbb{S}}\mathbb{A}_{\omega}^{[h]}(u,v)\frac{\xi^\omega}{\omega!}\right\} = \sum_{\omega=0}^{\infty}\omega\ {}_{\mathbb{S}}\mathbb{A}_{\omega}^{[h]}(u,v)\frac{\xi^{\omega-1}}{\omega!}. \tag{53}$$

On the usage of identity expression (51) and replacing $\omega \to \omega+1$ in the righthand side of previous expression (53), assertion (47) is established.

Further, in view of identity expression (51), we have

$$\frac{{}_v\Delta_h}{h}\left[{}_{\mathbb{S}}\mathbb{A}_{\omega}^{[h]}(u,v)\right] = \left[\omega\ {}_{\mathbb{S}}\mathbb{A}_{\omega-1}^{[h]}(u,v)\right], \tag{54}$$

which gives an expression for the derivative operator (48). □

Next, we deduce the differential equation for the Δ_h Legendre–Appell polynomials ${}_{\mathbb{S}}\mathbb{A}_{\omega}^{[h]}(u,v)$ by demonstrating the succeeding result:

Theorem 11. *The Δ_h Legendre–Appell polynomials ${}_{\mathbb{S}}\mathbb{A}_{\omega}^{[h]}(u,v)$ satisfy the differential equation:*

$$\left(\frac{v}{1+{}_{v}\Delta_h}+\frac{2\,D_u^{-1}{}_{v}\Delta_h}{h+{}_{v}\Delta_h{}^2}+\frac{\gamma'(\frac{v\Delta_h}{h})}{\gamma(\frac{v\Delta_h}{h})}-\frac{\omega h}{{}_{v}\Delta_h}\right){}_{\mathbb{S}}\mathbb{A}_{\omega}^{[h]}(u,v)=0. \tag{55}$$

Proof. Inserting expression (47) and (48) in the expression (44), the assertion (55) is proved. □

Next, we give the determinant form of Δ_h Legendre–Appell polynomials ${}_{\mathbb{S}}\mathbb{A}_{\omega}^{[h]}(u,v)$ in terms of Δ_h Legendre polynomials $\mathbb{S}_{\omega}^{[h]}(u,v)$ by proving the result listed below:

Theorem 12. *The Δ_h Legendre–Appell polynomials ${}_{\mathbb{S}}\mathbb{A}_{\omega}^{[h]}(u,v)$ give rise to the determinant represented by:*

$${}_{\mathbb{S}}\mathbb{A}_{\omega}^{[h]}(u,v)=\frac{(-1)^{\omega}}{(\gamma_{0,h})^{\omega+1}}\begin{vmatrix} 1 & \mathbb{S}_1^{[h]}(u,v) & \mathbb{S}_2^{[h]}(u,v) & \cdots & \mathbb{S}_{\omega-1}^{[h]}(u,v) & \mathbb{S}_{\omega}^{[h]}(u,v) \\ \gamma_{0,h} & \gamma_{1,h} & \gamma_{2,h} & \cdots & \gamma_{\omega-1,h} & \gamma_{\omega,h} \\ 0 & \gamma_{0,h} & \binom{2}{1}\gamma_{1,h} & \cdots & \binom{\omega-1}{1}\gamma_{\omega-2,h} & \binom{\omega}{1}\gamma_{\omega-1,h} \\ 0 & 0 & \gamma_{0,h} & \cdots & \binom{\omega-1}{2}\gamma_{\omega-3,h} & \binom{\omega}{2}\gamma_{\omega-2,h} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & \gamma_{0,h} & \binom{\omega}{\omega-1}\gamma_{1,h} \end{vmatrix}, \tag{56}$$

where

$$\gamma_{\omega,h},\ \omega=0,1,\cdots \text{are the coefficients of Maclaurin series of } \frac{1}{\gamma(\xi)}.$$

Proof. Multiplying both sides of Equation (13) by $\frac{1}{\gamma(\xi)}=\sum_{\omega=0}^{\infty}\gamma_{\omega,h}\frac{\xi^{\omega}}{\omega!}$, we find

$$\sum_{\omega=0}^{\infty}\mathbb{S}_{\omega}^{[h]}(u,v)\frac{\xi^{\omega}}{\omega!}=\sum_{\omega=0}^{\infty}\sum_{\nu=0}^{\infty}\gamma_{\nu,h}\frac{\xi^{\nu}}{\nu!}\,{}_{\mathbb{S}}\mathbb{A}_{\omega}^{[h]}(u,v)\frac{\xi^{\omega}}{\omega!}, \tag{57}$$

which, on using the Cauchy product rule, becomes

$$\mathbb{S}_{\omega}^{[h]}(u,v)=\sum_{\nu=0}^{\omega}\binom{\omega}{\nu}\gamma_{\nu,h}\,{}_{\mathbb{S}}\mathbb{A}_{\omega-\nu}^{[h]}(u,v). \tag{58}$$

This equality results in a set of ν equations with variables $\mathbb{S}_{\omega}^{[h]}(u,v)$, where $\omega = 0$, 1, 2, $\cdots$. Solving this set using Cramer's rule, and exploiting the denominator as the determinant of a lower triangular matrix with a determinant of $(\gamma_{0,h})^{\omega+1}$, while transposing the numerator and subsequently substituting the i-th row with the $(i+1)$-th position for $i=1,2,\cdots,n-1$ produces the desired outcome. □

5. Examples

The Appell polynomial family is diverse, spanning various members derived by selecting an appropriate function $\gamma(\xi)$. Each member boasts unique characteristics, including distinct names, generating functions, and associated numerical properties. These polynomials find applications across numerous mathematical domains due to their versatility and rich properties. The selection of $\gamma(\xi)$ plays a crucial role in defining the specific polynomial within the family, allowing for tailored solutions to various problems in mathematics and physics. Understanding the generating functions associated with these polynomials is essential for their practical utilization, enabling efficient computation and analysis. In the following sections, we delve into the intricacies of the generating functions that underpin the diverse set of Appell polynomials, shedding light on their mathematical elegance and practical significance in a wide array of applications. The generating function for the Δ_h Bernoulli polynomials $\beta_\omega^{[h]}(v)$ is given by

$$\frac{\log(1+h\xi)^{\frac{1}{h}}}{(1+h\xi)^{\frac{1}{h}}-1}(1+h\xi)^{\frac{v}{h}} = \sum_{\omega=0}^{\infty} \beta_\omega^{[h]}(v)\frac{\xi^\omega}{\omega!}, \quad |\,\xi\,| < 2\pi. \tag{59}$$

The generating expression for Δ_h Euler polynomials $E_\omega^{[h]}(v)$ is given by

$$\frac{2}{(1+h\xi)^{\frac{1}{h}}+1}(1+h\xi)^{\frac{v}{h}} = \sum_{\omega=0}^{\infty} E_\omega^{[h]}(v)\frac{\xi^\omega}{\omega!}, \quad |\,\xi\,| < \pi. \tag{60}$$

The generating expression for Δ_h Genocchi polynomials $G_\omega^{[h]}(v)$ is given by

$$\frac{2\log(1+h\xi)^{\frac{1}{h}}}{(1+h\xi)^{\frac{1}{h}}+1}(1+h\xi)^{\frac{v}{h}} = \sum_{\omega=0}^{\infty} G_\omega^{[h]}(v)\frac{\xi^\omega}{\omega!}, \quad |\,\xi\,| < \pi. \tag{61}$$

For $h \to 0$, these polynomials reduce to the $B_\omega(v)$, $E_\omega(v)$ and $G_\omega(v)$ polynomials [17].

The Bernoulli, Euler, and Genocchi numbers have found numerous applications in various areas of mathematics, including number theory, combinatorics, and numerical analysis. These applications extend to practical mathematics, where these polynomials and numbers are utilized to solve problems and derive mathematical formulas.

For instance, the Bernoulli numbers appear in various mathematical formulas, such as the Taylor expansion, trigonometric and hyperbolic tangent and cotangent functions, and sums of powers of natural numbers. These numbers play a crucial role in number theory, providing insights into patterns and relationships among integers.

Similarly, the Euler numbers arise in the Taylor expansion and have close connections to trigonometric and hyperbolic secant functions. They have applications in graph theory, automata theory, and calculating the number of up/down ascending sequences, contributing to the analysis of structures and patterns in discrete mathematics.

Moreover, the Genocchi numbers find utility in graph theory and automata theory. They are particularly valuable in counting the number of up/down ascending sequences, which involves studying the order and arrangement of elements in a sequence. Therefore, these Δ_h polynomials and numbers of Bernoulli, Euler, and Genocchi play a significant role in various mathematical domains, allowing for the exploration of mathematical relationships, the derivation of formulas, and the analysis of patterns and structures.

By appropriately choosing the function $\gamma(\xi)$ in Equation (13), we can establish the following generating functions for the Δ_h Legendre-based Bernoulli ${}_{\mathbb{S}}\mathbb{B}_\omega^{[h]}(u,v)$, Euler ${}_{\mathbb{S}}\mathbb{E}_\omega^{[h]}(u,v)$, and Genocchi ${}_{\mathbb{S}}\mathbb{G}_\omega^{[h]}(u,v)$ polynomials:

$$\frac{\log(1+h\xi)}{h(1+h\xi)^{\frac{1}{h}}-h}(1+h\xi)^{\frac{v}{h}}(1+h\xi^2)^{\frac{D_u^{-1}}{h}} = \sum_{\omega=0}^{\infty} {}_{\mathbb{S}}\mathbb{B}_\omega^{[h]}(u,v)\frac{\xi^\omega}{\omega!}, \tag{62}$$

$$\frac{2}{(1+h\xi)^{\frac{1}{h}}+1}(1+h\xi)^{\frac{v}{h}}(1+h\xi^2)^{\frac{D_u^{-1}}{h}} = \sum_{\omega=0}^{\infty} {}_{\mathbb{S}}\mathbb{E}_\omega^{[h]}(u,v)\frac{\xi^\omega}{\omega!}, \tag{63}$$

and

$$\frac{2\log(1+h\xi)}{h(1+h\xi)^{\frac{1}{h}}+h}(1+h\xi)^{\frac{v}{h}}(1+h\xi^2)^{\frac{D_u^{-1}}{h}} = \sum_{\omega=0}^{\infty} {}_{\mathbb{S}}\mathbb{G}_\omega^{[h]}(u,v)\frac{\xi^\omega}{\omega!}. \tag{64}$$

Further, in view of expression (22) and Table 1, the polynomials ${}_{\mathbb{S}}\mathbb{B}_{\omega}^{[h]}(u,v)$, ${}_{\mathbb{S}}\mathbb{E}_{\omega}^{[h]}(u,v)$ and ${}_{\mathbb{S}}\mathbb{G}_{\omega}^{[h]}(u,v)$ satisfy the following explicit form:

$$ {}_{\mathbb{S}}\mathbb{B}_{\omega}^{[h]}(u,v) = \sum_{\nu=0}^{\omega} \binom{\omega}{\nu} \mathbb{B}_{\nu,h}\, {}_{\mathbb{S}}\mathbb{A}_{\omega-\nu}^{[h]}(u,v), \tag{65} $$

$$ {}_{\mathbb{S}}\mathbb{E}_{\omega}^{[h]}(u,v) = \sum_{\nu=0}^{n} \binom{\omega}{\nu} \mathbb{E}_{\nu,h}\, {}_{\mathbb{S}}\mathbb{A}_{\omega-\nu}^{[h]}(u,v) \tag{66} $$

and

$$ {}_{\mathbb{S}}\mathbb{G}_{\omega}^{[h]}(u,v) = \sum_{\nu=0}^{\omega} \binom{\omega}{\nu} \mathbb{G}_{\nu,h}\, {}_{\mathbb{S}}\mathbb{A}_{\omega-\nu}^{[h]}(u,v). \tag{67} $$

Furthermore, in view of expressions (56), the polynomials ${}_{\mathbb{S}}\mathbb{B}_{\omega}^{[h]}(u,v)$, ${}_{\mathbb{S}}\mathbb{E}_{\omega}^{[h]}(u,v)$ and ${}_{\mathbb{S}}\mathbb{G}_{\omega}^{[h]}(u,v)$ satisfy the following determinant representations:

$$ {}_{\mathbb{S}}\mathbb{B}_{\omega}^{[h]}(u,v) = \frac{(-1)^{\omega}}{(\gamma_{0,h})^{\omega+1}} \begin{vmatrix} 1 & \mathbb{B}_{1}^{[h]}(u,v) & \mathbb{B}_{2}^{[h]}(u,v) & \cdots & \mathbb{B}_{\omega-1}^{[h]}(u,v) & \mathbb{B}_{\omega}^{[h]}(u,v) \\ \gamma_{0,h} & \gamma_{1,h} & \gamma_{2,h} & \cdots & \gamma_{\omega-1,h} & \gamma_{\omega,h} \\ 0 & \gamma_{0,h} & \binom{2}{1}\gamma_{1,h} & \cdots & \binom{\omega-1}{1}\gamma_{\omega-2,h} & \binom{\omega}{1}\gamma_{\omega-1,h} \\ 0 & 0 & \gamma_{0,h} & \cdots & \binom{\omega-1}{2}\gamma_{\omega-3,h} & \binom{\omega}{2}\gamma_{\omega-2,h} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & \gamma_{0,h} & \binom{\omega}{\omega-1}\gamma_{1,h} \end{vmatrix}, \tag{68} $$

$$ {}_{\mathbb{S}}\mathbb{E}_{\omega}^{[h]}(u,v) = \frac{(-1)^{\omega}}{(\gamma_{0,h})^{\omega+1}} \begin{vmatrix} 1 & \mathbb{E}_{1}^{[h]}(u,v) & \mathbb{E}_{2}^{[h]}(u,v) & \cdots & \mathbb{E}_{\omega-1}^{[h]}(u,v) & \mathbb{E}_{\omega}^{[h]}(u,v) \\ \gamma_{0,h} & \gamma_{1,h} & \gamma_{2,h} & \cdots & \gamma_{\omega-1,h} & \gamma_{\omega,h} \\ 0 & \gamma_{0,h} & \binom{2}{1}\gamma_{1,h} & \cdots & \binom{\omega-1}{1}\gamma_{\omega-2,h} & \binom{\omega}{1}\gamma_{\omega-1,h} \\ 0 & 0 & \gamma_{0,h} & \cdots & \binom{\omega-1}{2}\gamma_{\omega-3,h} & \binom{\omega}{2}\gamma_{\omega-2,h} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & \gamma_{0,h} & \binom{\omega}{\omega-1}\gamma_{1,h} \end{vmatrix}, \tag{69} $$

and

$$ {}_{\mathbb{S}}\mathbb{G}_{\omega}^{[h]}(u,v) = \frac{(-1)^{\omega}}{(\gamma_{0,h})^{\omega+1}} \begin{vmatrix} 1 & \mathbb{G}_{1}^{[h]}(u,v) & \mathbb{G}_{2}^{[h]}(u,v) & \cdots & \mathbb{G}_{\omega-1}^{[h]}(u,v) & \mathbb{G}_{\omega}^{[h]}(u,v) \\ \gamma_{0,h} & \gamma_{1,h} & \gamma_{2,h} & \cdots & \gamma_{\omega-1,h} & \gamma_{\omega,h} \\ 0 & \gamma_{0,h} & \binom{2}{1}\gamma_{1,h} & \cdots & \binom{\omega-1}{1}\gamma_{\omega-2,h} & \binom{\omega}{1}\gamma_{\omega-1,h} \\ 0 & 0 & \gamma_{0,h} & \cdots & \binom{\omega-1}{2}\gamma_{\omega-3,h} & \binom{\omega}{2}\gamma_{\omega-2,h} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & \gamma_{0,h} & \binom{\omega}{\omega-1}\gamma_{1,h} \end{vmatrix}. \tag{70} $$

Table 1. Several members of the Appell polynomials family.

S. No.	Appell Polynomials	Generating Function	$\mathcal{A}(\xi)$
I.	The Bernoulli polynomials [11]	$\dfrac{\xi}{e^{\xi}-1}e^{u\xi} = \sum_{\omega=0}^{\infty} \mathbb{B}_{\omega}(u)\frac{\xi^{\omega}}{\omega!}$	$\mathbb{A}(\xi) = \frac{\xi}{e^{\xi}-1}$
II.	The Euler polynomials [11]	$\dfrac{2}{e^{\xi}+1}e^{u\xi} = \sum_{\omega=0}^{\infty} \mathbb{E}_{\omega}(u)\frac{\xi^{\omega}}{\omega!}$	$\mathbb{A}(\xi) = \frac{2}{e^{\xi}+1}$
III.	The Genocchi polynomials [11]	$\dfrac{2\xi}{e^{\xi}+1}e^{u\xi} = \sum_{\omega=0}^{\infty} \mathbb{G}_{\omega}(u)\frac{\xi^{\omega}}{\omega!}$	$\mathbb{A}(\xi) = \frac{2\xi}{e^{\xi}+1}$

6. Conclusions

The introduction and exploration of Δ_h Legendre–Appell polynomials mark a significant advancement in polynomial theory, particularly in quantum mechanics and entropy modeling. Integrating the monomiality principle and operational rules, these polynomials offer fresh insights into uncharted mathematical territory. This research provides explicit formulas and elucidates fundamental properties, deepening our understanding of Legendre polynomials and linking them to established polynomial categories, enriching the mathematical landscape.

Future research could delve into structural properties and algebraic aspects, uncovering deeper insights and potential applications. Exploring their applicability in quantum mechanics and mathematical physics may reveal new research directions and practical implications. Additionally, bridging the gap between mathematical theory and real-world applications could maximize their potential, especially in statistical mechanics, information theory, and computational science. Collaborative interdisciplinary efforts could unlock the full potential of Δ_h hybrid polynomials across diverse domains.

Therefore, introducing and investigating hybrid Δ_h polynomials represent a significant milestone, fostering new research avenues and applications in various mathematical and scientific fields. Continued exploration and collaboration are essential for realizing their full potential and understanding their broader implications.

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