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Reparameterized Scale Mixture of Rayleigh Distribution Regression Models with Varying Precision

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Abstract: In this paper, we introduce a new parameterization for the scale mixture of the Rayleigh distribution, which uses a mean linear regression model indexed by mean and precision parameters to model asymmetric positive real data. To test the goodness of fit, we introduce two residuals for the new model. A Monte Carlo simulation study is performed to evaluate the parameter estimation of the proposed model. We compare our proposed model with existing alternatives and illustrate its advantages and usefulness using Gilgais data in R software version 4.2.3 with the gamlss package.

Keywords: scale mixture of Rayleigh distribution; maximum likelihood estimator; regression models; residuals

MSC: 62F10; 62E99

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1. Introduction

Generalized linear models (GLM) were introduced by Nelder and Wedderburn [1], and are widely used in statistical modeling. For example, Lange et al. [2] proposed robust statistical modeling using the Student's-t distribution as an alternative to the normal distribution. Smith [3] reviewed the regression models based on the Weibull distribution and gives various applications involving reliability data. Ferrari and Cribari-Neto [4] introduced the beta regression model for modeling situations where the variable of interest is continuous and restricted to the interval (0, 1), and is related to other variables through a regression structure. Rodríguez-Avi et al. [5] developed a regression model for count data based on the generalized Waring distribution. A methodology based on a reparameterized Birnbaum-Saunders regression model with variable accuracy was suggested by Santos-Neto et al. [6]. This methodology enhances the existing literature on the topic. Bourguignon and Gallardo [7] presented a mean linear regression model in which the response variable follows an inverse gamma distribution. This was achieved using a reparameterization of the distribution, indexed by average and precision parameters. Gallardo et al. [8] put forward a regression model where the response variable follows a reparameterized slashed Rayleigh distribution, indexed by mean and precision parameters. Mota et al. [9] proposed a regression model based on a weighted Lindley distribution, reparameterized in terms of mean and precision parameters, and Gómez et al. [10] discuss several aspects of the slashed half-normal distribution, reparameterizing the model based on the mean and comparing it with the known regression models for positive data. The models discussed show the versatility of generalized linear models in statistical analysis.

Additionally, in the context of scale mixture models, Guzmán et al. [11] proposed a linear regression model with censored responses based on skew scale mixtures of normal distributions. Ferreira et al. [12] constructed a linear mixed model that relies on skew scale combinations of normal distributions, highlighting the skew Student-t normal, skew–slash, and skew–contaminated normal distributions. A class of additive partially linear models based on scale mixtures of skew-normal under the centralized parameterization was suggested by De Freitas et al. [13], and Benites et al. [14] proposed a novel censored linear regression model where the random errors follow a finite mixture of scale mixtures of normal distribution. From count data to continuous variables constrained within specific intervals, various distributions and reparameterizations offer tailored solutions for different data types. The development of methodologies like the Reparameterized regression model with variable precision, demonstrates the advance of specialized techniques to address specific needs in statistical modeling, such as accommodating varying precision in the data; they allow for a more intuitive understanding of the models' parameters and can improve model performance. The ongoing development of regression models reflects the continuous evolution and refinement of statistical techniques to meet the evolving needs of data analysis.

In this paper we propose a reparametrized model based on the response variable. The model used is a Scale mixture of Rayleigh (SMR) distribution, proposed by Rivera et al. [15]. This model is based on the ratio of two independent random variables (RVs), the distributed Rayleigh and the Generalized Gamma distribution introduced by Stacy [16]. Let $T \sim SMR(\sigma, q)$, the probability density function (pdf) of T is given by

$$f_T(t;\sigma,q) = \frac{q\, t}{2\sigma\left(\frac{t^2}{2\sigma}+1\right)^{\frac{q}{2}+1}} \quad , \quad t > 0\,, \tag{1}$$

with $\sigma > 0$ and $q > 0$. The SMR distribution is known for its ability to model variability in data that may exhibit asymmetry and heavy tails, making it suitable for a wide range of applications in science, engineering, and other disciplines. In this context, we propose a reparametrization of the original model to enhance its interpretability and applicability across different data analysis scenarios. Our reparametrization focuses on facilitating the interpretation of model parameters and improving the computational efficiency of their estimation. Under this parameterization, we propose a reparameterized scale mixture of Rayleigh (RSMR) distribution to be used in a regression model framework, considering a regression structure for the mean. This reparametrized approach has the potential to enhance researchers' ability to comprehend and model data distributed according to the SMR, which could lead to significant advances in the analysis and interpretation of a wide variety of phenomena observed in practice.

The rest of the paper proceeds as follows. In Section 2, we introduce a new parameterization of the SMR distribution indexed by the mean and precision parameters. The RSMR regression model is analyzed in Section 3. In Section 4, a simulation analysis is carried out to test the performance of the estimates derived from our model. This evaluation of the performance serves to validate the robustness and reliability of our proposed approach under various scenarios and conditions. In Section 5, utilizing the gamlss function ([17]) from the corresponding package as documented by [18], we specifically focus on the Gilgais dataset. Through this empirical application, we aim to demonstrate the efficacy and practical utility of our proposed methodology in real data settings. We compare the new model with the reparameterized Birnbaum-Saunders distribution, the reparameterized Gamma distribution, and the reparameterized Weibull distribution, using the Akaike Information Criterion (AIC) Akaike [19] and the Bayesian Information Criterion (BIC), introduced in Schwarz [20]. Section 6 concludes the paper.

2. An SMR Distribution Parameterized by Its Mean and Precision Parameters

In this section, we consider an alternative parameterization of the SMR model based on mean and precision parameters. For $Y \sim SMR(\sigma, q)$, the mean and variance are given by

$$E[Y] = \sqrt{\frac{\pi\sigma}{2}}\,\frac{\Gamma\left(\frac{\phi+1}{2}\right)}{\Gamma\left(\frac{\phi+2}{2}\right)}, \qquad V[Y] = \mu^2\left[\frac{4\,\Gamma\left(\frac{\phi}{2}\right)\,\Gamma\left(\frac{\phi+2}{2}\right)}{\pi\,\Gamma\left(\frac{\phi+1}{2}\right)^2} - 1\right]. \tag{2}$$

We use the notation $Y \sim RSMR(\mu;\phi)$ to indicate that Y is a random variable following a reparameterized SMR distribution proposed in (1) with mean $\mu > 0$ and precision parameter $\phi > 0$.

Using the proposed parameterization, the RSMR density in (2) can be written as

$$f_Y(y;\mu,\phi) = \frac{(\phi+2)\,\pi\,\Gamma\left(\frac{\phi+1}{2}\right)^2 y}{4\,\mu^2\,\Gamma\left(\frac{\phi+2}{2}\right)^2\left(\frac{\pi\,\Gamma\left(\frac{\phi+1}{2}\right)^2\cdot y^2}{4\,\mu^2\,\Gamma\left(\frac{\phi+2}{2}\right)^2}+1\right)^{\frac{\phi}{2}+2}}, \quad y > 0, \tag{3}$$

and its cumulative distribution function can be written as

$$F_Y(y;\mu,\phi) = 1 - \left(\frac{\pi\,\Gamma\left(\frac{\phi+1}{2}\right)^2\cdot y^2}{4\,\mu^2\,\Gamma\left(\frac{\phi+2}{2}\right)^2}+1\right)^{-\frac{\phi+2}{2}}. \tag{4}$$

Then, the quantile function of the RSMR distribution is given by

$$Q(p) = \frac{2\,\mu\,\Gamma\left(\frac{\phi+2}{2}\right)}{\Gamma\left(\frac{\phi+1}{2}\right)}\sqrt{\frac{1}{\pi}\left[(1-p)^{-\frac{2}{\phi+2}}-1\right]}.$$

The expression of the noncentral moments μ_1, μ_2, μ_3, and μ_4 are represented by

$$\mu_1 = \sqrt{\frac{\sigma}{2}}\,B\left(\frac{1}{2},\frac{\phi+1}{2}\right), \qquad \mu_2 = \frac{2\mu^2\phi}{\pi^2}\,B^2\left(\frac{1}{2},\frac{\phi}{2}\right),$$

$$\mu_3 = 24\left[\frac{\mu\phi}{2\pi}\,B\left(\frac{1}{2},\frac{\phi}{2}\right)\right]^6 B\left(\frac{3}{2},\frac{\phi-1}{2}\right), \quad \phi > 1,$$

$$\mu_4 = 2\left[\frac{\mu\phi}{\pi}\,B\left(\frac{1}{2},\frac{\phi}{2}\right)\right]^4 B\left(2,\frac{\phi-2}{2}\right), \quad \phi > 2,$$

where B is the beta function.

The coefficient of variation (cv) of the RSMR distribution is given by

$$\text{cv} = \left[\frac{2\phi}{\pi^2}\,B^2\left(\frac{1}{2},\frac{\phi}{2}\right)-1\right]^{1/2}, \tag{5}$$

where B is the beta function (note that cv does not depend on the μ parameter).

The skewness and kurtosis coefficients of the RSMR distribution are given by

$$\sqrt{\beta_1} = \frac{24\left[\frac{\mu\phi}{2\pi}\,B\left(\frac{1}{2},\frac{\phi}{2}\right)\right]^6 B\left(\frac{3}{2},\frac{\phi-1}{2}\right) - \frac{24}{\phi}\left[\frac{\mu\phi}{2\pi}\,B\left(\frac{1}{2},\frac{\phi}{2}\right)\right]^2 + 2\mu^3}{\left[\mu^2\left[\frac{2\phi}{\pi^2}B^2\left(\frac{1}{2},\frac{\phi}{2}\right)-1\right]\right]^{3/2}} = \frac{24\mu^6 a_6\, b_{31} - \frac{24\mu^3}{\phi}a_2 + 2\mu^3}{\mu^3\left[\frac{8}{\phi}a_2 - 1\right]^{3/2}}$$

$$= \frac{2\mu^3(12\mu^3\phi\, a_6\, b_{31} - 12a_2 + \phi)}{\phi\mu^3\left(\frac{1}{\phi}\right)^{3/2}[8a_2-\phi]^{3/2}} = \frac{2\sqrt{\phi}(12\mu^3\phi\, a_6\, b_{31} - 12a_2 + \phi)}{[8a_2-\phi]^{3/2}},$$

and

$$\beta_2 = \frac{2\left[\frac{\mu\phi}{\pi}\,B\left(\frac{1}{2},\frac{\phi}{2}\right)\right]^4 B\left(2,\frac{\phi-2}{2}\right) - (4\mu)24\left[\frac{\mu\phi}{2\pi}\,B\left(\frac{1}{2},\frac{\phi}{2}\right)\right]^6 B\left(\frac{3}{2},\frac{\phi-1}{2}\right) + (6\mu^2)\frac{2\mu^2\phi}{\pi^2}\,B^2\left(\frac{1}{2},\frac{\phi}{2}\right) - 3\mu^4}{\left[\mu^2\left[\frac{2\phi}{\pi^2}B^2\left(\frac{1}{2},\frac{\phi}{2}\right)-1\right]\right]^2}$$

$$= \frac{32\mu^4 a_4\, b_{42} - 96\mu^7 a_6\, b_{31} + 48\frac{\mu^4 a_2}{\phi} - 3\mu^4}{\mu^4\left(\frac{1}{\phi^2}\right)[8a_2-\phi]^2} = \frac{\frac{\mu^4}{\phi}(32\phi\, a_4\, b_{42} - 96\mu^4\phi a_6\, b_{31} + 48a_2 - 3\phi)}{\mu^4\left(\frac{1}{\phi^2}\right)[8a_2-\phi]^2}$$

$$= \frac{\phi(32\phi\, a_4\, b_{42} - 96\mu^4\phi a_6\, b_{31} + 48a_2 - 3\phi)}{[8a_2-\phi]^2},$$

where $a_k = \left[\frac{\phi}{2\pi}\,B\left(\frac{1}{2},\frac{\phi}{2}\right)\right]^k$ and $b_{ij} = B\left(\frac{i}{2},\frac{\phi-j}{2}\right)$.

3. RSMR Regression Model

Let $Y_1,\dots,Y_n$ be n independent random variables, where each Y_i, $i=1,\dots,n$, follows the RMSR model with pdf provided in Equation (3) with mean μ_i and precision parameter ϕ. Suppose the mean of Y_i satisfies the following functional relation

$$g_1(\mu_i) = \eta_{1i} = \mathbf{x}_i^\top \boldsymbol{\beta} \qquad \text{and} \qquad g_2(\phi_i) = \eta_{2i} = \mathbf{z}_i^\top \boldsymbol{\nu}$$

where $\boldsymbol{\beta} = (\beta_1,\dots,\beta_p)^\top$ is a vector of unknown regression coefficients, $\boldsymbol{\beta} \in \mathbb{R}^p$, η_{1i} are linear predictors, $\mathbf{x}_i = (x_{i1},\dots,x_{ip})^\top$ are observations on p known covariates, for $i = 1,\dots,n$. We assume the matrix $\mathbf{X} = (\mathbf{x_1},\dots,\mathbf{x_n})^\top$ has rank p. The link function $g_1 : \mathbb{R} \to \mathbb{R}^+$; $\mu_i = g_1^{-1}(\mathbf{x}_i^\top\hat{\boldsymbol{\beta}})$, with $g_1^{-1}(\cdot)$ being the inverse functions of $g_1(\cdot)$.

The log-likelihood function has the form

$$\ell(\boldsymbol{\theta}) = \sum_{i=1}^{n} \ell(\mu_i,\phi_i)\,, \tag{6}$$

with

$$\ell(\mu_i,\phi_i) = \log(\phi+2) + 2\left[\log\Gamma\left(\frac{\phi+1}{2}\right) - \log(\mu) - \log\Gamma\left(\frac{\phi+2}{2}\right)\right] - \left(\frac{\phi}{2}+2\right)\log(Z(\boldsymbol{\theta})),$$

where $Z(\mu_i,\phi_i) = \dfrac{\pi\,\Gamma\left(\frac{\phi+1}{2}\right)^2\cdot t^2}{4\,\mu^2\,\Gamma\left(\frac{\phi+2}{2}\right)^2} + 1.$

3.1. Residuals

To check the validity of the RSMR regression model, in this subsection we discuss two kind of residuals for this model based on the randomized quantile (Dunn and Smyth [21]) and deviance residuals.

Randomized Quantile Residuals

The randomized quantile residual for the RSMR model are defined as

$$r_i^Q = \Phi^{-1}\left\{1 - \left(\frac{\pi\,\Gamma\left(\frac{\hat{\phi}_i+1}{2}\right)^2 \cdot y_i^2}{4\,\hat{\mu}_i^2\,\Gamma\left(\frac{\hat{\phi}_i+2}{2}\right)^2} + 1\right)^{-\frac{\hat{\phi}_i+2}{2}}\right\},$$

where $i = 1, \ldots, n$, $\Phi^{-1}(\cdot)$ denotes the inverse of the cdf for the standard normal model, $\hat{\mu}_i = g_1^{-1}(\mathbf{x}_i^\top \hat{\boldsymbol{\beta}})$, and $\hat{\phi}_i = g_2^{-1}(\mathbf{z}_i^\top \hat{\boldsymbol{\nu}})$.

If the model is correctly specified, $r_1^Q, r_2^Q, \ldots, r_n^Q$ is a random sample from the standard normal distribution. The latter expression is used to sketch the QQ-plots in the applications section.

3.2. Deviance Residuals

For the RSMR regression model, the deviance residual is given by

$$r_i^D = \text{sign}(1 + \log(S(y_i; \hat{\mu}_i, \hat{\phi}_i)))\left[-2\{1 + \log(S(y_i; \hat{\mu}_i, \hat{\phi}_i)) + \log(-\log(S(y_i; \hat{\mu}_i, \hat{\phi}_i)))\}\right]^{1/2}$$

where $i = 1, \ldots, n$, sign(x) is the sign function, and $S(y; \mu, \phi)$ corresponds to the survival function of the RSMR model, given by $S(y; \mu, \phi) = \left(\frac{\pi\,\Gamma\left(\frac{\phi+1}{2}\right)^2 \cdot y^2}{4\,\mu^2\,\Gamma\left(\frac{\phi+2}{2}\right)^2} + 1\right)^{-\frac{\phi+2}{2}}$.

4. Simulation Study

In this section, a Monte Carlo simulation study were conducted to assess the performance of the estimators for the RSMR regression model. All simulations were performed using the R programming language [18]. Specifically, to perform parameter estimation for the RSMR model, we used the gamlss package. The initial values used for optimization are $(\beta, \beta_1, \beta_2)$ and $\log(\phi)$ given by $(-0.3, 0.5, -0.6)$, $(-0.3, 0.25, -0.6)$, $(-0.3, 0, -0.6)$, $(-0.3, 0.5, -0.3)$, $(-0.3, 0.5, 0)$, $(-0.3, 0.25, -0.3)$, $(-0.3, 0, 0)$ with $\log(\phi) = 0.7$, and $(-0.2, 0.5, 0.8)$, $(-0.2, 0.25, 0.8)$, $(-0.2, 0, 0.8)$, $(-0.2, 0.5, -0.3)$, $(-0.2, 0.5, 0)$, $(-0.2, 0.25, -0.3)$, $(-0.2, 0, 0)$ with $\log(\phi) = 1.3$. This process was repeated 1000 times with sample size $n = 50$, 100, and 200. We calculated the average of the estimations, average of the errors (se), root of the mean squared error (RMSE), and the coverage (Cov) percentages, as shown in Table 1 below. The averages are close to the initial values, and the bias and RMSEs decrease as the sample size increases. The results are found in Table 1 and in the tables in Appendix A in this paper.

Table 1. Simulation study for RSMR model.

$\boldsymbol{\beta}$	$\boldsymbol{\beta_1}$	$\boldsymbol{\beta_2}$	$\log(\boldsymbol{\phi})$			**se**	**RMSE**	**% Cov.**
						$n = 50$		
−0.3	0.5	−0.6	0.7	$\widehat{\beta}$	−0.3114	0.0826	0.0683	0.94
				$\widehat{\beta}_1$	0.5910	0.0908	0.0792	0.92
				$\widehat{\beta}_2$	−0.4984	0.0818	0.0718	0.92
				$\log(\widehat{\phi})$	1.7274	1.1188	1.0343	1.00
						$n = 100$		
				$\widehat{\beta}$	−0.3254	0.0698	0.0609	0.91
				$\widehat{\beta}_1$	0.4862	0.0809	0.0647	0.94
				$\widehat{\beta}_2$	−0.5809	0.0689	0.0573	0.95
				$\log(\widehat{\phi})$	1.2398	0.6569	0.5543	0.99

Table 1. *Cont.*

β	β_1	β_2	$\log(\phi)$			se	RMSE	% Cov.
						$n = 200$		
				$\widehat{\beta}$	−0.3124	0.0522	0.0434	0.94
				$\widehat{\beta}_1$	0.4889	0.0556	0.0421	0.94
				$\widehat{\beta}_2$	−0.5957	0.0504	0.0406	0.96
				$\log(\widehat{\phi})$	0.9225	0.4250	0.2827	0.97
						$n = 50$		
−0.3	0.25	−0.6	0.7	$\widehat{\beta}$	−0.3409	0.0911	0.0889	0.88
				$\widehat{\beta}_1$	0.2481	0.1061	0.0921	0.92
				$\widehat{\beta}_2$	−0.5913	0.0865	0.0790	0.92
				$\log(\widehat{\phi})$	1.6989	1.1065	1.0057	1.00
						$n = 100$		
				$\widehat{\beta}$	−0.3229	0.0691	0.0609	0.92
				$\widehat{\beta}_1$	0.2437	0.0732	0.0621	0.93
				$\widehat{\beta}_2$	−0.5909	0.0740	0.0617	0.93
				$\log(\widehat{\phi})$	1.2421	0.6522	0.5542	0.99
						$n = 200$		
				$\widehat{\beta}$	−0.3132	0.0519	0.0419	0.94
				$\widehat{\beta}_1$	0.2464	0.0499	0.0429	0.93
				$\widehat{\beta}_2$	−0.5974	0.0516	0.0418	0.94
				$\log(\widehat{\phi})$	0.9607	0.4281	0.3264	0.95
						$n = 50$		
−0.3	0	−0.6	0.7	$\widehat{\beta}$	−0.3429	0.0909	0.0884	0.88
				$\widehat{\beta}_1$	−0.0038	0.1131	0.1006	0.93
				$\widehat{\beta}_2$	−0.5787	0.0913	0.0787	0.92
				$\log(\widehat{\phi})$	1.6921	1.1078	0.9989	1.00
						$n = 100$		
				$\widehat{\beta}$	−0.3232	0.0689	0.0609	0.92
				$\widehat{\beta}_1$	0.0006	0.0661	0.0552	0.94
				$\widehat{\beta}_2$	−0.5938	0.0648	0.0544	0.94
				$\log(\widehat{\phi})$	1.2792	0.6498	0.5904	0.99
						$n = 200$		
				$\widehat{\beta}$	−0.3125	0.0516	0.0439	0.94
				$\widehat{\beta}_1$	0.0002	0.0455	0.0378	0.94
				$\widehat{\beta}_2$	−0.5954	0.0464	0.0375	0.96
				$\log(\widehat{\phi})$	0.9785	0.4307	0.3426	0.94
						$n = 50$		
−0.3	0.5	−0.3	0.7	$\widehat{\beta}$	−0.3323	0.0923	0.0858	0.90
				$\widehat{\beta}_1$	0.4993	0.1092	0.0980	0.90
				$\widehat{\beta}_2$	−0.2986	0.0975	0.0823	0.94
				$\log(\widehat{\phi})$	1.6489	1.0871	0.9557	1.00
						$n = 100$		
				$\widehat{\beta}$	−0.3196	0.0697	0.0592	0.93
				$\widehat{\beta}_1$	0.4991	0.0775	0.0652	0.93
				$\widehat{\beta}_2$	−0.3045	0.0626	0.0522	0.94
				$\log(\widehat{\phi})$	1.2575	0.6506	0.5707	0.99

Table 1. *Cont.*

β	β_1	β_2	$\log(\phi)$			se	RMSE	% Cov.
						$n = 200$		
				$\widehat{\beta}$	−0.3143	0.0519	0.0424	0.94
				$\widehat{\beta}_1$	0.4983	0.0509	0.0395	0.95
				$\widehat{\beta}_2$	−0.2989	0.0497	0.0401	0.95
				$\log(\widehat{\phi})$	0.9739	0.4316	0.3335	0.94

5. Real Data Application

We consider the Gilgais (*R) dataset, collected on a line transect survey in Gilgai territory in New South Wales, Australia. The main aim of this study is to explain the electrical conductivity at depth 30–40 cm (ec_{30}) in terms of pH at depth 30–40 cm (pH_{30}) and electrical conductivity at depth 80–90 cm (ec_{80}). To use the dataset, the MASS package is required (see Venables and Ripley [22]).

Table 2 shows a summary for the variable ec_{30}. We compared the RSMR regression model with the RGA, reparameterized Birnbaum–Saunders (RBS), and reparameterized Weibull (RWE) regression models. The computational implementation to model mean and dispersion parameters with a set of covariates linked to both components in RGA and RWE models is implemented in the gamlss.dist package in R version 4.2.3 (see Rigby and Stasinopoulos [23] and Rigby and Stasinopoulos [24]), while the RBS model is discussed in Santos-Neto et al. [6].

Table 2. Descriptive statistics for ec_{30}.

n	Mean	Median	sd	cv	Skewness	Kurtosis
365	95.770	54	97.953	102.279	1.642	5.483

Table 3 shows the estimated parameters, standard errors (se), AIC (Akaike information criterion), presented in Akaike [19] and BIC (Bayesian information criterion), presented in Schwarz [20] for the models considered. Based on those criteria, the model which is preferable for this data set is the RSMR regression model.

Table 3. Estimated parameters and standard error (se) for RGA, RBS, RWE and RSMR regression models in Gilgais data set.

	RGA		RBS		RWE		RSMR	
Parameter	Estimate	se	Estimate	se	Estimate	se	Estimate	se
$\beta_{Intercept}$	8.0826	0.9266	4.9413	0.7576	10.1701	0.9637	7.7041	1.0611
$\beta_{pH_{30}}$	−0.5651	0.1014	−0.2012	0.0859	−0.7919	0.1046	−0.5269	0.1163
$\beta_{ec_{80}}$	0.0063	0.0004	0.0061	0.0003	0.0058	0.0004	0.0065	0.0004
$\log(\phi)$	−0.4639	0.0349	1.3385	0.0740	0.5149	0.0396	1.4591	0.3473
AIC	3726.30		3742.23		3736.92		3723.66	
BIC	3741.90		3757.83		3752.52		3739.26	
CVM	0.0451		<0.0001		0.0916		0.1376	
AD	0.0855		<0.0001		0.1389		0.1761	
SW	0.1403		<0.0001		0.0031		0.4559	

Figure 1 presents the QQ-plot for the randomized quantile residuals for the models considered. If the respective model is appropriate for the dataset, such residuals represent a random sample from the standard normal distribution. The model that provides the best fit for this dataset is the RSMR regression model.

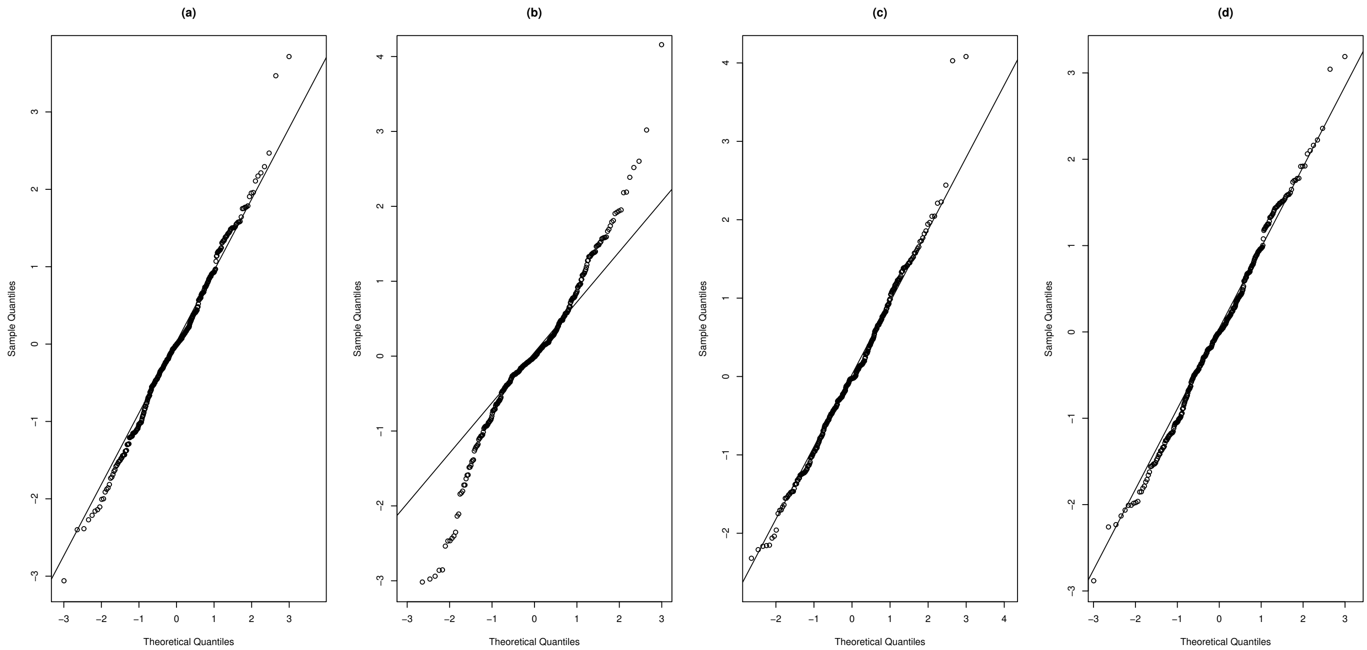

Figure 1. QQ-plot of the randomized quantile residuals of the (**a**) RGA, (**b**) RBS, (**c**) RWE, and (**d**) RSMR distributions for the dataset.

5.1. MASS Package

This is a functions and datasets package to support Venables and Ripley [22]. To fit this model, we use the Gilgais dataset from this package. The dataset was collected on a line transect survey in Gilgai territory in New South Wales, Australia.

The data collection was stimulated by the following question: are these patterns reflected in soil properties? These data have 9 columns and 365 observations for each of the following:

- pH00, pH30, and pH80: pH at depth 0–10, 30–40, and 80–90 cm, respectively;
- e00, e30, and e80: electrical conductivity in mS/cm (0–10 cm), (30–40 cm), and (80–90 cm), respectively;
- c00, c30, and c80: chloride content in ppm (0–10 cm), (30–40 cm), and (80–90 cm), respectively.

Samples were taken at each of the 365 sampling locations on a linear grid with 4 meter spacing, at depths of 0 to 10 cm, 30 to 40 cm, and 80 to 90 cm below the surface.

5.2. Gamlss Package: Fit

Functions for fitting the Generalized Additive Models for Location Scale and Shape are introduced in [23]. In this model, we use the gamlss function in the same package to fit the RSMR regression model, as follows:

```
fit <- gamlss(y∼y1+y2,sigma.fo=∼1,i.control = glim.control(cyc = 1000),
                    control = gamlss.control(n.cyc = 1000),
                    family=MRS(mu.link="log",sigma.link="identity"))
```

To analyze the summary of the adjustment that we made, we use `summary(fit)`, by which we obtain `Estimate`, `Std.Error`, `Global Deviance`, `AIC`, `BIC`, and other values of interest for parameter adjustment. The `plot(fit)` function displays four plots: residuals against values fitted, residuals against an index or a specific explanatory variable, a density plot of the residuals, and a Normal QQ plot of the residuals, as shown in Figure 2.

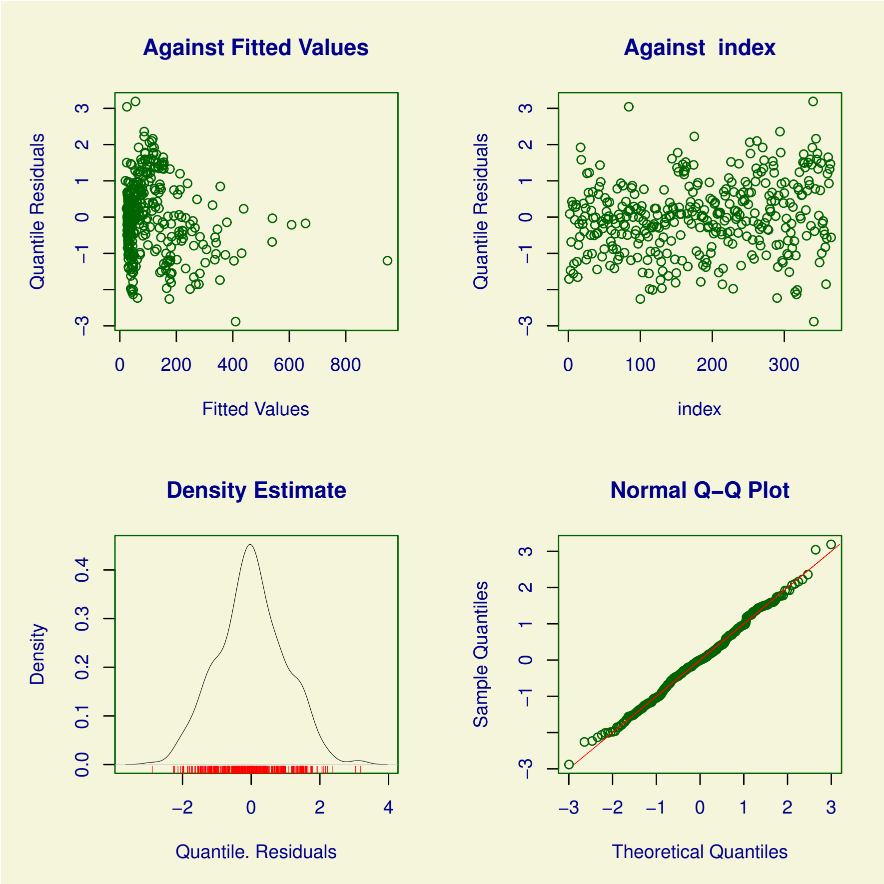

Figure 2. Plot by `plot(fit)` for fitting the dataset.

6. Conclusions

In this article, we introduce a novel SMR distribution characterized by mean and precision parameters. Following this characterization, we have put forth a unique regression model designed for modeling positively skewed real data. We assess the advantages and practicality of our model by comparing it to various alternative models also reparameterized in its mean. For the estimation of model parameters, we have employed maximum likelihood inference and evaluated its effective performance through Monte Carlo simulations. We also have carried out statistical modeling with actual data using our newly proposed model. The results from this application reveal the impact of covariates pH and ec_{80} on the mean and precision of observations. Our aspiration is that this innovative model will encourage broader applications in the field of regression analysis, as well other related context such as random effects models and error-in-variables models.

Author Contributions: Conceptualization, P.A.R. and H.W.G.; methodology, H.W.G. and D.I.G.; software, D.I.G. and E.G.-D.; validation, P.A.R. and E.G.-D.; formal analysis, O.V. and D.I.G.; investigation, P.A.R. and O.V.; writing—original draft preparation, P.A.R. and H.W.G.; writing—review and editing, D.I.G. and O.V.; funding acquisition, H.W.G. and O.V. All authors have read and agreed to the published version of the manuscript.

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Data Availability Statement: The MASS package version 7.3.60 is required to use this dataset (see Venables and Ripley [22]).

Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A. Simulated Results

Table A1. Simulation study for the RSMR model.

β	β_1	β_2	$\log(\phi)$			se	RMSE	% Cov.
						$n = 50$		
−0.3	0.5	0	0.7	$\widehat{\beta}$	−0.3380	0.0927	0.0862	0.91
				$\widehat{\beta}_1$	0.4797	0.0916	0.0762	0.93
				$\widehat{\beta}_2$	−0.0009	0.0942	0.0840	0.91
				$\log(\widehat{\phi})$	1.7051	1.1172	1.0119	1.00
						$n = 100$		
				$\widehat{\beta}$	−0.3245	0.0694	0.0611	0.92
				$\widehat{\beta}_1$	0.4935	0.0602	0.0489	0.94
				$\widehat{\beta}_2$	0.0025	0.0660	0.0560	0.92
				$\log(\widehat{\phi})$	1.2640	0.6573	0.5781	0.99
						$n = 200$		
				$\widehat{\beta}$	−0.3125	0.0517	0.0411	0.93
				$\widehat{\beta}_1$	0.4999	0.0468	0.0376	0.94
				$\widehat{\beta}_2$	−0.0013	0.0483	0.0402	0.94
				$\log(\widehat{\phi})$	0.9655	0.4306	0.3303	0.94
						$n = 50$		
−0.3	0.25	−0.3	0.7	$\widehat{\beta}$	−0.3419	0.0979	0.0917	0.89
				$\widehat{\beta}_1$	0.2543	0.1082	0.0993	0.90
				$\widehat{\beta}_2$	−0.2969	0.0933	0.0816	0.92
				$\log(\widehat{\phi})$	1.6772	1.1181	0.9840	1.00
						$n = 100$		
				$\widehat{\beta}$	−0.3280	0.0702	0.0624	0.91
				$\widehat{\beta}_1$	0.2478	0.0718	0.0589	0.94
				$\widehat{\beta}_2$	−0.2965	0.0569	0.0475	0.94
				$\log(\widehat{\phi})$	1.2624	0.6587	0.5760	0.99
						$n = 200$		
				$\widehat{\beta}$	−0.3134	0.0520	0.0436	0.94
				$\widehat{\beta}_1$	0.2545	0.0536	0.0424	0.95
				$\widehat{\beta}_2$	−0.2975	0.0485	0.0391	0.95
				$\log(\widehat{\phi})$	0.9610	0.4325	0.3256	0.94
						$n = 50$		
−0.3	0	0	0.7	$\widehat{\beta}$	−0.3366	0.0922	0.0875	0.90
				$\widehat{\beta}_1$	0.0016	0.0877	0.0756	0.93
				$\widehat{\beta}_2$	0.0047	0.0868	0.0774	0.92
				$\log(\widehat{\phi})$	1.6621	1.0891	0.9689	1.00
						$n = 100$		
				$\widehat{\beta}$	−0.3235	0.0696	0.0601	0.91
				$\widehat{\beta}_1$	0.0002	0.0686	0.0591	0.93
				$\widehat{\beta}_2$	−0.0009	0.0641	0.0546	0.93
				$\log(\widehat{\phi})$	1.2229	0.6447	0.5367	0.99
						$n = 200$		
				$\widehat{\beta}$	−0.3138	0.0518	0.0422	0.94
				$\widehat{\beta}_1$	−0.0006	0.0476	0.0379	0.94
				$\widehat{\beta}_2$	−0.0045	0.0523	0.0414	0.93
				$\log(\widehat{\phi})$	0.9591	0.4316	0.3241	0.95

Table A2. Simulation study for the RSMR model.

β	β_1	β_2	$\log(\phi)$			se	RMSE	% Cov.
						$n = 50$		
−0.2	0.5	0.8	1.3	$\widehat{\beta}$	−0.2133	0.0878	0.0778	0.92
				$\widehat{\beta}_1$	0.4929	0.0921	0.0760	0.93
				$\widehat{\beta}_2$	0.7743	0.0936	0.0745	0.94
				$\log(\widehat{\phi})$	2.0817	1.3548	0.8630	1.00
						$n = 100$		
				$\widehat{\beta}$	−0.1926	0.0667	0.0552	0.94
				$\widehat{\beta}_1$	0.4628	0.0763	0.0622	0.93
				$\widehat{\beta}_2$	0.7334	0.0608	0.0737	0.81
				$\log(\widehat{\phi})$	1.7633	0.9083	0.7928	0.99
						$n = 200$		
				$\widehat{\beta}$	−0.1967	0.0489	0.0396	0.95
				$\widehat{\beta}_1$	0.4858	0.0459	0.0319	0.95
				$\widehat{\beta}_2$	0.7709	0.0464	0.0426	0.91
				$\log(\widehat{\phi})$	1.3095	0.5066	0.4513	0.99
						$n = 50$		
−0.2	0.25	0.8	1.3	$\widehat{\beta}$	−0.2295	0.0875	0.0796	0.91
				$\widehat{\beta}_1$	0.2239	0.0965	0.0808	0.94
				$\widehat{\beta}_2$	0.7565	0.0983	0.0844	0.93
				$\log(\widehat{\phi})$	2.0389	1.3051	0.8243	1.00
						$n = 100$		
				$\widehat{\beta}$	−0.2055	0.0657	0.0532	0.95
				$\widehat{\beta}_1$	0.2499	0.0642	0.0512	0.96
				$\widehat{\beta}_2$	0.7775	0.0625	0.0503	0.95
				$\log(\widehat{\phi})$	1.6055	0.7559	0.5537	1.00
						$n = 200$		
				$\widehat{\beta}$	−0.1997	0.0474	0.0375	0.95
				$\widehat{\beta}_1$	0.2501	0.0497	0.0399	0.95
				$\widehat{\beta}_2$	0.7879	0.0505	0.0413	0.95
				$\log(\widehat{\phi})$	1.3712	0.4505	0.3176	0.99
						$n = 50$		
−0.2	0	0.8	1.3	$\widehat{\beta}$	−0.2173	0.0893	0.0756	0.94
				$\widehat{\beta}_1$	−0.0045	0.0953	0.0819	0.94
				$\widehat{\beta}_2$	0.7957	0.0967	0.0788	0.94
				$\log(\widehat{\phi})$	1.9084	1.2111	0.6987	1.00
						$n = 100$		
				$\widehat{\beta}$	−0.2069	0.0653	0.0549	0.93
				$\widehat{\beta}_1$	0.0049	0.0696	0.0561	0.95
				$\widehat{\beta}_2$	0.7879	0.0657	0.0531	0.94
				$\log(\widehat{\phi})$	1.6471	0.7601	0.5623	1.00
						$n = 200$		
				$\widehat{\beta}$	−0.1977	0.0484	0.0376	0.95
				$\widehat{\beta}_1$	0.0019	0.0474	0.0368	0.96
				$\widehat{\beta}_2$	0.7794	0.0425	0.0365	0.94
				$\log(\widehat{\phi})$	1.3764	0.4707	0.3918	0.99

Table A2. *Cont.*

β	β_1	β_2	$\log(\phi)$			se	RMSE	% Cov.
						$n = 50$		
−0.2	0.5	−0.3	1.3	$\widehat{\beta}$	−0.2155	0.0895	0.0759	0.93
				$\widehat{\beta}_1$	0.4868	0.0869	0.0692	0.95
				$\widehat{\beta}_2$	−0.2979	0.0937	0.0800	0.92
				$\log(\widehat{\phi})$	1.9269	1.2313	0.7285	1.00
						$n = 100$		
				$\widehat{\beta}$	−0.2105	0.0648	0.0546	0.94
				$\widehat{\beta}_1$	0.4968	0.0641	0.0526	0.95
				$\widehat{\beta}_2$	−0.2969	0.0578	0.0468	0.95
				$\log(\widehat{\phi})$	1.6587	0.7587	0.5527	1.00
						$n = 200$		
				$\widehat{\beta}$	−0.2018	0.0473	0.0382	0.93
				$\widehat{\beta}_1$	0.5003	0.0494	0.0404	0.94
				$\widehat{\beta}_2$	−0.3009	0.0463	0.0375	0.94
				$\log(\widehat{\phi})$	1.4277	0.4620	0.3456	0.99
						$n = 50$		
−0.2	0.5	0	1.3	$\widehat{\beta}$	−0.2196	0.0926	0.0782	0.93
				$\widehat{\beta}_1$	0.4987	0.0939	0.0784	0.95
				$\widehat{\beta}_2$	0.0048	0.0834	0.0685	0.94
				$\log(\widehat{\phi})$	1.8430	1.1669	0.6403	1.00
						$n = 100$		
				$\widehat{\beta}$	−0.2071	0.0649	0.0525	0.94
				$\widehat{\beta}_1$	0.4943	0.0682	0.0563	0.94
				$\widehat{\beta}_2$	−0.0048	0.0578	0.0499	0.93
				$\log(\widehat{\phi})$	1.6330	0.7588	0.5319	1.00
						$n = 200$		
				$\widehat{\beta}$	−0.2017	0.0471	0.0387	0.94
				$\widehat{\beta}_1$	0.5005	0.0471	0.0384	0.94
				$\widehat{\beta}_2$	−0.0014	0.0464	0.0353	0.95
				$\log(\widehat{\phi})$	1.4435	0.4633	0.3758	0.99
						$n = 50$		
−0.2	0.25	−0.3	1.3	$\widehat{\beta}$	−0.2135	0.0888	0.0746	0.93
				$\widehat{\beta}_1$	0.2459	0.1039	0.0903	0.92
				$\widehat{\beta}_2$	−0.2973	0.0908	0.0717	0.94
				$\log(\widehat{\phi})$	1.8519	1.1839	0.6639	1.00
						$n = 100$		
				$\widehat{\beta}$	−0.2047	0.0657	0.0544	0.95
				$\widehat{\beta}_1$	0.2509	0.0661	0.0543	0.94
				$\widehat{\beta}_2$	−0.3008	0.0609	0.0509	0.93
				$\log(\widehat{\phi})$	1.5863	0.7245	0.5005	1.00
						$n = 200$		
				$\widehat{\beta}$	−0.2051	0.0473	0.0385	0.95
				$\widehat{\beta}_1$	0.2502	0.0457	0.0367	0.95
				$\widehat{\beta}_2$	-0.3002	0.0449	0.0351	0.94
				$\log(\widehat{\phi})$	1.4318	0.4623	0.3612	0.99

Table A3. Simulation study for the RSMR model.

β	β_1	β_2	$\log(\phi)$			se	RMSE	% Cov.
						$n = 50$		
−0.2	0	0	1.3	$\widehat{\beta}$	−0.2158	0.0889	0.0741	0.94
				$\widehat{\beta}_1$	0.0071	0.0949	0.0786	0.94
				$\widehat{\beta}_2$	0.0041	0.0836	0.0688	0.94
				$\log(\widehat{\phi})$	1.8456	1.1793	0.6392	1.00
						$n = 100$		
				$\widehat{\beta}$	−0.2091	0.0651	0.0547	0.93
				$\widehat{\beta}_1$	0.0017	0.0646	0.0550	0.93
				$\widehat{\beta}_2$	0.0018	0.0667	0.0549	0.95
				$\log(\widehat{\phi})$	1.6153	0.7492	0.5129	1.00
						$n = 200$		
				$\widehat{\beta}$	−0.2037	0.0473	0.0382	0.95
				$\widehat{\beta}_1$	0.0006	0.0470	0.0401	0.94
				$\widehat{\beta}_2$	0.0001	0.0461	0.0380	0.94
				$\log(\widehat{\phi})$	1.4376	0.4622	0.3549	0.99
						$n = 50$		
0.1	−0.9	0.4	2.3	$\widehat{\beta}$	0.0816	0.0809	0.0703	0.92
				$\widehat{\beta}_1$	−0.8572	0.0829	0.0705	0.94
				$\widehat{\beta}_2$	0.3909	0.0771	0.0613	0.95
				$\log(\widehat{\phi})$	3.1047	2.1269	1.1778	0.99
						$n = 100$		
				$\widehat{\beta}$	0.0928	0.0561	0.0477	0.93
				$\widehat{\beta}_1$	−0.8727	0.0581	0.0485	0.94
				$\widehat{\beta}_2$	0.3989	0.0536	0.0448	0.93
				$\log(\widehat{\phi})$	3.3279	1.9031	1.2772	0.98
						$n = 200$		
				$\widehat{\beta}$	0.0969	0.0389	0.0318	0.94
				$\widehat{\beta}_1$	−0.8789	0.0411	0.0384	0.92
				$\widehat{\beta}_2$	0.3855	0.0415	0.0363	0.92
				$\log(\widehat{\phi})$	3.3714	1.5050	1.2099	0.99

Table A4. Simulation study for the RMRS model.

β	β_1	β_2	$\log(\phi)$			se	RMSE	% Cov.
						$n = 50$		
0.1	0.25	0.4	2.3	$\widehat{\beta}$	0.0889	0.0840	0.0672	0.95
				$\widehat{\beta}_1$	0.2455	0.0762	0.0623	0.93
				$\widehat{\beta}_2$	0.3959	0.0708	0.0566	0.96
				$\log(\widehat{\phi})$	2.3443	1.4810	0.6362	1.00
						$n = 100$		
				$\widehat{\beta}$	0.0931	0.0594	0.0465	0.96
				$\widehat{\beta}_1$	0.2513	0.0559	0.0452	0.95
				$\widehat{\beta}_2$	0.4040	0.0619	0.0489	0.95
				$\log(\widehat{\phi})$	2.3964	1.1167	0.5963	0.98

Table A4. *Cont.*

β	β_1	β_2	$\log(\phi)$			se	RMSE	% Cov.
						$n = 200$		
				$\widehat{\beta}$	0.0996	0.0409	0.0339	0.94
				$\widehat{\beta}_1$	0.2509	0.0404	0.0326	0.95
				$\widehat{\beta}_2$	0.3978	0.0375	0.0298	0.96
				$\log(\widehat{\phi})$	2.4670	0.7939	0.4702	0.98
						$n = 50$		
0.1	0	0.4	2.3	$\widehat{\beta}$	0.0911	0.0849	0.0668	0.95
				$\widehat{\beta}_1$	0.0005	0.0832	0.0638	0.96
				$\widehat{\beta}_2$	0.4035	0.0883	0.0703	0.95
				$\log(\widehat{\phi})$	2.3143	1.4615	0.6092	1.00
						$n = 100$		
				$\widehat{\beta}$	0.0944	0.0594	0.0485	0.94
				$\widehat{\beta}_1$	−0.0004	0.0627	0.0503	0.94
				$\widehat{\beta}_2$	0.3993	0.0633	0.0509	0.95
				$\log(\widehat{\phi})$	2.3285	1.0408	0.5454	0.98
						$n = 200$		
				$\widehat{\beta}$	0.0980	0.0415	0.0329	0.95
				$\widehat{\beta}_1$	−0.0014	0.0398	0.0323	0.94
				$\widehat{\beta}_2$	0.3972	0.0395	0.0323	0.95
				$\log(\widehat{\phi})$	2.3969	0.7638	0.4708	0.98
						$n = 50$		
0.1	−0.9	−0.3	2.3	$\widehat{\beta}$	0.0874	0.0854	0.0650	0.96
				$\widehat{\beta}_1$	−0.8848	0.1055	0.0809	0.96
				$\widehat{\beta}_2$	−0.2998	0.0889	0.0712	0.96
				$\log(\widehat{\phi})$	2.4723	1.5877	0.7407	1.00
						$n = 100$		
				$\widehat{\beta}$	0.0935	0.0559	0.0465	0.94
				$\widehat{\beta}_1$	−0.8674	0.0576	0.0525	0.91
				$\widehat{\beta}_2$	−0.2889	0.0520	0.0422	0.94
				$\log(\widehat{\phi})$	3.2666	1.8451	1.2295	0.98
						$n = 200$		
				$\widehat{\beta}$	0.0945	0.0396	0.0325	0.94
				$\widehat{\beta}_1$	−0.8833	0.0435	0.0367	0.94
				$\widehat{\beta}_2$	−0.2956	0.0391	0.0322	0.95
				$\log(\widehat{\phi})$	2.9612	1.1331	0.8559	0.98

Table A5. Simulation study for the RMRS model.

β	β_1	β_2	$\log(\phi)$			se	RMSE	% Cov.
						$n = 50$		
0.1	−0.9	0	2.3	$\widehat{\beta}$	0.0872	0.0836	0.0664	0.96
				$\widehat{\beta}_1$	−0.9020	0.0929	0.0709	0.95
				$\widehat{\beta}_2$	0.0050	0.0927	0.0701	0.97
				$\log(\widehat{\phi})$	2.3302	1.4452	0.6193	0.99

Table A5. *Cont.*

β	β_1	β_2	$\log(\phi)$			se	RMSE	% Cov.
						$n = 100$		
				$\widehat{\beta}$	0.0952	0.0573	0.0451	0.95
				$\widehat{\beta}_1$	−0.8897	0.0605	0.0489	0.94
				$\widehat{\beta}_2$	0.0054	0.0673	0.0526	0.96
				$\log(\widehat{\phi})$	2.7286	1.3207	0.7811	0.99
						$n = 200$		
				$\widehat{\beta}$	0.0955	0.0395	0.0330	0.93
				$\widehat{\beta}_1$	−0.8659	0.0422	0.0446	0.88
				$\widehat{\beta}_2$	−0.0024	0.0456	0.0338	0.94
				$\log(\widehat{\phi})$	3.2724	1.3848	1.1547	0.98
						$n = 50$		
0.1	0.25	−0.3	2.3	$\widehat{\beta}$	0.0876	0.0852	0.0673	0.95
				$\widehat{\beta}_1$	0.2536	0.0796	0.0632	0.96
				$\widehat{\beta}_2$	−0.2993	0.0981	0.0763	0.95
				$\log(\widehat{\phi})$	2.2752	1.4473	0.6438	0.99
						$n = 100$		
				$\widehat{\beta}$	0.0939	0.0596	0.0475	0.95
				$\widehat{\beta}_1$	0.2519	0.0615	0.0469	0.95
				$\widehat{\beta}_2$	−0.2970	0.0661	0.0518	0.95
				$\log(\widehat{\phi})$	2.3496	1.0559	0.5568	0.98
						$n = 200$		
				$\widehat{\beta}$	0.0962	0.0411	0.0327	0.96
				$\widehat{\beta}_1$	0.2509	0.0416	0.0329	0.95
				$\widehat{\beta}_2$	−0.3001	0.0414	0.0334	0.94
				$\log(\widehat{\phi})$	2.4067	0.7695	0.4752	0.98
						$n = 50$		
0.1	0	0	2.3	$\widehat{\beta}$	0.0875	0.0849	0.0634	0.95
				$\widehat{\beta}_1$	0.0003	0.0738	0.0588	0.95
				$\widehat{\beta}_2$	0.0028	0.0838	0.0655	0.96
				$\log(\widehat{\phi})$	2.2455	1.4284	0.6279	0.99
						$n = 100$		
				$\widehat{\beta}$	0.0959	0.0587	0.0461	0.95
				$\widehat{\beta}_1$	0.0009	0.0654	0.0527	0.94
				$\widehat{\beta}_2$	0.0011	0.0613	0.0478	0.95
				$\log(\widehat{\phi})$	2.3922	1.1244	0.5890	0.98
						$n = 200$		
				$\widehat{\beta}$	0.0989	0.0411	0.0330	0.95
				$\widehat{\beta}_1$	−0.0025	0.0409	0.0319	0.96
				$\widehat{\beta}_2$	−0.0011	0.0395	0.0312	0.95
				$\log(\widehat{\phi})$	2.4799	0.8204	0.5018	0.99

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