

Article

Robust Constrained Cooperative Control for Multiple Trains

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Abstract: This paper investigates robust constrained cooperative control for multiple trains, taking into account disturbances, velocity and control input constraints, and nonlinear operation resistances. A robust constrained cooperative control algorithm is employed, utilizing position information from neighboring trains to ensure each train operates within the desired formation. The effects of external disturbances are examined through multiple transformations and the convexity of stochastic matrices, resulting in an error bound for the final relative positions. This error boundary is correlated with the parameters of the system matrix, initial state conditions, and disturbance amplitudes. The theoretical findings are substantiated with a numerical example.

Keywords: multiple trains systems; robust constrained cooperative control; velocity and control input constraints; disturbances

MSC: 93A16

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1. Introduction

Railways play an important role in modern transportation [1]. Numerous research efforts have been dedicated to improving train control systems. Early works on train control primarily focused on cruise control for single trains. For instance, mixed control H_2/H_∞ [2], a robust adaptive control algorithm [3], adaptive iterative learning control [4], and velocity difference control [5] were derived for high-speed trains to study cruise control problems.

The control of multi-agent systems has been extensively studied. The multi-agent model was introduced to address the coordinated cruise control problem for multiple trains [6,7]. In [8], cooperative control was used in multiple train systems. In [9], the proposed event-triggered communication mechanism can reduce the effects of time delay on the coordinated control of multiple trains effectively. Under a moving block signaling mechanism, the adaptive control method was introduced to realize the coordinated control of multiple trains in [10]. Based on the model in [10], the braking distance and redundant safe distance were considered in [11]. Moreover, in [12], the robust constrained cooperative control problem for trains was transformed into an optimization problem that can be solved by the related optimization algorithms.

Additionally, addressing the cooperative control issue for multiple high-speed trains with constraints, [13] proposed a cooperative control strategy that considers constraints. In addition, there has been work focusing on energy-saving problems of coordinated control for multiple train systems in [14,15] and the discussion of cooperative time in [16].

However, most of the above results assumed that each train operates under ideal conditions without considering constraints and disturbances, which may lead to the poor performances of those algorithms under actual scenarios. In [4,17,18], constraints were addressed, but the results were given in the absence of external disturbances and uncertainties. As high-speed railways extend for hundreds of kilometers, the movement of high-speed trains is inevitably affected by complex environments [19]. Real-world factors,

such as operational resistance, sensor measurement errors, and environmental disturbances like wind and rain, impact the control performance. In [19–23], external disturbances and uncertainties were studied, but it was assumed that the trains are not subjected to the constraints of states and inputs. There are few works that address the coexistence of constraints and external disturbances simultaneously. Motivated by these practical challenges, we investigate robust constrained cooperative control for multiple train systems considering disturbances, velocity and control input constraints, and nonlinear operation resistances.

Our work extends the approach in [17] by incorporating external disturbances. The robust constrained cooperative control algorithm using information from neighboring trains can guarantee that each train runs in the desired formation. The effects of external disturbances are studied by using multiple transformations and the convexity of stochastic matrices, based on which an error bound is given for the final relative positions, which are correlated with the parameters of the system matrix, initial state conditions, and disturbance maximums. The main contribution is the simultaneous consideration of four different kinds of nonlinearities and the provision of an explicit upper bound for the effects of external disturbances on robust constrained cooperative control of multiple train systems. This consideration enhances the robustness and reliability of train control systems in practical scenarios.

2. Problem Description

In this section, we present the problem description of a multi-train system, considering disturbances, resistance, and constraints on velocity and control inputs.

2.1. Graph Theory

Let $\mathcal{G}(\mathcal{I}, \mathcal{E})$ be a directed graph composed of n nodes, where $\mathcal{I} = \{1, \dots, n\}$ and $\mathcal{E} \in \mathcal{I} \times \mathcal{I}$ denotes the edge set. Edge $(j, i) \in \mathcal{E}$ if node i can obtain node j 's information. Each edge weight satisfies that $a_{ij} \geq \xi_a$ for some constant $\xi_a > 0$ if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The neighbor set of node i is denoted by $\mathcal{N}_i = \{j \in \mathcal{I} : (i, j) \in \mathcal{E}\}$.

The Laplacian matrix of $\mathcal{G}$ is defined as $[L]_{ij} = -a_{ij}$ and $[L]_{ii} = \sum_{j=1}^n a_{ij}$ for all $i \neq j$. A path is composed of a series of edges such as $(i_1, i_2), (i_2, i_3), \dots$, where $i_j \in \mathcal{I}$. A strongly connected graph is one in which there is a directed path between every node [24]. The non-negative matrix $C \in \mathbb{R}^{n \times n}$ is stochastic if $C\mathbf{1} = \mathbf{1}$ [25].

2.2. Model

Suppose a multiple train system with n trains. Each train is considered as one node in graph $\mathcal{G}$ and is assumed to have the following dynamics

$$\begin{aligned} p_i((k+1)T) &= p_i(kT) + v_i(kT)T + \omega_{i1}(kT), \\ v_i((k+1)T) &= v_i(kT) + [u_i(kT) + \omega_{i3}(kT)]T - r_i(kT)T + \omega_{i2}(kT), \end{aligned} \quad (1)$$

where $p_i(kT), v_i(kT), u_i(kT) \in \mathbb{R}$ are the position, velocity and control input for train i , $\omega_{i1}(kT), \omega_{i2}(kT)$, and $\omega_{i3}(kT)$ represent the different disturbances, such as uncertainty in the system parameters, sensor measurement errors, and uncertainty in air resistance, $r_i(kT)$ denotes the operation resistance, k is the time index, and T represents the sampling period. These disturbances affect the position, velocity, and control input of the trains, respectively. Usually, the operation resistance consists of aerodynamic drag, and mechanical resistance is assumed to have the following form, as in [26],

$$r_i(kT) = c_0 + c_v v_i(kT) + c_a v_i^2(kT), \quad (2)$$

where the coefficients are $c_0 > 0$, $c_v > 0$, and $c_a > 0$. The first two terms represent the mechanical resistance, and the third term is the aerodynamic drag. For simplicity, we use “ (k) ” instead of “ (kT) ” in the following when no confusion can arise.

In actual situations, the velocity and input for all trains usually satisfy some constraints due to the equipment restrictions. To this end, it is assumed that $v_i(k) \in V_i = [0, \bar{\theta}_{1i}] \subset \mathbb{R}$

and $u_i(k) \in U_i = [\underline{\theta}_{2i}, \bar{\theta}_{2i}] \subset \mathbb{R}$ for all trains, where $\bar{\theta}_{1i} > 0$ denotes the maximum velocity, $\underline{\theta}_{2i} < 0$ denotes the maximum braking acceleration, and $\bar{\theta}_{2i} > 0$ denotes the maximum forward acceleration, respectively.

Let $S_{X_i}[\cdot]$ represents the saturation operator for a set $X_i = [\underline{\zeta}_i, \bar{\zeta}_i]$, such that

$$S_{X_i}[x_i] = \begin{cases} x_i, & \underline{\zeta}_i \leq x_i \leq \bar{\zeta}_i \\ \bar{\zeta}_i, & x_i > \bar{\zeta}_i \\ \underline{\zeta}_i, & x_i < \underline{\zeta}_i. \end{cases} \quad (3)$$

The system (1) with constraints is expressed as

$$\begin{aligned} p_i(k+1) &= p_i(k) + v_i(k)T + \omega_{i1}(k), \\ v_i(k+1) &= S_{V_i}[v_i(k) + S_{U_i}[u_i(k) + \omega_{i3}(k)]T - r_i(k)T + \omega_{i2}(k)]. \end{aligned} \quad (4)$$

According to reference [26], the operation resistance is far less than the maximum braking acceleration and forward acceleration for trains. For simplicity of discussion, the following assumption is given.

Assumption 1. Suppose that $|\bar{\theta}_{2i}| > \kappa|r_i(k)|$, $|\underline{\theta}_{2i}| > \kappa|r_i(k)|$ for some constant $\kappa > 10$.

Our objective of this paper is to analyze the effects of the disturbances in an explicit manner when the trains move in the desired formation, i.e.,

$$\lim_{k \rightarrow \infty} |p_i(k) - p_j(k) - d_{ij}| = 0 \quad (5)$$

for all i and j . From reference [17], d_{ij} can be regarded as $d_{ij} = d_i - d_j$ for all $i, j \in \mathcal{I}$, where d_i and d_j are two constants.

3. Main Results

3.1. Robust Constrained Cooperative Control Algorithm

For all trains, the algorithm proposed in [17] is adopted as

$$\begin{aligned} u_i(k) &= u_{ri}(k) + u_{ci}(k) \\ u_{ri}(k) &= c_0 + c_v v_i(k) + c_a v_i^2(k) \\ u_{ci}(k) &= -h_i(v_i(k) - v_0) + \sum_{j \in \mathcal{N}_i} a_{ij}(p_j(k) - p_i(k) + d_{ij}), \end{aligned} \quad (6)$$

where $h_i > 0$ is a constant. It is assumed here that the desired velocity v_0 is known to all trains because of the high requirements of safety.

$u_{ri}(k)$ is used to counteract $r_i(k)$ for all trains, and $u_{ci}(k)$ is used to ensure all trains move in a desired formation by tracking the desired velocity v_0 .

3.2. Model Transformations

In this subsection, we introduce transformations and scaling factors to handle constraints and resistance.

3.2.1. Constraint Set Transformations

In system (4), due to input constraints, $u_{ri}(k)$ cannot compensate the operation resistance directly. Here, we introduce the following equivalent transformations.

Let

$$\bar{V}_i = [-v_0, \bar{\theta}_{1i} - v_0]. \quad (7)$$

From reference [17,27], it follows that $S_{V_i}[v_i(k)] - v_0 = S_{\bar{V}_i}[v_i(k) - v_0]$.

From the definitions of v_0 and $\bar{V}_i$, we have $|S_{\bar{V}_i}[x]| \geq \underline{\beta}_{1i} \triangleq \min\{v_0, \bar{\theta}_{1i} - v_0\}$ and $|S_{\bar{V}_i}[x]| \leq \bar{\beta}_{1i} \triangleq \max\{v_0, \bar{\theta}_{1i} - v_0\}$ if $S_{\bar{V}_i}(x) \neq x$. Define

$$\bar{U}_i(k) = [\underline{\theta}_{2i} - r_i(k), \bar{\theta}_{2i} - r_i(k)]. \quad (8)$$

Then, we have $S_{U_i}[u_i(k)] - r_i(k) = S_{\bar{U}_i(k)}[u_i(k) - r_i(k)]$. From (2) and Assumption 1, we have $c_0 \leq r_i(k) \leq c_0 + c_v \bar{\theta}_{1i} + c_a \bar{\theta}_{1i}^2$.

Similarly, if $S_{\bar{U}_i(k)}[x] \neq x$, we have $|S_{\bar{U}_i(k)}[x]| \geq \underline{\beta}_{2i} \triangleq \min\{|\bar{\theta}_{2i} - c_0|, |\underline{\theta}_{2i} - (c_0 + c_v \bar{\theta}_{1i} + c_a \bar{\theta}_{1i}^2)|\}$ and $|S_{\bar{U}_i(k)}[x]| \leq \bar{\beta}_{2i} \triangleq \max\{|\bar{\theta}_{2i} - c_0|, |\underline{\theta}_{2i} - (c_0 + c_v \bar{\theta}_{1i} + c_a \bar{\theta}_{1i}^2)|\}$.

3.2.2. Model Transformations

Define $\hat{p}_i(k) = p_i(k) - v_0 kT - d_i$, and $\bar{v}_i(k) = v_i(k) - v_0$.

It follows that

$$\begin{aligned} \hat{p}_i(k+1) &= p_i(k+1) - v_0(k+1)T - d_i \\ &= p_i(k) + v_i(k)T + \omega_{i1}(k) - v_0(k+1)T - d_i \\ &= \hat{p}_i(k) + \bar{v}_i(k)T + \omega_{i1}(k) \\ \bar{v}_i(k+1) &= v_i(k+1) - v_0 \\ &= S_{V_i}[v_i(k) + S_{\bar{U}_i(k)}[u_{ri}(k) + u_{ci}(k) \\ &\quad + \omega_{i3}(k) - r_i(k)]T + \omega_{i2}(k)] - v_0 \\ &= S_{V_i}[\bar{v}_i(k) + S_{\bar{U}_i(k)}[-h_i \bar{v}_i(k) + \omega_{i3}(k) \\ &\quad + \sum_{j \in \mathcal{N}_i} a_{ij}(\hat{p}_j(k) - \hat{p}_i(k))]T + \omega_{i2}(k)]. \end{aligned} \quad (9)$$

To remove the constraint operator $S_{F_i}[\cdot]$, the following scaling factors $\rho_{1i}(k)$ and $\rho_{2i}(k)$ are introduced for all i and $k \geq 0$,

$$\rho_{1i}(k) = \begin{cases} \frac{S_{\bar{V}_i}[f_{1i}(k)]}{f_{1i}(k)} & , f_{1i}(k) \neq 0, \\ 1 & , f_{1i}(k) = 0, \end{cases} \quad (10)$$

and

$$\rho_{2i}(k) = \begin{cases} \frac{S_{\bar{U}_i(k)}[f_{2i}(k)]}{f_{2i}(k)} & , f_{2i}(k) \neq 0, \\ 1 & , f_{2i}(k) = 0, \end{cases} \quad (11)$$

where $f_{1i}(k) = \bar{v}_i(k) + S_{\bar{U}_i(k)}[u_{ci}(k) + \omega_{i3}(k)]T + \omega_{i2}(k)$, $f_{2i}(k) = u_{ci}(k) + \omega_{i3}(k)$. According to the definition of $\rho_{1i}(k)$, we have $\|S_{\bar{V}_i}[f_{1i}(k)]\| \leq \|f_{1i}(k)\|$. Note that $\|S_{\bar{V}_i}[f_{1i}(k)]\| > 0$ and $\|f_{1i}(k)\| > 0$. It follows that $0 < \rho_{1i}(k) \leq 1$. Analogously, we have $0 < \rho_{2i}(k) \leq 1$.

Applying algorithm (6), system (4) can be expressed as

$$\begin{aligned} \hat{p}_i(k+1) &= \hat{p}_i(k) + \bar{v}_i(k)T + \omega_{i1}(k), \\ \bar{v}_i(k+1) &= \rho_{1i}(k)\bar{v}_i(k) - \rho_{1i}(k)\rho_{2i}(k)h_i\bar{v}_i(k)T \\ &\quad + \rho_{1i}(k)\rho_{2i}(k)\alpha_i(k)T + \rho_{1i}(k)\omega_{i2}(k) \\ &\quad + \rho_{1i}(k)\rho_{2i}(k)\omega_{i3}(k)T, \end{aligned} \quad (12)$$

where $\alpha_i(k) = \sum_{j \in \mathcal{N}_i} a_{ij}(\hat{p}_j(k) - \hat{p}_i(k))$. In system (12), the constraints of the velocities and the inputs are removed by introducing $\rho_{1i}(k)$ and $\rho_{2i}(k)$. However, there still exists an integral relationship between the states $\hat{p}_i(k)$ and $\bar{v}_i(k)$, which makes the system analysis hard to be proceeded. To this end, a further model transformation needs to be made.

Let

$$\hat{v}_i(k) = \hat{p}_i(k) + c_i \bar{v}_i(k), \quad (13)$$

where c_i is a constant. It follows that

$$\begin{aligned}\hat{p}_i(k+1) &= (1 - \frac{T}{c_i})\hat{p}_i(k) + \frac{T}{c_i}\hat{v}_i(k) + \omega_{i1}(k), \\ \hat{v}_i(k+1) &= (1 - \frac{T}{c_i} - \rho_{1i}(k)(1 - \rho_{2i}(k)h_iT))\hat{p}_i(k) \\ &\quad + (\frac{T}{c_i} + \rho_{1i}(k)(1 - \rho_{2i}(k)h_iT))\hat{v}_i(k) \\ &\quad + c_i\rho_{1i}(k)\rho_{2i}(k)\alpha_i(k)T + c_i\rho_{1i}(k)\omega_{i2}(k) \\ &\quad + c_i\rho_{1i}(k)\rho_{2i}(k)\omega_{i3}(k)T + \omega_{i1}(k).\end{aligned}\quad (14)$$

Let $\psi(k) = [\hat{p}_1(k), \hat{v}_1(k), \dots, \hat{p}_n(k), \hat{v}_n(k)]^T$, and $M(k) = \text{diag}\{M_1(k), \dots, M_n(k)\}$ with

$$M_i(k) = \begin{bmatrix} 1 - \frac{T}{c_i} & \frac{T}{c_i} \\ M_{1i}(k) & M_{2i}(k) \end{bmatrix}, \quad (15)$$

where $\text{diag}\{H_1, \dots, H_m\}$ denotes a block diagonal matrix with H_i as the i th diagonal matrix block for all $i \in \{1, 2, \dots, m\}$. Let $M_{1i}(k) = 1 - \frac{T}{c_i} - \rho_{1i}(k)(1 - \rho_{2i}(k)h_iT)$, $M_{2i}(k) = \frac{T}{c_i} + \rho_{1i}(k)(1 - \rho_{2i}(k)h_iT)$, $Q = \text{diag}\{Q_1, \dots, Q_n\}$ with $Q_i = \begin{bmatrix} 0 & 0 \\ 0 & c_i \end{bmatrix}$, $P_1(k) = \text{diag}\{\rho_{11}(k), \dots, \rho_{1n}(k)\}$, $P_2(k) = \text{diag}\{\rho_{21}(k), \dots, \rho_{2n}(k)\}$ and $P(k) = P_1(k)P_2(k) = \text{diag}\{\rho_{11}(k)\rho_{21}(k), \dots, \rho_{1n}(k)\rho_{2n}(k)\}$, $F = \begin{bmatrix} 0 & 0 \\ T & 0 \end{bmatrix}$, $\omega_1(k) = [\omega_{11}(k), \dots, \omega_{n1}(k)]^T$, $\omega_2(k) = [\omega_{12}(k), \dots, \omega_{n2}(k)]^T$, and $\omega_3(k) = [\omega_{13}(k), \dots, \omega_{n3}(k)]^T$.

Write system (14) in a matrix form:

$$\psi(k+1) = \{M(k) - Q[P(k)L \otimes F]\}\psi(k) + \omega(k), \quad (16)$$

where $\omega(k) = \omega_1(k) \otimes [1, 1]^T + Q\{[P_1\omega_2(k) + TP\omega_3(k)] \otimes [0, 1]^T\}$, and $\otimes$ denotes the Kronecker product.

3.3. Convergence Analysis

In this subsection, we analyze the convergence of the robust constrained cooperative control algorithm for train systems.

Before the main analysis, some results are presented.

Assumption 2. Suppose that $\omega_1 = \max_{i,k}\{|\omega_{i1}(k)|\}$, $\omega_2 = \max_{i,k}\{|\omega_{i2}(k)|\}$, $\omega_3 = \max_{i,k}\{|\omega_{i3}(k)|\}$, $0 < h_iT < 1$ and $q_i = \max\{L\}_{ii} < \frac{\eta_{2i}^2 h_i^2}{4}$ for all i and $k \geq 0$, where $\eta_{2i} = \min\{\frac{\beta_{2i}}{T} / (\frac{\beta_{1i}}{T} + nq_i(\max_{l,s}\{|\psi_l(0) - \psi_s(0)|\} + 4n(\omega_1 + c_i\omega_2 + c_i\omega_3T)) + \omega_3), 1\}$. Let $c_i = \frac{2}{\eta_{2i}h_i}$; it is clear that $q_i < \frac{1}{c_i^2}$.

Lemma 1. Under Assumptions 1 and 2,

$$\eta_{1i} = \frac{\beta_{1i}}{\beta_{1i} + \beta_{2i}T + \omega_2} \leq \rho_{1i}(k) \leq 1, \quad (17)$$

and

$$\eta_{2i} \leq \rho_{2i}(0) \leq 1. \quad (18)$$

Proof. Since $S_{\bar{U}_i(k)}[-h_i\bar{v}_i(k) + \alpha_i(k) + \omega_{i3}(k)] \in \bar{U}_i(k)$ and $\bar{v}_i(k) \in \bar{V}_i$, it is clear that $\bar{v}_i(k) \leq \bar{\beta}_{1i}$, $S_{\bar{U}_i(k)}[-h_i\bar{v}_i(k) + \alpha_i(k) + \omega_{i3}(k)] \leq \bar{\beta}_{2i}$, and $\bar{v}_i(k) + S_{\bar{U}_i(k)}[-h_i\bar{v}_i(k) + \alpha_i(k) + \omega_{i3}(k)]T + \omega_{i2}(k) \leq \bar{\beta}_{1i} + \bar{\beta}_{2i}T + \omega_2$. $\square$

Therefore, we have

$$\frac{\underline{\beta}_{1i}}{\underline{\beta}_{1i} + \underline{\beta}_{2i}T + \omega_2} \leq \rho_{1i}(k) \leq 1, \quad (19)$$

for all i and $k \geq 0$. Also, from the definition of $\rho_{2i}(k)$, $\rho_{2i}(k) = 1$ if $S_{\bar{U}_i(k)}[-h_i\bar{v}_i(k) + \alpha_i(k) + \omega_{i3}(k)] = -h_i\bar{v}_i(k) + \alpha_i(k) + \omega_{i3}(k)$, and $S_{\bar{U}_i(k)}[-h_i\bar{v}_i(k) + \alpha_i(k) + \omega_{i3}(k)] \geq \underline{\beta}_{2i}$ if $S_{\bar{U}_i(k)}[-h_i\bar{v}_i(k) + \alpha_i(k) + \omega_{i3}(k)] \neq -h_i\bar{v}_i(k) + \alpha_i(k) + \omega_{i3}(k)$. Note that $|S_{\bar{V}_i}(x)| \leq \underline{\beta}_{1i}$; therefore, we have $\rho_{2i}(0) \geq \min\{\underline{\beta}_{2i}/(\frac{\underline{\beta}_{1i}}{T} + nq_i \max_{l,s}\{|\psi_l(0) - \psi_s(0)|\} + 4n(\omega_1 + c_i\omega_2 + c_i\omega_3T) + \omega_3), 1\}$.

Hence,

$$\rho_{2i}(0) \geq \eta_{2i}, \quad (20)$$

for all i .

Lemma 2. Under Assumptions 1 and 2, the system matrix $\Psi(0,0)$ is a stochastic matrix with all nonzero elements larger than a positive constant.

Proof. Under Assumptions 1 and 2, from Lemma 1, we have $c_i = \frac{2}{\eta_{2i}h_i} \geq \frac{2}{\rho_{2i}(0)h_i}$. It is found that $\frac{T}{c_i} \leq \frac{\rho_{2i}(0)h_iT}{2} \leq \frac{1}{2}$ and

$$\begin{aligned} & 1 - \frac{T}{c_i} - \rho_{1i}(0)(1 - \rho_{2i}(0)h_iT) \\ & \geq -\frac{T}{c_i} + \rho_{2i}(0)h_iT \\ & \geq \frac{\rho_{2i}(0)h_iT}{2} \\ & > 0. \end{aligned} \quad (21)$$

Consider that $[L]_{ii} \leq q_i < \frac{1}{c_i^2}$. It is found that

$$\begin{aligned} & 1 - \frac{T}{c_i} - \rho_{1i}(0)(1 - \rho_{2i}(0)h_iT) - c_i\rho_{1i}(0)\rho_{2i}(0)T[L]_{ii} \\ & > \frac{\rho_{2i}(0)h_iT}{2} - c_iq_iT \\ & \geq \frac{T}{c_i} - \frac{T}{c_i} \\ & = 0. \end{aligned} \quad (22)$$

Note that $a_{ij} \geq \xi_a$ if $a_{ij} \neq 0$. Thus, all elements of $\Psi(0,0)$ are non-negative. Also, note that the sum of each row in $M(k)$ is 1 and $L\mathbf{1} = \mathbf{0}$. It follows that $\Psi(0,0)\mathbf{1} = M(0)\mathbf{1} = \mathbf{1}$. Therefore, $\Psi(0,0)$ is a stochastic matrix with all nonzero elements larger than a positive constant. $\square$

3.4. The Effects of External Disturbances

In this subsection, we discuss the system in the presence of external disturbances, i.e., $\omega_1 \neq 0, \omega_2 \neq 0$ and $\omega_3 \neq 0$. By considering disturbance models, we assessed the algorithm's robustness and its ability to ensure that the trains reach a desired formation with bounded relative position errors despite external disturbances.

Let $\bar{\mu} = \min_i\{\frac{\eta_{2i}h_iT}{2} - \frac{2q_iT}{\eta_{2i}h_i}, \frac{2\eta_{1i}\xi_aT}{h_i}\}$, $\lambda_1 = \bar{\mu}^{2n}$ and $\lambda_2 = 2n(\omega_1 + c_i\omega_2 + c_i\omega_3T)$. Let $\Psi(k,s) = \Pi_{l=s}^k\{M(l) - Q[P(l)L \otimes F]\}$. System (16) is equivalent to

$$\psi(k+1) = \Psi(k,s)\psi(s) + \sum_{l=s}^{k-1} (\Psi(k,l+1)w(l)) + w(k), \quad (23)$$

for all $k > s$.

Lemma 3. Under Assumptions 1 and 2, $\rho_{2i}(k) \geq \eta_{2i}$ for all i and the system matrix $\Psi(k, k)$ is a stochastic matrix with all nonzero elements larger than a positive constant $\bar{\mu}$ for $0 \leq k < 2n$.

Proof. Under Assumptions 1 and 2, according to Lemma 2, $\Psi(0, 0)$ is a stochastic matrix. It follows that $\max_i \psi_i(1) - \min_i \psi_i(1) \leq \max_i \psi_i(0) - \min_i \psi_i(0) + 2(\omega_1 + c_i \omega_2 + c_i \omega_3 T) \leq \max_{l,s} \{|\psi_l(0) - \psi_s(0)|\} + 4n(\omega_1 + c_i \omega_2 + c_i \omega_3 T)$, and we have $\rho_{2i}(1) \geq \eta_{2i}$, similar to Lemmas 2, $\Psi(1, 1)$ is a stochastic matrix with all nonzero elements larger than $\bar{\mu}$. By analogy, $\rho_{2i}(k) \geq \eta_{2i}$ for all i and $\Psi(k, k)$ is a stochastic matrix with all nonzero elements larger than $\bar{\mu}$ for $0 \leq k < 2n$. $\square$

Theorem 1. Consider that $\mathcal{G}$ is strongly connected and $\max_{l,s} \{|\psi_l(0) - \psi_s(0)|\} > \frac{\lambda_2}{\lambda_1} + 2\lambda_2$. Under Assumptions 1 and 2, applying control algorithm (6), all trains finally run in a desired formation with some certain relative position errors, i.e., $-\frac{\lambda_2}{\lambda_1} - 2\lambda_2 \leq \lim_{k \rightarrow \infty} [p_i(k) - p_j(k) - d_{ij}] \leq \frac{\lambda_2}{\lambda_1} + 2\lambda_2$.

Proof. From Lemma 3, $\rho_{2i}(k) \geq \eta_{2i}$ for all i , and $\Psi(k, k)$ is a stochastic matrix with all nonzero elements larger than $\bar{\mu}$ for $0 \leq k < 2n$. It is proven that $\Psi(2n - 1, 0)$ is a stochastic matrix and from [reference [28], Lemma 2], we have

$$[\Psi(2n - 1, 0)]_{ij} \geq \lambda_1 = \bar{\mu}^{2n}, \quad (24)$$

for all $i, j \in \{1, 2, \dots, 2n\}$. It follows that

$$\psi(2n) = \Psi(2n - 1, 0)\psi(0) + \sum_{l=0}^{2n-2} (\Psi(2n - 1, l + 1)w(l)) + w(2n - 1). \quad (25)$$

Note that $\max_i (\sum_{l=0}^{2n-2} (\Psi(2n - 1, l + 1)w(l)) + w(2n - 1)) \leq \lambda_2$.

Construct a Lyapunov function as

$$V(k) = \max_s \{\psi_l(k)\} - \min_s \{\psi_s(k)\}. \quad (26)$$

From (24), it follows that

$$\max_s \{\psi_s(2n)\} \leq (1 - \lambda_1) \max_s \{\psi_s(0)\} + \lambda_1 \min_s \{\psi_s(0)\} + \lambda_2, \quad (27)$$

and

$$\min_s \{\psi_s(2n)\} \geq (1 - \lambda_1) \min_s \{\psi_s(0)\} + \lambda_1 \max_s \{\psi_s(0)\} - \lambda_2. \quad (28)$$

As a consequence,

$$\begin{aligned} V(2n) &= \max_s \{\psi_s(2n)\} - \min_s \{\psi_s(2n)\} \\ &\leq (1 - 2\lambda_1)(\max_s \{\psi_s(0)\} - \min_s \{\psi_s(0)\}) + 2\lambda_2 \\ &= (1 - 2\lambda_1)V(0) + 2\lambda_2. \end{aligned} \quad (29)$$

Then,

$$V(2n) - V(0) \leq -2\lambda_1 V(0) + 2\lambda_2 = -2\lambda_1 [V(0) - \lambda_2 / \lambda_1]. \quad (30)$$

$\square$

It follows that $V(2n) - V(0) < 0$, $\rho_{2i}(2n) \geq \eta_{2i}$ and $\Psi(2n, 2n)$ is a stochastic matrix with all nonzero elements larger than $\bar{\mu}$. $\Psi(2n, 1)$ is also a stochastic matrix with all

elements larger than $\bar{\mu}^{2n}$. Similar to (27)–(29), it can be proven that $V(2n+1) - V(1) \leq -2\lambda_1[V(1) - \lambda_2/\lambda_1]$.

Note that when $V(1) > \lambda_2/\lambda_1$, $V(2n+1) - V(1) < 0$. When $V(1) \leq \lambda_2/\lambda_1$, from the stochasticity of $\Psi(2n, 1)$, under Assumption 2, $\max_i \psi_i(2n+1) - \min_i \psi_i(2n+1) \leq \max_i \psi_i(1) - \min_i \psi_i(1) + 2(\omega_1 + c_i\omega_2 + c_i\omega_3 T) < \lambda_2/\lambda_1 + 2\lambda_2 \leq \max_i \psi_i(0) - \min_i \psi_i(0) + 4n(\omega_1 + c_i\omega_2 + c_i\omega_3 T)$ and $\rho_{2i}(2n+1) \geq \eta_{2i}$. $\Psi(2n+1, 2n+1)$ is a stochastic matrix with all nonzero elements larger than $\bar{\mu}$.

By analogy, $\rho_{2i}(k) \geq \eta_{2i}$, and $\Psi(k, k)$ is a stochastic matrix with all nonzero elements larger than $\bar{\mu}$ for all i and $k \geq 2n$. We have $\lim_{\bar{k} \rightarrow \infty} V(2\bar{k}n) \leq \frac{\lambda_2}{\lambda_1} + 2\lambda_2$ and $\lim_{\bar{k} \rightarrow \infty} V(2\bar{k}n + 1) \leq \frac{\lambda_2}{\lambda_1} + 2\lambda_2$. By analogy, it can be proven that $\lim_{\bar{k} \rightarrow \infty} V(2\bar{k}n + \bar{k}) \leq \frac{\lambda_2}{\lambda_1} + 2\lambda_2$, where $0 \leq \bar{k} \leq 2n-1$. Hence, $\lim_{k \rightarrow \infty} V(k) \leq \frac{\lambda_2}{\lambda_1} + 2\lambda_2$. Finally, from the relationship of $p_i(k)$, $\psi_i(k)$, and $V(k)$, it follows that $-\frac{\lambda_2}{\lambda_1} - 2\lambda_2 \leq \lim_{k \rightarrow \infty} [p_i(k) - p_j(k) - d_{ij}] \leq \frac{\lambda_2}{\lambda_1} + 2\lambda_2$ for all i, j .

4. Simulation

Consider a multiple train system with four trains. Suppose that the communication graph is as given in Figure 1, where the weight of each edge is 0.01. The maximum velocity of the trains, denoted as $\bar{\theta}_{1i}$, is 85 m/s. The control input parameters h_i and a_{ij} are set according to selection rule given by Assumption 2. The control input constraint set is $U_i = [-2 \text{ m/s}^2, 2 \text{ m/s}^2]$. The coefficients of $r_i(k)$ are $c_0 = 1.176 \times 10^{-2} \text{ N/kg}$, $c_v = 7.7616 \times 10^{-4} \text{ N} \cdot \text{s}/(\text{m} \cdot \text{kg})$ and $c_a = 1.6 \times 10^{-5} \text{ N} \cdot \text{s}^2/(\text{m}^2 \cdot \text{kg})$. The desired relative positions of all trains are set as $d_{12} = d_{23} = d_{34} = 8000 \text{ m}$. The sampling time T is 0.8 s. The initial states of four trains are given by $p_1(0) = 27,000 \text{ m}$, $p_2(0) = 18,500 \text{ m}$, $p_3(0) = 7500 \text{ m}$, $p_4(0) = 0 \text{ m}$, and $v_1(0) = v_2(0) = v_3(0) = v_4(0) = 0 \text{ m/s}$ and $v_0 = 80 \text{ m/s}$. The feedback gain is $h_i = 0.3$.

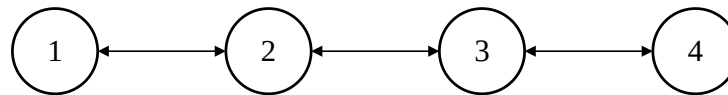


Figure 1. The directed communication graph.

Based on Theorem 1, when the controller coefficients are determined, the amplitude of disturbances needs to establish $\frac{\lambda_2}{\lambda_1} + 2\lambda_2 < d_{ij}$ for any $d_{ij} > 0$. Accordingly, suppose that the disturbances are $\omega_{11}(k) = 0.1 \text{ m}$, $\omega_{12}(k) = 0.1 \text{ m/s}$, $\omega_{13}(k) = 0.01 \text{ m/s}^2$, $\omega_{21}(k) = -0.1 \text{ m}$, $\omega_{22}(k) = 0.2 \text{ m/s}$, $\omega_{23}(k) = -0.01 \text{ m/s}^2$, $\omega_{31}(k) = 0.1 \text{ m}$, $\omega_{32}(k) = -0.1 \text{ m/s}$, $\omega_{33}(k) = 0.01 \text{ m/s}^2$, $\omega_{41}(k) = -0.1 \text{ m}$, $\omega_{42}(k) = -0.2 \text{ m/s}$, and $\omega_{43}(k) = -0.01 \text{ m/s}^2$.

Figures 2 and 3 show the states and control inputs of all trains. The distances between neighboring trains are shown in Figures 4 and 5. The results indicate that all trains will finally move in a desired formation with some certain relative position errors. By calculation, $\lim_{k \rightarrow \infty} |p_i(k) - p_j(k) - d_{ij}| \leq 7694 \text{ m}$. The original relative position of 8000 m plus the maximum potential error of 7694 m, resulting in a total distance of 15,694 m, is acceptable because it meets the current safety requirements based on practical scheduling guidelines.

By taking disturbances into consideration, we assessed the algorithm's robustness and its ability to ensure that the trains reach a desired formation with bounded relative position errors despite external disturbances. The simulation results indicate that trains successfully keep the desired formation with bounded relative position errors despite external disturbances, demonstrating its effectiveness in disturbance scenarios.

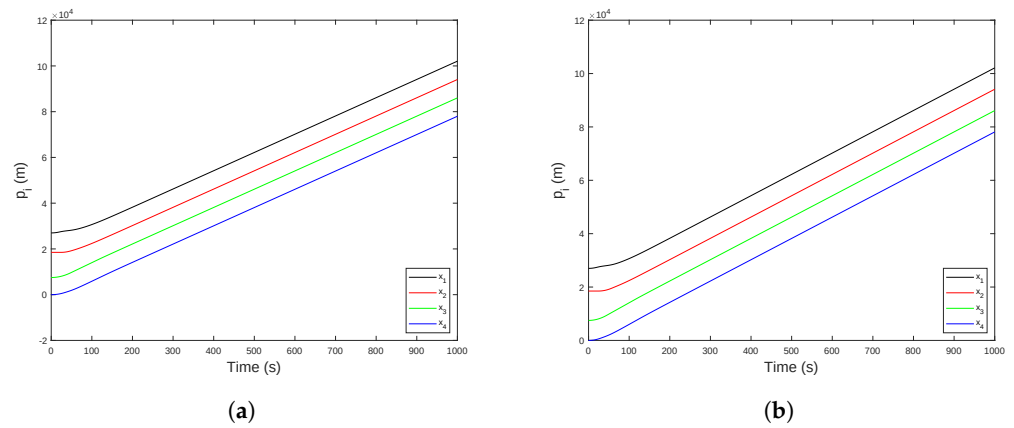


Figure 2. The position trajectories of all trains (a) in the presence of disturbances and (b) in the absence of disturbances.

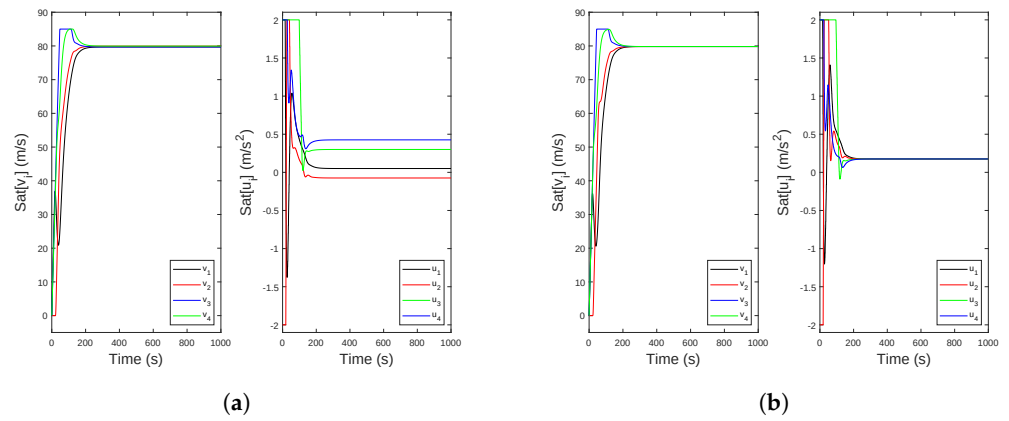


Figure 3. The velocity and control input trajectories of all trains (a) in the presence of disturbances and (b) in the absence of disturbances.

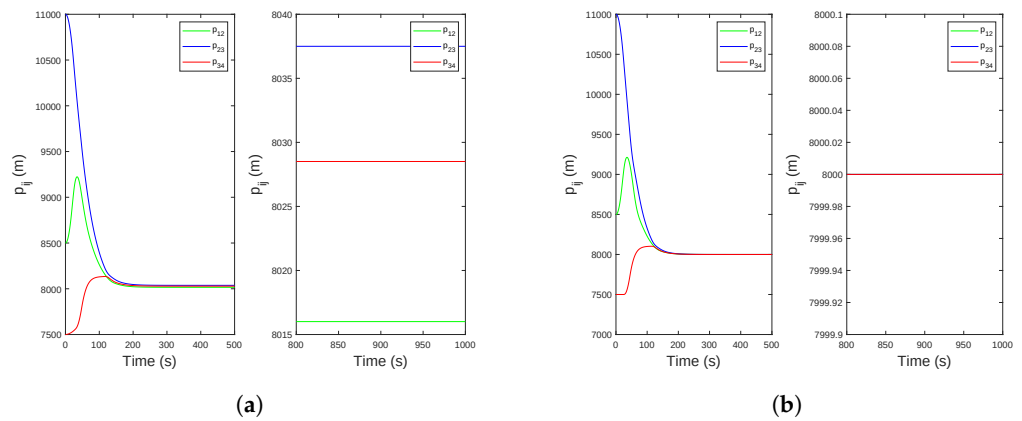


Figure 4. Relative position trajectories between neighboring trains (a) in the presence of disturbances and (b) in the absence of disturbances.

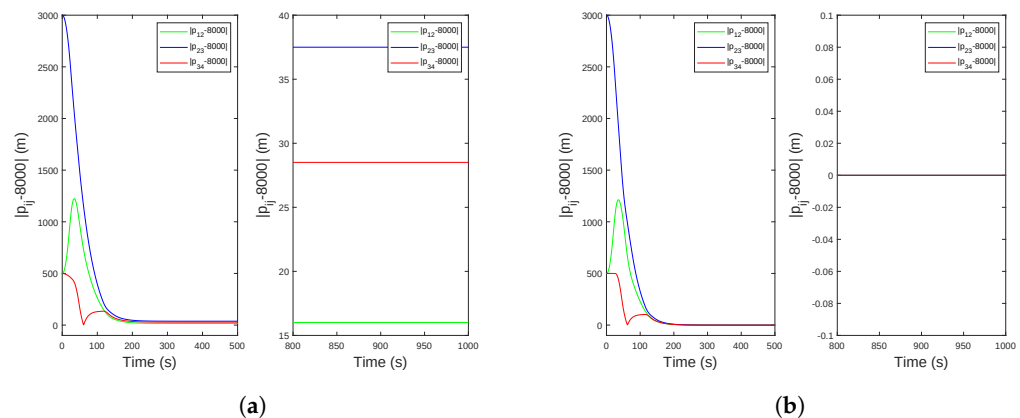


Figure 5. The error trajectories $|p_{ij} - d_{ij}|$ (a) in the presence of disturbances and (b) in the absence of disturbances.

5. Conclusions

This paper studied robust constrained cooperative control for multiple train systems with disturbances. A robust constrained cooperative control algorithm using position information of neighboring trains is employed to guarantee that all trains can run in a desired formation. The effects of disturbances on robust cooperative control, input constraints, and velocity constraints for multiple train systems were considered, based on which an explicit upper bound is given for the final relative positions, which are related to the initial state conditions and the disturbance maximums. This work might be the first effort to identify the specific impact boundaries for multiple train systems under the influence of external disturbances. This paper demonstrates that the robust constrained cooperative control method enables the safe operation of multiple trains systems under these nonlinearities, enhancing the robustness and reliability of train control systems in practical scenarios. Future work could be directed to consider communication delays and collision avoidance.

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