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Convergence and Almost Sure Polynomial Stability of Partially Truncated Split-Step Theta Method for Stochastic Pantograph Models with Lévy Jumps

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Abstract: This paper addresses a stochastic pantograph model with Lévy leaps where non-jump coefficients exceed linearity. The partially truncated split-step theta method is introduced and applied to the proposed model. The finite time $\mathcal{L}^{\hat{q}}(\hat{q} \geq 2)$ convergence rate of the numerical scheme is obtained. Furthermore, the almost sure polynomial stability of the numerical scheme is investigated and numerical examples are presented to endorse the addressed theorems.

Keywords: stochastic pantograph models; Lévy jumps; split-step theta method; convergence rate; almost sure polynomial stability

MSC: 60H10; 65C30; 60H35



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1. Introduction

Stochastic pantograph models are much more substantial, and they have been harnessed to describe dynamical behaviours with haphazard manners. They are used in many majors such as finance and quantum mechanics [1,2]. Also, the Weiner process is not always the optimal direction for modelling dynamical cases that have sudden changes and bizarre events. In these scenarios, jump models are much more convenient for handling these situations [3,4]. Merton [5] updated the Black–Scholes model [6] by augmenting the Poisson process. In this paper, we want to take into account history data [7–9] and also try to apply a more generic and wide jump process, such as a Lévy process [10]. Therefore, we shine light on the stochastic pantograph model with Lévy jumps. Stochastic pantograph models with Lévy jumps can be applied in real-life applications such as financial markets, where the partially truncated split-step theta method can be applied for capturing stock price behaviour, allowing for better pricing and risk management in financial markets. Furthermore, in some financial situations we need to approximate the variance or the higher moment of the solution. In these cases, we need to have the convergence in $\mathcal{L}^{\hat{q}}(\hat{q} \geq 2)$ sense. Also, stochastic pantograph models can be employed to study the spread of infectious diseases and to analyse the effectiveness of control strategies, where the applicability of the proposed scheme can be utilized to simulate an epidemic's progression accurately, capturing the impact of delays and sudden changes in the infection rate, and aiding in designing effective intervention strategies.

Also, it is not always an easy task to figure out the exact solution of stochastic models. Therefore, numerical schemes [11–13] are indispensable. Mao [14] addressed the truncated Euler–Maruyama algorithm and studied its convergence rate and stability. In recent years, great attention has been paid to split-step theta methods because of their abilities to possess desirable stability characteristics and rates of convergence [15–21]. Higham et al. [22]

introduced the implicit split-step variant of the Euler–Maruyama technique for stochastic models, where the drift and diffusion coefficients satisfy the one-sided Lipschitz and the global Lipschitz conditions, respectively. Wang and Liu [23] discussed the split-step backward balanced Milstein techniques for stiff Itô stochastic models. Moreover, these attempts were further updated and evolved by Wang and Li [24], who introduced the fully explicit split-step forward techniques for solving Itô stochastic models under the global Lipschitz and linear growth conditions. According to some authors [25–27], the split-step theta scheme has the ability of attaining the stability of a system with a free choice of the step size. Furthermore, inclusion of the Euler–Maruyama scheme and split-step backward Euler scheme inside it by adjusting the value of theta to $\theta = 0$ or $\theta = 1$, respectively, can also be considered as an advantage. The split-step backward Euler technique applied to linear stochastic delay models was investigated by Zhang et al. [28]. Cao et al. [29] applied the split-step theta scheme to stochastic delay models and showed that the numerical scheme has an order of convergence of half, and an exponential mean square stability. Bao and Hu [30] focused their research on stochastic variable delay models; they applied the split-step theta scheme to the model and studied its convergence and mean square stability. Wang et al. [23] showed the drifting split-step backward Milstein technique. The split-step Adams–Moulton Milstein technique was discussed by Voss et al. [31]. A multi-step Maruyama technique for stochastic delay models was discussed by Bauckwar et al. [32]. Lu et al. [33] applied a split-step composite theta method to stochastic delay models and studied the convergence and stability of them. The stability of a split-step composite theta technique for stochastic models with non-variable delay was discussed in [34].

To the best of our knowledge, there are not many studies on split-step theta schemes applied to stochastic models that take into consideration history data and are interspersed with the Lévy process where coefficients act super linearly. In this paper, following the approach of the partially truncated technique [35], we address the partially truncated split-step theta algorithm and apply it to our aforementioned model, where coefficients could exceed linearity. Then, we study the convergence rate in $\mathcal{L}^{\hat{q}}(\hat{q} \geq 2)$ and the almost sure polynomial stability of the proposed scheme.

The set up of the upcoming parts will be as follows. The model structure, the partially truncated SS θ -method, and its properties will be presented in Section 2. Section 3 addresses the convergence rate. The almost sure polynomial stability of the numerical algorithm will be depicted in Section 4. Section 5 will present some examples. Finally, we will wrap up Section 6 with some conclusions.

2. Partially Truncated Split-Step Theta Method

Let C be a real positive constant changing throughout the analysis and ϵ^* represent the expectation operator. Then, consider the following o -dimensional stochastic model

$$ds(g) = u(s(g), s(\eta g))dg + r(s(g), s(\eta g))dB(g) + \int_M w(s(g), s(\eta g), m)\tilde{J}(dg, dm), \quad (1)$$

defined on $0 \leq g \leq G$ with $0 < \eta < 1$ and $s(0) = s_0$. Here, $B(g)$ is d -dimensional Brownian motion and $\tilde{J}(dg, dm) = J(dg, dm) - \pi(dm)dg$ is the compensated Poisson random measure independent of $B(g)$ with Lévy measure π defined on set $M \in \mathbb{R}^o \setminus \{0\}$ with $\pi(M) = \lambda$. $u : \mathbb{R}^o \times \mathbb{R}^o \rightarrow \mathbb{R}^o$, $r : \mathbb{R}^o \times \mathbb{R}^o \rightarrow \mathbb{R}^{o \times d}$ and $w : \mathbb{R}^o \times \mathbb{R}^o \times M \rightarrow \mathbb{R}^o$, $o, d \in \mathbb{N}^+$. It is assumed that the coefficients u and r can be factored as $u(\iota_1, \bar{\iota}_1) = u_1(\iota_1, \bar{\iota}_1) + u^*(\iota_1, \bar{\iota}_1)$ and $r(\iota_1, \bar{\iota}_1) = r_1(\iota_1, \bar{\iota}_1) + r^*(\iota_1, \bar{\iota}_1)$, where $u_1, u^* : \mathbb{R}^o \times \mathbb{R}^o \rightarrow \mathbb{R}^o$ and $r_1, r^* : \mathbb{R}^o \times \mathbb{R}^o \rightarrow \mathbb{R}^{o \times d}$.

Assumption 1. *There exist constants $p \geq 2$, $k_2 > 0$ and $k_3 > 0$, such that*

$$|Y(\iota, \bar{\iota}) - Y(\iota_2, \bar{\iota}_2)|^2 \leq k_2(|\iota_1 - \iota_2|^2 + |\bar{\iota}_1 - \bar{\iota}_2|^2),$$

and

$$\int_M |w(\iota_1, \bar{\iota}_1, m) - w(\iota_2, \bar{\iota}_2, m)|^p \pi(dm) \leq k_3(|\iota_1 - \iota_2|^p + |\bar{\iota}_1 - \bar{\iota}_2|^p),$$

for all $\iota_1, \iota_2, \bar{\iota}_1, \bar{\iota}_2 \in \mathbb{R}^0$, $m \in M$ and $Y = u_1$ or r_1 .

Assumption 2 (Khasminskii-type condition). *There exists a constant $\bar{q} > 2$, such that*

$$\langle \iota, u^*(\iota, \bar{\iota}) \rangle + \frac{\bar{q} - 1}{2} |r^*(\iota, \bar{\iota})|^2 \leq C(1 + |\iota|^2 + |\bar{\iota}|^2),$$

for all $\iota, \bar{\iota} \in \mathbb{R}^0$.

By utilizing Assumptions 1 and 2, it can be concluded that

$$\langle \iota, u(\iota, \bar{\iota}) \rangle + \frac{\hat{q} - 1}{2} |r(\iota, \bar{\iota})|^2 \leq C(1 + |\iota|^2 + |\bar{\iota}|^2), \quad (2)$$

for all $\iota, \bar{\iota} \in \mathbb{R}^0$ and $\hat{q} \in [2, \bar{q}]$. Under Assumptions 1 and 2, there exists a unique solution to Equation (1) (see, e.g., [36,37]). To define the partially truncated split-step theta (PTSS θ) scheme, we utilize the truncation technique discussed in [14], where we pick up $\Delta^* \in (0, 1]$, increasing function $\Gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and decreasing function $\varepsilon : (0, \Delta^*] \rightarrow (0, \infty)$, such that

$$\sup_{|\iota| \vee |\bar{\iota}| \leq r^*} |\Phi^*(\iota, \bar{\iota})| \leq \Gamma(r^*), \quad \forall r^* \geq 1, \quad (3)$$

$$(\Delta^{1/\bar{q}} \vee \Delta^{1/4}) \varepsilon(\Delta) \leq 1, \quad \forall \Delta \in (0, \Delta^*], \quad (4)$$

where $\Phi^* = u^*$ or r^* . Then, our partially truncated coefficients are given by

$$u_\Delta^*(\iota, \bar{\iota}) = u^*(T_\Delta^*(\iota), T_\Delta^*(\bar{\iota})), \quad r_\Delta^*(\iota, \bar{\iota}) = r^*(T_\Delta^*(\iota), T_\Delta^*(\bar{\iota})), \quad (5)$$

where $\iota, \bar{\iota} \in \mathbb{R}^0$, step size $\Delta \in (0, \Delta^*]$, and $T_\Delta^*(\iota) = (|\iota| \wedge \Gamma^{-1}(\varepsilon(\Delta))) \frac{\iota}{|\iota|}$ is the truncation mapping from $\mathbb{R}^0$ to the closed ball $\{\iota \in \mathbb{R}^0 : |\iota| \leq \Gamma^{-1}(\varepsilon(\Delta))\}$.

Lemma 1. *Under Assumption 2,*

$$\langle \iota, u_\Delta^*(\iota, \bar{\iota}) \rangle + \frac{\bar{q} - 1}{2} |r_\Delta^*(\iota, \bar{\iota})|^2 \leq C(1 + |\iota|^2 + |\bar{\iota}|^2), \quad \forall \iota, \bar{\iota} \in \mathbb{R}^0 \quad (6)$$

Proof. The concept of the proof is similar to that discussed in [38]. $\square$

Now, the partially truncated split-step theta (PTSS θ) scheme applied to Equation (1) is defined by $Q_0 = s_0$ and for $n = 0, 1, \dots$

$$v_n^* = Q_n + \theta \Delta u_\Delta(v_n^*, v_{[\eta n]}^*), \quad (7)$$

$$\begin{aligned} Q_{n+1} &= Q_n + \Delta u_\Delta(v_n^*, v_{[\eta n]}^*) + r_\Delta(v_n^*, v_{[\eta n]}^*) \Delta B_n \\ &\quad + \int_M w(v_n^*, v_{[\eta n]}^*, m) \Delta \tilde{f}(dm), \end{aligned} \quad (8)$$

where $\theta \in [0, 1]$, $[\iota]$ takes the integer part of ι and $Q_n \approx s(g_n)$ at $g_n = n\Delta$, $\Delta B_n := B(g_{n+1}) - B(g_n)$ and $\Delta \tilde{f}(dm) = \tilde{f}((0, (n+1)\Delta], dm) - \tilde{f}((0, n\Delta], dm)$. Here,

$$u_\Delta(v_n^*, v_{[\eta n]}^*) = u_1(v_n^*, v_{[\eta n]}^*) + u_\Delta^*(v_n^*, v_{[\eta n]}^*), \quad (9)$$

$$r_\Delta(v_n^*, v_{[\eta n]}^*) = r_1(v_n^*, v_{[\eta n]}^*) + r_\Delta^*(v_n^*, v_{[\eta n]}^*). \quad (10)$$

By utilizing Assumptions 1 and 2 and proceeding the same as in [39], it can be concluded that for all $\Delta \in (0, \Delta^*]$ and $\hat{q} \in [2, \bar{q}]$,

$$\langle \iota, u_\Delta(\iota, \bar{\iota}) \rangle + \frac{\hat{q} - 1}{2} |r_\Delta(\iota, \bar{\iota})|^2 \leq C(1 + |\iota|^2 + |\bar{\iota}|^2), \quad \forall \iota, \bar{\iota} \in \mathbb{R}^0. \quad (11)$$

For all $t \in [g_n, g_{n+1})$ and $\Delta \in (0, \Delta^*]$, we define continuous time approximation

$$\begin{aligned}\vartheta_{\Delta}(g) &:= Q_n + (g - g_n)u_{\Delta}(v_n^*, v_{[\eta n]}^*) + r_{\Delta}(v_n^*, v_{[\eta n]}^*)(B(g) - B(g_n)) \\ &\quad + \int_{g_n}^g \int_M w(v_n^*, v_{[\eta n]}^*, m) \tilde{J}(dg, dm),\end{aligned}\quad (12)$$

and denote $l_1(g) = \sum_{\bar{r}=0}^{\infty} v_{\bar{r}}^* 1_{[g_{\bar{r}}, g_{\bar{r}+1})}(g)$ and $l_2(g) = \sum_{\bar{r}=0}^{\infty} v_{[\eta \bar{r}]}^* 1_{[g_{\bar{r}}, g_{\bar{r}+1})}(g)$. Therefore, Equation (12) can be defined in an integral form as follows

$$\begin{aligned}\vartheta_{\Delta}(g) &= s_0 + \int_0^g u_{\Delta}(l_1(s), l_2(s)) ds + \int_0^g r_{\Delta}(l_1(s), l_2(s)) dB(s) \\ &\quad + \int_0^g \int_M w(l_1(s), l_2(s), m) \tilde{J}(ds, dm),\end{aligned}\quad (13)$$

3. Convergence Rate of Partially Truncated Split-Step Theta Method in $\mathcal{L}^{\hat{q}}(\hat{q} \geq 2)$

In this section, we address how quickly $s(g)$ and $\vartheta_{\Delta}(g)$ come close to each other in the presence of non-linear coefficients.

Assumption 3. $s(g)$ and $\vartheta(g)$ have $\hat{q}$ th moment bounds for $\hat{q} \geq 2$ and up to $\bar{q}$, i.e., for $\hat{q} \in [2, \bar{q})$

$$\sup_{0 \leq g \leq G} \epsilon^* |s(g)|^{\hat{q}} \vee \sup_{0 < \Delta \leq \Delta^*} \sup_{0 \leq g \leq G} \epsilon^* |\vartheta_{\Delta}(g)|^{\hat{q}} \leq C, \quad \forall G > 0. \quad (14)$$

Assumption 4. There exists a constant $\bar{m} > 2$, such that

$$\begin{aligned}\langle (\iota_1 - \iota_2), (u^*(\iota_1, \bar{\iota}_1) - u^*(\iota_2, \bar{\iota}_2)) \rangle + \frac{\bar{m} - 1}{2} |r^*(\iota_1, \bar{\iota}_1) - r^*(\iota_2, \bar{\iota}_2)|^2 \\ \leq C(|\iota_1 - \iota_2|^2 + |\bar{\iota}_1 - \bar{\iota}_2|^2),\end{aligned}$$

for all $\iota_1, \iota_2, \bar{\iota}_1, \bar{\iota}_2 \in \mathbb{R}^o$.

By utilizing Assumption 3, Chebyshev's inequality for arbitrary $k \in \mathbb{R}$, $k > |s_0|$, and $\Delta \in (0, \Delta^*]$, we are able to define stopping times $\tau_k = \inf\{g \geq 0 : |s(g)| \geq k\}$ and $\bar{\tau}_{\Delta, k} = \inf\{g \geq 0 : |\vartheta_{\Delta}(g)| \geq k\}$, such that

$$\mathbb{P}(\tau_k \leq G) \leq \frac{C}{k^{\hat{q}}} \quad \text{and} \quad \mathbb{P}(\bar{\tau}_{\Delta, k} \leq G) \leq \frac{C}{k^{\hat{q}}} \quad (15)$$

Lemma 2. Let Assumptions 1–4 be satisfied and suppose that $q \geq 2$. Consider $k > |s_0|$, $\Delta \in (0, \Delta^*]$ to be small s.t. $\Gamma^{-1}(\epsilon(\Delta)) \geq k$. Then,

$$\epsilon^* |f_{\Delta}(g \wedge \tau)|^q \leq C((\epsilon(\Delta))^q \Delta^{q/2} + \Delta), \quad (16)$$

where $f_{\Delta}(g) = s(g) - \vartheta_{\Delta}(g)$ and $\tau = \tau_k \wedge \bar{\tau}_{\Delta, k}$.

Proof. The proof of this lemma can be attained by following the same approach as in [40]. $\square$

Theorem 1. Suppose that Assumptions 1–4 hold with $\hat{q} > q \geq 2$. Consider also for small values of $\Delta \in (0, \Delta^*]$

$$\epsilon(\Delta) \geq \Gamma([(\epsilon(\Delta))^q \Delta^{q/2}]^{-1/(\hat{q}-q)}). \quad (17)$$

Then,

$$\epsilon^* |f_{\Delta}(G)|^q \leq C((\epsilon(\Delta))^q \Delta^{q/2} + \Delta). \quad (18)$$

Proof. By applying the Young inequality [22], for any $\rho > 0$ and $q \in [2, \hat{q})$ we have

$$\begin{aligned}\epsilon^* |f_{\Delta}(G)|^q &= \epsilon^* (|f_{\Delta}(G)|^q 1_{\{\tau > G\}}) + \epsilon^* (|f_{\Delta}(G)|^q 1_{\{\tau \leq G\}}) \\ &\leq \epsilon^* (|f_{\Delta}(G)|^q 1_{\{\tau > G\}}) + \frac{\rho q}{\hat{q}} \epsilon^* |f_{\Delta}(G)|^{\hat{q}} + \frac{\hat{q} - q}{\hat{q} \rho^{\frac{q}{\hat{q}-q}}} \mathbb{P}(\tau \leq G).\end{aligned}\quad (19)$$

By utilizing Assumption 3 and Equation (15), we have

$$\epsilon^* |f_\Delta(G)|^{\hat{\ell}} \leq 2^{\hat{\ell}-1} \epsilon^* (|s(G)|^{\hat{\ell}} + |\vartheta_\Delta(G)|^{\hat{\ell}}) \leq C, \quad (20)$$

and

$$\mathbb{P}(\tau \leq G) \leq \mathbb{P}(\bar{\tau}_{\Delta,k} \leq G) + \mathbb{P}(\tau_k \leq G) \leq \frac{C}{k^{\hat{\ell}}}. \quad (21)$$

By plugging in (20) and (21) into (19), we obtain

$$\epsilon^* |f_\Delta(G)|^q \leq \epsilon^* |f_\Delta(G \wedge \tau)|^q + \frac{C\rho q}{\hat{\ell}} + \frac{C(\hat{\ell} - q)}{\hat{\ell} k^{\hat{\ell}} \rho^{\frac{q}{\hat{\ell}-q}}}. \quad (22)$$

Upon selecting $\rho = (\varepsilon(\Delta))^q \Delta^{q/2}$, $k = \rho^{-1/(\hat{\ell}-q)}$ and utilizing condition (17) and Lemma 2, and then substituting in (22), the required assertion (18) is attained. $\square$

4. Stability Analysis of Partially Truncated Split-Step Theta Method

In this section, we will show that the partially truncated SS θ -method can reproduce the almost sure polynomial stability of Equation (1). It will be assumed that

$$u^*(0,0) = u_1(0,0) = r^*(0,0) = r_1(0,0) = w(0,0,m) = 0 \quad (23)$$

for all $m \in M$.

Assumption 5. There exist constants $\alpha \in [0, \infty]$, $\delta_1 > \delta_3 + 2k_3 \geq 0$, $\delta_2 \geq 0$ and $\delta_4 \geq 0$, such that

$$2\langle \iota, u_1(\iota, \bar{\iota}) \rangle + (1 + \alpha) |r_1(\iota, \bar{\iota})|^2 \leq -\delta_1 |\iota|^2 + \eta \delta_2 |\bar{\iota}|^2,$$

and

$$2\langle \iota, u^*(\iota, \bar{\iota}) \rangle + (1 + \alpha^{-1}) |r^*(\iota, \bar{\iota})|^2 \leq \delta_3 |\iota|^2 + \eta \delta_4 |\bar{\iota}|^2,$$

For all $\iota, \bar{\iota} \in \mathbb{R}^0$ and $\eta \in (0, 1)$, where during the following we select $\alpha = 0$ and put $\alpha^{-1} |r(\iota, \bar{\iota})|^2 = 0$ when the term $r^*(\iota, \bar{\iota})$ does not exist in $r(\iota, \bar{\iota})$, while $\alpha = \infty$ and we put $\alpha |r^*(\iota, \bar{\iota})|^2 = 0$ when the term $r_1(\iota, \bar{\iota})$ does not exist in $r(\iota, \bar{\iota})$. By utilizing Assumptions 1 and 5, it can be concluded that

$$\begin{aligned} 2\langle \iota, u(\iota, \bar{\iota}) \rangle + |r(\iota, \bar{\iota})|^2 + \int_M |w(\iota, \bar{\iota}, m)|^2 \pi(dm) \\ \leq -(\delta_1 - \delta_3 - k_3) |\iota|^2 + (\eta(\delta_2 + \delta_4) + k_3) |\bar{\iota}|^2 \end{aligned} \quad (24)$$

Lemma 3. Let Assumptions 1 and 5 hold; then, for all $\Delta \in (0, \Delta^*]$

$$\begin{aligned} 2\langle \iota, u_\Delta(\iota, \bar{\iota}) \rangle + |r_\Delta(\iota, \bar{\iota})|^2 + \int_M |w(\iota, \bar{\iota}, m)|^2 \pi(dm) \\ \leq -(\delta_1 - \delta_3 - 2k_3) |\iota|^2 + (\eta(\delta_2 + \delta_4) + 2k_3) |\bar{\iota}|^2 \end{aligned} \quad (25)$$

Proof. By utilizing the inequality $|\iota + \bar{\iota}|^2 \leq 2(|\iota|^2 + |\bar{\iota}|^2)$ (23) and Assumption 1, we have

$$\begin{aligned} \int_M |w(\iota, \bar{\iota}, m)|^2 \pi(dm) &= \int_M |w(\iota, \bar{\iota}, m) - w(0,0,m) + w(0,0,m)|^2 \pi(dm) \\ &\leq 2 \int_M |w(\iota, \bar{\iota}, m) - w(0,0,m)|^2 \pi(dm) + 2 \int_M |w(0,0,m)|^2 \pi(dm) \\ &\leq 2k_3(|\iota|^2 + |\bar{\iota}|^2). \end{aligned} \quad (26)$$

By considering Definitions (9) and (10) and then utilizing Assumption 5 and (23), the required assertion (25) is obtained. $\square$

Lemma 4 ([41]). Suppose that Assumptions 1 and 5 hold. Then, for any s_0 and $\zeta > 0$, the solution $s(g)$ to Equation (1) possesses the property of almost sure polynomial stability, i.e.,

$$\limsup_{g \rightarrow \infty} \frac{\log |s(g)|}{\log(1+g)} \leq -\frac{\zeta}{2} \quad a.s. \quad (27)$$

Theorem 2. Let Assumptions 1 and 5 hold and $\theta \in [0, \frac{1}{2}]$. Assume also the existence of a positive constant κ , such that

$$\Delta |u_\Delta(\iota, \bar{\iota})|^2 \leq \kappa(|\iota|^2 + |\bar{\iota}|^2) \quad (28)$$

and

$$\delta_1 > \delta_3 + 2k_3 + (\eta(\delta_2 + \delta_4) + 2k_3)(1 + [1/\eta]). \quad (29)$$

for all $a, b \in \mathbb{R}^0$, let ζ^* be a unique positive root of following equation

$$\begin{aligned} &\zeta(1 + \theta\Delta)(1 + \theta\kappa) - \delta_1 + \delta_3 + 2k_3 + \kappa(1 - 2\theta) \\ &+ (\zeta(1 + \theta\Delta)\theta\Delta + \eta(\delta_2 + \delta_4) + 2k_3 + \kappa(1 - 2\theta))(1 + [1/\eta])\eta^{-\zeta} = 0, \end{aligned} \quad (30)$$

with κ satisfying

$$\kappa < \frac{\delta_1 - \delta_3 - 2k_3 - (\eta(\delta_2 + \delta_4) + 2k_3)(1 + [1/\eta])}{(1 - 2\theta)(2 + [1/\eta])}. \quad (31)$$

Then, there exists a constant $\bar{\Delta}$, such that for all $\Delta \in (0, \bar{\Delta})$ and any initial value Q_0 , the approximate solution defined by Equation (8) possesses the property of almost sure polynomial stability, i.e.,

$$\limsup_{n \rightarrow \infty} \frac{\log |Q_n|}{\log(1 + n\Delta)} \leq -\frac{\zeta^*}{2} \quad a.s. \quad (32)$$

Proof. For $\zeta > 0$,

$$\begin{aligned} &(1 + (n+1)\Delta)^\zeta |Q_{n+1}|^2 - (1 + n\Delta)^\zeta |Q_n|^2 \\ &= (1 + (n+1)\Delta)^\zeta (|Q_{n+1}|^2 - |Q_n|^2) + (1 + (n+1)\Delta)^\zeta \left[1 - \left(\frac{1 + n\Delta}{1 + (n+1)\Delta} \right)^\zeta \right] |Q_n|^2. \end{aligned} \quad (33)$$

For $\zeta > 0$ and $\zeta \geq 0$, we have $1 - \zeta^\zeta \leq -\zeta \log \zeta$. Therefore,

$$1 - \left(\frac{1 + n\Delta}{1 + (n+1)\Delta} \right)^\zeta \leq \zeta \log \left(\frac{1 + (n+1)\Delta}{1 + n\Delta} \right) \leq \zeta \Delta. \quad (34)$$

From (8), we have

$$\begin{aligned} |Q_{n+1}|^2 &= |Q_n|^2 + 2\langle v_n^*, u_\Delta(v_n^*, v_{[\eta n]}^*) \Delta \rangle + |r_\Delta(v_n^*, v_{[\eta n]}^*)|^2 \Delta \\ &+ \Delta \int_M |w(v_n^*, v_{[\eta n]}^*, m)|^2 \pi(dm) + (1 - 2\theta) |u_\Delta(v_n^*, v_{[\eta n]}^*)|^2 \Delta^2 + N_n, \end{aligned} \quad (35)$$

where

$$\begin{aligned} N_n &= |r_\Delta(v_n^*, v_{[\eta n]}^*)|^2 (|\Delta B_n|^2 - \Delta) - \Delta \int_M |w(v_n^*, v_{[\eta n]}^*, m)|^2 \pi(dm) \\ &+ \left| \int_{g_n}^{g_{n+1}} \int_M w(v_n^*, v_{[\eta n]}^*, m) \tilde{f}(dg, dm) \right|^2 + 2\langle Q_n + u_\Delta(v_n^*, v_{[\eta n]}^*) \Delta, r_\Delta(v_n^*, v_{[\eta n]}^*) \Delta B_n \rangle \\ &+ 2\langle Q_n + u_\Delta(v_n^*, v_{[\eta n]}^*) \Delta, \int_{g_n}^{g_{n+1}} \int_M w(v_n^*, v_{[\eta n]}^*, u) \tilde{f}(ds, dm) \rangle \\ &+ 2\langle r_\Delta(v_n^*, v_{[\eta n]}^*) \Delta B_n, \int_{g_n}^{g_{n+1}} \int_M w(v_n^*, v_{[\eta n]}^*, m) \tilde{f}(ds, dm) \rangle. \end{aligned} \quad (36)$$

Then by plugging (34) and (35) into (33), we obtain

$$\begin{aligned}
& (1 + (n+1)\Delta)^\zeta |Q_{n+1}|^2 - (1 + n\Delta)^\zeta |Q_n|^2 \\
& \leq (1 + (n+1)\Delta)^\zeta (1 - 2\theta) |u_\Delta(v_n^*, v_{[\eta n]}^*)|^2 \Delta^2 \\
& \quad + (1 + (n+1)\Delta)^\zeta \left(2\langle (v_n^*), \varphi_\Delta(v_n^*, v_{[\eta n]}^*) \rangle \Delta \right. \\
& \quad \left. + |r_\Delta(v_n^*, v_{[\eta n]}^*)|^2 \Delta + \Delta \int_M |w(v_n^*, v_{[\eta n]}^*, m)|^2 \pi(dm) \right) \\
& \quad + (1 + (n+1)\Delta)^\zeta N_n + (1 + (n+1)\Delta)^\zeta \zeta \Delta |Q_n|^2.
\end{aligned} \tag{37}$$

By utilizing (7) and (28), we obtain

$$\begin{aligned}
|Q_n|^2 & \leq (1 + \theta\Delta) |v_n^*|^2 + (1 + \theta\Delta) \theta \Delta |u_\Delta(v_n^*, v_{[\eta n]}^*)|^2 \\
& \leq (1 + \theta\Delta) (1 + \theta\kappa) |v_n^*|^2 + (1 + \theta\Delta) \theta \kappa |v_{[\eta n]}^*|^2.
\end{aligned} \tag{38}$$

Then by applying Lemma 3 and substituting with (38) into (37), we obtain

$$\begin{aligned}
& (1 + (n+1)\Delta)^\zeta |Q_{n+1}|^2 - (1 + n\Delta)^\zeta |Q_n|^2 \\
& \leq (1 + (n+1)\Delta)^\zeta \left(A_1(\zeta, \Delta) \Delta |v_n^*|^2 + A_2(\zeta, \Delta) \Delta |v_{[\eta n]}^*|^2 + N_n \right),
\end{aligned} \tag{39}$$

where

$$A_1(\zeta, \Delta) = \zeta(1 + \theta\Delta)(1 + \theta\kappa) - \delta_1 + \delta_3 + 2k_3 + \kappa(1 - 2\theta),$$

and

$$A_2(\zeta, \Delta) = \zeta(1 + \theta\Delta) \theta \kappa + \eta(\delta_2 + \delta_4) + 2k_3 + \kappa(1 - 2\theta).$$

Taking the summation from $j = 0$ to $j = n - 1$ to both sides of (39) yields

$$\begin{aligned}
(1 + n\Delta)^\zeta |Q_n|^2 & \leq |Q_0|^2 + A_1(\zeta, \Delta) \Delta \sum_{j=0}^{n-1} (1 + (j+1)\Delta)^\zeta |v_j^*|^2 \\
& \quad + A_2(\zeta, \Delta) \Delta \sum_{j=0}^{n-1} (1 + (j+1)\Delta)^\zeta |v_{[\eta j]}^*|^2 + \sum_{j=0}^{n-1} (1 + (j+1)\Delta)^\zeta N_j.
\end{aligned} \tag{40}$$

Also,

$$\begin{aligned}
\sum_{j=0}^{n-1} (1 + (j+1)\Delta)^\zeta |v_{[\eta j]}^*|^2 & = |v_0^*|^2 \sum_{0 \leq j \leq n-1: [\eta j]=0} (1 + (j+1)\Delta)^\zeta \\
& \quad + |v_1^*|^2 \sum_{0 \leq j \leq n-1: [\eta j]=1} (1 + (j+1)\Delta)^\zeta \\
& \quad + \dots + |v_{[\eta(n-1)]}^*|^2 \sum_{0 \leq j \leq n-1: [\eta j]=[\eta(n-1)]} (1 + (j+1)\Delta)^\zeta.
\end{aligned} \tag{41}$$

By following the same approach as in [42] and noting that for any $l \in \{0, 1, \dots, [\eta(n-1)]\}$, $[\eta j] = l \Leftrightarrow l \leq j < l+1$, it is deduced that the maximum number of $j \in \{0, 1, \dots, n-1\}$ satisfying $[\eta j] = l$ is $1 + [1/\eta]$. Then, by assuming that $\Delta \in (0, \Delta_0)$, $\Delta_0 = 1/\eta - 1$, we have

$$\begin{aligned}
\sum_{j=0}^{n-1} (1 + (j+1)\Delta)^\zeta |v_{[\eta j]}^*|^2 & \leq (1 + [1/\eta]) \eta^{-\zeta} \left(\sum_{j=0}^{n-1} (1 + (j+1)\Delta)^\zeta |v_j^*|^2 \right. \\
& \quad \left. - \sum_{j=[\eta(n-1)]+1}^{n-1} (1 + (j+1)\Delta)^\zeta |v_j^*|^2 \right).
\end{aligned} \tag{42}$$

By plugging in (42) into (40), we obtain

$$\begin{aligned}
& (1+n\Delta)^\zeta |Q_n|^2 + A_2(\zeta, \Delta)(1 + [1/\eta])\eta^{-\zeta} \Delta \sum_{j=[\eta(n-1)]+1}^{n-1} (1+(j+1)\Delta)^\zeta |v_j^*|^2 \\
& \leq |Q_0|^2 + \sum_{j=0}^{n-1} (1+(j+1)\Delta)^\zeta N_j + B(\zeta, \Delta) \Delta \sum_{j=0}^{n-1} (1+(j+1)\Delta)^\zeta |v_j^*|^2,
\end{aligned} \tag{43}$$

where

$$B(\zeta, \Delta) = A_1(\zeta, \Delta) + A_2(\zeta, \Delta)(1 + [1/\eta])\eta^{-\zeta}. \tag{44}$$

Then, for any $\eta \in (0, 1)$, $\zeta \geq 0$ and $\Delta \in (0, \Delta_0]$; it is noticeable that $\frac{d}{d\zeta} B(\zeta, \Delta) > 0$, and recalling (31), we have $B(0, \Delta) < 0$. Therefore, there exists a unique $\zeta^* > 0$, such that $B(\zeta^*, \Delta) = 0$, and this leads to

$$\begin{aligned}
& \limsup_{n \rightarrow \infty} (1+n\Delta)^{\zeta^*} |Q_n|^2 \\
& \leq \limsup_{n \rightarrow \infty} (1+n\Delta)^{\zeta^*} |Q_n|^2 \\
& \quad + A_2(\zeta^*, \Delta)(1 + [1/\eta])\eta^{-\zeta^*} \Delta \sum_{j=[\eta(n-1)]+1}^{n-1} (1+(j+1)\Delta)^{\zeta^*} |v_j^*|^2 \\
& \leq D_n,
\end{aligned} \tag{45}$$

where $D_n = |Q_0|^2 + \sum_{j=0}^{n-1} (1+(j+1)\Delta)^{\zeta^*} N_j$. By applying the discrete semi-martingale convergence theorem [43], we obtain for any initial data v_0 , $\limsup_{n \rightarrow \infty} D_n < \infty$ a.s., which yields

$$\limsup_{n \rightarrow \infty} \frac{\log((1+n\Delta)^{\zeta^*} |Q_n|^2)}{\log(1+n\Delta)} \leq \limsup_{n \rightarrow \infty} \frac{\log D_n}{\log(1+n\Delta)} = 0. \tag{46}$$

Therefore,

$$\limsup_{n \rightarrow \infty} \frac{\log |Q_n|}{\log(1+n\Delta)} \leq -\frac{\zeta^*}{2} \quad a.s., \tag{47}$$

which implies the required assertion (32). The proof is complete. $\square$

Theorem 3. Let Assumptions 1 and 5 hold and $\theta \in (\frac{1}{2}, 1]$. Let ζ^* be a positive root of the following equation

$$\delta_1 - \delta_3 - 2k_3 - (\eta(\delta_2 + \delta_4) + 2k_3)(1 + [1/\eta])\eta^{-\zeta} - \frac{(2\theta - 1)\zeta}{(2\theta - 1) - \theta^2 \zeta \Delta} = 0 \tag{48}$$

and assume that

$$\delta_1 > \delta_3 + 2k_3 + (\eta(\delta_2 + \delta_4) + 2k_3)(1 + [1/\eta]). \tag{49}$$

Then, for any step size $\Delta > 0$ and initial data Q_0 , the approximate solution defined by Equation (8) possesses the property of almost sure polynomial stability, i.e.,

$$\limsup_{n \rightarrow \infty} \frac{\log |Q_n|}{\log(1+n\Delta)} \leq -\frac{\zeta^*}{2} \quad a.s. \tag{50}$$

Proof. It is noted that from Equation (7)

$$\Delta u_\Delta(v_n^*, v_{[\eta n]}^*) = \frac{1}{\theta} (v_n^* - Q_n). \tag{51}$$

By utilizing Lemma 3 and (51), we can rewrite (8) as follows

$$\begin{aligned}
|Q_{n+1}|^2 & \leq \left(1 + \frac{1-2\theta}{\theta^2}\right) |Q_n|^2 - \left(\frac{2\theta-1}{\theta^2} + (\delta_1 - \delta_3 - 2k_3)\Delta\right) |v_n^*|^2 \\
& \quad + (\eta(\delta_2 + \delta_4) + 2k_3)\Delta |v_{[\eta n]}^*|^2 + \frac{2\theta-1}{\theta^2} 2\langle Q_n, v_n^* \rangle + N_n,
\end{aligned} \tag{52}$$

where N_n is the same as defined before. By utilizing (33) and (34), we have for any constant $\zeta > 0$

$$\begin{aligned} & (1 + (n+1)\Delta)^\zeta |Q_{n+1}|^2 - (1 + n\Delta)^\zeta |Q_n|^2 \\ & \leq (1 + (n+1)\Delta)^\zeta \left\{ \left(\frac{1-2\theta}{\theta^2} + \zeta\Delta \right) |Q_n|^2 - \left(\frac{2\theta-1}{\theta^2} + (\delta_1 - \delta_3 - 2k_3)\Delta \right) |v_n^*|^2 \right. \\ & \quad \left. + (\eta(\delta_2 + \delta_4) + 2k_3)\Delta |v_{[n]}^*|^2 + \frac{2\theta-1}{\theta^2} 2\langle Q_n, v_n^* \rangle + N_n \right\}. \end{aligned} \quad (53)$$

By summing up the above inequality (53) from $j = 0$ to $j = n-1$ and proceeding the same as it was done to get inequality (43) from inequality (40), we can obtain the following

$$\begin{aligned} & (1 + n\Delta)^\zeta |Q_n|^2 \\ & \leq |Y_0|^2 + \left(\frac{1-2\theta}{\theta^2} + \zeta\Delta \right) \sum_{j=0}^{n-1} (1 + (j+1)\Delta)^\zeta |Q_j|^2 - \left(\frac{2\theta-1}{\theta^2} + (\delta_1 - \delta_3 - 2k_3) \right. \\ & \quad \left. - (\eta(\delta_2 + \delta_4) + 2k_3)(1 + [1/\eta])\eta^{-\zeta}\Delta \right) \sum_{j=0}^{n-1} (1 + (j+1)\Delta)^\zeta |v_j^*|^2 \\ & \quad - (\eta(\delta_2 + \delta_4) + 2k_3)(1 + [1/\eta])\eta^{-\zeta}\Delta \sum_{j=[\eta(n-1)]+1}^{n-1} (1 + (j+1)\Delta)^\zeta |v_j^*|^2 \\ & \quad + \frac{2\theta-1}{\theta^2} \sum_{j=0}^{n-1} (1 + (j+1)\Delta)^\zeta 2\langle Q_j, v_j^* \rangle + \sum_{j=0}^{n-1} (1 + (j+1)\Delta)^\zeta N_j. \end{aligned} \quad (54)$$

Then by choosing $\tilde{\zeta} > 0$ small enough to achieve $\frac{1-2\theta}{\theta^2} + \tilde{\zeta}\Delta < 0$, it can be concluded that, for any $\zeta \in (0, \tilde{\zeta})$

$$H(\zeta) := \frac{2\theta-1}{\theta^2} - \zeta\Delta > 0. \quad (55)$$

It is also noted that

$$\begin{aligned} \frac{2\theta-1}{\theta^2} 2\langle Q_j, v_j^* \rangle & \leq \frac{2\theta-1}{\theta^2} \left\{ \frac{\theta^2 H(\zeta)}{2\theta-1} |Q_j|^2 + \frac{2\theta-1}{\theta^2 H(\zeta)} |v_j^*|^2 \right\} \\ & = H(\zeta) |Q_j|^2 + \frac{(2\theta-1)^2}{\theta^4 H(\zeta)} |v_j^*|^2 \end{aligned} \quad (56)$$

and

$$\frac{1}{\Delta} \left(\frac{2\theta-1}{\theta^2} - \frac{(2\theta-1)^2}{\theta^4 H(\zeta)} \right) = -\frac{(2\theta-1)\zeta}{(2\theta-1) - \theta^2 \zeta \Delta}. \quad (57)$$

Therefore, by utilizing (56) and (57), inequality (54) can be rewritten as

$$\begin{aligned} (1 + n\Delta)^\zeta |Q_n|^2 & \leq |Q_0|^2 + \sum_{j=0}^{n-1} (1 + (j+1)\Delta)^\zeta N_j \\ & \quad + \phi(\zeta, \Delta) \sum_{j=0}^{n-1} (1 + (j+1)\Delta)^\zeta |v_j^*|^2, \end{aligned} \quad (58)$$

where

$$\begin{aligned} \phi(\zeta, \Delta) & = - \left(\delta_1 - \delta_3 - 2k_3 - (\eta(\delta_2 + \delta_4) + 2k_3)(1 + [1/\eta])\eta^{-\zeta} \right) \\ & \quad + \frac{(2\theta-1)\zeta}{(2\theta-1) - \theta^2 \zeta \Delta}. \end{aligned} \quad (59)$$

From (59), we have $\phi(0, \Delta) < 0$. It is also noted that $\frac{d}{d\zeta}\phi(\zeta, \Delta) > 0$. So, there exists a unique constant $\zeta^* > 0$ satisfying $\phi(\zeta^*, \Delta) = 0$, and this yields $\limsup_{n \rightarrow \infty} (1 + n\Delta)^{\zeta^*} |Q_n|^2 \leq \limsup_{n \rightarrow \infty} C_n$, where $C_n = |Q_0|^2 + \sum_{j=0}^{n-1} (1 + (j+1)\Delta)^{\zeta^*} N_j$. Then, by applying the discrete semi-martingale convergence theorem and proceeding the same as we did before in Theorem 2, the required assertion (50) will be attained. $\square$

5. Numerical Examples

In this section, we will present numerical examples to illustrate the theoretical results discussed. All of the upcoming computations were performed using Python 3.7. To manifest the computational merits of the partially truncated split-step theta (PTSST) method, we will present a comparison with the truncated Euler–Maruyama (TEM) method [13]

$$Q_{n+1} = Q_n + \Delta u_\Delta(Q_n, Q_{[\eta n]}) + r_\Delta(Q_n, Q_{[\eta n]})\Delta B_n + \int_M w(Q_n, Q_{[\eta n]}, m)\Delta\tilde{J}(dm), \quad (60)$$

where $f_\Delta(l, \bar{l}) = f\left(\left(|l| \wedge \Gamma^{-1}(\varepsilon(\Delta))\right)\frac{l}{|l|}, \left|\bar{l}\right| \wedge \Gamma^{-1}(\varepsilon(\Delta))\frac{\bar{l}}{|\bar{l}|}\right)$ for $f = u$ or r .

The tamed explicit (TE) method [44] is defined by

$$Q_{n+1} = Q_n + \Delta \frac{u(Q_n, Q_{[\eta n]})}{1 + \Delta |u(Q_n, Q_{[\eta n]})|} + r(Q_n, Q_{[\eta n]})\Delta B_n + \int_M w(Q_n, Q_{[\eta n]}, m)\Delta\tilde{J}(dm), \quad (61)$$

and the split-step backward Euler (SSBE) method [45] is given by

$$v_n^* = Q_n + \Delta u(v_n^*, v_{[\eta n]}^*), \quad (62)$$

$$Q_{n+1} = Q_n + \Delta u(v_n^*, v_{[\eta n]}^*) + r(v_n^*, v_{[\eta n]}^*)\Delta B_n + \int_M w(v_n^*, v_{[\eta n]}^*, m)\Delta\tilde{J}(dm). \quad (63)$$

Also, discrete Brownian paths are generated over interval $[0, 1]$ with step size $dt = 2^{-13}$, and for step sizes $\Delta = 2^{q+1}dt$, where $(4 \leq q \leq 7)$, 2000 sample paths are used to simulate the numerical solution. For the i th sample path, let the exact solution be represented by $Q(g_n, \omega_i)$ at time $g = g_n$, and let $Q_n(\omega_i)$ represent numerical approximation of the numerical method at the n th step. Let ϵ represent the error in the p th moment, and, by the law of large numbers, the error in the $\hat{p}$ -th moments at final time $g = G$ will be

$$\epsilon(G) \approx \frac{1}{2000} \sum_{i=0}^{2000} |Q(G, \omega_i) - Q_N(\omega_i)|^{\hat{p}}$$

Example 1. Consider the following stochastic pantograph model with jumps of the form

$$ds(g) = (3s(g) - 2s^5(g) - s(0.5g))dg + (s^2(g) + s(g)\sin^2(s(0.5g)))dB(g) + \int_M (s(g) + s(\eta g)m)\tilde{J}(dg, dm), \quad (64)$$

with $s_0 = 1$ and $\pi(dm)dg = \lambda f(m)dm dt$, where $\lambda = 2$,

$$f(m) = \frac{1}{\sqrt{2\pi m}} e^{-\frac{(\ln m)^2}{2}}, \quad 0 \leq m < \infty.$$

It can be deduced that $u_1(s(g), s(0.5g)) = 3s(g) - s(0.5g)$, $u^*(s(g), s(0.5g)) = -2s^5(g)$, $r_1(s(g), s(0.5g)) = 0$, $r^*(s(g), s(0.5g)) = s^2(g) + s(g)\sin^2(s(0.5g))$, and $w(s(g), s(0.5g), u) = (s(g) + s(0.5g))u$. Then, Assumption 1 is satisfied. For Assumption 4,

$$\begin{aligned}
& (\iota_1 - \iota_2)(u^*(\iota_1, \bar{\iota}_1) - u^*(\iota_2, \bar{\iota}_2)) + \frac{\bar{m} - 1}{2} |r^*(\iota_1, \bar{\iota}_1) - r^*(\iota_2, \bar{\iota}_2)|^2 \\
&= (\iota_1 - \iota_2)(-2(\iota_1^5 - \iota_2^5)) + \frac{\bar{m} - 1}{2} |\iota_1^2 + \iota_1 \sin^2 \bar{\iota}_1 - \iota_2^2 - \iota_2 \sin^2 \bar{\iota}_2|^2 \\
&\leq (\iota_1 - \iota_2)(-2(\iota_1 - \iota_2)(\iota_1^4 + \iota_1^3 \iota_2 + \iota_1^2 \iota_2^2 + \iota_1 \iota_2^3 + \iota_2^4)) \\
&\quad + (\bar{m} - 1)(|\iota_1^2 - \iota_2^2|^2 + |(\iota_1 - \iota_2) \sin^2 \bar{\iota}_1 + \iota_2(\sin^2 \bar{\iota}_1 - \sin^2 \bar{\iota}_2)|^2).
\end{aligned}$$

Note that

$$-(\iota_1^3 \iota_2 + \iota_1^2 \iota_2^2 + \iota_1 \iota_2^3) \leq 0.5(\iota_1^4 + \iota_2^4)$$

Then,

$$\begin{aligned}
& (\iota_1 - \iota_2)(u^*(\iota_1, \bar{\iota}_1) - u^*(\iota_2, \bar{\iota}_2)) + \frac{\bar{m} - 1}{2} |r^*(\iota_1, \bar{\iota}_1) - r^*(\iota_2, \bar{\iota}_2)|^2 \\
&\leq \left[-(\iota_1^4 + \iota_2^4) + 2(\bar{m} - 1)(\iota_1^2 + \iota_2^2) + (\bar{m} - 1)(1 + \iota_1^2 + \iota_2^2) \right] \\
&\quad \times (|\iota_1 - \iota_2|^2 + |\bar{\iota}_1 - \bar{\iota}_2|^2) \\
&\leq \left[\frac{1}{4} + 6(\bar{m} - 1)^2 \right] (|\iota_1 - \iota_2|^2 + |\bar{\iota}_1 - \bar{\iota}_2|^2).
\end{aligned}$$

Therefore, Assumption 4 is satisfied for any value of $\bar{m}$. Furthermore,

$$\begin{aligned}
\langle \iota, u^*(\iota, \bar{\iota}) \rangle + \frac{\bar{q} - 1}{2} |r^*(\iota, \bar{\iota})|^2 &= -2\iota^6 + \frac{\bar{q} - 1}{2} |\iota^2 + \iota \sin^2 \bar{\iota}|^2 \\
&\leq -2\iota^6 + (\bar{q} - 1)(|\iota|^4 + |\iota|^2) \\
&\leq -2\iota^2 \left[\left(\iota^2 - \frac{\bar{q} - 1}{4} \right)^2 - \left(\frac{\bar{q} - 1}{4} \right)^2 \right] + (\bar{q} - 1)|\iota|^2 \\
&\leq \frac{9}{8} (\bar{q} - 1)^2 (1 + |\iota|^2 + |\bar{\iota}|^2).
\end{aligned} \tag{65}$$

Hence, Assumption 2 is hold for any $\bar{q}$. Also,

$$\sup_{|\iota| \vee |\bar{\iota}| \leq b} (|u^*(\iota, \bar{\iota})| \vee |r^*(\iota, \bar{\iota})|) \leq 2b^5, \quad \forall b \geq 1.$$

Therefore, we can select $\Gamma(b) = 2(b)^5$ and $\varepsilon(\Delta) = \Delta^{-1/10}$, such that with these selected functions Γ and ε , the partially truncated split-step theta (PTSS θ) method (8) can be applied to obtain the numerical solution of Equation (64). Because it is difficult to obtain the analytical solution of Equation (64), it is needed to solve Equation (64) by using the partially truncated split-step theta (PTSS θ) method with a sufficiently small step size ($dt = 2^{-13}$) and to identify its output as the exact solution for the error comparison. The mean square errors of the partially truncated split-step theta (PTSS θ) method are calculated at the final time $G = 1$ with $\theta = 0, 0.5$, and 1 . Figure 1 depicts the plot of the mean square errors against Δ on a log-log scale and, as a reference, a dashed line of slope $1/2$ is added. The convergence rate is approximately one half. Furthermore, the numerical approximations (60), (61), and (63) have been applied to Equation (64), and the mean square errors of these approximations at the final time $G = 1$ have been calculated and listed in Table 1, where the exact solutions are taken as the corresponding numerical schemes with small step sizes ($dt = 2^{-13}$). From Table 1, it can be seen that the partially truncated split-step theta (PTSS θ) method with $\theta = 0.4, 0.6$, and 0.8 has a higher accuracy and less mean square error than the other schemes. Also, when the step size decreases, the mean square error decreases and the value of the mean square error changes with different selections of θ .

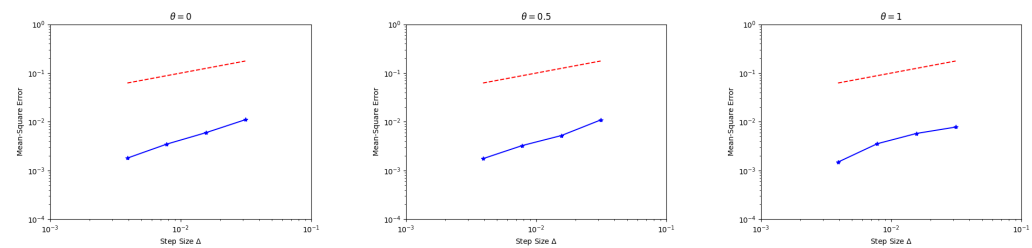


Figure 1. Log-log plot of mean square errors of the PTSS θ method versus step size Δ for Equation (64), with $\theta = 0, 0.5$, and 1.

Table 1. Mean square errors of the numerical approximations for Equation (64).

Step Size	SSBE Method	TEM Method	TE Method	PTSS θ Method		
				$\theta = 0.4$	$\theta = 0.6$	$\theta = 0.8$
2^{-5}	0.1761	0.1522	0.2341	0.0217	0.0331	0.0486
2^{-6}	0.1421	0.0763	0.1594	0.0134	0.0232	0.0256
2^{-7}	0.0928	0.0541	0.0953	0.0038	0.0117	0.0143
2^{-8}	0.0654	0.0324	0.0721	0.0018	0.0032	0.0041

Example 2. Consider the following stochastic Lévy jump model of the form

$$ds(g) = (-7s(g) - s^3(g) - s^5(g))dg + s(g) \cos^2(s(0.25g))dB(t) + \int_M (0.5s(g) + 0.1s(0.25g))u\tilde{f}(dg, dm), \quad (66)$$

with $s_0 = 1$ and $\pi(dm)dg = \lambda f(u)dmdt$ where $\lambda = 2$,

$$f(m) = \frac{1}{\sqrt{2\pi}} e^{-\frac{m^2}{2}}, \quad -\infty < m < \infty.$$

It can be deduced that $u_1(s(g), s(0.25g)) = -7s(g)$, $u^*(s(g), s(0.25g)) = -s^3(g) - s^5(g)$, $r_1(s(g), s(0.25g)) = 0$, $r^*(s(g), s(0.25g)) = s(g) \cos^2(s(0.25g))$, and $w(s(g), s(0.25g), m) = (0.5s(g) + 0.1s(0.25g))m$. Because of the absence of $r_1(l, \bar{l})$, we select $\alpha = \infty$. Then,

$$2\langle l, u_1(l, \bar{l}) \rangle + (1 + \alpha)|r_1(l, \bar{l})|^2 = -14|l|^2,$$

$$\begin{aligned} 2\langle l, u^*(l, \bar{l}) \rangle + (1 + \alpha^{-1})|r^*(l, \bar{l})|^2 &= -2l^6 - 2l^4 + |l \cos^2 \bar{l}|^2 \\ &\leq -2l^2 \left[\left(l^2 + \frac{1}{2} \right)^2 - \frac{1}{4} \right] + |l|^2 \\ &\leq -2l^2 \left(l^2 + \frac{1}{2} \right)^2 + \frac{3}{2}|l|^2 \leq \frac{3}{2}|l|^2 \end{aligned}$$

and

$$\int_M |w(l, \bar{l}, m)|^2 \pi(dm) = \lambda \int_M |0.5l + 0.1\bar{l}|^2 m^2 f(m) dm \leq |l|^2 + |\bar{l}|^2.$$

Therefore, inequality (24) is satisfied with $\delta_1 = 14$, $\delta_3 = 3/2$, $\delta_2 = \delta_4 = 0$, and $k_3 = 1$. By Lemma 4, the solution of Equation (66) possesses the property of almost sure polynomial stability. Then, we select $\Gamma(b) = b^5$ and $\varepsilon(\Delta) = \Delta^{-1/4}$ to define the numerical solution of Equation (66) by using the partially truncated split-step theta (PTSS θ) method (8). Moreover, the conditions of Theorem 2 are satisfied. Therefore, it can be concluded that for any $\kappa \in (0, \frac{7}{4(1-2\theta)})$, there exists a constant $\bar{\Delta}$, such that for every $\Delta \in (0, \bar{\Delta})$ and any initial value s_0 , the partially truncated split-step theta (PTSS θ) method applied to Equation (66) has almost sure polynomial stability when selecting $\theta \in [0, 1/2]$. To manifest the almost sure polynomial stability of the numerical solution of (66), several trajectories of the partially truncated split-step theta (PTSS θ)-solutions are generated and

plotted in Figure 2. It is obvious that the numerical solutions possess the property of almost sure polynomial stability.

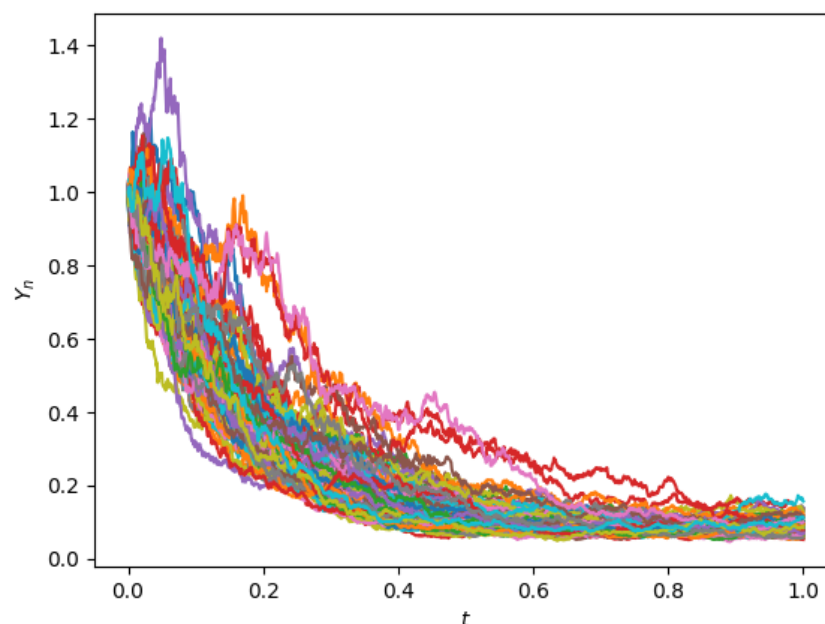


Figure 2. Trajectories of the partially truncated split-step theta (PTSS θ)-solutions for Equation (66), with $\theta = 0.25$ and $\Delta = 0.05$.

Example 3. Consider the following stochastic pantograph model with Lévy jumps of the form

$$ds(g) = (-8s(g) - s^3(g) + \frac{s(0.9g)}{2 + 2s^2(0.9g)})dg + s(g) \sin^2(s(0.9g))dB(g) + \int_M 0.1(s(g) + s(0.9g))u\tilde{f}(dg, dm), \quad (67)$$

with $s_0 = 1$, and the compensator is also the same as in Example 1 with $\lambda = 5$. By proceeding the same as previously, it can be deduced that $u_1(s(g), s(0.9g)) = -8s(g)$, $u^*(s(g), s(0.9g)) = -s^3(g) + \frac{s(0.9g)}{1+s^2(0.9g)}$, $r_1(s(g), s(0.9g)) = 0$, $r^*(s(g), s(0.9g)) = s(g) \sin^2(s(0.9g))$, and $w(s(g), s(0.9g), m) = 0.1(s(g) + s(0.9g))m$. Because of the absence of $r_1(\iota, \bar{\iota})$, we select $\alpha = \infty$. Then,

$$\begin{aligned} 2\langle \iota, u_1(\iota, \bar{\iota}) \rangle + (1 + \alpha)|r_1(\iota, \bar{\iota})|^2 &= -16|\iota|^2, \\ 2\langle \iota, u^*(\iota, \bar{\iota}) \rangle + (1 + \alpha^{-1})|r^*(\iota, \bar{\iota})|^2 &= -2\iota^4 + \iota \frac{\bar{\iota}}{1 + \bar{\iota}^2} + |\iota \sin^2 \bar{\iota}|^2 \\ &\leq -2\iota^4 + \bar{\iota} + |\iota|^2 \leq \frac{3}{2}|\iota|^2 + \frac{1}{2}|\bar{\iota}|^2 \end{aligned}$$

and

$$\int_M |w(\iota, \bar{\iota}, m)|^2 \pi(dm) = \lambda \int_M |0.1\iota + 0.1\bar{\iota}m|^2 f(m)dm \leq 0.1e^2(|\iota|^2 + |\bar{\iota}|^2).$$

Therefore, inequality (24) is satisfied with $\delta_1 = 16$, $\delta_3 = 3/2$, $\delta_2 = 0$, $\delta_4 = 0.5$, and $k_3 = 0.1e^2$. By Lemma 4, the solution of (67) possesses the property of almost sure polynomial stability. Then, we select $\Gamma(b) = b^5$ and $\varepsilon(\Delta) = \Delta^{-1/4}$ to define the numerical solution of Equation (67) by using the partially truncated split-step theta (PTSS θ) method (8). Moreover, the conditions of Theorem 3 are satisfied. Therefore the partially truncated split-step theta method applied to Equation (67) has almost sure polynomial stability when selecting $\theta \in (1/2, 1]$. Figure 3 displays several trajectories of the partially truncated split-

step theta (PTSS θ) solutions (67), and it is clear that the numerical approximations have almost sure polynomial stability.

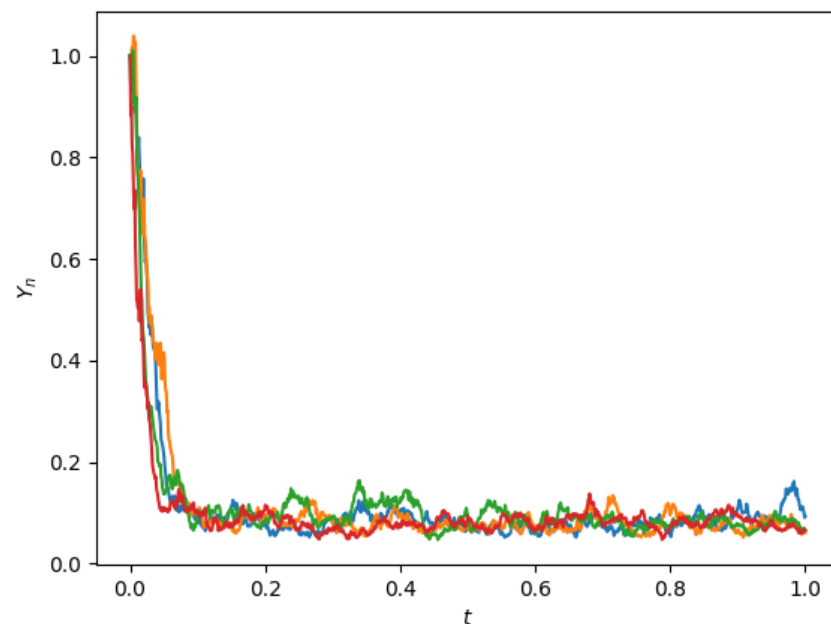


Figure 3. Trajectories of the partially truncated split-step theta (PTSS θ) solutions for Equation (67), with $\theta = 0.6$ and $\Delta = 0.7$.

6. Conclusions

In this paper, the stochastic pantograph model with Lévy jumps is presented and the partially truncated split-step theta method is applied to it. The finite time $\mathcal{L}^{\hat{q}}(\hat{q} \geq 2)$ convergence rate was obtained, under both drift and diffusion coefficients, satisfying the super-linear growth condition while the jump coefficient grows linearly, and it was close to half. By setting theta equal to zero, the proposed scheme can be viewed as the partially truncated Euler–Maruyama method, which indicates that our results are more general. By utilizing the discrete semi-martingale convergence theorem, the numerical scheme reproduced the almost sure polynomial stability of the exact solution. The conditions imposed on the numerical scheme to reproduce the almost sure polynomial stability were sufficient, but not necessary. Therefore, our future work will focus on finding out the sufficient and necessary conditions. Other interesting topics would be to study the convergence rate with all coefficients growing super-linearly and trying to improve the convergence order of the proposed scheme.

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