mathematics

Article

On a Local and Nonlocal Second-Order Boundary Value Problem with In-Homogeneous Cauchy–Neumann Boundary Conditions—Applications in Engineering and Industry

Tudor Barbu [1], Alain Miranville [2] and Costică Moroşanu [3,*]

[1] Institute of Computer Science of the Romanian Academy, Iasi Branch, 700481 Iași, Romania; tudor.barbu@iit.academiaromana-is.ro
[2] Laboratoire de Mathématiques et Applications, UMR CNRS 7348, SP2MI, Université de Poitiers, 86962 Poitiers, CEDEX, France; alain.miranville@math.univ-poitiers.fr
[3] Department of Mathematics, "Alexandru Ioan Cuza" University, Bd. Carol I, 11, 700506 Iaşi, Romania
* Correspondence: costica.morosanu@uaic.ro

Abstract: A qualitative study for a second-order boundary value problem with local or nonlocal diffusion and a cubic nonlinear reaction term, endowed with in-homogeneous Cauchy–Neumann (Robin) boundary conditions, is addressed in the present paper. Provided that the initial data meet appropriate regularity conditions, the existence of solutions to the nonlocal problem is given at the beginning in a function space suitably chosen. Next, under certain assumptions on the known data, we prove the well posedness (the existence, a priori estimates, regularity, uniqueness) of the classical solution to the local problem. At the end, we present a particularization of the local and nonlocal problems, with applications for image processing (reconstruction, segmentation, etc.). Some conclusions are given, as well as new directions to extend the results and methods presented in this paper.

Keywords: qualitative properties of solutions; nonlinear PDE of parabolic type; reaction–diffusion equations; fixed points; Leray–Schauder degree theory; diffusion processes; image analysis; applications in engineering and industry; existence of solutions; optimization; phase changes; sensitivity; stability; parametric optimization; biomedical imaging; image processing (compression, reconstruction, segmentation, etc.)

MSC: 35B99; 35K55; 35K57; 37C25; 47H11; 60J60; 62H35; 62P30; 74H20; 74P99; 80A22; 90C31; 92C55; 94A08

check for updates

Citation: Barbu, T.; Miranville, A.; Moroşanu, C. On a Local and Nonlocal Second-Order Boundary Value Problem with In-Homogeneous Cauchy–Neumann Boundary Conditions—Applications in Engineering and Industry. *Mathematics* **2024**, *12*, 2050. https://doi.org/10.3390/math12132050

Academic Editor: Mario Versaci

Received: 23 May 2024
Revised: 24 June 2024
Accepted: 27 June 2024
Published: 30 June 2024

1. Introduction

Industrial image processing plays a very important role in ensuring quality control and automation across many sectors [1–3]. As part of the vast image processing domain, image segmentation is one of the main building blocks of other image processing tasks such as object detection and image classification [4–7]. Image segmentation is simply the process of partitioning images into meaningful subregions, and although the techniques used for segmentation operations are constantly improving, their accurate implementation in certain industrial scenarios remains challenging.

Industrial imaging systems have noise characteristics and constraints that complicate segmentation tasks. Existing research is often based on heuristic methods [4,8] or machine learning algorithms [9–12], which, while effective in certain cases, may need more fine-tuning to adapt to the specific challenges posed by industrial environments. For example, image segmentation is widely used in a lot of applications involving medical imaging [13,14]. Many of the noninvasive imaging techniques developed for medicine, like radiography, computer tomography (CT), and near-infrared spectroscopy (NIRS) used for neuroimaging [15]; magnetic resonance imaging (MRI) [16–19]; elastography; tomography; and echocardiography, have scientific and biomedical applications [20].

The present research addresses some of the limitations of current segmentation methods in industrial image processing by proposing an alternative approach that integrates analytical modeling principles [21–26] with computational techniques [27,28].

We aim to provide a theoretical foundation for image segmentation frameworks in order to make them more robust and adaptable to the complexities encountered in industrial applications.

In the following sections, we will describe the new proposed mathematical framework for our problem. Let $\Omega \subset \mathbb{R}^2$ be a bounded, closed domain of Lebesgue measures, $|\Omega|$, with a C^2 boundary, $\partial\Omega$, and $[0,T]$, $T > 0$, be a generic time interval. We consider the following nonlocal and nonlinear second-order boundary value problem:

$$\begin{cases} \frac{\partial}{\partial t}V(t,x) = M(t,x,\nabla V(t,x))N(t,x,\nabla V(t,x))\int\limits_{\Omega} J(x-y)\left[V(t,y)-V(t,x)\right]dy \\ \quad + \int\limits_{\partial\Omega} G(x-y_s)\left[g_{fr}(t,y_s) - p_t V(t,x)\right]dy_s \\ \quad + M(t,x,\nabla V(t,x))\nabla N(t,x,\nabla V(t,x)) \cdot \nabla V(t,x) \\ \quad + p_r\left[V(t,x) - V^3(t,x)\right] + g_d(t,x), \quad (t,x) \in Q = (0,T] \times \Omega, \\ V(0,x) = V_0(x), \;\; x \in \Omega, \end{cases} \tag{1}$$

where

- $t \in (0,T]$, with $T > 0$ and $x = (x_1, x_2)$, varies in Ω;
- $p_r > 0$ and $p_t > 0$ are physical parameters (measures of the interface's thickness and the heat transfer coefficient, for example);
- $\frac{\partial}{\partial t}V(t,x)$ is the partial derivative of $V(t,x)$ with respect to t;
- In the following equation, we denote the gradient of $V(t,x)$ in x by $\nabla V(t,x) = V_x(t,x)$ ($\nabla V = V_x$ in short); that is,

$$\nabla V(t,x) = \left(\frac{\partial}{\partial x_1}V(t,x), \frac{\partial}{\partial x_2}V(t,x))\right).$$

  We set $\frac{\partial}{\partial x_i}V = V_{x_i}$, with $i = 1,2$, and so $V_x = \nabla V = \left(V_{x_1}, V_{x_2}\right)$;
- $g_d(t,\cdot) \in L^\infty(\Omega)$ is a given real function (the **d***istributed control*);
- $g_{fr}(t,x) \in L^\infty((0,T], L^\infty(\partial\Omega))$ is a given real function (the *boundary control*);
- dy represents the volume element $dy_1 dy_2$, $y = (y_1, y_2) \in \mathbb{R}^2$;
- dy_s represents the surface element in the surface integral;
- $V(t,x)$ (**hereafter**, V) is an unknown function. J and G are symmetric, continuous, nonnegative real functions defined on $\mathbb{R}^2$, compactly supported in the unit ball such that $\int\limits_{\mathbb{R}^2} J(z)dz = 1$ and $\int\limits_{\mathbb{R}^2} G(z)dz = 1$. We consider

$$m_1 = \max_{x\in\mathbb{R}^2} |J(x)| \quad \text{and} \quad m_2 = \max_{x\in\mathbb{R}^2} |G(x)|; \tag{2}$$

- $M\big(t,x,\nabla V(t,x)\big)$ and $N\big(t,x,\nabla V(t,x)\big)$ are positive and bounded nonlinear real functions of class $C^1(Q)$, attached to the solutions, $V(t,x)$, of problem (1), with bounded derivatives and the role of *controlling the speed of the diffusion process and enhancing the edges* (e.g., in the evolving image), and they are assumed to satisfy

$$\begin{cases} \exists\, L_1 \in (0,\infty) \quad \text{such that} \quad |M(t,x,z)| \le L_1, \\ \exists\, L_2 \in (0,\infty) \quad \text{such that} \quad |N(t,x,z)| \le L_2, \end{cases} \quad \forall (t,x,z) \in Q \times \mathbb{R}^2. \tag{3}$$

Moreover, let us note that the functions M and N depend on t, x, and $\nabla V(t,x)$. A particular case where M and N depend on $\nabla V(t,x)$ can be found in [29];

- $V_0(x) \in L^\infty(\Omega)$ is the initial condition.

We denote this by

$$\hat{m} = 2|\Omega| L_1 L_2 m_1 + c p_t m_2 + p_r, \tag{4}$$

where $c > 0$ in (4) comes from the continuous embedding $L^\infty(\Omega) \subset L^\infty(\partial\Omega)$, which implies the inequality $\|V\|_{L^\infty(\partial\Omega)} \le c\|V\|_{L^\infty(\Omega)}$ (Ω being assumed to be closed and bounded).

Let $m_3, m_4 \in (0,\infty)$ such that

$$\begin{cases} \sup\limits_{x\in\Omega} |g_d(t,x)| \le m_3, \\ \sup\limits_{x\in\Omega} \left|M\big(t,x,\nabla V(t,x)\big)\nabla N\big(t,x,\nabla V(t,x)\big)\cdot \nabla V(t,x)\right| \le m_4, \end{cases} \quad \forall t \in (0,T]. \tag{5}$$

Problem (1) is a *nonlocal* one due to the diffusion of the density, $V(t,x)$, which depends on all values of V through the convolution-like term (see [1,13,21,29,30])

$$(J*V)(t,x) = \int_\Omega J(x-y)V(t,y)dy. \tag{6}$$

Numerical approximations of the solutions to the nonlocal problem (1) can be found in [12,22,27,31,32], while for the study of well posedness, we guide the reader to the studies [21,30,33].

Details about the terms $J(x-y)$, $\int_\Omega J(x-y)V(t,y)dy$, and $-\int_\Omega J(x-y)V(t,x)dy$ from (1) can be found in [13,33,34] and the references therein.

The second integral in (1) takes into consideration the prescribed bidirectional flux at the boundary $\partial\Omega$ (according to the sign of $g_{fr}(t,x)$, $(t,x) \in \Sigma = (0,T] \times \partial\Omega$). Thus, the density $V(t,x)$ verifies Equation (1) without any internal or external sources.

The nonlocal reaction–diffusion problem (1) can be seen as similar to the local reaction–diffusion equation with in-homogeneous Cauchy–Neumann boundary conditions (see [3,20,29,31]), namely

$$\begin{cases} \dfrac{\partial}{\partial t}V(t,x) - M\big(t,x,\nabla V(t,x)\big)\mathrm{div}\Big(N\big(t,x,\nabla V(t,x)\big)\ \nabla V(t,x)\Big) \\ \qquad = p_r\Big[V(t,x) - V^3(t,x)\Big] + g_d(t,x) & \text{in } Q \\ \dfrac{\partial}{\partial \mathbf{n}}V(t,x) + p_t V(t,x) = g_{fr}(t,x) & \text{on } \Sigma \\ V(0,x) = V_0(x) & \text{on } \Omega. \end{cases} \tag{7}$$

where $\mathbf{n} = \mathbf{n}$(x,y) is a vector of the outward unit normal to the surface Σ and the initial condition, $V_0(x)$, is assumed to verify the compatibility condition:

$$\frac{\partial}{\partial \mathbf{n}} V_0(x) + p_{_t} V_0(x) = g_{fr}(0,x) \quad \text{on } \partial\Omega. \tag{8}$$

For the readers' convenience, we recall that the connection between problems (1) and (7) derives from the following relation:

$$\begin{aligned} &M\big(t,x,\nabla V(t,x)\big)\mathrm{div}\Big(N\big(t,x,\nabla V(t,x)\big)\ \nabla V(t,x)\Big) \\ &\quad = M\big(t,x,\nabla V(t,x)\big)\Big[N\big(t,x,\nabla V(t,x)\big)\Delta V(t,x) + \nabla N\big(t,x,\nabla V(t,x)\big)\cdot \nabla V(t,x)\Big], \end{aligned}$$

with the mention that the Laplace operator, $\Delta V(t,x)$, is approximated by $\int_\Omega J(x-y)\big[V(t,y)-V(t,x)\big]dy$ (see [33] and references therein).

Concerning Equation $(7)_1$, we recall that

$$\begin{aligned} a_{ij}\big(t,x,V_x(t,x)\big) &= \frac{\partial}{\partial V_{x_j}} M\big(t,x,V_x(t,x)\big)N\big(t,x,V_x(t,x)\big)V_{x_i}(t,x), \quad i = 1,2, \\ a\big(t,x,v(t,x),v_x(t,x)\big) &= -\frac{\partial}{\partial x_i} M\big(t,x,V_x(t,x)\big)N\big(t,x,V_x(t,x)\big)V_{x_i}(t,x) \\ &\quad - p_{_r}\Big[V(t,x) - V^3(t,x)\Big] - g_{_d}(t,x), \end{aligned}$$

while $(7)_2$ is expressed so that

$$\Big[a_{ij}\big(t,x,V_x(t,x)\big)V_{x_j}(t,x)\, cos\, \alpha_i + p_{_t} V(t,x) - g_{fr}(t,x)\Big]_\Sigma = 0$$

(see [31,35] and references therein).

As one can see, the local problem (7) is a more general case of the Allen–Cahn equation (see [5,22,32,33,36,37] for more details). For more general assumptions (with various types of boundary conditions), Equation (7) has been numerically investigated in, e.g., [22,27,30,31] or [29]. For the existence, estimate, uniqueness, and regularity of a solution in the Sobolev space, $W_p^{1,2}(Q)$, characterized by the presence of some new physical parameters, the principal part being in *divergence form*, and by considering the cubic nonlinearity $p_{_r}\big[V(t,x) - V^3(t,x)\big]$, satisfying for $n \in \{1,2,3\}$ the assumption H_0 in [38], that is,

$$H_0: \ (V - V^3)|V|^{3p-4}V \le 1 + |V|^{3p-1} - |V|^{3p}, \ \ p \ge 2,$$

we refer to [31,35,36] .

Since the local model cannot be applied when the wavelength of the microstructure is very small, the nonlocal model is a good alternative [13,19,21,23,39–41].

The nonlinear second-order problem (1) (or (7)) is important for modeling a variety of phenomena of life sciences, including in biology, biochemistry, economics, medicine, and physics. For details on the qualitative and quantitative analyses, we direct the reader to the studies [35,42]. In addition, nonlinear problems of type $(7)_1$ occur in the phase-field transition system (e.g., [42]) where the phase function, $V(t,x)$, describes the transition between the solid and liquid phases in the solidification process of a material occupying a region Ω.

The main novelty of our problem $(1)_1$ refers to the cubic nonlinearity $V - V^3$, thus increasing the chance of better capturing the complexity of the phenomena that surround us (see [13,21,31,36] and references therein).

Another important novelty concerns the nonhomogeneous Cauchy–Neumann boundary conditions, which can be seen as *boundary control* in industry (see [13,15,28,36,37,42–44] for details).

This paper is organized as follow. In Section 2, the existence of solutions to the nonlocal and nonlinear problem (1) are proved via the fixed-point method. A serious mathematical treatment for the local problem (7) is performed in Section 3, with its well posedness also being investigated. New (nonlocal and local) second-order anisotropic reaction–diffusion equations for image segmentation are presented in Section 4. Our new PDE models, (36) and (40), respectively, outperform the Perona–Malik technique (see [31] and references therein) and many other state-of-the-art image processing models [20,30,32]. In Section 5, we briefly summarize our results and discuss future directions and challenges in this illustrious field of research.

2. Existence of Solutions to the Nonlocal and Nonlinear Problem (1)

The starting points of this section are derived from the paper [30]. Here, we consider a nonconstant mobility variable $M\big(t,x,\nabla V(t,x)\big)$ that depends on the nonlinear real function $\nabla V(t,x)$, which leads to changes in the working assumptions. Moreover, we propose here more general boundary conditions, which are more appropriate to model life science phenomena.

Let us note at the beginning that, following [30] (Lemma 4.1, p. 13), we can find a small enough $t^* > 0$, depending on $\|V_0\|_{L^\infty(\Omega)}$ and $\|g_{fr}\|_{FR}$, such that

$$\begin{cases} \hat{m}t^* < 1, \\ m_2 t^* \|g_{fr}\|_{FR} + (m_3 + m_4)t^* + \|V_0\|_{L^\infty(\Omega)} < \dfrac{2}{3}\sqrt{\dfrac{(1-\hat{m}t^*)^3}{3p_r t^*}}, \end{cases} \tag{9}$$

with $\hat{m}$ being given by (4) and $FR = L^\infty((0,T], L^\infty(\partial\Omega))$.

We are looking for solutions to problem (1) in the space

$$Y = C([0,t^*], C^1(\Omega)) \quad \text{and} \quad W = C([0,t^*], L^\infty(\Omega)), \quad Y \subset W, \tag{10}$$

with the corresponding norm

$$\|V\| = \max_{t\in[0,t^*]} \|V(t,\cdot)\|_{L^\infty(\Omega)} = \max_{t\in[0,t^*]} \operatorname*{ess\,sup}_{x\in\Omega} \big|V(t,x)\big|.$$

For the readers' convenience, we remember that

$$W = C([0,t^*], L^\infty(\Omega)) = \Big\{F : [0,t^*] \to L^\infty(\Omega),\ F \text{ continue on } [0,t^*]\Big\};$$

that is, $F(t,\cdot) \in L^\infty(\Omega)$ and $\exists m_5 \in (0,\infty)$ such that

$$\|F(t,\cdot)\|_{L^\infty(\Omega)} \le m_5, \forall t \in [0,t^*].$$

Next, by integrating (1) on $[0,t]$, $0 < t \le t^*$, we obtain

$$
\begin{aligned}
V(t,x) = {} & V_0(x) \\
& + \int_0^t M\big(s,x,\nabla V(s,x)\big) N\big(s,x,\nabla V(s,x)\big) \int_\Omega J(x-y) V(s,y)\,dy\,ds \\
& - \int_0^t M\big(s,x,\nabla V(s,x)\big) N\big(s,x,\nabla V(s,x)\big) \int_\Omega J(x-y) V(s,x)\,dy\,ds \\
& + \int_0^t \left\{ \int_{\partial\Omega} G(x-y)\big[g_{fr}(s,y) - p_t V(s,x)\big]\,dy_s \right\} ds \\
& + \int_0^t M\big(s,x,\nabla V(s,x)\big) \nabla N\big(s,x,\nabla V(s,x)\big) \cdot \nabla V(s,x)\,ds \\
& + p_r \int_0^t \big[V(s,x) - V^3(s,x)\big]\,ds + \int_0^t g_d(s,x)\,ds.
\end{aligned} \tag{11}
$$

Definition 1. *For any $V(t,x) \in Y \subset W$, we define*

$$
\begin{aligned}
(HV)(t,x) = {} & V_0(x) \\
& + \int_0^t M\big(s,x,\nabla V(s,x)\big) N\big(s,x,\nabla V(s,x)\big) \int_\Omega J(x-y) V(s,y)\,dy\,ds \\
& - \int_0^t M\big(s,x,\nabla V(s,x)\big) N\big(s,x,\nabla V(s,x)\big) \int_\Omega J(x-y) V(s,x)\,dy\,ds \\
& + \int_0^t \left\{ \int_{\partial\Omega} G(x-y)\big[g_{fr}(s,y) - p_t V(s,x)\big]\,dy_s \right\} ds \\
& + \int_0^t M\big(s,x,\nabla V(s,x)\big) \nabla N\big(s,x,\nabla V(s,x)\big) \cdot \nabla V(s,x)\,ds \\
& + p_r \int_0^t \big[V(s,x) - V^3(s,x)\big]\,ds + \int_0^t g_d(s,x)\,ds
\end{aligned} \tag{12}
$$

and

$$(HV)(0,x) = V_0(x) \in L^\infty(\Omega), \;\; x \in \Omega. \tag{13}$$

It is easy to verify that $\forall V(t,x) \in Y \subset W;$ then, $(HV)(t,x) \in W, \;\; (t,x) \in Q$. Indeed, let us denote $F_1(t,x) = \int_0^t M\big(s,x,\nabla V(s,x)\big) N\big(s,x,\nabla V(s,x)\big) \int_\Omega J(x-y) V(s,y)\,dy\,ds$. We have

$$
\begin{aligned}
\big|F_1(t,x)\big| &= \left| \int_0^t M\big(s,x,\nabla V(s,x)\big) N\big(s,x,\nabla V(s,x)\big) \int_\Omega J(x-y) V(s,y)\,dy\,ds \right| \\
&\le \int_0^t \left| M\big(s,x,\nabla V(s,x)\big) N\big(s,x,\nabla V(s,x)\big) \int_\Omega J(x-y) V(s,y)\,dy \right| ds
\end{aligned}
$$

$$= \int_0^t \left| M(s,x,\nabla V(s,x)) N(s,x,\nabla V(s,x)) \right| \left| \int_\Omega J(x-y)V(s,y)dy \right| ds$$

$$\le L_1 L_2 \int_0^t \left| \int_\Omega J(x-y)V(s,y)dy \right| ds = L_1 L_2 \int_0^t \int_\Omega \left| J(x-y) \right| \left| V(s,y) \right| dy\, ds$$

$$\le L_1 L_2 m_1 \int_0^t \int_\Omega \left| V(s,y) \right| dy\, ds \le L_1 L_2 m_1 \int_0^t \int_\Omega \operatorname*{ess\,sup}_{y\in\Omega} |V(t,y)| dy\, ds$$

$$= L_1 L_2 m_1 \int_0^t \left[\operatorname*{ess\,sup}_{y\in\Omega} |V(t,y)| \int_\Omega dy \right] ds = L_1 L_2 m_1 \int_0^t \left[\operatorname*{ess\,sup}_{y\in\Omega} |V(t,y)| |\Omega| \right] ds$$

$$= L_1 L_2 m_1 |\Omega| \int_0^t \|V(t,\cdot)\|_{L^\infty(\Omega)} ds \le L_1 L_2 m_1 |\Omega| m_5 \int_0^t ds \le L_1 L_2 m_1 |\Omega| m_5 t^*.$$

So $|F_1(t,x)| \le L_1 L_2 m_1 |\Omega| m_5 t^*$, $\forall t \in [0,t^*]$, a.p.t. $x \in \Omega$, and thus

$$\operatorname*{ess\,sup}_{x\in\Omega} |F_1(t,x)| < \infty, \forall t \in [0,t^*], \text{ a.p.t. } x \in \Omega, \text{ i.e., } F_1(t,\cdot) \in L^\infty(\Omega), \forall t \in [0,t^*].$$

Through similar reasoning, it can be shown that the remaining terms on the right-hand side of (12) also belong to $L^\infty(\Omega), \forall t \in [0,t^*]$, which means that

$$(HV)(t,x) \in W, \quad \forall V(t,x) \in Y \subset W, \quad (t,x) \in Q.$$

Lemma 1. *If $V_0(x) \in L^\infty(\Omega)$, $g_d(t,\cdot) \in L^\infty(\Omega)$, and $g_{fr}(t,x) \in L^\infty((0,t^*], L^\infty(\partial\Omega))$, then the operator*

$$H : Y \subset W \to W, \tag{14}$$

given by (12) and (13), *is well defined.*

Proof. Let us consider $V(\cdot,\cdot) \in Y \subset W$ (see (10)) and $t_1, t_2 \in [0,t^*]$, with $t_1 < t_2$. Then, we have (see (11) and (12))

$$\|(HV)(t_1,\cdot) - (HV)(t_2,\cdot)\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega} |(HV)(t_1,x) - (HV)(t_2,x)|$$

$$= \operatorname*{ess\,sup}_{x\in\Omega} \Bigg| \int_{t_1}^{t_2} M(s,x,\nabla V(s,x)) \left\{ N(s,x,\nabla V(s,x)) \int_\Omega J(x-y)\left[V(s,y) - V(s,x)\right] dy \right\} ds$$

$$+ \int_{t_1}^{t_2} \left\{ \int_{\partial\Omega} G(x-y)\left[g_{fr}(s,y) - p_t V(s,x)\right] dy_s \right\} ds$$

$$+ \int_{t_1}^{t_2} M(s,x,\nabla V(s,x)) \nabla N(s,x,\nabla V(s,x)) \cdot \nabla V(s,x) ds$$

$$+ p_r \int_0^t \left[V(s,x) - V^3(s,x)\right] ds + \int_0^t g_d(s,x) ds \Bigg|$$

$$\le L(t_2 - t_1),$$

where

$$L = 2|\Omega| L_1 L_2 m_1 \|V\| + m_2 \|g_{fr}\|_{FR} + c p_t m_2 \|V\| + p_r \left[\|V\| + \|V\|^3 \right] + m_3 + m_4. \tag{15}$$

Here, we have used the continuous embedding $L^\infty(\Omega) \subset L^\infty(\partial\Omega)$, which implies, for a positive constant c, that

$$\|V\|_{L^\infty(\partial\Omega)} \le c\|V\|_{L^\infty(\Omega)}. \tag{16}$$

Therefore, the operator H in (14) is Lipschitz continuous on $(0, t^*]$, with the Lipschitz constant L given by (15), where the positive parameters $m_1, m_2, L_1, L_2, c, m_3, m_4, p_r, p_t$ are defined in (2), (3), and (5).

For $t = 0$, we obtain

$$\|(HV)(t, \cdot) - V_0(x)\|_{L^\infty(\Omega)} \leq tL. \tag{17}$$

Thus, from the last two inequalities, we can conclude that the operator H is continuous for $\forall t \in [0, t^*]$; accordingly, H is well defined. □

In the following theorem, we shall use an important result obtained and proven in [30] (Lemma 4.1, p. 13).

Theorem 1. *There exists $S = S(t^*) > 0$ such that*

$$\|V_0\|_{L^\infty(\Omega)} + \left(\hat{m}S + p_r S^3 + m_2 \|g_{fr}\|_{FR} + m_3 + m_4\right) t^* \leq S$$

and

$$[3p_r S^2 + \hat{m}]t^* < 1,$$

where $t^* > 0$ satisfies (9).

Theorem 2. *The operator* H *in* (14) *maps the closed ball $B(0, S)$ of $Y \subset W$ into itself, where t^* and S are given in Theorem 1.*

Proof. For any $V \in Y \subset W$ and $t_1, t_2 \in [0, t^*]$, with $t_1 < t_2$, we know (see Lemma 1)

$$\begin{aligned} &\|(HV)(t_1, \cdot) - (HV)(t_2, \cdot)\|_{L^\infty(\Omega)} \\ &\leq \left[\left(2|\Omega|L_1 L_2 m_1 + c p_t m_2 + p_r\right)\|V\| + p_r\|V\|^3 + m_2\|g_{fr}\|_{FR} + m_3 + m_4\right](t_2 - t_1). \end{aligned}$$

Now, consider $V \in B(0, S)$, i.e., $\|V\| \leq S$, with S given as in Theorem 1. It follows that

$$\begin{aligned} &\|(HV)(t, \cdot) - (HV)(0, \cdot)\|_{L^\infty(\Omega)} \\ &\leq \left[\left(2|\Omega|L_1 L_2 m_1 + c p_t m_2 + p_r\right)\|V\| + p_r\|V\|^3 + m_2\|g_{fr}\|_{FR} + m_3 + m_4\right]t \\ &\leq \left[\left(2|\Omega|L_1 L_2 m_1 + c p_t m_2 + p_r\right)S + p_r S^3 + m_2\|g_{fr}\|_{FR} + m_3 + m_4\right]t, \ \ \forall t, \ 0 \leq t \leq t^*. \end{aligned}$$

Owing to relation (4) and Theorem 1, it follows that

$$\|HV(t^*, \cdot)\|_{L^\infty(\Omega)} \leq \|V_0\|_{L^\infty(\Omega)} + \left(\hat{m}S + p_r S^3 + m_2\|g_{fr}\|_{FR} + m_3 + m_4\right)t^* \leq S,$$

which completes the proof. □

Therefore, the fixed point of the operator H defined by (14) is the solution to problem (1) on $(0, t^*]$.

Remark 1. *We can extend the already obtained results to find the solution on an interval larger than $(0, t^*]$, with t^* satisfying (9). To do this, we consider the same problem (1), but with the initial condition $u^*(x) = u(t^*, x)$, where $x \in \Omega$. We can now find the solution on $[t^*, t^* + \bar{t}_1]$, where $\bar{t}_1$ also satisfies the relations in (9). If we continue this procedure, we obtain a solution defined on some time interval $(0, T]$* (for more details, see [30] (Section 3.1, p. 11)).

In following section, we present a characterization of the solution to problem (1).

Theorem 3. *Let* $w(x) = \int\limits_{\Omega} J(x-y)dy$. *Then, the function* $V \in C\big((0,t^*], C^1(\Omega)\big)$ *is a solution to problem* (1) *if and only if for all* $(t,x) \in (0,t^*] \times \Omega$, *it holds that*

$$\begin{aligned}
V(t,x) \ &= e^{-w(x)t}V_0(x) \\
&\quad + \int\limits_0^t M\big(s,x,\nabla V(s,x)\big)N\big(s,x,\nabla V(s,x)\big)\int\limits_{\Omega} J(x-y)e^{-w(x)(t-s)}V(s,y)dyds \\
&\quad + \int\limits_0^t \int\limits_{\partial\Omega} e^{-w(x)(t-s)}G(x-y)\big[g_{fr}(s,y_s) - p_t V(s,x)\big]dy_s ds \\
&\quad + \int\limits_0^t e^{-w(x)(t-s)}M\big(s,x,\nabla V(s,x)\big)\nabla N\big(s,x,\nabla V(s,x)\big)\cdot\nabla V(s,x)ds \\
&\quad + p_r\int\limits_0^t e^{-w(x)(t-s)}\Big[V(s,x) - V^3(s,x)\Big]ds + \int\limits_0^t e^{-w(x)(t-s)}g_d(s,x)ds.
\end{aligned} \tag{18}$$

Proof. From (1), it follows that

$$\begin{aligned}
& e^{w(x)s}\frac{\partial}{\partial s}V(s,x) \\
&\quad = M\big(s,x,\nabla V(s,x)\big)N\big(s,x,\nabla V(s,x)\big)\int\limits_{\Omega} e^{w(x)s}J(x-y)\big[V(s,y) - V(s,x)\big]dy \\
&\qquad + \int\limits_{\partial\Omega} e^{w(x)s}G(x-y)\big[g_{fr}(s,y_s) - p_t V(s,x)\big]dy_s \\
&\qquad + e^{w(x)s}M\big(s,x,\nabla V(s,x)\big)\nabla N\big(s,x,\nabla V(s,x)\big)\cdot\nabla V(s,x) \\
&\qquad + p_r e^{w(x)s}\Big[V(s,x) - V^3(s,x)\Big] + e^{w(x)s}g_d(s,x).
\end{aligned}$$

The above permits us to write

$$\begin{aligned}
& \frac{\partial}{\partial s}\big(e^{w(x)s}V(s,x)\big) \\
&\quad = e^{w(x)s}\frac{\partial}{\partial s}V(s,x) + e^{w(x)s}w(x)V(s,x) \\
&\quad = M\big(s,x,\nabla V(s,x)\big)N\big(s,x,\nabla V(s,x)\big)\int\limits_{\Omega} e^{w(x)s}J(x-y)V(s,y)dy \\
&\qquad + \int\limits_{\partial\Omega} e^{w(x)s}G(x-y)\big[g_{fr}(s,y_s) - p_t V(s,x)\big]dy_s \\
&\qquad + e^{w(x)s}M\big(s,x,\nabla V(s,x)\big)\nabla N\big(s,x,\nabla V(s,x)\big)\cdot\nabla V(s,x) \\
&\qquad + p_r e^{w(x)s}\Big[V(s,x) - V^3(s,x)\Big] + e^{w(x)s}g_d(s,x),
\end{aligned}$$

which leads to

$$
\begin{aligned}
&e^{w(x)t}V(t,x)-V_0(x)\\
&\qquad=\int_0^t M\big(s,x,\nabla V(s,x)\big)N\big(s,x,\nabla V(s,x)\big)\int_\Omega e^{w(x)s}J(x-y)V(s,y)dyds\\
&\qquad+\int_0^t\int_{\partial\Omega} e^{w(x)s}G(x-y)\big[g_{fr}(s,y_s)-p_t V(s,x)\big]dy_s ds\\
&\qquad+\int_0^t e^{w(x)s}M\big(s,x,\nabla V(s,x)\big)\nabla N\big(s,x,\nabla V(s,x)\big)\cdot\nabla V(s,x)ds\\
&\qquad+p_r\int_0^t e^{w(x)s}\Big[V(s,x)-V^3(s,x)\Big]ds+\int_0^t e^{w(x)s}g_d(s,x)ds.
\end{aligned}
$$

Multiplicating by $e^{-w(x)t}$, we obtain (18). $\square$

3. Existence and Uniqueness of the Solution to Problem (7)

In this section, we adapt some techniques from [29,31,45] to our problem (7), in which the nonconstant diffusion coefficients have a particular form, i.e.,

$$
\Phi(t,x,V(t,x),V_x(t,x))=M\big(t,x,V_x(t,x)\big) \text{ and } K(t,x,V(t,x),V_x(t,x))=N\big(t,x,V_x(t,x)\big)
$$

(see A. Miranville and C Moroşanu [45], for example).

Let us write problem (7) in an equivalent form:

$$
\begin{cases}
\dfrac{\partial}{\partial t}V(t,x)-M\big(t,x,\nabla V(t,x)\big)\dfrac{\partial}{\partial V_{x_j}}(N\big(t,x,\nabla V(t,x)\big)V_{x_i})V_{x_jx_i} & \\
\qquad=A(t,x,V_{x_i})+p_r\big[V(t,x)-V^3(t,x)\big]+g_d(t,x) & \text{in } Q\\
\dfrac{\partial}{\partial \mathbf{n}}V(t,x)+p_tV(t,x)=g_{fr}(t,x) & \text{in } \Sigma\\
V(0,x)=V_0(x) & \text{on } \Omega,
\end{cases}
\tag{19}
$$

with $V_{x_jx_i}=\dfrac{\partial^2}{\partial x_j\partial x_i}V$, where $i,j=1,2$, and

$$
A(t,x,V_{x_i})=M\big(t,x,\nabla V(t,x)\big)\frac{\partial}{\partial x_i}(N(t,x,\nabla V(t,x))V_{x_i}),\, i=1,2.
$$

As in [35], we recall that Equation $(7)_1$ is a quasi-linear one with the principal part being in divergence form and

$$
a_i\big(t,x,V_x(t,x)\big)=N(t,x,\nabla V(t,x))V_{x_i},\ \ i=1,2,
$$

$$
a(t,x,V_x)=-p_r\big[V(t,x)-V^3(t,x)\big]-g_d(t,x).
$$

On the other hand, problem (19) is a quasi-linear one (see [29]) with

$$
a_{ij}(t,x,V_x)=\frac{\partial}{\partial V_{x_j}}a_i(t,x,V_x)=\frac{\partial}{\partial V_{x_j}}N(t,x,\nabla V(t,x))V_{x_i},\ \ i=1,2
$$

and

$$
a(t,x,V_x)=-A(t,x,V_x)-p_r(V-V^3)-g_d(t,x).
$$

The boundary conditions (7)$_2$ are of the second type (see [36] or [29] for details). In addition, unless otherwise stated, we assume that Equations (7)$_1$ and (19)$_1$ are *uniformly parabolic*, which means the following conditions are fulfilled:

$$\nu_1(|V|)\xi^2 \le \frac{\partial a_i(t,x,z)}{\partial z_j}\xi_i\xi_j \le \nu_2(|V|)\xi^2 \tag{20}$$

for an arbitrary V, z, and $\xi = (\xi_1, \xi_2)$, an arbitrary real vector, where $\nu_1(r)$ and $\nu_2(r)$ are positive (nonincreasing and nondecreasing, respectively) continuous functions for $r \ge 0$.

Definition 2. *A solution, $V(t,x)$, of the second-order boundary value problem (19) is a classical solution if it is continuous in $\bar{Q}$, has continuous derivatives V_t, V_x, and V_{xx} in Q, satisfies Equation (19)$_1$ at all points, $(t,x) \in Q$, and satisfies conditions (19)$_2$ and (19)$_3$ for $(t,x) \in \Sigma$ and $t = 0$, respectively.*

In the present paper, we will investigate the solvability of the second-order boundary value problems of the form of (7) in the class $W_p^{1,2}(Q)$. We will adapt the results from [31,45] in order to prove the existence, the regularity, and the uniqueness of the solutions to the new nonlinear parabolic problem expressed in (7).

3.1. Well Posedness of Solutions to Problem (7)

We will establish the dependence of the solution, $V(t,x)$, to problem (7) on the terms $g_d(t,x)$ and $g_{fr}(t,x)$ using the Leray–Schauder degree theory (see, e.g., C. Moroşanu [37]), the L^p-theory of linear and quasi-linear parabolic equations (see [20]), and the Lions and Peetre embedding theorem ([35], p. 14), $W_p^{1,2}(Q) \subset L^\mu(Q)$, where

$$\mu = \begin{cases} \text{any positive number} \ge 3p & \text{if } \frac{1}{p} \le \frac{1}{2}, \\ \left(\frac{1}{p} - \frac{1}{2}\right)^{-1} & \text{if } \frac{1}{p} > \frac{1}{2}, \end{cases} \tag{21}$$

and, for a given positive integer k and $1 \le p \le \infty$, $W_p^{k,2k}(Q)$ denotes the Sobolev space on Q:

$$W_p^{k,2k}(Q) = \left\{ y \in L^p(Q) : \frac{\partial^r}{\partial t^r}\frac{\partial^q}{\partial x^q}\, y \in L^p(Q) \text{ for } 2r + q \le 2k \right\},$$

(see [30,36,45] for details).

The main result for the well posedness in problem (7) (or (19)) is given in the next theorem.

Theorem 4. *Suppose $V(t,x) \in C^{1,2}(Q)$ is a classical solution to problem (7) and for positive values of c_1, C, C_0, C_1, C_2, C_3, and C_4, one has* $\mathbf{I}_1$*. For $|V(t,x)| < C$ $\forall (t,x) \in Q$ and $\forall t,x,z$, the map $N(t,x,z)$ is continuous and differentiable with respect to x and z, and its x-derivatives and z-derivatives are measurable and bounded, satisfying (20) and*

$$0 < N_{min} \le N(t,x,V_x(t,x)) < N_{max}, \quad for \;\; (t,x) \in Q, \tag{22}$$

$$\begin{aligned} &|N(t,x,z)V_{x_i}|(1+|z|) + \left|\frac{\partial}{\partial x_1}(N(t,x,z)V_{x_1})\right| + \left|\frac{\partial}{\partial x_2}(N(t,x,z)V_{x_1})\right| \\ &+ \left|\frac{\partial}{\partial x_1}(N(t,x,z)V_{x_2})\right| + \left|\frac{\partial}{\partial x_2}(N(t,x,z)V_{x_2})\right| + |V(t,x_1,x_2)| \le C_0(1+|z|)^2. \end{aligned} \tag{23}$$

$\mathbf{I}_2$. *$M(t,x,V_x(t,x))$ is a positive and bounded nonlinear real function of class $C^1(Q)$ with bounded derivatives and*

$$0 < c_1 \le M(t,x,V_x(t,x)) \le C_1.$$

In addition, for every $\varepsilon > 0$, the functions $V(t,x)$ and $N(t,x,V_x(t,x))V_{x_i}$ satisfy the relations

$$\|V\|_{L^s(Q)} \le C_2, \qquad \|N(t,x,V_x)V_{x_i}\|_{L^r(Q)} < C_3, \qquad i = 1,2,$$

where

$$r = \begin{cases} \max\{p,4\} & p \neq 4 \\ 4+\varepsilon & p = 4, \end{cases} \qquad s = \begin{cases} \max\{p,2\} & p \neq 2 \\ 2+\varepsilon & p = 2. \end{cases}$$

Then, for $\forall g_d \in L^p(Q)$ and $\forall V_0 \in W_p^{2-\frac{2}{p}}(\Omega)$, with $p \neq \frac{3}{2}$, problem (7) has a solution, $V \in W_p^{1,2}(Q)$, and the following estimate holds:

$$\|V\|_{W_p^{1,2}(Q)} \le c\left[1 + \|V_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + \|V_0\|^{3-\frac{2}{p}}_{L^{3p-2}(\Omega)} \right. \\ \left. + \|g_d\|_{L^p(Q)} + \|g_{fr}\|^{3-\frac{2}{p}}_{L^{3p-2}(\Sigma)} + \|g_{fr}\|_{W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)}\right], \tag{24}$$

where $c > 0$ is independent of V, g_d, and g_{fr}.

If $V^1, V^2 \in W_p^{1,2}(Q)$ are two solutions to (7), corresponding to $\{g_d^1, g_{fr}^1, V_0^1\}$ and $\{g_d^2, g_{fr}^2, V_0^2\}$, respectively, such that $\|V^1\|_{W_p^{1,2}(Q)} \le C_4$, $\|V^2\|_{W_p^{1,2}(Q)} \le C_4$, then the following estimate holds:

$$\max_{(t,x)\in Q} |V^1 - V^2| \le c e^{\bar{c}T} \max\left[\max_{(t,x)\in Q} |g_d^1 - g_d^2|, \max_{(t,x)\in\Sigma} |g_{fr}^1 - g_{fr}^2|, \max_{(t,x)\in\Omega} |V_0^1 - V_0^2|\right], \tag{25}$$

where the constant c, $\bar{c} > 0$ does not depend on $\{V^1, g_d^1, g_{fr}^1, V_0^1\}$ and $\{V^2, g_d^2, g_{fr}^2, V_0^2\}$. In particular, this solution to problem (7) is unique.

3.2. The Proof of Theorem 4

Let the Banach space $B = W_p^{0,1}(Q) \cap L^{3p}(Q)$, equipped with the norm

$$\|V\|_B = \|V\|_{L^p(Q)} + \|V_x\|_{L^p(Q)}$$

and a nonlinear operator $\mathcal{O} : B \times [0,1] \to B$, defined by

$$v = v(V,\lambda) = \mathcal{O}(V,\lambda) \qquad \forall (V,\lambda) \in W_p^{0,1}(Q) \cap L^{3p}(Q) \times [0,1], \tag{26}$$

where $v(V,\lambda)$ is a unique solution to the problem

$$\begin{cases} \dfrac{\partial}{\partial t} v(t,x) - \left[\lambda M(t,x,V_x(t,x)) \dfrac{\partial}{\partial V_{x_j}} (N(t,x,V_x(t,x)) V_{x_i}) + (1-\lambda)\delta_i^j \right] v_{x_i x_j} \\ \qquad = \lambda \Big\{ A(t,x,V_x) + p_r [V(t,x) - V^3(t,x)] + g_d(t,x) \Big\} & \text{in } Q \\ \dfrac{\partial}{\partial \nu} v(t,x) + p_t v(t,x) = \lambda g_{fr}(t,x) & \text{on } \Sigma \\ v(0,x) = \lambda v_0(x), & \text{on } \Omega. \end{cases} \tag{27}$$

with $A(t,x,V_x(t,x)) = M(t,x,V_x(t,x)) \, \nabla N(t,x,V_x(t,x)) \cdot \nabla V(t,x), \forall\, (t,x) \in Q$.

First of all, let us prove the following lemma:

Lemma 2. *We assume hypotheses* $\mathbf{I}_1$ *and* $\mathbf{I}_2$ *to be valid for* $\forall V \in W_p^{1,2}(Q) \subset W_p^{0,1}(Q) \cap L^{3p}(Q)$. *Then,*

$$A\big(t,x,V_{x_i}(t,x)\big) + p_r[V(t,x) - V^3(t,x)] + g_d(t,x) \in L^p(Q). \tag{28}$$

Proof. Indeed, since $V \in W_p^{1,2}(Q) \subset L^{\mu}(Q) \subset L^{3p}(Q)$ (see (21)), then $\|V\|_{L^{3p}(Q)} \leq Konst$, and thus

$$\|V^3\|_{L^p(Q)} = \left(\int_Q |V^3|^p dxdt\right)^{\frac{1}{p}} = \left[\left(\int_Q |V|^{3p} dxdt\right)^{\frac{1}{3p}}\right]^{3p\frac{1}{p}} = \|V\|^3_{L^{3p}(Q)} \leq (Konst)^3,$$

i.e., the nonlinear term in (28) belongs to $L^p(Q)$, $\forall V \in W_p^{1,2}(Q) \subset W_p^{0,1}(Q) \cap L^{3p}(Q)$ (see Miranville and Moroşanu [45]).

Next, we prove that $A(t,x,V_{x_i}) \in L^p(Q)$ for $\forall V \in W_p^{1,2}(Q) \subset W_p^{0,1}(Q) \cap L^{3p}(Q)$. Using (19), we obtain ($V_{x_i} = \dfrac{\partial}{\partial x_i} V(t,x)$, $i = 1,2$)

$$\begin{aligned}
A(t,x,V_{x_i}) &= M\big(t,x,\nabla V(t,x)\big)\frac{\partial}{\partial x_i}\big[N\big(t,x,\nabla V(t,x)\big)V_{x_i}\big] \\
&= M\big(t,x,\nabla V(t,x)\big)\left\{\frac{\partial}{\partial x_i}\big[N\big(t,x,\nabla V(t,x)\big)\big]V_{x_i}\right. \\
&\qquad \left. + N\big(t,x,\nabla V(t,x)\big)\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_i}V(t,x)\right)\right\} \\
&= M\big(t,x,\nabla V(t,x)\big)\left\{\frac{\partial}{\partial x_i}N\big(t,x,\nabla V(t,x)\big)\right. \\
&\qquad \left. + \sum_{j=1}^{2}\frac{\partial}{\partial V_{x_j}}N\big(t,x,\nabla V(t,x)\big)V^2_{x_j x_i}\right\}V_{x_i} + N\big(t,x,\nabla V(t,x)\big)V^2_{x_i x_i}.
\end{aligned}$$

We denote

$$\begin{aligned}
T_1 &= \frac{\partial}{\partial x_i}\big[N\big(t,x,\nabla V(t,x)\big)\big]V_{x_i}, \\
T_2 &= N\big(t,x,\nabla V(t,x)\big)V^2_{x_i x_i}, \\
G_j &= V_{x_i}\frac{\partial}{\partial V_{x_j}}N\big(t,x,\nabla V(t,x)\big)V^2_{x_j x_i}, j = 1,2.
\end{aligned}$$

According to the hypothesis, we have ($i,j = 1,2$)

- **i.** $\dfrac{\partial}{\partial x_i}N\big(t,x,\nabla V(t,x)\big)$ is measurable and bounded, and $V_{x_i} \in L^p(Q)$;
- **ii.** $N\big(t,x,\nabla V(t,x)\big)$ is measurable and bounded (see $(23)_1$), and $V^2_{x_i x_i}$ is continuous;
- **iii.** $\dfrac{\partial}{\partial V_{x_j}}N\big(t,x,\nabla V(t,x)\big)$ is measurable and bounded, and V_{x_i} and $V^2_{x_j x_i}$ are continuous.

Using classical measure theory, from **i.–iii.**, it results that T_1, T_2, and G_j, with $j = 1,2$, are in $L^p(Q)$, and thus $A(t,x,V_{x_i}) \in L^p(Q)$, with $i = 1,2$.

Owing to the above outcomes and knowing that $g_d(t,x) \in L^p(Q)$, we can easily conclude that the statement in (28) is true. □

3.3. The Proof of Theorem 4 (Continued)

We shall prove that the nonlinear operator $\mathcal{O}$ defined by (26) is well defined, continuous, and compact.

Proof. From the right-hand side of (27)$_1$ and owing to (28), we can use the L^p-theory of linear parabolic equations (see [35]) to conclude a the solution, v, to problem (27) exists, and it is unique with

$$v = v(V,\lambda) \in W_p^{1,2}(Q), \quad \forall\, v \in W_p^{0,1}(Q) \cap L^{3p}(Q),\ \forall\, \lambda \in [0,1]. \tag{29}$$

Using now the continuous inclusions (see [31] and references therein)

$$W_p^{1,2}(Q) \subset W_p^{0,1}(Q) \cap L^{3p}(Q), \tag{30}$$

we obtain that $\mathcal{O}(V,\lambda) = v \in W_p^{0,1}(Q) \cap L^{3p}(Q)$ for all $V \in W_p^{0,1}(Q) \cap L^{3p}(Q)$ and $\forall\, \lambda \in [0,1]$, which means that the nonlinear operator $\mathcal{O}$ is well defined.

Let $V_n \to V$ in $W_p^{0,1}(Q) \cap L^{3p}(Q)$ and $\lambda_n \to \lambda$ in $[0,1]$. We denote $v^{n,\lambda_n} = \mathcal{O}(V^n,\lambda_n)$, $v^{n,\lambda} = \mathcal{O}(V^n,\lambda)$, and $v^{1,\lambda} = \mathcal{O}(V,\lambda)$. Using ideas from [31,35,36,45], we obtain

$$\|v^{n,\lambda_n} - v^{n,\lambda}\|_{W_p^{1,2}(Q)} \to 0 \quad as \ \ n \to \infty, \tag{31}$$

$$\|v^{n,\lambda} - v^{1,\lambda}\|_{W_p^{1,2}(Q)} \to 0 \quad as \ \ n \to \infty. \tag{32}$$

Making use of the continuous embedding (30) and relations (31) and (32), we derive the continuity of the nonlinear operator $\mathcal{O}$ defined in (26). Furthermore, $\mathcal{O}$ is compact. Indeed, since $\mu > 3p$ (see (21)), the inclusion $W_p^{1,2}(Q) \hookrightarrow W_p^{0,1}(Q) \cap L^{3p}(Q)$ is compact (see [35], p. 14). Moreover, writing $\mathcal{O}$ as the composition

$$B \times [0,1] \to W_p^{1,2}(Q) \hookrightarrow W_p^{0,1}(Q) \cap L^{3p}(Q) = B,$$

the compactness of $\mathcal{O}$ immediately follows. □

3.3.1. The Regularity of the Solution $V(t,x)$

We establish now the existence of a number $\delta > 0$ such that

$$(V,\lambda) \in W_p^{0,1}(Q) \cap L^{3p}(Q) \times [0,1] \ \text{ with } \ V = \mathcal{O}(V,\lambda) \implies \|V\|_B < \delta. \tag{33}$$

The equality $V = \mathcal{O}(V,\lambda)$ in (33) is equivalent to

$$\begin{cases} \dfrac{\partial}{\partial t}V(t,x) - \lambda M\big(t,x,V_x(t,x)\big)\mathrm{div}\Big(N\big(t,x,V_x(t,x)\big)\cdot\nabla V\Big) - (1-\lambda)\Delta V & \\ \qquad = \lambda\Big\{A(t,x,V_x) + p_r\left[V(t,x) - V^3(t,x)\right] + g_d(t,x)\Big\} & \text{in } Q \\ \dfrac{\partial}{\partial \nu}V(t,x) + p_t V(t,x) = \lambda g_{fr}(t,x) & \text{on } \Sigma \\ V(0,x) = \lambda V_0(x), & \text{on } \Omega. \end{cases} \tag{34}$$

(see (7) and (27)).

Multiplying the first equation in (34) by $|V|^{3p-4}V$ and integrating over $Q_t := (0,t) \times \Omega$, $t \in (0,T]$, we obtain

$$\int\limits_{Q_t} \frac{\partial}{\partial t}|V(\tau,x)|^{3p-2}d\tau dx - \lambda \int\limits_{Q_t} M(t,x,V_x(t,x))\,\mathrm{div}\Big(N(t,x,V_x(t,x))\cdot\nabla V\Big)|V|^{3p-4}V d\tau dx$$
$$-(1-\lambda)\int\limits_{Q_t} \Delta V\,|V|^{3p-4}v\,d\tau dx$$
$$= \lambda p_r \int\limits_{Q_t}(V-V^3)|V|^{3p-4}V\,d\tau dx + \lambda\int\limits_{Q_t} g_d|V|^{3p-4}V\,d\tau dx.$$

Following the same line of proof as in the studies [17,27,31], we finally obtain

$$\begin{aligned}\|V\|_{W_p^{1,2}(Q)} \le c\Big\{& 1 + \|V_0\|_{W_\infty^{2-\frac{2}{p}}(\Omega)} + \|V_0\|^{\frac{3p-2}{p}}_{L^{3p-2}(\Omega)} \\ &+ \|g_d\|_{L^p(Q)} + \|g_{fr}\|_{W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)} + \|g_{fr}\|^{\frac{3p-2}{p}}_{L^{3p-2}(\Sigma_t)}\Big\}.\end{aligned} \tag{35}$$

The continuous embedding $W_p^{1,2}(Q) \subset B = W_p^{0,1}(Q) \cap L^{3p}(Q)$ ensures that

$$\|V\|_B \le c\|V\|_{W_p^{1,2}(Q)},$$

which, owing to (35), ensures that a constant $\delta > 0$ can be found such that the property expressed in (33) is true.

We denote this as

$$B_\delta := \Big\{V \in B:\ \|V\|_B < \delta\Big\}.$$

From (33), we derive that

$$\mathcal{O}(V,\lambda) \ne V \quad \forall V \in \partial B_\delta,\ \ \forall \lambda \in [0,1],$$

provided that $\delta > 0$ is sufficiently large (see also [38]). Furthermore, following the same reasoning as in [29–31,35,36,38,45], we conclude that problem (7) has a solution $V \in W_p^{1,2}(Q)$. The estimate (24) results from (35), and the proof of the first part in Theorem 4 is finished.

3.3.2. The Uniqueness of the Solution $V(t,x)$

To establish the estimate in (24) and, as a consequence, the uniqueness of the solution to problem (7) or $(19)_1$ and $(7)_{2,3}$, we refer to [29,31,35] and the references therein.

As a consequence, this shows the uniqueness of the solution to the nonlinear problem (7).

Corollary 1. *For $V_0^1 = V_0^2$, problem (7) possesses a unique solution in $W_p^{1,2}(Q)$.*

Proof. Let $g_d^1 = g_d^2 = g_d$ and $g_{fr}^1 = g_{fr}^2 = g_{fr}$ in Theorem 4. Then, (25) demonstrates the corollary (see also [38] and references therein). $\square$

Remark 2. *The nonlinear operator $\mathcal{O}$ in (26) depends on $\lambda \in [0,1]$, and its fixed points for $\lambda = 1$ are solutions to (34).*

4. A Novel Nonlocal and Nonlinear Second-Order Anisotropic Reaction–Diffusion Model in Image Segmentation

The nonlocal and nonlinear parabolic second-order PDE problem (1) can be applied, for example, to image segmentation, denoising, enhancement, and restoration. Here, we consider the particularization of this mathematical model, namely

$$\begin{aligned}
\frac{\partial}{\partial t}V(t,x) &= M(\|\nabla V(t,x)\|)N(\|\nabla V(t,x)\|)\times \\
&\left\{\int_{\Omega} J(x-y)\Big[V(t,y)-V(t,x)\Big]dy + \int_{\partial\Omega} G(x-y_s)\Big[g_{fr}(t,y_s)-p_t V(t,x)\Big]dy_s\right\} \\
&+M(\|\nabla V(t,x)\|)\nabla N(\|\nabla V(t,x)\|)\cdot\nabla V(t,x) \\
&+p_r\Big[V(t,x)-V^3(t,x)\Big]+g_d(t,x),
\end{aligned} \tag{36}$$

with the initial condition

$$V(0,x) = V_0(x). \tag{37}$$

The first function $M : [0,\infty) \to (0,\infty)$ in the PDE-based model (36) has the form

$$M(s) = \frac{(\alpha s^{\kappa}+\nu)^{1/(\kappa+1)}}{\xi}, \tag{38}$$

where $\alpha, \nu \in (0,4]$, $\xi \geq 1.5$, and $\kappa \in (0,2]$. It is worth noting that the term $M(\|\nabla V(t,x)\|)$ controls the speed of this diffusion process and enhances the edges of the corresponding image.

The edge-stopping (diffusivity) function $N : [0,\infty) \to (0,\infty)$ in (36) has the form

$$N(s) = \varepsilon\left(\frac{\delta(V)}{\beta\ln(\delta(V))+\gamma s^2}\right)^{1/3}, \tag{39}$$

where $\varepsilon \in (0,2)$, $\gamma \in (1,5]$, and $\beta \in (0,1)$, and the conductance parameter δ is defined by

$$\delta(V) := |r\mu(\|\nabla V\|)+\zeta\mathcal{M}(\|\nabla V\|)|, \quad r>0, \quad \zeta\in(0,1),$$

with the respective averaging and median operators μ and $\mathcal{M}$.

The function N in (39) satisfies the main requirements for successful restoration [31], e.g., it is positive and monotonically decreasing in $(0,\infty)$ and $\lim_{s\to\infty} N(s) = 0$.

The nonlocal PDE model given by (36) and (37) admits a solution in the space $C([0,t^*],C^1(\Omega))$ (see Theorem 2), which represents the recovered image. The solution to this problem can be derived by an iterative algorithm, and it can be determined by numerically solving the nonlinear diffusion-based model given by (36) and (37) using the finite-differences method [29–31].

The local anisotropic reaction–diffusion model corresponding to (36) can be written as follows:

$$\begin{cases} \dfrac{\partial}{\partial t}V(t,x) - M\big(\|\nabla V(t,x)\|\big)\,\mathrm{div}\Big(N\big(\|\nabla V(t,x)\|\big)\cdot \nabla V(t,x)\Big) & \\ \qquad\qquad = p_r\left[V(t,x) - V^3(t,x)\right] + g_d f(t,x) & \text{in } Q \\ \dfrac{\partial}{\partial \mathbf{n}}V(t,x) + p_t V(t,x) = g_{fr}(t,x) & \text{on } \Sigma \\ V(0,x) = V_0(x) & \text{on } \Omega, \end{cases} \tag{40}$$

$$\nabla V(t,x) = V_x = \big(V_{x_1}(t,x), V_{x_2}(t,x)\big).$$

The edge-stopping (diffusivity) function in $(36)_2$ is positive, monotonically decreasing, and converges to zero (see [3,12,29–31]), thus satisfying the conditions imposed by proper diffusion. Moreover, it is easy to check that M and N in (36) satisfy assumptions I_1 and I_2 in Theorem 4, and thus, the nonlinear anisotropic reaction–diffusion model (40) is well posed, as proven in the previous section. Consequently, it admits a unique classical solution, $V(t,x) \in W_p^{1,2}(Q)$, that represents an evolving image of the observed image $V(0,x) = V_0(x)$.

Because of the presence of the term $M\big(\|\nabla V(t,x)\|\big)$, the nonlinear operator in (40) does not represent the gradient of the energy function. Therefore, the proposed second-order nonlinear diffusion-based scheme cannot be obtained from the minimization of any energy cost function, so this scheme is not a variational PDE model.

5. Conclusions

This paper deals with a qualitative study for some second-order boundary value problems, with *local* or *nonlocal* diffusion and a cubic nonlinear reaction term, endowed with in-homogeneous Cauchy–Neumann (Robin) boundary conditions. Here, we are focused on finding concrete cases of functions corresponding to the general cases $\Phi(t,x,V(t,x),V_x(t,x))$ and $K(t,x,V(t,x),V_x(t,x))$, which were introduced for the first time in a paper by Miranville, Moroşanu, and Pavăl [30], whose work represents a major challenge for both theory and applications. In our new nonlinear, second-order, anisotropic, reaction–diffusion problems given by (1) and (7), we consider the following settings: $\Phi(t,x,V(t,x),V_x(t,x)) = M\big(t,x,\nabla V(t,x)\big)$ and $K(t,x,V(t,x),V_x(t,x)) = N\big(t,x,\nabla V(t,x)\big)$.

First, provided that the initial data meet appropriate regularity conditions, we prove the existence of a solution in the space $C([0,t^*],C^1(\Omega))$ of the nonlocal and nonlinear second-order boundary value problem (1) (in particular, (36) and (37)).

Secondly, under certain assumptions about the input data, $g_d(t,x)$, $g_{fr}(t,x)$, and $V_0(x)$, we study the well posedness (the existence, a priori estimates, regularity, uniqueness) of a classical solution in the Sobolev space, $W_p^{1,2}(Q)$, of the local and nonlinear second-order boundary value problem (7) (in particular, (40)). Precisely, the Leray–Schauder principle is applied to prove the existence of solutions to the nonlinear problem in question, while the L^p theory of linear and quasi-linear parabolic equations is used in order to derive regularity properties for the solutions. Moreover, the a priori estimates are made in $L^p(Q)$, which leads to better estimates for unknown functions $V(t,x)$ (for more details in this respect, see [13,26,30,31,35,38,39,46] and references therein). This approach could be applied in future to study other kinds of first- and second-order boundary value problems.

We note that due to the presence of the nonlinear coefficients $M\big(t,x,\nabla V(t,x)\big)$ and $N\big(t,x,\nabla V(t,x)\big)$ (see (1) and (7)), the proposed second-order nonlinear reaction–diffusion schemes (36), (37), and (40) represent nonvariational PDE models (see [29–31]). Therefore, they cannot be obtained from the minimization of any energy cost function; thus, these new schemes are not variational PDE models.

These models describe a great variety of phenomena that appear in many sciences, like physics, biology, chemistry, image processing, etc. From this, we can deduce the importance of developing numerical methods that bring efficiency to the process of approximating solutions (accuracy, computer time, etc.). Thus, the construction of numerical approximation schemes to approximate unique solutions to the mathematical models (1) and (7) represents an urgent problem to be solved in the future.

To conclude, we can highlight some questions that could be addressed in future work, namely

- Would it be possible to discuss such results for other classes of models? What restrictions should be imposed?
- Can we use such results in the study of distributed and/or boundary nonlinear optimal control problems governed by such a nonlinear equation (see [36])?
- To illustrate the effectiveness of theory and applications equally, a nonuniform grid of points for the time interval $[0, T]$ could be constructed (see Remark 1) and used for both the nonlocal problems (36) and (37) and the local problem (40).

Author Contributions: Conceptualization, C.M.; Methodology, A.M.; Validation, T.B., A.M. and C.M.; Formal analysis, T.B. and A.M.; Investigation, T.B., A.M. and C.M.; Writing—original draft, C.M.; Visualization, T.B. and A.M.; Project administration, C.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research work was supported by a grant from the Romanian Academy, GAR-2023, Project code 19.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Angenent, S.B.; Pichon, E.; Tannenbaum, A. Mathematical methods in medical image processing. *Bull. Am. Math. Soc.* **2006**, *43*, 365–396. https://doi.org/10.1090/S0273-0979-06-01104-9.
2. Demant, C.; Streicher-Abel, B.; Garnica, C. *Industrial Image Processing. Visual Quality Control in Manufacturing*; Springer: Berlin/Heidelberg, Germany, 2013; ISBN 978-3-642-33904-2.
3. Morfu, S. On some applications of diffusion processes for image processing. *Phys. Lett.* **2009**, *373*, 2438–2444. https://doi.org/10.1016/j.physleta.2009.04.076.
4. Chan, T.F.; Vese, L.A. Active contours without edges. *IEEE Trans. Image Process.* **2001**, *10*, 266–277. https://doi.org/10.1109/83.902291.
5. Lee, D.; Lee, S. Image Segmentation Based on Modified Fractional Allen–Cahn Equation. *Math. Probl. Eng.* **2019**, *2019*, 3980181. https://doi.org/10.1155/2019/3980181.
6. Song, M. Application Research of Image Segmentation Technology in Digital Image Processing. *Electron. Technol. Softw. Eng.* **2017**, *1*, 75.
7. Vijayakrishna, R.; Kumar, B.V.R.; Halim, A. A PDE Based Image Segmentation Using Fourier Spectral Method. *Differ. Equ. Dyn. Syst.* **2018**, *7*, 629–642. https://doi.org/10.1007/s12591-018-0414-x.
8. Ma, L.; Zeng, T.; Li, G. Hybrid variational model for texture image restoration. *East Asian J. Appl. Math.* **2017**, *7*, 629–642.
9. Gilboa, G.; Osher, S. Nonlocal Linear Image Regularization and Supervised Segmentation. *Multiscale Model. Simul.* **2007**, *6*, 595–630. https://doi.org/10.1137/060669358.
10. Hu, Y.; Jacob, M. Higher degree total variation (HDTV) regularization for image recovery. *IEEE Trans. Image Process.* **2012**, *21*, 2559–2571.
11. Long, J.; Shelhamer, E.; Darrell, T. Fully Convolutional Networks for Semantic Segmentation. In Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition, Boston, MA, USA, 7–12 June 2015.
12. Pavăl, S.-D.; Craus, M. Reaction-diffusion model applied to enhancing U-Net accuracy for semantic image segmentation. *Discret. Contin. Dyn. Syst.* **2023**, *16*, 54–74. https://doi.org/10.3934/dcdss.2022142.
13. Banerjee, M.; Kuznetsov, M.; Udovenko, O.; Volpert, V. Nonlocal Reaction–Diffusion Equations in Biomedical Applications. *Acta Biotheor.* **2022**, *70*, 12. https://doi.org/10.1007/s10441-022-09436-4.
14. Lysaker, M.; Lundervold, A.; Tai, X.C. Noise removal using fourth-order partial differential equation with applications to medical magnetic resonance images in space and time. *IEEE Trans. Image Process.* **2003**, *12*, 1579–1590.
15. James, A.P.; Dasarathy, B.V. Medical Image Fusion: A survey of state of the art. *Inf. Fusion* **2014**, *19*, 4–19. https://doi.org/10.1016/j.inffus.2013.12.002.
16. Dubey, R.B.; Hanmandlu, M.; Gupta, S.K.; Gupta, S.K. The brain MR Image segmentation techniques and use of diagnostic packages. *Acad Radiol.* **2010**, *17* 658–671. https://doi.org/10.1016/j.acra.2009.12.017.

17. Gordillo, N.; Montseny, E.; Sobrevilla, P. State of the art survey on MRI brain tumor segmentation. *Magn. Reson. Imaging* **2013**, *31*, 1426–1438. https://doi.org/10.1016/j.mri.2013.05.002.
18. Prastawa, M.; Bullitt, E.; Gerig, G. Simulation of brain tumors in MR images for evaluation of segmentation efficacy. *Med. Image Anal.* **2009**,*13*, 297–311. https://doi.org/10.1016/j.media.2008.11.002.
19. Strachota, P. Implementation of the MR Tractography Visualization Kit Based on the Anisotropic Allen–Cahn Equation. *Kybernetika* **2009**, *45*, 657–669
20. Sarti, A.; Mikula, K.; Sgallari, F.; Lamberti, C. Evolutionary partial differential equations for biomedical image processing. *J. Biomed. Inform.* **2002**, *35*, 77–91. https://doi.org/10.1016/S1532-0464(02)00502-6.
21. Andreu-Vaillo, F.; Mazón, J.M.; Rossi, J.D.; Toledo-Melero, J.J. *Nonlocal Diffusion Problems*; American Mathematical Society: Providence, PI, USA, 2010; Volume 165.
22. Bates, P.W.; Brown, S.; Han, J. Numerical analysis for a nonlocal Allen-Cahn equation. *Int. J. Numer. Anal. Model.* **2009**, *6*, 33–49.
23. Cortazar, C.; Elgueta, M.; Rossi, J.D.; Wolanski, N. Boundary fluxes for nonlocal diffusion. *J. Differ. Equ.* **2007**, *234*, 360–390.
24. Guichard, F.; Moisan, L.; Morel, J.M. A review of P.D.E. models in image processing and image analysis. *J. Phys.* **2001**, *4*, 137–154.
25. Pérez-Llanos, M.; Rossi J. D. Blow-up for a non-local diffusion problem with Neumann boundary conditions and a reaction term. *Nonlinear Anal.* **2009**, *70*, 1629–1636
26. Siddiqi, K.; Lauzière, Y.B.; Tannenbaum, A.; Zucker, S.W. Area and length minimizing flows for shape segmentation. *IEEE Trans. Image Process.* **1998**, *7*, 433–443.
27. Craus, M.; Paval, S.-D. An Accelerating Numerical Computation of the Diffusion Term in a Nonlocal Reaction-Diffusion Equation. *Mathematics* **2020**, *8*, 2111.
28. Wang, L.-L.; Gu, Y. Efficient Dual Algorithms for Image Segmentation Using TV-Allen-Cahn Type Models. *Commun. Comput. Phys.* **2011**, *9*, 859–877. https://doi.org/10.4208/cicp.221109.290710a.
29. Moroşanu, C.; Pavăl, S.D. Rigorous Mathematical Investigation of a Nonlocal and Nonlinear Second-Order Anisotropic Reaction-Diffusion Model: Applications on Image Segmentation. *Mathematics* **2021** *9*, 91. https://doi.org/10.3390/math9010091.
30. Miranville, A.; Moroşanu, C.; Pavăl, S.-D. Preface: Qualitative and Quantitative Analysis of Nonlinear and Nonlocal Reaction-Diffusion Models. Applications. *Discret. Contin. Dyn. Syst.* **2023**, *16*, i–ii. https://doi.org/10.3934/dcdss.2022208.
31. Barbu, T.; Miranville, A.; Moroşanu, C. A qualitative analysis and numerical simulations of a nonlinear second-order anisotropic diffusion problem with non-homogeneous Cauchy-Neumann boundary conditions. *Appl. Math. Comput.* **2019**, *350*, 170–180. https://doi.org/10.1016/j.amc.2019.01.004.
32. Benes, M.; Chalupecky, V.; Mikula, K. Geometrical image segmentation by the Allen–Cahn equation. *Appl. Numer. Math.* **2004**, *51*, 187–205.
33. Bogoya, M.; Gómez, J. On a nonlocal diffusion model with Neumann boundary conditions. *Nonlinear Anal.* **2012**, *75*, 3198–3209.
34. Wang, H.; Wang, Y.; Ren, W. Image Denoising Using Anisotropic Second and Fourth Order Diffusions Based on Gradient Vector Convolution. *Comput. Sci. Inf. Syst.* **2012**, *9*, 1493–1512.
35. Miranville, A.; Moroşanu, C. Qualitative and quantitative analysis for the mathematical models of phase separation and transition. Aplications. *AIMS-Am. Inst. Math. Sci. Differ. Equ. Dyn. Syst.* **2020**, *7*.
36. Moroşanu, C. *Analysis and Optimal Control of Phase-Field Transition System: Fractional Steps Methods*; Bentham Science Publishers: Sharjah, United Arab Emirates, 2012. http://dx.doi.org/10.2174/97816080535061120101.
37. Taylor, J.E.; Cahn, J.W. Diffuse interfaces with sharp corners and facets: Phase-field models with strongly anisotropic surfaces. *Physica D* **1998**, *112*, 381–411.
38. Moroşanu, C.; Motreanu, D. The phase field system with a general nonlinearity. *Int. J. Differ. Equ. Appl.* **2000**, *1*, 187–204.
39. Ignat, L.I.; Rossi, J.D. A nonlocal convection-diffusion equation. *J. Funct. Anal.* **2007**, *251*, 399–437.
40. Ning, H.E.; Ke, L.U.A. A Non Local Feature-Preserving Strategy for Image Denoising. *Chin. J. Electron.* **2012**, *21*, 651–656.
41. Rubinstein, J.; Sternberg, P. Nonlocal reaction-diffusion equations and nucleation. *IMA J. Appl. Math.* **1992**, *48*, 249–264.
42. Moroşanu, C. Modeling of the continuous casting process of steel via phase-field transition system. Fractional steps method. *AIMS Math.* **2019**, *4*, 648–662. https://doi.org/10.3934/math.2019.3.648.
43. Guimarães, S.J.F.; Couprie, M. Video segmentation based on 2D image analysis. *Pattern Recognit. Lett.* **2003**, *24*, 947–957.
44. Tekalp, A.M. *Digital Video Processing;* Prentice-Hall: Hoboken, NJ, USA, 1995.
45. Miranville, A.; Moroşanu, C. A Qualitative Analysis of a Nonlinear Second-Order Anisotropic Diffusion Problem with Non-homogeneous Cauchy–Stefan–Boltzmann Boundary Conditions. *Appl. Math. Optim.* **2021**, *84*, 227–244. https://doi.org/10.1007/s00245-019-09643-5.
46. Volpert, V. *Elliptic Partial Differential Equations. Volume 2: Reaction-Diffusion Equations;* Birkhäuser: Villeurbanne, France, 2014.