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Article

Studies on the Marchenko–Pastur Law

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Abstract: In free probability, the theory of Cauchy–Stieltjes Kernel (CSK) families has recently been introduced. This theory is about a set of probability measures defined using the Cauchy kernel similarly to natural exponential families in classical probability that are defined by means of the exponential kernel. Within the context of CSK families, this article presents certain features of the Marchenko–Pastur law based on the Fermi convolution and the t-deformed free convolution. The Marchenko–Pastur law holds significant theoretical and practical implications in various fields, particularly in the analysis of random matrices and their applications in statistics, signal processing, and machine learning. In the specific context of CSK families, our study of the Marchenko–Pastur law is summarized as follows: Let $\mathcal{K}_+(\mu) = \{Q_m^\mu(dx);\ \ m \in (m_0^\mu, m_+^\mu)\}$ be the CSK family generated by a non-degenerate probability measure μ with support bounded from above. Denote by $\left(Q_m^\mu\right)^{\bullet s}$ the Fermi convolution power of order $s > 0$ of the measure Q_m^μ. We prove that if $\left(Q_m^\mu\right)^{\bullet s} \in \mathcal{K}_+(\mu)$, then μ is of the Marchenko–Pastur type law. The same result is obtained if we replace the Fermi convolution $\bullet$ with the t-deformed free convolution $\boxed{t}$.

Keywords: variance function; Cauchy–Stieltjes transform; Marchenko–Pastur law

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1. Introduction

In free probability, the Marchenko–Pastur law plays the same role that the Poisson law plays in classical probability. In mathematical random matrices theory for large rectangular random matrices, the Marchenko–Pastur measure describes the asymptotic behavior of the corresponding singular values. However, many properties and characterizations have been given regarding Marchenko–Pastur law in the literature. In [1], the Lukacs type characterization of Marchenko–Pastur law is studied in free probability. In [2], Marchenko–Pastur law was characterized in the context of Cauchy–Stieltjes Kernel (CSK) families based on Boolean additive convolution. Furthermore, a short proof for the Marchenko–Pastur theorem was given in [3]. Further results related to the Marchenko–Pastur measure are given in [4–10]. In the present article, our study on the Marchenko–Pastur law is related to CSK families. In this study, we involve two kinds of convolutions of importance in free probability, namely the Fermi convolution introduced in [11] and the t-deformed free convolution defined in [12,13]. In fact, the study of the stability of a given CSK family with respect to a Fermi convolution (or a t-deformed free convolution) power leads to the result that the generating measure of the CSK family is of the Marchenko–Pastur type law. To clarify our results, we need to present some fundamental notions on CSK families as a basis for the reader. We also discuss certain concepts of Fermi convolution and t-deformed free convolution.

It is well known that the theory of natural exponential families (NEFs) in classical probability is based on the exponential kernel $\exp(\vartheta y)$. The CSK family in free probability is introduced in a way similar to NEFs by using the Cauchy–Stieltjes kernel $(1-\vartheta y)^{-1}$. Some properties of CSK families are given in [14] involving measures with compact support. Extended properties of CSK families are provided in [15] to cover probability measure having support bounded from one side, say, from above. $\mathbf{P}_{ba}$ denotes the set of non-degenerate real probability measures having support bounded from above. Suppose $\mu \in \mathbf{P}_{ba}$, then

$$\mathcal{M}_\mu(\vartheta) = \int \frac{1}{1-\vartheta y}\mu(dy) \tag{1}$$

is defined as $\forall\, \vartheta \in [0, \vartheta_+^\mu)$ with $\frac{1}{\vartheta_+^\mu} = \max\{0, \sup \mathrm{supp}(\mu)\}$. The set

$$\mathcal{K}_+(\mu) = \{\mathbb{P}_\vartheta^\mu(dy) = \frac{1}{\mathcal{M}_\mu(\vartheta)(1-\vartheta y)}\mu(dy) : \ \vartheta \in (0, \vartheta_+^\mu)\}$$

represents the one-sided CSK family generated by μ.

The mean function $\vartheta \mapsto \mathbf{K}_\mu(\vartheta) = \int y \mathbb{P}_\vartheta^\mu(dy)$ is strictly increasing on $(0, \vartheta_+^\mu)$ (see [15]). The interval $(m_0^\mu, m_+^\mu) = \mathbf{K}_\mu((0, \vartheta_+^\mu))$ represents the (one-sided) mean domain of $\mathcal{K}_+(\mu)$. Consider $\chi_\mu(\cdot)$ to be the inverse of $\mathbf{K}_\mu(\cdot)$; for $m \in (m_0^\mu, m_+^\mu)$, write $Q_m^\mu(dy) = \mathbb{P}_{\chi_\mu(m)}^\mu(dy)$. A mean parametrization is then provided for $\mathcal{K}_+(\mu)$ as

$$\mathcal{K}_+(\mu) = \{Q_m^\mu(dy); \ m \in (m_0^\mu, m_+^\mu)\}.$$

It was proven in [15] that

$$m_0^\mu = \lim_{\vartheta \to 0^+} \mathbf{K}_\mu(\vartheta) \quad \text{ and } \quad m_+^\mu = B - \lim_{z \to B^+} \frac{1}{\mathcal{G}_\mu(z)}, \tag{2}$$

where

$$B = B(\mu) = \max\{0, \sup \mathrm{supp}(\mu)\} = \frac{1}{\vartheta_+^\mu}, \tag{3}$$

and

$$\mathcal{G}_\mu(w) = \int \frac{1}{w-y}\mu(dy), \quad \text{ for } w \in \mathbb{C} \setminus \mathrm{supp}(\mu) \tag{4}$$

represent the Cauchy–Stieltjes transform of μ.

The CSK family is denoted as $\mathcal{K}_-(\mu)$ when the support of μ is bounded from below. We have $\vartheta_-^\mu < \vartheta < 0$, where ϑ_-^μ is equal to $1/A(\mu)$ or $-\infty$ with $A = A(\mu) = \min\{0, \inf supp(\mu)\}$. The interval (m_-^μ, m_0^μ) represents the mean domain for $\mathcal{K}_-(\mu)$ where $m_-^\mu = A - 1/\mathcal{G}_\mu(A)$. If the support of μ is compact, then $\vartheta \in (\vartheta_-^\mu, \vartheta_+^\mu)$ and $\mathcal{K}(\mu) = \mathcal{K}_-(\mu) \cup \{\mu\} \cup \mathcal{K}_+(\mu)$ is the two-sided CSK family.

The function

$$m \mapsto \mathcal{V}_\mu(m) = \int (x-m)^2 Q_m^\mu(dx), \tag{5}$$

is called a variance function of $\mathcal{K}_+(\mu)$ (see [14]). If μ does not have a moment of order 1, then all measures in $\mathcal{K}_+(\mu)$ have infinite variance. The authors in [15] introduced the concept of a pseudo-variance function $\mathbb{V}_\mu(\cdot)$ as

$$\mathbb{V}_\mu(m) = m\left(\frac{1}{\chi_\mu(m)} - m\right). \tag{6}$$

If $m_0^\mu = \int y\mu(dy)$ is finite, then $\mathcal{V}_\mu(.)$ exists and (see [15])

$$\mathbb{V}_\mu(m) = \frac{m}{m - m_0^\mu}\mathcal{V}_\mu(m). \tag{7}$$

Remark 1. *(i) The law $Q_m^\mu(dy)$ can be written as $Q_m^\mu(dy) = h_\mu(y,m)\mu(dy)$ with*

$$h_\mu(y,m) := \begin{cases} \frac{\mathbb{V}_\mu(m)}{\mathbb{V}_\mu(m)+m(m-y)}, & m \neq 0 \quad ; \\ 1, & m = 0,\ \mathbb{V}_\mu(0) \neq 0 \quad ; \\ \frac{\mathbb{V}'_\mu(0)}{\mathbb{V}'_\mu(0)-y}, & m = 0,\ \mathbb{V}_\mu(0) = 0 \quad . \end{cases} \tag{8}$$

(ii) μ is characterized by $\mathbb{V}_\mu(\cdot)$. If we consider

$$\varpi = \varpi(m) = m + \frac{\mathbb{V}_\mu(m)}{m}, \tag{9}$$

then

$$\mathcal{G}_\mu(\varpi) = \frac{m}{\mathbb{V}_\mu(m)}. \tag{10}$$

(iii) Consider $\varphi(\mu)$ as the image of μ by $\varphi : y \longmapsto \alpha y + \beta$ where $\alpha \neq 0$ and $\beta \in \mathbb{R}$. Then, $\forall\, m$ close enough to $m_0^{\varphi(\mu)} = \varphi(m_0^\mu) = \alpha m_0^\mu + \beta$,

$$\mathbb{V}_{\varphi(\mu)}(m) = \frac{\alpha^2 m}{m-\beta}\mathbb{V}_\mu\left(\frac{m-\beta}{\alpha}\right). \tag{11}$$

When $\mathcal{V}_\mu(.)$ exists,

$$\mathcal{V}_{\varphi(\mu)}(m) = \alpha^2\mathcal{V}_\mu\left(\frac{m-\beta}{\alpha}\right). \tag{12}$$

(iv) For $a \neq 0$, the Marchenko–Pastur measure is

$$\boldsymbol{MP}_a(dy) = \frac{\sqrt{((a+1)^2-y)(y-(a-1)^2)}}{2\pi a^2 y}\mathbf{1}_{((a-1)^2,(a+1)^2)}(y)dy + (1-1/a^2)^+\delta_0 \tag{13}$$

with $m_0^{\boldsymbol{MP}_a} = 1$. We have

$$\mathbb{V}_{\boldsymbol{MP}_a}(m) = \frac{a^2m^2}{m-1}, \tag{14}$$

and

$$(m_-^{\boldsymbol{MP}_a}, m_+^{\boldsymbol{MP}_a}) = \begin{cases} (1-|a|, 1+|a|), & \text{if } a^2 \leq 1; \\ (0, 1+|a|), & \text{if } a^2 > 1. \end{cases} \tag{15}$$

For more details, see ([2] Section 3).

We now introduce the notion of Fermi convolution. Denote the set of real probability measures (the subsets of measures from $\mathbf{P}$ that have finite mean and variance and with compact support, respectively) by $\mathbf{P}$ ($\mathbf{P}^2$ and $\mathbf{P}_c$, respectively). For $\rho \in \mathbf{P}^2$, the $\mathcal{B}$-transform is defined in [11] by

$$\mathcal{B}_\rho(z) = m_0^\rho z + z\mathcal{E}_{\rho^0}\left(\frac{1}{z}\right), \tag{16}$$

where m_0^ρ is the mean of ρ, ρ^0 is the zero mean shift of ρ and

$$\mathcal{E}_\rho(z) = z - \frac{1}{\mathcal{G}_\rho(z)}, \quad \text{for } z \in \mathbb{C}^+. \tag{17}$$

Since $\rho \in \mathbf{P}^2$ is determined by $\mathcal{G}_\mu(\cdot)$, it is also determined by $\mathcal{B}_\mu(\cdot)$.

Let $\rho_1,\ \rho_2 \in \mathbf{P}^2$ and the Fermi convolution of ρ_1 and ρ_2 be denoted $\rho = \rho_1 \bullet \rho_2$; then, we have,

$$\mathcal{B}_\rho(z) = \mathcal{B}_{\rho_1}(z) + \mathcal{B}_{\rho_2}(z) \tag{18}$$

(see ([11] Theorem 3.1)). In addition, $\rho \in \mathbf{P}^2$ and $m_0^\rho = m_0^{\rho_1} + m_0^{\rho_2}$.

$\rho \in \mathbf{P}^2$ is $\bullet$-infinitely divisible if for each $q \in \mathbb{N}$, there is $\rho_q \in \mathbf{P}^2$ so that

$$\rho = \underbrace{\rho_q \bullet \ldots\ldots \bullet \rho_q}_{q \text{ times}}.$$

All measures $\rho \in \mathbf{P}^2$ are $\bullet$-infinitely divisible (see ([11] Remark 3.2)). The Fermi convolution was studied from a combinatoric point of view in [11], and the Fermionic Poisson limit theorem was proven in ([11] Theorem 3.2). In ([16] Theorem 1), the variance function is expressed via Fermi convolution power.

Next, we describe some facts concerning t-deformed free convolution. In fact, Bożejko and Wysoczański [12,13] examined a deformation of the Cauchy–Stieltjes transform of a probability measure λ in the following manner: Let $t > 0$ and $\lambda \in \mathbf{P}$; based on the Nevanlinna theorem, the function $\mathcal{G}_{\lambda_t}(.)$ is defined by

$$\frac{1}{\mathcal{G}_{\lambda_t}(z)} = \frac{t}{\mathcal{G}_\lambda(z)} + (1-t)z, \tag{19}$$

is the Cauchy–Stieltjes transform of some probability measure indicated by $U_t(\lambda) := \lambda_t$. In [12,13], a new type of convolution, called a t-deformed free convolution (or a t-free convolution) and denoted by $\boxed{t}$-convolution was introduced, that is, for λ_1 and $\lambda_2 \in \mathbf{P}$

$$\lambda_1 \boxed{t} \lambda_2 = U_{1/t}(U_t(\lambda_1) \boxplus U_t(\lambda_2)). \tag{20}$$

For $\lambda \in \mathbf{P}_c$, the free cumulant transform $\mathcal{R}_\lambda(\cdot)$ of λ is provided by

$$\mathcal{R}_\lambda(\mathcal{G}_\lambda(\xi)) = \xi - \frac{1}{\mathcal{G}_\lambda(\xi)}, \qquad \text{for all } \xi \text{ in a neighborhood of } 0. \tag{21}$$

The t-deformed free cumulant transform, denoted by $\mathcal{R}_\lambda^t(\cdot)$, is given by

$$\mathcal{R}_\lambda^t(\xi) := \frac{1}{t}\mathcal{R}_{U_t(\lambda)}(\xi).$$

For λ_1 and λ_2 $\in \mathbf{P}_c$, we have

$$\mathcal{R}^t_{\lambda_1 \boxed{t} \lambda_2} = \mathcal{R}^t_{\lambda_1}(\xi) + \mathcal{R}^t_{\lambda_2}(\xi). \tag{22}$$

It is well known that the t-deformed free cumulant transform is a particular case of the (a,b)-deformed free cumulant transform, introduced in [17], by considering $t = a = b > 0$. One see that

$$\lim_{\xi \to 0} \mathcal{R}^t_\lambda(\xi) = m_0^\lambda. \tag{23}$$

$\lambda \in \mathbf{P}_c$ is $\boxed{t}$-infinitely divisible, if for each $q \in \mathbb{N}$, there is $\lambda_q \in \mathbf{P}_c$ such that

$$\lambda = \underbrace{\lambda_q \boxed{t} \ldots\ldots \boxed{t} \lambda_q}_{q \text{ times}}.$$

Let $\lambda^{\boxed{t}r}$ represent the r-fold $\boxed{t}$-convolution of λ with itself. This operation is well defined $\forall\, r \geq 1$, see [18], and

$$\mathcal{R}^t_{\lambda^{\boxed{t}r}}(\xi) = r\mathcal{R}^t_\lambda(\xi). \tag{24}$$

The proof of the central limit theorem related to $\boxed{t}$-convolution is provided. The limit measure is called the t-deformed free Gaussian law. A Poisson-type limit theorem, related to $\boxed{t}$-convolution, is also demonstrated. The limiting measure is termed a t-deformed free Poisson distribution; see [12,13] for more details. In addition, ref. [19] (Corollary 1) provides an intriguing formula for the variance function when considering the power of the $\boxed{t}$-convolution of the generating measure.

This article continues the investigation of Fermi convolution and $\boxed{t}$-convolution from the perspective of CSK families. The remaining sections of this article are grouped as follows: In Section 2, for $\rho \in \mathbf{P}^2$, we introduce the family of measures:

$$\mathbf{F} = \{\left(Q^\rho_m(dy)\right)^{\bullet s};\;\; m \in (m^\rho_0, m^\rho_+)\}. \tag{25}$$

We prove that if $\mathbf{F}$ is a re-parametrization of $\mathcal{K}_+(\rho)$, then, up to scale transformation, ρ is of the Marchenko–Pastur type law provided by (13). The same result is obtained in Section 3 (with different tools) if we replace the Fermi convolution $\bullet$ in (25) with the $\boxed{t}$-convolution.

2. A Property of MP_a Based on Fermi Convolution

Let $\mu \in \mathbf{P}^2$. For the clarity of the results in this section, instead of the $\mathcal{B}$-transform, we consider the following H-transform:

$$H_\mu(z) = z\mathcal{B}_\mu\left(\frac{1}{z}\right) = m^\mu_0 + \mathcal{E}_{\mu^0}(z) = m^\mu_0 + z - \frac{1}{\mathcal{G}_{\mu^0}(z)}. \tag{26}$$

We have

$$H_{\mu^{\bullet s}}(z) = sH_\mu(z) \qquad \text{for all } \; s > 0. \tag{27}$$

Now, we state and prove the following result about the H-transform. This is important in proving the primary result of this section, presented by Theorem 1.

Lemma 1. *Let $\mu \in \mathbf{P}^2$ be non degenerate with $b = \sup\operatorname{supp}(\mu) < \infty$. For $z \in \mathbb{C}\backslash supp(\mu)$ such that $z \neq \mathbb{V}_\mu(m)/m$, we have*

$$H_{Q^\mu_m}(z) = \frac{\left(m + \frac{\mathbb{V}_\mu(m)}{m}\right)H_\mu(z + m - m^\mu_0) - m(z+m)}{\frac{\mathbb{V}_\mu(m)}{m} - (z + m - H_\mu(z + m - m^\mu_0))}. \tag{28}$$

Proof. We have that

$$H_\mu(z) = \mathcal{E}_\mu(z + m^\mu_0) = z + m^\mu_0 - \frac{1}{\mathcal{G}_\mu(z + m^\mu_0)}. \tag{29}$$

According to ([20] Lemma 2.3), for $\xi \in \mathbb{C}\backslash supp(\mu)$ such that $\xi \neq m + \mathbb{V}_\mu(m)/m$, the Cauchy–Stieltjes transform of $Q^\mu_m \in \mathcal{K}_+(\mu)$ is given by

$$\mathcal{G}_{Q^\mu_m}(\xi) = \frac{1}{m + \mathbb{V}_\mu(m)/m - \xi}\left(\frac{\mathbb{V}_\mu(m)}{m}\mathcal{G}_\mu(\xi) - 1\right). \tag{30}$$

Combining (29) and (30), we get for z such that $z + m \in \mathbb{C}\backslash supp(\mu)$ and $z \neq \mathbb{V}_\mu(m)/m$

$$H_{Q^\mu_m}(z) = z + m - \frac{1}{\mathcal{G}_{Q^\mu_m}(z+m)} = \frac{(z+m)\mathcal{G}_\mu(z+m)\mathbb{V}_\mu(m)/m - (m + \mathbb{V}_\mu(m)/m)}{\mathcal{G}_\mu(z+m)\mathbb{V}_\mu(m)/m - 1}. \tag{31}$$

From (29), one see that

$$\mathcal{G}_\mu(z+m) = \frac{1}{z+m-H_\mu(z+m-m_0^\mu)}. \tag{32}$$

Combining (31) and (32), we obtain (28). □

To support the proof of Theorem 1, we provide and show the following proposition.

Proposition 1. *Let $\mu \in \mathbf{P}^2$ be non-degenerate with $b = \sup\operatorname{supp}(\mu) < \infty$. Then,*

(i) $\displaystyle\lim_{z\longrightarrow+\infty} \frac{H_\mu(z+m-m_0^\mu)}{z} = 0.$

(ii) $\displaystyle\lim_{z\longrightarrow+\infty} \frac{H_\mu(z+m-m_0^\mu)(z+m-H_\mu(z+m-m_0^\mu))}{z} = m_0^\mu.$

Proof. The proof follows from ([2] Proposition 3.2) and the relation (29). □

The major outcome of this section has now been stated and proved.

Theorem 1. *Let $\mu \in \mathbf{P}^2$ be non-degenerate with $b = \sup\operatorname{supp}(\mu) < \infty$. If $(Q_m^\mu)^{\bullet s} = Q^\mu_{g(m,s)}$ provided by (8) where $g(m,s)$ depends on $m \in (m_0^\mu, m_+^\mu)$ and $s > 0$, then $m_0^\mu \neq 0$, $g(m,s) = sm$, and μ is the image by $y \longmapsto m_0^\mu y$ of $\boldsymbol{MP}_a$ provided by (13) for $a^2 > 1$ such that $|a|$ is sufficiently large.*

Proof. We have that $(Q_m^\mu)^{\bullet s} = Q^\mu_{g(m,s)}$. That is, $\forall\, z \in (b - m_0^\mu, +\infty)$

$$sH_{Q_m^\mu}(z) = H_{Q^\mu_{g(m,s)}}(z). \tag{33}$$

From ([16] Proposition 3 (iii)), we have that

$$\lim_{z\longrightarrow+\infty} H_\mu(z) = m_0^\mu. \tag{34}$$

So,

$$g(m,s) = m_0^{Q^\mu_{g(m,s)}} = \lim_{z\longrightarrow+\infty} H_{Q^\mu_{g(m,s)}}(z) = \lim_{z\longrightarrow+\infty} sH_{Q_m^\mu}(z) = sm_0^{Q_m^\mu} = sm. \tag{35}$$

Using (28) and (35), Equation (33) becomes

$$\frac{s\left(m+\frac{\mathbb{V}_\mu(m)}{m}\right)H_\mu(z+m-m_0^\mu) - sm(z+m)}{\frac{\mathbb{V}_\mu(m)}{m} - (z+m-H_\mu(z+m-m_0^\mu))} = \frac{\left(sm+\frac{\mathbb{V}_\mu(sm)}{sm}\right)H_\mu(z+sm-m_0^\mu) - sm(z+sm)}{\frac{\mathbb{V}_\mu(sm)}{sm} - (z+sm-H_\mu(z+sm-m_0^\mu))}. \tag{36}$$

After some calculations, Equation (36) is

$$\begin{aligned}
&s\left(m+\frac{\mathbb{V}_\mu(m)}{m}\right)H_\mu(z+m-m_0^\mu)\frac{\mathbb{V}_\mu(sm)}{sm} - (z+m)\mathbb{V}_\mu(sm) - sm(z+m)H_\mu(z+sm-m_0^\mu)\\
&\qquad - s\left(m+\frac{\mathbb{V}_\mu(m)}{m}\right)H_\mu(z+m-m_0^\mu)(z+sm-H_\mu(z+sm-m_0^\mu))\\
&= \frac{\mathbb{V}_\mu(m)}{m}\left(sm+\frac{\mathbb{V}_\mu(sm)}{sm}\right)H_\mu(z+sm-m_0^\mu) - sm(z+sm)\frac{\mathbb{V}_\mu(m)}{m} - sm(z+sm)H_\mu(z+m-m_0^\mu)\\
&\qquad - \left(sm+\frac{\mathbb{V}_\mu(sm)}{sm}\right)H_\mu(z+sm-m_0^\mu)(z+m-H_\mu(z+sm-m_0^\mu)).
\end{aligned} \tag{37}$$

In both sides of (37), we divide by z and let $z \longrightarrow +\infty$. Recalling Proposition 1 and relation (34), we obtain

$$\left(\frac{tm - m_0^\mu}{tm}\right)\mathbb{V}_\mu(tm) = t\mathbb{V}_\mu(m)\left(\frac{m - m_0^\mu}{m}\right). \tag{38}$$

Combining (38) with (7), we obtain

$$\mathcal{V}_\mu(tm) = t\mathcal{V}_\mu(m) \quad \text{for all } \ m \in (m_0^\mu, m_+^\mu) \ \text{ and all } \ t > 0. \tag{39}$$

Note that $\mathcal{V}_\mu(\cdot) \neq 0$, as μ is non-degenerate by assumption. Then, $\mathcal{V}_\mu(m) = \gamma m$ for $\gamma > 0$.

- ✓ If $m_0^\mu = 0$, then $\mathcal{V}(m) = \gamma m$ with $\gamma > 0$ can not serve as a variance function, see ([2] page 6).
- ✓ If $m_0^\mu \neq 0$, then according to ([14] Theorem 3.2), μ is the image by $y \mapsto m_0^\mu y$ of $\mathbf{MP}_a$ given by (13). In this case, $\gamma = a^2 m_0^\mu$.

Remark 2. *For $m \in (m_0^\mu, m_+^\mu)$, we must have $g(m,s) = sm \in (m_0^\mu, m_+^\mu)$. The law μ is the image by $\phi : y \mapsto m_0^\mu y$ of $\mathbf{MP}_a$ given by (13). If $m_0^\mu > 0$, we have $m_+^\mu = \phi(m_+^{\mathbf{MP}_a}) = m_0^\mu(1 + |a|)$. If $m_0^\mu < 0$, $\mathcal{K}_-(\mu)$ is the CSK family and (m_-^μ, m_0^μ) is the domain of means with $m_-^\mu = m_0^\mu(1 + |a|)$. In all cases, we should have $a^2 > 1$ so that $|a|$ is sufficiently large to be sure that $g(m,s) = sm$ remains in the mean domain.*

We now show that, in Theorem 1, the inverse implication is not valid. Assume that $m_0^\mu > 0$. We know that $g(m,s) = sm$, and μ is the image by $y \mapsto m_0^\mu y$ of $\mathbf{MP}_a$ provided by (13) for $a^2 > 1$ such that $|a|$ is sufficiently large. The interval $(m_0^\mu, m_0^\mu(1 + |a|))$ is the mean domain of $\mathcal{K}_+(\mu)$. For $|a|$ values that are sufficiently large, we have that $sm \in (m_0^\mu, m_0^\mu(1 + |a|))$. We have to prove that

$$(Q_m^\mu)^{\bullet s} \neq Q_{sm}^\mu. \tag{40}$$

That is, for $x > sm$ close enough to sm,

$$\mathbb{V}_{(Q_m^\mu)^{\bullet s}}(x) \neq \mathbb{V}_{Q_{sm}^\mu}(x). \tag{41}$$

So, (40) is concluded from Remark 1(ii).

We have that $m_0^{Q_{sm}^\mu} = sm$ and from ([16] Theorem 1 (ii)) we also have $m_0^{(Q_m^\mu)^{\bullet s}} = sm_0^{Q_m^\mu} = tm$. Then, $\varepsilon > 0$ exists such that $\mathbb{V}_{(Q_m^\mu)^{\bullet s}}(\cdot)$ and $\mathbb{V}_{Q_{sm}^\mu}(\cdot)$ are well defined on $(sm, sm + \varepsilon)$. Furthermore, from ([2] Formula (3.24)), we have that $\forall\, p > m$ sufficiently close to m,

$$\mathbb{V}_{Q_m^\mu}(p) = p\left(\frac{ap(m_0^\mu)^2}{p - m} + p\left[\frac{m_0^\mu}{m} - 1\right]\right). \tag{42}$$

Using ([16] Theorem 1 (ii)) and (42) we have, $\forall\, 0 < s \neq 1$ and $\forall\, x \in (sm, sm + \varepsilon)$,

$$\begin{aligned}
\mathbb{V}_{(Q_m^\mu)^{\bullet s}}(x) &= s\mathbb{V}_{Q_m^\mu}(x/s) + x^2(1/s - 1) + m_0^{Q_m^\mu}(s-1)x \\
&= x\left(\frac{ax(m_0^\mu)^2}{x - sm} + x\left[\frac{m_0^\mu}{sm} - 1\right] + (s-1)m\right). \\
&\neq x\left(\frac{ax(m_0^\mu)^2}{x - sm} + x\left[\frac{m_0^\mu}{sm} - 1\right]\right) = \mathbb{V}_{Q_{sm}^\mu}(x).
\end{aligned}$$

A proof of (40) is achieved. □

3. A Property of $\mathbf{MP}_a$ Based on $\boxed{t}$-Convolution

Theorem 2. *Let $\mu \in \mathbf{P}_c$. For $0 < \alpha \neq 1$ such that $\left(Q_m^\mu\right)^{\boxed{t}\alpha}$ is defined, if $(Q_m^\mu)^{\boxed{t}\alpha} = Q_{k(m,t,\alpha)}^\mu$ provided by (8) where $k(m,t,\alpha)$ depends on $m \in (m_0^\mu, m_+^\mu)$, $t > 0$, and $0 < \alpha \neq 1$, then $m_0^\mu \neq 0$, $k(m,t,\alpha) = \alpha m$, and μ is the image by $y \mapsto m_0^\mu y$ of $\boldsymbol{MP}_a$ given by (13) for $a^2 > 1$ such that $|a|$ is large enough.*

Proof. For $0 < \alpha \neq 1$ such that $\left(Q_m^\mu\right)^{\boxed{t}\alpha}$ is defined, we have $\left(Q_m^\mu\right)^{\boxed{t}\alpha} = Q_{g(m,t,\alpha)}^\mu$. Equivalently, there is $\delta > 0$ such that $\forall\, z \in (-\delta, \delta)$

$$\mathcal{R}^t_{(Q_m^\mu)^{\boxed{t}\alpha}}(z) = \mathcal{R}^t_{Q^\mu_{k(m,t,\alpha)}}(z). \tag{43}$$

Using (23) and (43), we obtain

$$k(m,t,\alpha) = m_0^{Q^\mu_{g(m,t,\alpha)}} = \lim_{z\longrightarrow 0} \mathcal{R}^t_{Q^\mu_{g(m,t,\alpha)}}(z) = \lim_{z\longrightarrow 0} \mathcal{R}^t_{(Q_m^\mu)^{\boxed{t}\alpha}}(z) = \alpha \lim_{z\longrightarrow 0} \mathcal{R}^t_{Q_m^\mu}(z) = \alpha m. \tag{44}$$

The $\mathcal{R}$-transform of Q_m^μ can be expressed as

$$\mathcal{R}_{Q_m^\mu}(z) = c_1(Q_m^\mu) + c_2(Q_m^\mu)z + z\varepsilon(z). \tag{45}$$

where $c_1(Q_m^\mu)$ and $c_2(Q_m^\mu)$ denote, respectively, the first and the second free cumulants of Q_m^μ and $\lim_{z\to 0}\varepsilon(z) = 0$. That is

$$\mathcal{R}_{Q_m^\mu}(z) = m + \mathcal{V}_\mu(m)z + z\varepsilon(z). \tag{46}$$

Using (45), the $\mathcal{R}^t$-transform of Q_m^μ may be written as

$$\begin{aligned}\mathcal{R}^t_{Q_m^\mu}(z) = \frac{1}{t}\mathcal{R}_{U_t(Q_m^\mu)}(z) &= \frac{1}{t}[c_1(U_t(Q_m^\mu)) + c_2(U_t(Q_m^\mu))z + z\varepsilon(z)] \\ &= \frac{1}{t}[tm + t\mathcal{V}_\mu(m)z + z\varepsilon(z)] = m + \mathcal{V}_\mu(m)z + \frac{z}{t}\varepsilon(z).\end{aligned} \tag{47}$$

Using (47), we obtain

$$\mathcal{R}^t_{(Q_m^\mu)^{\boxed{t}\alpha}}(z) = \alpha\mathcal{R}^t_{Q_m^\mu}(z) = \alpha m + \alpha\mathcal{V}_\mu(m)z + \alpha\frac{z}{t}\varepsilon(z), \tag{48}$$

and

$$\mathcal{R}^t_{Q^\mu_{\alpha m}}(z) = \alpha m + \mathcal{V}_\mu(\alpha m)z + z\varepsilon(z). \tag{49}$$

Combining (43), (44), (48) and (49), we obtain

$$\mathcal{V}_\mu(\alpha m) = \alpha\mathcal{V}_\mu(m), \quad \text{for all } \; m \in (m_0^\mu, m_+^\mu) \; \text{ and } \; 0 < \alpha \neq 1. \tag{50}$$

Note that $\mathcal{V}_\mu(\cdot) \neq 0$ as μ is non-degenerate by assumption. So, $\mathcal{V}_\mu(m) = \sigma m$ for $\sigma > 0$.

✓ If $m_0^\mu = 0$, then $\mathcal{V}(m) = \sigma m$ with $\sigma > 0$ cannot be a variance function (see [2] (page 6)).

✓ If $m_0^\mu \neq 0$, then according to [14] (Theorem 3.2), μ is the image by $y \mapsto m_0^\mu y$ of $\mathbf{MP}_a$ provided by (13) and we have $\sigma = a^2 m_0^\mu$.

Remark 3. *For $m \in (m_0^\mu, m_+^\mu)$, we must have $k(m,t,\alpha) = \alpha m \in (m_0^\mu, m_+^\mu)$. Recall Remark 2: If $m_0^\mu > 0$, we have $m_+^\mu = m_0^\mu(1 + |a|)$. If $m_0^\mu < 0$, then (m_-^μ, m_0^μ) is the mean domain with $m_-^\mu = m_0^\mu(1 + |a|)$. In all cases, we should have that $a^2 > 1$ for $|a|$ is sufficiently large such that $k(m,t,\alpha) = \alpha m$ exists in the mean domain.*

We now establish that, in Theorem 2, the inverse implication is not valid. Assume that $m_0^\mu > 0$. We have that $k(m,t,\alpha) = \alpha m$ and μ is the image by $y \mapsto m_0^\mu y$ of $\mathbf{MP}_a$ given by (13) for $a^2 > 1$ with $|a|$ large enough. The interval $(m_0^\mu, m_0^\mu(1+|a|))$ is the mean domain of $\mathcal{K}_+(\mu)$. For sufficiently large $|a|$, one see that $\alpha m \in (m_0^\mu, m_0^\mu(1+|a|))$. We have to prove that

$$(Q_m^\mu)^{\boxed{t}\alpha} \neq Q_{\alpha m}^\mu. \tag{51}$$

Equivalently, for $x > \alpha m$ sufficiently close to αm,

$$\mathbb{V}_{(Q_m^\mu)^{\boxed{t}\alpha}}(x) \neq \mathbb{V}_{Q_{\alpha m}^\mu}(x). \tag{52}$$

So, (51) is deduced from Remark 1(ii).

We have that $m_0^{Q_{\alpha m}^\mu} = \alpha m$ and from [19] (Corollary 1) we also have $m_0^{(Q_m^\mu)^{\boxed{t}\alpha}} = \alpha m_0^{Q_m^\mu} = \alpha m$. Then, there is $\varepsilon > 0$ such that $\mathbb{V}_{(Q_m^\mu)^{\boxed{t}\alpha}}(\cdot)$ and $\mathbb{V}_{Q_{\alpha m}^\mu}(\cdot)$ are well defined on $(\alpha m, \alpha m + \varepsilon)$.

Using [19] (Corollary 1) and (42) we have, $\forall\, x \in (\alpha m, \alpha m + \varepsilon)$,

$$\begin{aligned}
\mathbb{V}_{(Q_m^\mu)^{\boxed{t}\alpha}}(x) &= \alpha \mathbb{V}_{Q_m^\mu}(x/\alpha) + x^2\left(\frac{1-t}{\alpha} + t - 1\right) \\
&= x\left(\frac{ax(m_0^\mu)^2}{x-\alpha m} + x\left[\frac{m_0^\mu}{\alpha m} - \frac{t}{\alpha} + t - 1\right]\right). \\
&\neq x\left(\frac{ax(m_0^\mu)^2}{x-\alpha m} + x\left[\frac{m_0^\mu}{\alpha m} - 1\right]\right) = \mathbb{V}_{Q_{\alpha m}^\mu}(x).
\end{aligned}$$

A proof of (51) is achieved. □

4. Conclusions

In this article, we have investigated two kinds of convolutions of importance in free probability: the Fermi convolution, denoted as $\bullet$, and the t-deformed free convolution, denoted as $\boxed{t}$. For $\rho \in \mathbf{P}_c$, we introduce the family of measures

$$\mathbf{T} = \{\left(Q_m^\rho(dy)\right)^{\boxed{t}\alpha};\ m \in (m_-^\rho, m_+^\rho)\}, \tag{53}$$

for $0 < \alpha \neq 1$. We have proven that if the family $\mathbf{T}$ is a re-parametrization of the CSK family $\mathcal{K}(\rho)$, then the measure ρ is of the Marchenko–Pastur type law. The proof is based on the properties of the t-deformed free cumulant transform, and the variance function plays an important role here. A similar property related to the Marchenko–Pastur law is obtained with different concepts by considering the Fermi convolution instead of the $\boxed{t}$-convolution. These results provide new insights into the structure of probability measures related to the Marchenko–Pastur law and may have implications for applications in statistical mechanics and random matrix theory.

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