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Forecast Horizon of Dynamic Lot Sizing Problem with Perishable Inventory and Multiple Chain Stores: Shipping and Stockout Cost

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Abstract: Perishable products are very common, but managing inventory of perishable products can be very challenging for firms, especially in distribution systems, including multiple chain stores. In this environment, we consider a dynamic lot-sizing problem faced by a distribution center that dis-patches a single perishable product to multiple chain stores. Demand cannot be backlogged, but it does not have to be satisfied; unsatisfied demand means stockout (lost sale). The first step is to transform the total profit function into a special total cost function. Our next step is to explore the properties of the optimal solution and use them to formulate a dynamic programming algorithm to solve the problem. Furthermore, we establish forecast and decision horizon results, which help the operation manager to decide the precise forecast horizon in a rolling decision-making process. Based on the model setting and the methods of dynamic programming, we obtained two interesting findings: (1) the maximized profit objective function is equivalent to the minimized cost objective function, and (2) the famous zero inventory property conditionally holds in the inventory management of perishable products. On an extensive test bed, useful insights were obtained on the impact of the lifetime of the product and cost parameters on the total cost and length of the forecast horizon. Thus, the contributions of this study are as follows: (1) we explore two structure policies in an optimal solution to devise efficient algorithms to reduce computational complexity; (2) we provide a sufficient condition for forecasting and decision horizons; and (3) we determine that, for a given fixed cost, the median forecast horizon first increases with the lifetime of the product and stockout cost and then remains invariable when it reaches a certain level.

Keywords: forecast horizon; dynamic lot size; perishable product; multiple chain stores

MSC: 90C26; 90C29; 90C31

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1. Introduction

Perishable products are products with short sales cycles and low residual values at the end of the period due to the characteristics of the products themselves or consumer preferences [1,2]. In recent years, with the increasing competition in the market and the continuous improvement of product systems, many enterprises have started to increase product models and improve the product iteration speed to gain competitive advantages, which has led to an increasing number of products showing the characteristics of perishable products, with electronics, fashion, and apparel as the main products. However, the consequent increase in inventory costs leads to an increase in total costs; thus, enterprises, in order to reduce the total cost, are not always timely and efficient in meeting customer demand, but, rather, choose to delay delivery or even to run out of stock. From the point of view of the economies of scale of production and the credibility of the point of view of their considerations, enterprises tend to more often choose to be out of stock (Kevin and José 2023) [3]. Therefore, considering the fickle nature of products, the investigation of dynamic lot-size decision making in fickle goods operating enterprises holds significant theoretical and practical implications.

In addition, multifrequency and small-batch distribution have become the primary indicators of logistics activities due to the increase in consumers' requirements for quality of life and the development of a logistics economy. Especially for perishable goods, the demand of each chain store is more dispersed in time and space, and the demand for goods is small in quantity and large in frequency, which leads to a larger proportion of the total cost of distribution and delivery [4]. In order to reduce costs, the distribution of perishable goods has developed a trend of common and mixed loading; that is, in a single chain store, the distribution number cannot reach the effective load of the vehicle [5,6], and the enterprise can bring together the demand for goods from different chains to match the loading. This enables the integration of logistics resources to make full use of the capacity, reduce transportation costs, and improve corporate profits [7]. Based on this, in order for enterprises to operate under minimum cost, customer demand cannot be met in a timely and effective manner, and although out-of-stock costs will increase, this will reduce the enterprise's production fixed costs and inventory holding costs; this also requires a trade-off between the level of service and transportation costs, and, at this time, the enterprise's operational decision making becomes complex and difficult or even idle. Methods of coordinating transportation costs, out-of-stock costs, and inventory costs become key to decision making. Thus, the study of distribution centers' distribution of perishable goods to multiple chain stores under the dynamic batch problem has become an urgent problem to solve.

In this study, we address a dynamic problem for a distribution center that distributes a single perishable product to multiple chain stores. We investigate two cases of managing the inventory: one is that the distribution center manages the inventory, and the chain stores do not carry excess inventory; the other is that the distribution center and chain stores manage the inventory jointly. The two ways of managing inventory have their respective advantages and disadvantages. The first one has an important similarity with the vendor-managed inventory (VMI) model: we know that the VMI can reduce and eliminate the bullwhip effect, improving the agility degree and flexibility in the supply chain, and so this policy of managing the inventory by the distribution center gains popularity, while, in reality, the distribution center may be restricted by its storage capacity, and so the distribution center and chain stores manage the inventory jointly. According to our research, only one DLS model in the literature explicitly considers one producer directly distributing the product to multiple specific customers [8]. The differences between our model and that of Chand et al. [8] are as follows: (i) Chand et al. assume that the inventory can be carried for an infinite amount of time and do not consider the short lifetime of products; (i) Chand et al. assume that customers have a tolerance for temporary product shortages, which means that all demands are ultimately satisfied, while we assume that customers are not willing to wait for orders that arrive late, so the demand in a period cannot be backlogged, but stockout (lost sale) is permitted.

In a multiperiod, dynamic decision-making environment, the forecast horizon is a period with the property that the data for periods beyond it are not required in order to determine an optimal first decision. The latest period covered by this first decision is called the decision horizon. In numerous studies, a decision horizon is also called a planning horizon. More formally, a decision horizon refers to the number of the next few periods, say, for which decisions must be made in the current period. An integer is referred to as a forecast horizon corresponding to the decision horizon if the data beyond the period do not influence the optimal decisions for the first periods in any period problem. Forecast and decision horizons were first introduced by Wagner and Whitin [9] and have been widely applied in production planning and inventory management. Simultaneously, the forecast horizon theory has been extended to broader areas, including capacity expansion, machine replacement, plant location, cash balance, and bond refunding.

The remainder of the article is structured as follows: Section 2 provides the literature review. Section 3 formulates the problem faced by a distribution center that dispatches a single perishable product to multiple chain stores with shipping and stockout costs. In Section 4, the properties of an optimal solution are outlined, and an efficient algorithm for solving the problem is developed. The forecast horizon results are established in Section 5. Section 6 extends the results from Section 4 by considering the case in which chain stores are allowed to carry excess inventory. Section 7 gives an overview of the computational results and managerial insights. The conclusions and suggestions for future research can be found in Section 8.

2. Literature Reviews

The earliest version of a dynamic lot-sizing (DLS) model was introduced by Wagner and Whitin [9]. This model can be described as follows: the demand in each period is known and must be satisfied. There is no restriction on production capacity or inventory capacity. The unit production cost remains constant, even though setup and unit-holding costs may vary from one period to another. The objective of the problem is to minimize the total production and holding costs. An important contribution of their study is the demonstration of the optimality of zero inventory property (ZIP). Based on ZIP, they developed a forward dynamic programming (DP) algorithm to solve the problem. The DLS model has been the subject of intensive research in the last few years. These extensions include DLS problems with backlogging [10], stockout [11,12], and production capacity [13–15], and they solve N-hard problems through optimization. Shi [12] found the optimal production rate that minimizes the expected discounted system cost subject to a given risk level of stockout. Bunn and Ventura [13] studied the multiproduct dynamic lot-sizing problem with capacity constraints and batch ordering. They defined three mixed-integer linear programming (MILP) models and applied Lagrangian relaxation to formulate the corresponding dual problems by relaxing the capacity constraints, which used one of two heuristics to find good feasible solutions. In addition, Hwang et al. [16], Fan and Qu [17], and Akbalik et al. [18] considered a class of dynamic lot-sizing problems with one-way and two-way product substitution modes for durable and perishable products and developed an efficient approximate DP algorithm to solve the problem with multiple perishable products. Cha and Moon [19], Li et al. [20], and Chang [21] developed a novel heuristic algorithm to solve a single-warehouse multiple retailer problem, an NP-hard problem, and modeled a modified all-unit discount cost structure close to the optimal solution. Cannella et al. [22], Zhang et al. [23], and Hwang et al. [24] considered the multi-level lot-sizing problem with production capacities (MLSP-PC). They developed the first polynomial algorithm for the MLSP-PC with general concave costs at all of the stages and introduced a novel approach to overcome the limitations of previous approaches. In short, the DLS model has been extensively studied from different perspectives, mainly focusing on inventory bounds [16–18], quantity discount [19–23], multi-echelon [22–24], the learning effect [25–27], and remanufacturing [28–30]. We refer to the generalizations cited above in relation to the classical DLS models.

Chand et al. [31] presented a comprehensive classified bibliography of the vast literature on the theory and applications of forecast horizons. From 2002 till now, the research on forecast horizons mainly includes the aspects outlined below. Cheevaprawatdomrong and Smith [32] established the existence of forecast horizons under the stochastic demand under the following assumptions: (i) costs and revenues are time-varying and linear, and (ii) demand is never eventually zero. Dawande et al. [33] used integer programming (IP) to compute the minimal forecast horizon for a specific class of DLS problems under the assumption that the future demands are integer multiples of a given positive real number. Ghate and Smith [34] established forecast horizon results for a DLS model with backlogging under convex costs. Dawande et al. [35] and Bardhan et al. [36] investigated the forecast horizon for a two-product DLS model under demand substitution in one direction and production changeovers and then proved the problem as the shortest path problem.

Teyarachakul [37] developed the forecast horizon results for a DLS model with learning and forgetting in setups.

It is assumed that demands can be aggregated in each period in the classical DLS problems, but this is not suitable for the situation where the distribution center or pro-ducer directly distributes the products to the different chain stores. Because of direct shipping, the transportation cost to distribute the products to the different chain stores could vary significantly due to the distance traveled. In addition, there are various prices for the same product in different chain stores because of the different consumer groups in the same period. For the above reasons, the assumption of aggregate demand in the classical DLS models may not be realistic. The distribution system, which integrates procurement (production), inventory, and transportation decisions, is similar to the basic supply chain. In recent years, the supply chain strategy has gained increasing popularity. For instance, Apple and Nike factory stores use production-distribution methods, while Walmart utilizes procurement–distribution methods.

There are some gaps between our study and the traditional literature. It is assumed that the inventory can be carried for an infinite number of periods in the classical DLS models. However, there are many situations where the assumption does not hold in reality. A wide variety of commodities have a short lifetime, such as fruits, vegetables, meat, and pharmaceuticals. Simultaneously with the fierce competition in the market and the development of technology, the speed of product replacement is accelerated, and the lifetime of the product is shorter than ever. Even some durable goods have a short lifetime, such as consumer electronics, seasonal fashion, and journals. In many articles, these products are also defined as perishable products. We have a clear understanding that ZIP is the key to solving the DLS problem, as it enables unlimited inventory transportation. The vast majority of extensions for DLS problems are based on ZIP. However, ZIP cannot be used when inventory cannot be held for an infinite number of periods (for DLS models with perishable inventory, see Hsu [38], Hsu [39], Chu et al. [40], and Sargut and Isik [41]). Other research on inventory management of perishable products includes recent works by Jing and Chao [42] and Claassen et al. [43].

Thus, the research questions are as follows: (1) How can we devise efficient algorithms to solve the dynamic optimization models? (2) How can we obtain the forecast horizons, and what are the sufficient conditions? (3) How do the parameters influence the forecast horizons?

3. Model Formulation

We follow the assumption of a perishable inventory outlined in the previous literature. The lifetime of the product is periods, there is no deterioration, and the value of the product will not reduce within the lifetime. In contrast, the product's value will be zero, and there is no disposal cost beyond the lifetime. We study this case. Without loss of generality, we assume that the procurement lead time (production) is zero. Furthermore, the following notations are used in our model (Table 1).

Table 1. Notation table.

T: The total number of time periods, which are indexed as $t = 1, 2, \ldots, T$;
N: The total number of different chain stores, which are indexed as $i = 1, 2, \ldots, N$;
d_{it}: Demand of chain store i in period t, $i = 1, 2, \ldots, N$, $t = 1, 2, \ldots, T$;
σ_t: Fixed cost of procurement of the distribution center in period t, $t = 1, 2, \ldots, T$;
c_t: Unit procurement cost in period t, $t = 1, 2, \ldots, T$;
p_{it}: Unit selling price of chain store i in period t, $i = 1, 2, \ldots, N$, $t = 1, 2, \ldots, T$;
h_t: Unit holding cost of the distribution center in period t, $t = 1, 2, \ldots, T$;
y_{it}: Unit shipping cost to distribute the product in period t to chain store i, $i = 1, 2, \ldots, N$, $t = 1, 2, \ldots, T$;
x_t: The amount of procurement at the beginning of period t, $t = 1, 2, \ldots, T$;
I_t: The amount of inventory at the end of period t, $t = 1, 2, \ldots, T$;
S_{it}: The amount of stockout (lost sale) of chain store i incurred in period t, $t = 1, 2, \ldots, T$;
Y_{it}: The amount of product distributed in period t to satisfy the demand of chain store i, $i = 1, 2, \ldots, N$, $t = 1, 2, \ldots, T$;
$\delta(x_t)$: Binary variables, $\delta(x_t) = \begin{cases} 1 & \text{if } x_t > 0 \\ 0 & \text{othewise} \end{cases}$.

A period t is called a procurement point (period) if $x_t > 0$. In addition, we make the following assumptions:

1. Any demand not satisfied in its period is considered stockout (lost sale), and backlogging is prohibited. Thus, we can focus on analyzing the effects of stockouts on firms' performance. In future research, we can study the interaction between stockout and backlog.

2. The gross marginal profit $(p_{it} - c_t)$ is non-negative for all chain stores i in any period t, $i = 1, 2, \ldots, N$; $t = 1, 2, \ldots, T$. This is a realistic assumption; otherwise, firms will not conduct operations.

We first present a profit-maximization formulation for our model. The profit of the supply chain is found by subtracting the total cost from the total realized revenue, where the total cost includes the shipping cost, fixed cost, variable cost of procurement, and inventory holding cost. The revenue for selling the product to the chain store i in period t results from multiplying the realized sales by the corresponding unit's selling price. The realized sales from chain store i in period t are given by $(d_{it} - S_{it})$. Revenues are summed from all chain stores and over all periods to yield the supply chain's realized revenue.

The profit maximization objective is as follows:

$$\max\left\{\sum_{i=1}^{N}\sum_{t=1}^{T} p_{it}(d_{it} - S_{it}) - \sum_{i=1}^{N}\sum_{t=1}^{T} y_{it}(d_{it} - S_{it}) - \sum_{t=1}^{T}[\delta(x_t)\sigma_t + c_t x_t + h_t I_t]\right\};$$

simplifying the above expression, we have

$$\max\left\{\sum_{i=1}^{N}\sum_{t=1}^{T} p_{it}d_{it} - \sum_{i=1}^{N}\sum_{t=1}^{T} p_{it}S_{it} - \sum_{i=1}^{N}\sum_{t=1}^{T} y_{it}(d_{it} - S_{it}) - \sum_{t=1}^{T}[\delta(x_t)\sigma_t + c_t x_t + h_t I_t]\right\}.$$

Let $Y_{it} = d_{it} - S_{it}$; note that the term $\sum_{i=1}^{N}\sum_{t=1}^{T} p_{it}d_{it}$ is constant and can be dropped from the objective function. Therefore, the profit-maximization objective mentioned above is equivalent to the cost-minimization formulation below:

$$\min\sum_{t=1}^{T}\left(\sigma_t\delta(x_t) + c_t x_t + h_t I_t + \sum_{i=1}^{N} p_{it}S_{it} + \sum_{i=1}^{N} y_{it}Y_{it}\right) \tag{1}$$

Subject to

$$I_t = I_{t-1} + x_t - \sum_{i=1}^{N} Y_{it} \quad \text{for } i = 1, 2, \ldots, N,\ t = 1, 2, \ldots, T \tag{2}$$

$$Y_{it} + S_{it} - d_{it} = 0 \quad \text{for } i = 1, 2, \ldots, N,\ t = 1, 2, \ldots, T \tag{3}$$

$$x_t \le \sum_{i=1}^{N}\sum_{i=t}^{t+m-1} d_{it} \quad \text{for } i = 1, 2, \ldots, N,\ t = 1, 2, \ldots, T \tag{4}$$

$$x_t, I_t, Y_{it}, S_{it} \ge 0 \quad \text{for } i = 1, 2, \ldots, N,\ t = 1, 2, \ldots, T \tag{5}$$

$$I_0 = I_T = 0 \tag{6}$$

Note that p_{it} is the stockout cost of chain store i in period t in the cost-minimization formulation. Constraint (2) represents the balance of inventory, and Constraint (3) means that the demands of customer i in period t are satisfied by shipping Y_{it} and stockout S_{it}. Constraint (4) means that the amount of production cannot exceed the sum demands of all customers within the product's lifetime. Constraint (5) requires that the production, inventory, shipping, and stockout quantities are non-negative. Without losing generality, we assume zero inventory at the beginning of period 1 and at the end of the period T

(Constraint (6)). We would like to denote the problem as a multiple stores problem, that is, MSP.

The MSP can be a formulated as a concave cost network flow problem. Let nodes D and S be supply nodes that contain the total demand over T periods, and let P_t and L_t be trans-shipment nodes that transit, respectively. The destination node C_{it} is the demand node, each corresponding to a demand d_{it} from chain store i in period t. The objective is to minimize the cost of flows. The network G corresponds to N chain stores, and the T periods problem is shown in Figure 1. More specially, the arc set consists of the following arc subsets:

(i) Directed arcs (D, P_t) for $i = 1, 2, \ldots, N$ have zero shipping cost for zero arc flow and a cost of $\sigma_t + c_t x_t$ if the arc's flow $x_t > 0$.

(ii) Directed arcs (S, L_t) for $t = 1, 2, \ldots, T$ ship flows at zero cost and have upper bounds of $+\infty$.

(iii) Directed arcs (P_t, P_{t+1}) for $t = 1, 2, \ldots, T-1$ ship flows at a holding cost of h_t per unit flow.

(iv) Directed arcs (P_t, C_{it}) for $i = 1, 2, \ldots, N$, $t = 1, 2, \ldots, T$ ship flows at a delivery cost of y_{it} per unit flow.

(v) Directed arcs (L_t, C_{it}) for $i = 1, 2, \ldots, N$, $t = 1, 2, \ldots, T$ ship flows at a stockout cost of p_{it} per unit flow.

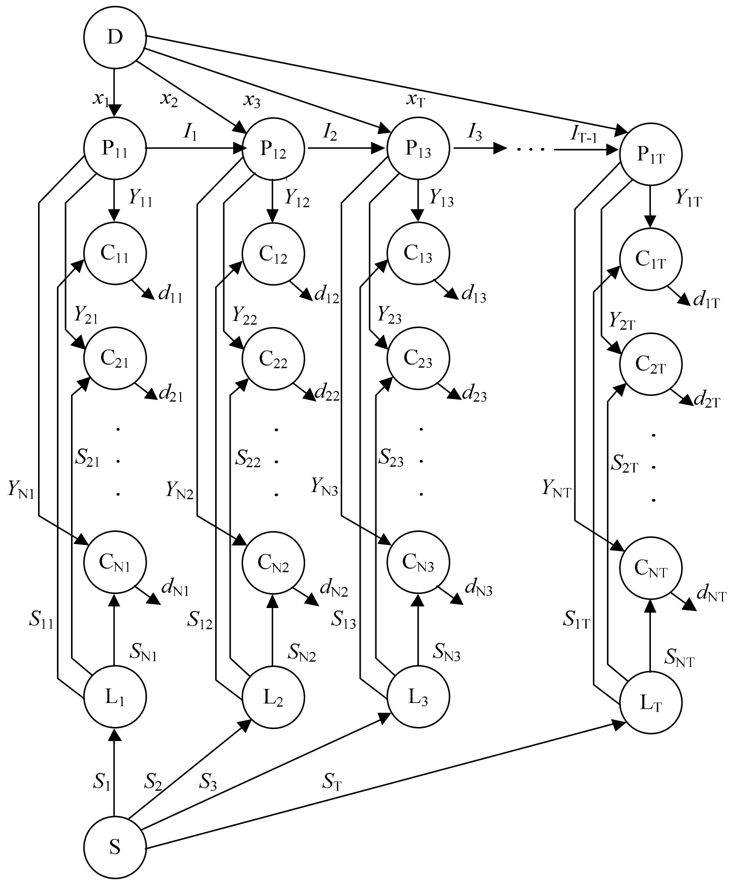


Figure 1. Network representation of the MSP.

Figure 1 provides a network representation of the MSP; please refer to Aksen et al. [44,45] for a step-by-step walkthrough of how the optimal solution is derived. Later, some useful properties of the optimal solution of the MSP will be derived using the network. Demand differentiation based on the different selling prices and shipping costs is an important generalization of the classical DLS model that acquires multifarious management practices from reality. The next illustration serves as an example of this viewpoint.

Example 1. *Consider two chain stores 1 and 2; the demand vectors for the first five periods are* $(3,4,4,4,2,7)$ *and* $(3,4,3,2,2,6)$*, respectively. The other parameters are as follows:* $m = 3$; $c_t = (8,9,13,10,11,9)$; $h_t = (3,1,1,1,3,2)$; $y_{1t} = (1,3,1,1,2,2)$; $y_{2t} = (1,2,1,2,4,3)$; $p_{1t} = (15,13,16,17,19,18)$; $p_{2t} = (17,18,18,20,18,21)$; $\sigma_t = (20,100,20,60,50,30)$. *The optimal solution of the problem is to purchase in period 1 and period 3 and period 6. The procurement in period 1 is used to satisfy demands* d_{11}, d_{13} *and* d_{21}, d_{22}, d_{23}*. The procurement in period 3 is used to satisfy demands* d_{14}, d_{15} *and* d_{24}*. The procurement in period 6 is used to satisfy demands* d_{16} *and* d_{26}*.*

In the above solution, two phenomena do not exist in the classical DLS model. First, note that $I_2 \cdot x_3 \neq 0$; it is not true for purchases only when the entering inventory is zero. This happens because of perishable inventory and fluctuating costs. With a speculative motive cost structure, for example, $c_t + \sum_{i=t}^{t'-1} h_i < c_{t'}$, the distribution center will purchase more to avoid a higher cost, At the same time, the inventory cannot be carried for an infinite number of periods, so ZIP does not hold.

Secondly, notice that in periods 2 and 5, the demand d_{12} (d_{25}) is stockout, while the demand d_{22} (d_{15}) is satisfied by procurement in period 1. In an aggregate demand DLS model, stockout in a certain period means that all demands are lost; however, in reality, some demands will be satisfied, and some demands could be lost in the same period. This happens because of two reasons: the first is due to the fact that different chain stores set different prices for various customer bases; the second is that the shipping cost is nonstationary.

4. Properties and Dynamic Programming Algorithm

In this section, we will investigate two important properties of the optimal solution. Later, they will be used to develop a DP algorithm.

Before we consider the problem with no restriction on cost structure, it should be first noted that if there is no speculative motive for holding inventory, that is $c_t + \sum_{i=t}^{t'-1} h_i > c_{t'}$, $\forall t < t'$, ZIP still holds for the perishable inventory. This is a special case in which the variable cost per unit of procurement cost c_t is constant or fluctuates slightly in the market. We drop the unrealistic assumption and investigate the MSP for perishable products under a more general cost structure. In the more general cost structure, the unit of procurement cost is not restricted; thus, it is more practical.

Theorem 1. *In an optimal solution, there is* $(d_{it} - Y_{it})(d_{it} - S_{it}) = 0$, $i = 1,2,\ldots,N$, $t = 1,2,\ldots,T$*; that is, the demand is either satisfied entirely by a shipment in exact one period, or completely lost.*

Proof. According to Aksen et al. [44], the network representation of the problem shows the characteristics of a concave cost network with a single source node. Accordingly, the basic optimal solution for the problem will be an extreme point solution corresponding to a spanning tree of the network. Furthermore, a basic optimal solution will have the property that arborescent flows have positive flow in at least one inward arc at each node. Next, we consider the chain store node C_{it}; if $(d_{it} - Y_{it})(d_{it} - S_{it}) > 0$, then node C_{it} would have two positive inward arcs, one from shipment and another one from stockout in that period, contradicting the property that any node has at most one positive inward arc. □

Now we are ready to show the next property (see the Appendix A for the proof):

Theorem 2. *There exists an optimal solution* Ω^* *to the MS; if* $l < \lambda$ *are two procurement periods and* d_{it} *is satisfied by procurement in period* λ*, then for any* $t^* > t$*, the procurement amount in period* l *for satisfying* d_{it^*} *is zero.*

Theorem 2 suggests that there is an optimal solution to the MSP where perishable stocks are used to satisfy demands in a first-in, first-out fashion. Such a solution is called a FIFO solution.

It is well known that there exists an optimal solution to many classical DLS problems that satisfies ZIP. This property is stated as follows:

Zero inventory property: In an optimal solution, $I_{t-1}x_t = 0$ for all t.

With perishable inventory, ZIP may not hold in an optimal solution. This was demonstrated in Example 1. This is an important difference between perishable products and common products. The existing forward DP algorithm for resolving DLS models typically depends on ZIP, while it cannot be used to resolve the DLS problem with perishable inventory. Based on Theorem 1 and Theorem 2, we use the following DP recursion to solve the problem. For any t, $1 \leq t \leq T$, let $F(t)$ be the minimum costs needed to satisfy demands from period 1 to t. Let $F(l,k,t)$ be the costs in the t-period problem in which the last procurement occurs at the beginning of period l for satisfying the demands from period k to t. For periods l, k, and t, $1 \leq l \leq k \leq t$; note that the limitation of the lifetime of the product, period l, and t must satisfy $t - l + 1 \leq m$. Denote $g_i(l,k,t)$ as the minimum variable cost to satisfy the demand from chain store i in periods k through t. We have the following dynamic programming recursion:

$$F(t) = \min_{r \leq l \leq k \leq t} F(l,k,t) = \min_{r \leq l \leq k \leq t} \left\{ F(k-1) + \sigma_l + \sum_{i=1}^{N} g_i(l,k,t) \right\}$$

where $r = \max\{t - m + 1, 1\}$ and $F(0) = 0$.

Now we discuss the computation of $g_i(l,k,t)$, first introducing the following equation:

$$R_i(l,k,t') = c_l + \sum_{j=l}^{t'-1} h_j + y_{it'} - p_{it'}$$

From the equation, $R_i(l,k,t')$ can be considered as a kind of marginal cost associated with one item lost in the period t'. The demands t' will be completely lost in an optimal solution if $R_i(l,k,t') > 0$. For any period t' ($k \leq t' \leq t$), if $R_i(l,k,t') \leq 0$, we argue that the period t' is in the period set U_1, if $R_i(l,k,t') > 0$, then the period t' is in the period set U_2.

By definition above, we can obtain that

$$g_i(l,k,t) = (c_l + \sum_{u \in U_1} y_{iu}) \cdot \sum_{v \in U_1} d_{iv} + \sum_{u=l}^{k-1} h_u \cdot \sum_{v \in U_1} d_{iv} + \sum_{u=k}^{t} h_u \cdot [\sum_{v \in U_1} d_{iv} - \sum_{w=k}^{v} (d_{iw} - S_{iw})] + \sum_{u \in U_2} p_{iu} d_{iu}$$

Let $MC_i(l,t)$ denote the trade-off between the cost of losing one unit demand of chain store i in period t and meeting one unit demand by a procurement in period l. The value of $MC_i(l,t)$ is determined by the following expression:

$$MC_i(l,t) = \min \left\{ c_l + \sum_{u=l}^{t-1} h_u + y_{it}, p_t \right\}$$

Using the fact that $F(l,k,t) = F(l,k,t-1) + \sum_{i=1}^{N} MC_i[l(t),t]$, we will illustrate how the DP algorithm solves Example 1.

We start by solving the 1-period MSP with an optimal value $F(1)$, from the above functional equation, $r = 1$.

$$F(1) = F(1,1,1) = F(0) + 20 + 54 = 74$$

Then, we proceed to solve the 2-period MSP. Again, $r = 1$.

$$F(1,1,2) = F(1,1,1) + 104 = 178,\ F(2,2,2) = F(1) + 100 + 92 = 266,$$
$$F(1,2) = F(1,1,2) = 178,\ F(2,2) = F(2,2,2) = 266,$$
$$F(2) = \min\{F(1,2); F(2,2)\} = F(1,1,2) = 178.$$

The 3-period MSP is solved in the following.

$$F(1,1,3) = F(1,1,2) + 104 = 269,$$
$$F(2,2,3) = F(2,2,2) + 77 = 343,$$
$$F(2,3,3) = F(2) + 100 + 77 = 355,$$
$$F(3,3,3) = F(2) + 20 + 98 = 296,$$
$$F(1,3) = F(1,1,3) = 269,$$
$$F(2,3) = \min\{F(2,2,3); F(2,3,3)\} = F(2,2,3) = 343,$$
$$F(3,3) = F(3,3,3) = 296, F(3) = \min\{F(1,3); F(2,3); F(3,3)\} = F(1,1,3) = 269$$

When $F(4)$ is computed, r is equal to 2.

$$F(2,2,4) = F(2,2,3) + 74 = 417,\ F(2,3,4) = F(2,3,3) + 74 = 429,$$
$$F(2,4,4) = F(3) + 100 + 74 = 443,\ F(3,3,4) = F(3,3,3) + 92 = 388,$$
$$F(3,4,4) = F(3) + 20 + 92 = 381,\ F(4,4,4) = F(3) + 60 + 92 = 421,$$
$$F(2,4) = \min\{F(2,2,4); F(2,3,4); F(2,4,4)\} = F(2,2,4) = 417$$
$$F(3,4) = \min\{F(3,3,4); F(3,4,4)\} = F(3,4,4) = 381$$
$$F(4,4) = F(4,4,4) = 421,\ F(4) = \min\{F(2,4); F(3,4); F(4,4)\} = F(3,4) = 381,$$

Continuing in the same manner, we obtain

$$F(5) = \min\{F(3,5); F(4,5); F(5,5)\} = F(3,5) = 451,\ F(6) = F(6,6,6) = 451 + 179 = 630.$$

From this example, we can see that when the number of chain stores is large, then the computational time will increase hugely. This is a limitation of this DP algorithm. In the future research, we can develop effective approximate algorithms for solving large-scale problems.

5. Detection of Forecast Horizon

In this section, we present a sufficient condition that determines the decision and forecast horizon.

Definition 1. *If x_j^t is the optimal period j procurement amount in the t-period MSP, then the sequence $\{x_1^t, x_2^t, \ldots, x_t^t\}$ is called an optimal procurement sequence.*

Definition 2. *If the sequence $\{x_1^t, x_2^t, \ldots, x_t^t\}$ is an optimal procurement sequence for the t-period MSP, then the sequence $\{x_1^t, x_2^t, \ldots, x_{t'}^t\}$ is an optimal procurement subsequence, provided that $t' \leq t$.*

Definition 3. *$l(t)$ is the period in which the last procurement takes place in an optimal solution to t-period problem.*

Theorem 3. *Suppose the optimal solution for a t-period problem is given by $F[l(t), k(t), t]$. Consider problems $MSP(t-m)$, $MSP(t-m+1)$, …, $MSP(t-1)$. If $x_j^{t-m} = x_j^{t-m+1} = \ldots = x_j^{t-1}$ for $j = 1, 2, \ldots, t'$ $(1 \leq t' \leq t-m)$, then period t is a forecast horizon and period t' is a decision horizon.*

Proof. $F[l(t_1), k(t_1), t_1]$ is the optimal solution for the t_1 $(t_1 > t)$ period problem. If $k(t_1) > t$, then examine the optimal solution to the $k(t_1) - 1$ period problem. Let $k(t_1) - 1 = t_2$; if $k(t_2) > t$, we continue the process until we find an optimal solution to a t_n period problem such that $k(t_n) \le t < k(t_{n-1})$. Note that $k(t_{n-1}) - 1 = t_n$, $F[l(t_n), k(t_n), t_n]$ is the optimal solution to the t_n period problem. Since the lifetime of the product is m periods, we have $l(t_n) > t - m$. Given the analysis above, it is known that the optimal solution of the t_n period problem is a part of the optimal solution of the longer t_1 period problem.

If $x_j^{t-m} = x_j^{t-m+1} = \ldots = x_j^{t-1}$, for $j = 1, 2, \ldots, t'$ $(1 \le t' \le t - m)$, that means that each problem $MSP(t-m)$, $MSP(t-m+1)$, …, $MSP(t-1)$ has the same optimal procurement subsequence for the first t' periods. Hence the first t' periods must be part of the optimal solution to any problem of length t^*, where $t \le t^* \le T$. Therefore, period t' is a decision horizon since it needs t periods of information to determine the decision horizon, and period t is the corresponding forecast horizon. □

More formally, the detection of forecast horizon can be summarized as follows:

STEP 1

Let $F(0) = 0$, and compute the matrix $g_i(l, k, t)$.

STEP 2

Compute

$$F(l, k, t) = F(k-1) + \sigma_l + \sum_{i=1}^{N} g_i(l, k, t),$$

which satisfies the following conditions:

$$r \le l \le k \le t,\ r = \max\{t - m + 1, 1\}$$

STEP 3

Compute $F(l, t) = {}_k\min F(l, k, t)$ for all l, k which satisfy the conditions in Step 2.

STEP 4

Compute $F(t) = {}_l\min F(l, t)$ for all l which satisfy the conditions in Step 2. Set $l(t) = l$ and $k(t) = k$ for that l, k which minimizes $F(l, k, t)$.

STEP 5

Check for the decision horizon:

If $x_j^{t-m} = x_j^{t-m+1} = \ldots = x_j^{t-1}$, for $j = 1, 2, 3, \ldots, t'$ $(1 \le t' \le t - m)$, then period t is the forecast horizon, and period t' is the decision horizon.

If Step 5 does not hold, then set $t = t + 1$; go to Step 2.

Using the above steps, we found that Period 6 is the forecast horizon, and the corresponding decision horizon is Period 5.

6. Model Extension to Allow Chain Stores Inventories

We now consider the extension model whereby chain stores are allowed to carry inventory; two extra notations are used in this extension model.

h_{it}: Unit holding cost of chain store i in period t, $i = 1, 2, \ldots, N$, $t = 1, 2, \ldots, T$;

I_{it}: The amount of inventory at the end of period t held by chain store i, $i = 1, 2, \ldots, N$, $t = 1, 2, \ldots, T$;

$$\min \sum_{t=1}^{T} \left(\sigma_t \delta(x_t) + c_t x_t + h_t I_t + \sum_{i=1}^{N} h_{it} I_{it} + \sum_{i=1}^{N} p_{it} S_{it} + \sum_{i=1}^{N} y_{it} Y_{it} \right).$$

The extended problem is equivalent to the minimum cost network flow problem, which is a modified version of the original network in Figure 1. It needs to add an arc from node $C_{i,t-1}$ to node $C_{i,t}$ with an inventory cost $h_{it} I_{it}$. Theorem 4, used for the extended problem derived from the modified network, is as follows.

Theorem 4. *In an optimal solution, there exists $S_{it}(d_{it} - S_{it}) = 0$, $i = 1, 2, \ldots, N$, $t = 1, 2, \ldots, T$; that is, the demand d_{it} is either satisfied entirely by a shipment in period t or is completely lost.*

Proof. Similar to Theorem 1, consider the chain store node C_{it}, if $S_{it}(d_{it} - S_{it}) > 0$, then node C_{it} would have at least two positive inward arcs, one from shipment in period t or from inventory at the end of the previous period and another from stockout in period t, contradicting the property that any node has at most one positive inward arc. □

Theorem 5. *There exists an optimal solution Ω^* to the MSP; if $l < \lambda$ are two procurement periods and d_{it} is satisfied by procurement in period λ, then for any $t^* > t$, the procurement amount in period l for satisfying d_{it^*} is zero.*

Proof. The proof is similar to Theorem 2; hence, its proof is omitted. □

The dynamic programming recursion is consistent with Section 3, except for the computation of $g_i(l,k,t)$:

$$F(t) = \min_{r \le l \le k \le t} F(l,k,t) = \min_{r \le l \le k \le t} \left\{ F(k-1) + \sigma_l + \sum_{i=1}^{N} g_i(l,k,t) \right\}$$

where $r = \max\{t - m + 1, 1\}$ and $F(0) = 0$.

Similarly, we define the following equation:

$$H_i(l,k,t') = \min_{l \le w_i \le k} \left\{ c_l + \sum_{u=l}^{w_i - 1} h_u + y_{iw_i} + \sum_{u=w_i}^{t'} h_{iu} \right\}$$

If $H_i(l,k,t') > p_i$, the demands of period t' are completely lost, so we can obtain the variable costs that satisfy the demands from period k to t.

That is similar to the situation where only the distribution center holds inventory in Section 4. $H_i(l,k,t')$ can be considered as a kind of marginal cost associated with one item lost in the period t'. The demands $d_{it'}$ are completely lost in an optimal solution if $H_i(l,k,t') > 0$. For any period t' ($k \le t' \le t$), if $H_i(l,k,t') \le 0$, we argue that the period t' is in the period set U_1; if $H_i(l,k,t') > 0$, then the period t' is in the period set U_2.

Using the definition above, we can determine that

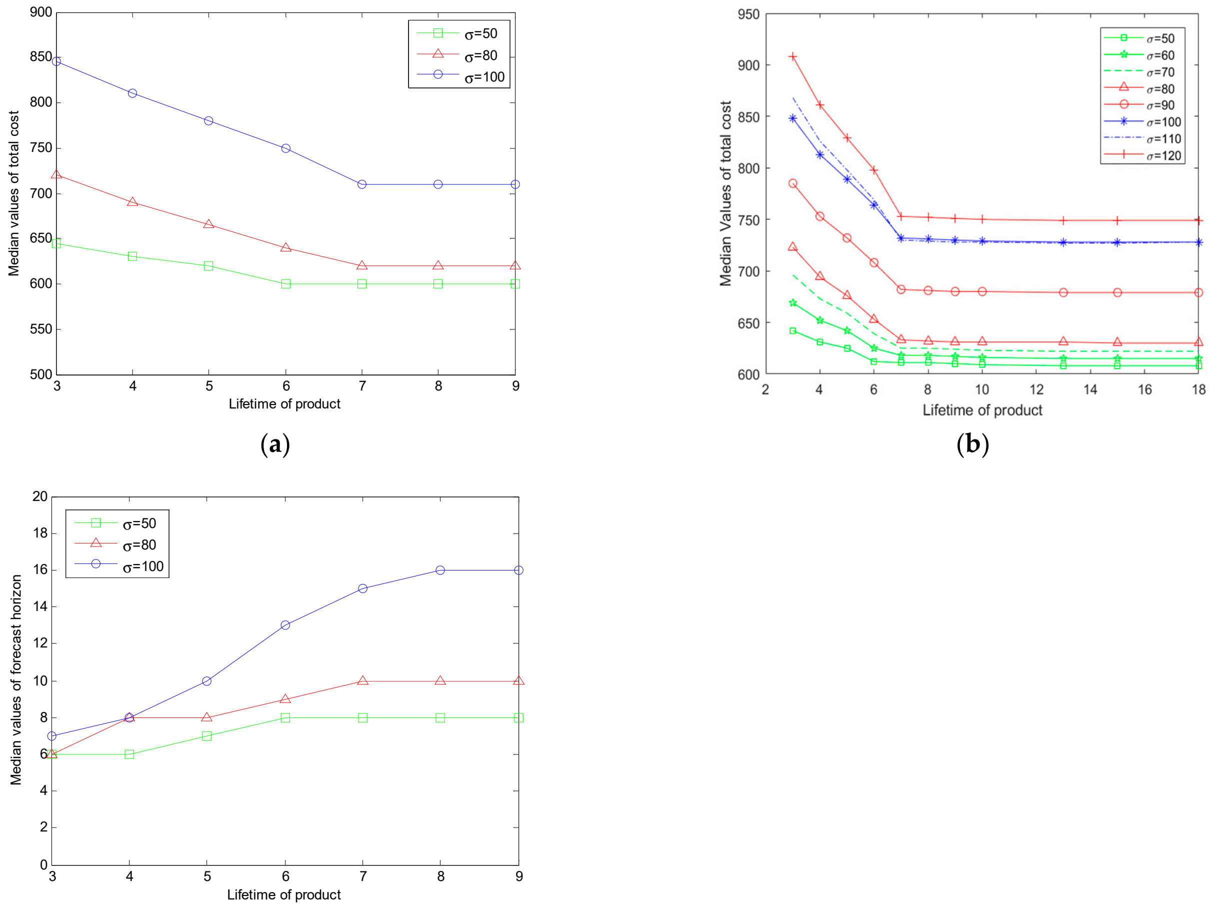
$$g_i(l,k,t) = (c_l + \sum_{u=l}^{w_i-1} h_u + y_{iw_i} + \sum_{u=w_i}^{k-1} h_{iu}) \sum_{v \in U_1} d_v + \sum_{u=k}^{t-1} h_u [\sum_{v \in U_1} d_{iv} - \sum_{r=k}^{u} (d_{ir} - S_{ir})] + \sum_{u \in U_2} p_{iu} d_{iu}$$

7. Computational Results and Managerial Insights

In this section, we compare optimal solutions from our model without a stockout policy. We also demonstrate some interesting phenomena regarding the effect of the product's lifetime, stockout cost, fixed cost, and holding cost on the forecast horizon. The test bed we use is comparable to those recommended by Dawande et al. [33,35] and Bardhan et al. [36]. The number of chain stores is set at 2. It is important to note that more stores will lead to a larger amount of computation, but the results are not affected by the number of stores. Therefore, for simplicity, we study the case of two stores. The chain stores are not allowed to hold excess inventory. In the work by Dawande et al. [35], they assume that the demands are normally distributed, with a mean of 10 and a standard deviation of 5. According to the characteristics of our study and the previous literature, we assume that the demands are normally distributed, where the mean is set to 5 and the standard deviation is set to 2. At any time, if a demand generated is less than zero or zero, it is set to one. The test implementation is carried out using MATLAB R2017b.

Test 1. For $i = 1,2$, $t = 1,2,\ldots,T$, the unit's selling price (stockout cost) p_{it} is set at 20; the unit's procurement cost c_t is set at 10; the unit's holding cost h_t is set at 2; the unit's shipping cost y_{it} is set at 2; the fixed cost of procurement σ_t takes three values: 50, 80, and 100; and the lifetime of the product takes seven values: 3, 4, 5, 6, 7, 8, and 9. First, we computed the total cost for six periods under different fixed costs and the lifetime of the product. Then, we computed the forecast horizon for the T period problem. For each combination of the setup cost and the lifetime of the product, we generated 11 instances (to enable the easy identification of the median total cost and forecast horizon). The total number of instances in the test bed is $3 \times 7 \times 11 = 231$.

Figure 2a shows a plot of the total cost as a function of the product's lifetime under three fixed-cost values. For a given fixed cost, the median total cost increases with the product's lifetime and then remains constant. We offer the following explanation: the additional flexibility for bigger products' lifetime expands the set of feasible actions. When the product's lifetime is very large, the set of feasible actions remains unchanged.



(**a**) (**b**) (**c**)

Figure 2. (**a**). The median total cost as a function of lifetime and fixed cost. (**b**). Sensibility analysis of the median total cost as a function of lifetime and fixed cost. (**c**) The median forecast horizon as a function of lifetime and fixed cost.

Figure 2b shows a plot of the sensitivity analysis about the median total cost as a function of the lifetime of the product under eight values of fixed cost, which shows that the median total cost decreases with the increase in the lifetime of the product and increases with the increase in the fixed cost of procurement σ_t. Subsequent sensitivity about the median forecast horizon as the lifetime of the product and fixed cost of procurement is similar. In order to more clearly see the relationship between them, Figure 2c takes three values, 50, 80, and 100; the lifetime of the product takes seven values, 3, 4, 5, 6, 7, 8, and 9; and the specific content is shown in Figure 2c.

Figure 2c shows a plot of the total cost as a function of the product's lifetime under three values of fixed cost. For a given fixed cost, the median total cost increases with the lifetime of the product, then remains constant. We offer the following explanation: the additional flexibility for a bigger product lifetime expands the set of feasible actions. When the product's lifetime is very long, the set of feasible actions remains unchanged.

Figure 2c plots the median forecast horizon as a function of the product's lifetime for the three values of fixed cost. For a given setup cost, the median forecast horizon first increases with the product's lifetime and then remains constant. In general, a shorter product lifespan leads to more procurement periods, leading to a shorter forecast horizon. When other parameters are fixed and the lifetime of the product is very long, the perishable product will be equal to the durable product; the forecast horizon cannot be affected by the lifetime. As the fixed cost increases, the median forecast horizon for a given product's lifetime increases. Higher fixed costs lead to shorter procurement periods, leading to a longer forecast horizon.

Test 2. For $i = 1, 2, t = 1, 2, \ldots, T$, the fixed cost of procurement σ_t is set at 50; the unit's procurement cost c_t is set at 10; the unit's shipping cost y_{it} is set at 2; the unit's holding cost h_t takes three values: 1, 2, 4; the unit's selling price (stockout cost) p_{it} takes 6 values: 18, 19, 20, 25, 30, and 35; and the lifetime of the product is 5 periods. Similarly, we first compute the total cost for six periods under different holding stockout costs. Then, we compute the forecast horizon for the T period problem. For each combination of the holding cost and stockout cost, 11 instances are generated, and the total number of instances in the test bed is $3 \times 6 \times 11 = 191$.

Figure 3 plots the median forecast horizon as a function of the stockout cost for the three values of holding cost. For a given holding cost, the median forecast horizon first increases with the stockout cost and then remains constant. Similar to Figure 2c, a higher stockout cost results in more procurement periods and, consequently, a shorter forecast horizon. When other parameters are fixed and the stockout cost is very high, we do not choose the option of stockout. Therefore, the forecast horizon remains unchanged. For a given stockout cost, the median forecast horizon typically decreases with the holding cost. A higher holding cost results in more procurement periods and, consequently, a shorter forecast horizon.

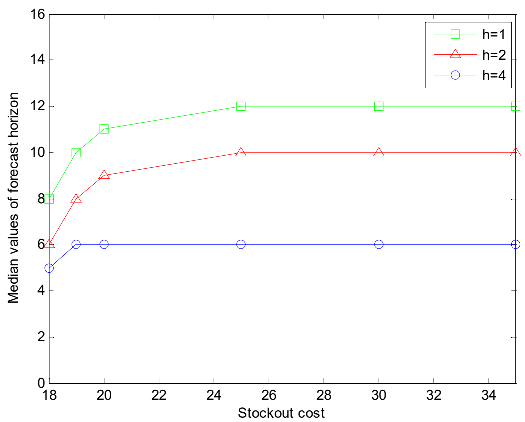


Figure 3. The median forecast horizon as a function of stockout and holding cost.

8. Conclusions and Suggestions for Future Research

The existing papers on DLS problems assume that the lifetime of products is infinite. In addition, previous studies assume that demands can be aggregated in each period. Several structural policies of an optimal solution do not hold if we relax the abovementioned assumptions. This motivates us to investigate a novel problem. This study presents an important variant of the DLS problem faced by a distribution center that supplies a single perishable product to multiple chain stores, which incurs shipping and stockout costs. We first consider a case where only the distribution center can carry inventory. We also analyze an extension of the problem where chain stores can hold inventory.

The theoretical implications are as follows: in this study, we explored the structural properties of the optimal solution. For example, the demand is either satisfied entirely by a shipment in exact one period, or completely lost; perishable stocks are used to satisfy demands in first-in, first-out fashion. We further used them to develop an efficient DP algorithm to solve the multiperiod optimization model. Furthermore, we established forecast and decision horizon results, which help the operation manager analyze the appropriate data information periods. Moreover, the managerial implications are as follows: using a comprehensive test bed, we obtained useful managerial insights into the impact of the lifetime of the product and cost parameters on the total cost and the length of the forecast horizon. For example, the median total cost increases with the lifetime of the product and then remains constant. At the same time, the median forecast horizon increases with the lifetime of the product and then remains constant.

This study does not consider the possibility of trans-shipment between two stores and the capacity constraints. In addition, we do not consider the time windows of procurement and delivery. Thus, the research can be extended in various directions. One extension of our work would be to study the general case where the inventory of one chain store can be shipped to other chain stores.

Secondly, the capacity constraint is an important reason for stockouts. Thus, another interesting extension of our work would be to consider capacity constraints, including the procurement capacity, inventory capacity, and shipping capacity. For example, in each period, the number of procurements cannot exceed the largest capacity C_t, that is, $x_t \leq C_t$.

Thirdly, the customer offers a grace period, which is the demand time window during which specific demand can be satisfied with no penalty in reality. That is associated with each demand i, in which the customer specifies the earliest and latest delivery time, denoted by E_i and L_i, respectively, where $E_i \leq L_i$. Hence, the interval $[E_i, L_i]$ represents the time window corresponding to demand i. The inclusion of the procurement or delivery time window can also provide opportunities for further analysis.

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Appendix A

Before presenting Theorem 2, first, we show a property of optimal solution.

Lemma A1. *Suppose that $l < \lambda$ are two procurement periods in the optimal solution to problem MSP and procure in period λ for satisfying d_{it} and d_{it^*} for some $\lambda \leq t < t^*$. Then, if $c_l + \sum_{u=l}^{t^*-1} h_u + y_{it*} > p_{it^*}$, there exists $c_l + \sum_{u=l}^{\lambda-1} h_u > c_\lambda$; if $c_l + \sum_{u=l}^{t^*-1} h_u + y_{it^*} \leq p_{it'}$, there exists $(c_l + \sum_{u=l}^{\lambda-1} h_u - c_\lambda) \geq 0$.*

Proof. From the assumption, procure in period λ for satisfying d_{it} and $d_{it'}$ for some $\lambda \leq t < t^*$, we have $c_\lambda + \sum_{u=\lambda}^{t-1} h_u + y_{it} < p_{it}$ and $c_\lambda + \sum_{u=\lambda}^{t^*-1} h_u + y_{it^*} < p_{it^*}$. If $c_l + \sum_{u=l}^{t*-1} h_u + y_{it^*} > p_{it^*}$, then $c_l + \sum_{u=l}^{t^*-1} h_u + y_{it^*} > c_\lambda + \sum_{u=\lambda}^{t^*-1} h_u + y_{it^*}$, so we have $c_l + \sum_{u=l}^{\lambda-1} h_u > c_\lambda$.

If $c_l + \sum_{u=l}^{t^*-1} h_u + y_{it^*} \leq p_{it^*}$, the total costs in the optimal solution Ω^+ from period l to period t^*, which include procurement cost, inventory cost, shipping cost to satisfy the demand of N chain stores, and stockout cost for N chain stores, can be expressed as follows:

$$\sigma_l + c_l x_l^+ + \sum_{u=l}^{\lambda-1} h_u I_u^+ + \sigma_\lambda + c_\lambda x_\lambda^+ + \sum_{v=\lambda}^{t^*-1} h_v I_v^+ + y_{it} d_{it} + y_{it^*} d_{it^*} + F^+(l, t^*)$$

We modify optimal solution Ω^+ to feasible solution Ω^* by satisfying the demand of chain store i in period t' by procurement in period l. Thus, we have

$$x_l^* = x_l^+ + d_{it^*};\ x_\lambda^* = x_\lambda^+ - d_{it^*};\ I_u^* = I_u^+ + d_{it^*},\ u = l, l+1, \ldots, \lambda-1;\ I_v^* = I_v^+,\ v = \lambda, \lambda+1, \ldots, t^*-1$$

The costs in the modified solution Ω^* from period l to period $t*$ can be expressed as follows:

$$\sigma_l + c_l(x_l^+ + d_{it^*}) + \sum_{u=l}^{\lambda-1} h_u(I_u^+ + d_{it^*}) + \sigma_\lambda + c_\lambda(x_\lambda^+ - d_{it^*}) + \sum_{v=\lambda}^{t^*-1} h_v I_v^+ + y_{it} d_{it} + y_{it^*} d_{it^*} + F^*(l, t^*)$$

Note that it remains unchanged in the solution Ω^+ and Ω^* from the above analysis, that is, $F^+(k, t*) = F^*(k, t*)$. The perturbed solution cannot decrease cost below the optimal solution, so we have

$$\sigma_l + c_l x_l^+ + \sum_{u=l}^{\lambda-1} h_u I_u^+ + \sigma_\lambda + c_\lambda x_\lambda^+ + \sum_{v=\lambda}^{t^*-1} h_v I_v^+ + y_{it} d_{it} + y_{it^*} d_{it^*} + F^+(l, t^*) \leq$$
$$\sigma_l + c_l(x_l^+ + d_{it^*}) + \sum_{u=l}^{\lambda-1} h_u(I_u^+ + d_{it^*}) + \sigma_\lambda + c_\lambda(x_\lambda^+ - d_{it^*}) + \sum_{v=\lambda}^{t'-1} h_v I_v^+ + y_{it} d_{it} + y_{it^*} d_{it^*} + F^*(l, t^*)$$

Simplifying the above expression, we have $(c_l + \sum_{u=l}^{\lambda-1} h_u - c_\lambda) d_{it^*} \geq 0$. □

Now we are ready to prove Theorem 2.

Proof. Suppose we have an optimal solution Ω^+ where $l < \lambda$ are two procurement periods and d_{it} is satisfied by procurement in period λ. d_{it^*} is satisfied by procurement in period l, so we have $c_\lambda + \sum_{u=\lambda}^{t-1} h_u + y_{it} \leq p_{it}$ and $c_l + \sum_{u=l}^{t^*-1} h_u + y_{it^*} \leq p_{it^*}$. If $c_\lambda + \sum_{u=\lambda}^{t^*-1} h_u + y_{it^*} \geq p_{it^*}$, then $c_\lambda + \sum_{u=\lambda}^{t^*-1} h_u + y_{it^*} \geq c_l + \sum_{u=l}^{t^*-1} h_u + y_{it^*}$, and we have $c_\lambda \geq c_l + \sum_{u=l}^{\lambda-1} h_u$, so the costs are not increased by rescheduling the program to let d_{it} be also satisfied by procurement in period l, while this contradicts the assumption of optimal solution Ω^+, so we have $c_\lambda + \sum_{u=\lambda}^{t^*-1} h_u + y_{it^*} < p_{it^*}$. Under $c_\lambda + \sum_{u=\lambda}^{t^*-1} h_u + y_{it^*} < p_{it^*}$, we modify this optimal solution Ω^+ to obtain another feasible solution Ω^*, where d_{it^*} is satisfied by procurement in period λ instead of by procurement in period l, as in Ω^+. In the modified solution Ω^*, we have

$$x_l^* = x_l^+ - d_{it^*};\ x_\lambda^* = x_\lambda^+ + d_{it^*};\ I_u^* = I_u^+ - d_{it^*},\ u = l, l+1, \ldots, \lambda-1;\ I_v^* = I_v^+,\ v = \lambda, \lambda+1, \ldots, t^*-1$$

$V(\Omega^+)$ represents the total costs in the optimal solution Ω^+ from period l to period t^*,

$$V(\Omega^+) = (\sigma_l + c_l x_l^+) + \sum_{u=l}^{\lambda-1} h_u I_u + (\sigma_\lambda + c_\lambda x_\lambda^+) + \sum_{v=\lambda}^{t^*-1} h_u I_u + y_{it} d_{it} + y_{it^*} d_{it^*} + F^+(l, t^*)$$

$V(\Omega^*)$ represents the total costs in the modified solution Ω^* from period l to period t^*,

$$V(\Omega^*) = [\sigma_l + c_l(x_l^+ - d_{it^*})] + \sum_{u=l}^{\lambda-1} h_u(I_u - d_{it^*}) + [\sigma_\lambda + c_\lambda(x_\lambda^+ + d_{it^*})] + \sum_{v=\lambda}^{t^*-1} h_u I_u \\ + y_{it} d_{it} + y_{it^*} d_{it^*} + F^*(l, t^*)$$

To complete the proof, we need to show that $V(\Omega^+) \geq V(\Omega^*)$.

$$V(\Omega^+) - V(\Omega^*) = (c_l + \sum_{u=l}^{\lambda-1} h_u - c_\lambda) d_{it*} + F^+(l, t*) - F^*(l, t*)$$

We know that $F^+(k, t^*) = F^*(k, t^*)$, from Lemma A1, and $d_{it^*} \geq 0$, so we have $V(\Omega^+) \geq V(\Omega^*)$. □

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