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Article

Exponential Convergence-(t, s)-Weak Tractability of Approximation in Weighted Hilbert Spaces

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Abstract: We study L_2-approximation problems in the weighted Hilbert spaces in the worst case setting. Three interesting weighted Hilbert spaces appear in this paper, whose weights are equipped with two positive parameters γ_j and α_j for $j \in \mathbb{N}$. We consider algorithms using the class of arbitrary linear functionals. We discuss the exponential convergence-(t, s)-weak tractability of these L_2-approximation problems. In particular, we obtain the sufficient and necessary conditions on the weights for exponential convergence-weak tractability and exponential convergence-$(t, 1)$-weak tractability with $t < 1$.

Keywords: L_2-approximation; information complexity; tractability; weighted Hilbert spaces

MSC: 41A81; 47A58; 47B02

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1. Introduction

We study multivariate approximation problems $\text{APP} = \{\text{APP}_d\}_{d \in \mathbb{N}}$ of functions defined over Hilbert spaces with large or huge d in the worst case setting (approximation error by the worst case error). Such problems appear in quantum physics (see [1]), computational chemistry (see [2]), and economics (see [3]). We consider algorithms using the class of arbitrary linear functionals. The information complexity $n(\varepsilon, \text{APP}_d)$ is the minimal number n of linear functionals for which the approximation error of some algorithm is at most ε. Tractability describes the dependence of the information complexity $n(\varepsilon, \text{APP}_d)$ on the threshold ε and the dimension d. We consider the classical tractability which is polynomially convergent, and the exponential convergence-tractability (EC-tractability) which is exponentially convergent. Recently many authors discuss classical tractability and EC-tractability in weighted Hilbert spaces (see [4] by linear information, ref. [5] by standard information for functionals, and [6] by standard information for operators), especially in analytic Korobov spaces, such as exponential convergence and uniform exponential convergence (see [7]), classical tractability (see [8]) and EC-tractability for L_2-approximation (see [9] for exponential convergence-(t, s)-weak tractability and [10] for other EC-tractability results by algorithms using continuous linear functionals, and see [11] for EC-tractability by algorithms using function values), and EC-tractability for L_p-approximation with $1 \le p \le \infty$ by algorithms using continuous linear functionals (see [12]). Some authors consider tractability in weighted Hilbert spaces, such as classical tractability in weighted Korobov spaces (see [13] for strong polynomial tractability and polynomial tractability, [14] for other classical tractability results by algorithms using continuous linear functionals, and [15] by algorithms using function values), EC-tractability in weighted Korobov spaces (see [16]), and classical tractability in weighted Gaussian ANOVA spaces (see [17,18] with different weights, respectively).

In this paper, we investigate EC-tractability of L_2-approximation problems from the weighted Hilbert spaces with some weights. Let $H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ be a Hilbert space with weight $R_{d,\boldsymbol{\alpha},\gamma}$, where $\boldsymbol{\gamma} = \{\gamma_j\}_{j\in\mathbb{N}}$ and $\boldsymbol{\alpha} = \{\alpha_j\}_{j\in\mathbb{N}}$ are two positive sequences satisfying $1 \geq \gamma_1 \geq \gamma_2 \geq \cdots \geq 0$ and $1 < \alpha_1 \leq \alpha_2 \leq \cdots$. In the worst case setting, we consider the L_2-approximation problem

$$\mathrm{APP}_d : H(K_{R_{d,\boldsymbol{\alpha},\gamma}}) \to L_2([0,1]^d) \ \text{ with } \ \mathrm{APP}_d(f) = f.$$

The classical tractability for L_2-approximation problem $\mathrm{APP} = \{\mathrm{APP}_d\}$ in weighted Korobov spaces $H(K_{R_{d,\boldsymbol{\alpha},\gamma}})$ such as strong polynomial tractability and polynomial tractability were discussed in [13,15,17]; quasi-polynomial tractability, uniform weak tractability, weak tractability and (t,s)-weak tractability were investigated in [14,17]. Additionally, ref. [17] also discussed classical tractability in several weighted Hilbert spaces, including weighted Korobov spaces and weighted Gaussian ANOVA spaces. The EC-tractability of the problem $\mathrm{APP} = \{\mathrm{APP}_d\}$ in weighted Korobov spaces such as EC-$(t,1)$-weak tractability for $0 < t \leq 1$ were studied in [16]. However, the above weighted Hilbert spaces $H(K_{R_{d,\boldsymbol{\alpha},\gamma}})$ with weights $R_{d,\boldsymbol{\alpha},\gamma}$ satisfy $1 \geq \gamma_1 \geq \gamma_2 \geq \cdots \geq 0$ and $1 < \alpha_1 = \alpha_2 = \cdots$.

In this paper we present three cases of weighted Hilbert spaces $H(K_{R_{d,\boldsymbol{\alpha},\gamma}})$ with weights $R_{d,\boldsymbol{\alpha},\gamma}$ for $1 \geq \gamma_1 \geq \gamma_2 \geq \cdots \geq 0$ and $1 < \alpha_1 \leq \alpha_2 \leq \cdots$ that appear in the reference [18]. These weighted Hilbert spaces are similar but also different. The authors in [18] studied the polynomial tractability, strong polynomial tractability, weak tractability, and (t,s)-weak tractability for $t > 1$ and $s > 0$ of the problems $\mathrm{APP} = \{\mathrm{APP}_d\}$ in these three weighted Hilbert spaces. However, there are no results about EC-tractability of the approximation problems $\mathrm{APP} = \{\mathrm{APP}_d\}$ in the above three weighted Hilbert spaces. We will study exponential convergence-(t,s)-weak tractability (EC-(t,s)-WT) for some $t > 0$, $s > 0$ and obtain the complete sufficient and necessary conditions for $t = s = 1$ and $t < 1$, $s = 1$, respectively.

The paper is structured in the following ways. We present three cases of weighted Hilbert spaces in Section 2. Section 3 gives preliminaries about the L_2-approximation problem in the weighted Hilbert space. Section 4.1 is devoted to recall some notions about the tractability, such as classical tractability and exponential convergence-tractability and state the main results. In Section 4.2 we give the proof of Theorem 1. In Section 5 we present a summary.

2. Weighted Reproducing Kernel Hilbert Spaces

In this section we consider weighted reproducing kernel Hilbert spaces with different weights.

Let $H(K_d)$ be a Hilbert space defined in $[0,1]^d$. The function $K_d(\boldsymbol{x},\boldsymbol{y})$ of $\boldsymbol{x},\boldsymbol{y} \in [0,1]^d$ is called a reproducing kernel of $H(K_d)$ if for every $\boldsymbol{y} \in [0,1]^d$ and every $f \in H(K_d)$,

$$f(\boldsymbol{y}) = \langle f(\boldsymbol{x}), K_d(\boldsymbol{x},\boldsymbol{y})\rangle_{H(K_d)}.$$

The Hilbert space is a so-called reproducing kernel Hilbert space. We can study more details on reproducing kernel Hilbert spaces in the reference [19].

In this paper, let $\boldsymbol{\gamma} = \{\gamma_j\}_{j\in\mathbb{N}}$ and $\boldsymbol{\alpha} = \{\alpha_j\}_{j\in\mathbb{N}}$ be two positive sequences of the Hilbert space $H(K_{R_{d,\boldsymbol{\alpha},\gamma}})$ with $R_{d,\boldsymbol{\alpha},\gamma}$ satisfying

$$1 \geq \gamma_1 \geq \gamma_2 \geq \cdots \geq 0, \ \text{ and } \ 1 < \alpha_1 \leq \alpha_2 \leq \cdots. \tag{1}$$

Assume that the function $K_{R_{d,\boldsymbol{\alpha},\gamma}}$ of the space $H(K_{R_{d,\boldsymbol{\alpha},\gamma}})$ with $K_{R_{d,\boldsymbol{\alpha},\gamma}} : [0,1]^d \times [0,1]^d \to \mathbb{C}$ is of product form

$$K_{R_{d,\boldsymbol{\alpha},\gamma}}(\boldsymbol{x},\boldsymbol{y}) := \prod_{k=1}^{d} K_{R_{1,\alpha_k,\gamma_k}}(x_k,y_k),$$

where $K_{R_{1,\alpha,\gamma}} : [0,1] \times [0,1] \to \mathbb{C}$ is a universal weighted function

$$K_{R_{1,\alpha,\gamma}}(x,y) := \sum_{h\in\mathbb{N}_0} R_{\alpha,\gamma}(h) \exp(2\pi i h(x-y)), \ \ x,\ y \in [0,1].$$

Here, let weight $R_{\alpha,\gamma} : \mathbb{N}_0 \to \mathbb{R}^+$ be a summable function, i.e., $\sum_{k\in\mathbb{N}_0} R_{\alpha,\gamma}(k) < \infty$. Then we have

$$K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}(\boldsymbol{x},\boldsymbol{y}) = \sum_{\boldsymbol{h}\in\mathbb{N}_0^d} R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{h}) \exp(2\pi i\boldsymbol{h}\cdot(\boldsymbol{x}-\boldsymbol{y})),\ \boldsymbol{x},\ \boldsymbol{y} \in [0,1]^d, \tag{2}$$

the inner product

$$\langle f,g\rangle_{H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})} = \sum_{\boldsymbol{h}\in\in\mathbb{N}_0^d} \frac{1}{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{h})}\widehat{f}(\boldsymbol{h})\overline{\widehat{g}}(\boldsymbol{h}), \tag{3}$$

and

$$||f||_{H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})} = \sqrt{\langle f,f\rangle_{H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})}},$$

where

$$R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{h}) := \prod_{j=1}^{d} R_{\alpha_j,\gamma_j}(h_j),\ \boldsymbol{h} = (h_1,h_2,\ldots,h_d) \in \mathbb{N}_0^d,$$

$$\boldsymbol{x}\cdot\boldsymbol{y} := \sum_{h=1}^{d} x_h\cdot y_h,\ \ \boldsymbol{x} = (x_1,x_2,\ldots,x_d),\ \boldsymbol{y} = (y_1,y_2,\ldots,y_d) \in [0,1]^d,$$

and

$$\widehat{f}(\boldsymbol{h}) = \int_{[0,1]^d} f(\boldsymbol{x}) \exp(-2\pi i\boldsymbol{h}\cdot\boldsymbol{x})d\boldsymbol{x}.$$

We can ascertain that $K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}(\boldsymbol{x},\boldsymbol{y})$ is well defined for $1 < \alpha_1 \le \alpha_2 \le \cdots$ and for all $\boldsymbol{x},\ \boldsymbol{y} \in [0,1]^d$, since

$$|K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}(\boldsymbol{x},\boldsymbol{y})| \le \sum_{\boldsymbol{h}\in\mathbb{N}_0^d} R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{h}) = \prod_{j=1}^{d}(\sum_{h\in\mathbb{N}_0} R_{\alpha_j,\gamma_j}(h)) < \infty.$$

Note that the Hilbert space $H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ is a reproducing kernel Hilbert space with the reproducing kernel $K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}$. Indeed, for every $f \in H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ we have

$$f(\boldsymbol{y}) = \langle f(\boldsymbol{x}), K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}(\boldsymbol{x},\boldsymbol{y})\rangle_{H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})}.$$

The kernel $K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}$ with weight $R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}$ is called a weighted reproducing kernel and the space $H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ is called a weighted reproducing kernel Hilbert space. If $\gamma_1 = \gamma_2 = \cdots = 1$ and $1 < \alpha_1 = \alpha_2 = \cdots$, then the space $H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ is called unweighted space. Here, $\mathbb{N}_0 = \{0,\ 1,\ \ldots\}$ and $\mathbb{N} = \{1,\ 2,\ \ldots\}$.

There are many ways to introduce weighted reproducing kernel Hilbert spaces with weights $R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}$. In this paper we consider three weights like the cases in the reference [18].

2.1. A Weighted Korobov Space

Let $\boldsymbol{\alpha} = \{\alpha_j\}_{j\in\mathbb{N}}$ and $\boldsymbol{\gamma} = \{\gamma_j\}_{j\in\mathbb{N}}$ be two sequences satisfying (1). We consider a weighted Korobov space $H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ with weight

$$R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{h}) = r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{h}) := \prod_{j=1}^{d} r_{\alpha_j,\gamma_j}(h_j),$$

where

$$r_{\alpha,\gamma}(h) := \begin{cases} 1, & \text{for } h = 0, \\ \frac{\gamma}{h^{\lceil\alpha\rceil}}, & \text{for } h \ge 1 \end{cases}$$

for $\alpha > 1$ and $\gamma \in (0,1]$. We can see the case in the references [18,20]. Then we have the kernel function (2) with

$$K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}(\boldsymbol{x},\boldsymbol{y}) = K_{r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}(\boldsymbol{x},\boldsymbol{y}) = \sum_{\boldsymbol{h}\in\mathbb{N}_0^d} r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{h}) \exp(2\pi i \boldsymbol{h}\cdot(\boldsymbol{x}-\boldsymbol{y}))$$

for $\boldsymbol{x},\ \boldsymbol{y} \in [0,1]^d$, and the inner product (3) with

$$\langle f,g\rangle_{H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})} = \langle f,g\rangle_{H(K_{r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})} = \sum_{\boldsymbol{h}\in\in\mathbb{N}_0^d} \frac{1}{r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{h})}\widehat{f}(\boldsymbol{h})\overline{\widehat{g}}(\boldsymbol{h}).$$

Remark 1. *Obviously, the kernel $K_{r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}(\boldsymbol{x},\boldsymbol{y})$ is well defined for $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ satisfying (1), due to*

$$|K_{r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}(\boldsymbol{x},\boldsymbol{y})| \le \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{k}) = \prod_{j=1}^{d}(1+\zeta(\lceil\alpha_j\rceil)\gamma_j) < \infty,$$

where $\zeta(\cdot)$ is the Riemann zeta function.

2.2. A First Variant of the Weighted Korobov Space

Let $\boldsymbol{\alpha} = \{\alpha_j\}_{j\in\mathbb{N}}$ and $\boldsymbol{\gamma} = \{\gamma_j\}_{j\in\mathbb{N}}$ be two sequences satisfying (1). We discuss a first variant of the weighted Korobov space with weight

$$R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{h}) = \psi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{h}) := \prod_{j=1}^{d}\psi_{\alpha_j,\gamma_j}(h_j),$$

where

$$\psi_{\alpha,\gamma}(h) := \begin{cases} 1, & \text{for } h = 0, \\ \frac{\gamma}{h!}, & \text{for } 1 \le h < \lceil\alpha\rceil, \\ \frac{\gamma(h-\lceil\alpha\rceil)!}{h!}, & \text{for } h \ge \lceil\alpha\rceil \end{cases}$$

for $\alpha > 1$ and $\gamma \in (0,1]$.

Then we have the kernel function (2) with

$$K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}(\boldsymbol{x},\boldsymbol{y}) = K_{\psi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}(\boldsymbol{x},\boldsymbol{y}) = \sum_{\boldsymbol{h}\in\mathbb{N}_0^d} \psi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{h}) \exp(2\pi i \boldsymbol{h}\cdot(\boldsymbol{x}-\boldsymbol{y}))$$

for $\boldsymbol{x},\ \boldsymbol{y} \in [0,1]^d$ and the inner product (3) with

$$\langle f,g\rangle_{H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})} = \langle f,g\rangle_{H(K_{\psi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})} = \sum_{\boldsymbol{h}\in\in\mathbb{N}_0^d} \frac{1}{\psi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{h})}\widehat{f}(\boldsymbol{h})\overline{\widehat{g}}(\boldsymbol{h}).$$

Lemma 1 ([18] Lemma 2)**.** *For all $j,\ k \in \mathbb{N}$ we have*

$$\psi_{\alpha_j,\gamma_j}(k) \le \lceil\alpha_j\rceil^{\lceil\alpha_j\rceil} r_{\alpha_j,\gamma_j}(k).$$

Remark 2. *From Lemma 1 and* $1 < \alpha_1 \le \alpha_2 \le \cdots$ *we get*

$$\begin{aligned}|K_{\psi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}(\boldsymbol{x},\boldsymbol{y})| &\le \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} \psi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{k}) = \prod_{j=1}^{d}(1+\sum_{k\in\mathbb{N}}\psi_{\alpha_j,\gamma_j}(k)) \\ &\le \prod_{j=1}^{d}(1+\sum_{k\in\mathbb{N}}\lceil\alpha_j\rceil^{\lceil\alpha_j\rceil} r_{\alpha_j,\gamma_j}(k)) \\ &= \prod_{j=1}^{d}(1+\lceil\alpha_j\rceil^{\lceil\alpha_j\rceil}\zeta(\lceil\alpha_j\rceil)\gamma_j) \\ &< \infty.\end{aligned}$$

Hence, the kernel $K_{\psi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}(\boldsymbol{x},\boldsymbol{y})$ *is well defined.*

2.3. A Second Variant of the Weighted Korobov Space

Let $\boldsymbol{\alpha} = \{\alpha_j\}_{j\in\mathbb{N}}$ and $\boldsymbol{\gamma} = \{\gamma_j\}_{j\in\mathbb{N}}$ be two sequences satisfying (1). We study a second variant of the weighted Korobov space $H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ (see the references [18,21]) with weight

$$R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{h}) = \omega_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{h}) := \prod_{j=1}^{d}\omega_{\alpha_j,\gamma_j}(h_j),$$

where

$$\omega_{\alpha,\gamma}(h) := \left(1+\frac{1}{\gamma}\sum_{l=1}^{\lceil\alpha\rceil}\theta_l(h)\right)^{-1}$$

for $\alpha > 1$ and $\gamma \in (0,1]$ and

$$\theta_l(h) := \begin{cases} \frac{h!}{(h-l)!}, & \text{for } \ h \ge l, \\ 0, & \text{for } \ 0 \le h < l. \end{cases}$$

Then we have the kernel function (2) with

$$K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}(\boldsymbol{x},\boldsymbol{y}) = K_{\omega_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}(\boldsymbol{x},\boldsymbol{y}) = \sum_{\boldsymbol{h}\in\mathbb{N}_0^d}\omega_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{h})\exp(2\pi i\boldsymbol{h}\cdot(\boldsymbol{x}-\boldsymbol{y}))$$

for $\boldsymbol{x},\ \boldsymbol{y} \in [0,1]^d$, and the inner product (3) with

$$\langle f,g\rangle_{H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})} = \langle f,g\rangle_{H(K_{\omega_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})} = \sum_{\boldsymbol{h}\in\mathbb{N}_0^d}\frac{1}{\omega_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{h})}\widehat{f}(\boldsymbol{h})\overline{\widehat{g}}(\boldsymbol{h}).$$

Lemma 2 ([18] Lemma 3)**.** *For all* $j,\ k \in \mathbb{N}$ *we have*

$$\omega_{\alpha_j,\gamma_j}(k) \le \lceil\alpha_j\rceil^{\lceil\alpha_j\rceil} r_{\alpha_j,\gamma_j}(k).$$

Remark 3. *We note that the kernel $K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}(\boldsymbol{x},\boldsymbol{y})$ is also well defined. Indeed, it follows from Lemma 2 and $1 < \alpha_1 \le \alpha_2 \le \cdots$ that*

$$\begin{aligned}
|K_{\omega_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}(\boldsymbol{x},\boldsymbol{y})| &\le \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} \omega_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{k}) = \prod_{j=1}^{d}(1+\sum_{k\in\mathbb{N}}\omega_{\alpha_j,\gamma_j}(k)) \\
&\le \prod_{j=1}^{d}(1+\sum_{k\in\mathbb{N}}\lceil\alpha_j\rceil^{\lceil\alpha_j\rceil} r_{\alpha_j,\gamma_j}(k)) \\
&= \prod_{j=1}^{d}(1+\lceil\alpha_j\rceil^{\lceil\alpha_j\rceil}\zeta(\lceil\alpha_j\rceil)\gamma_j) \\
&< \infty.
\end{aligned}$$

Lemma 3. *Let $R_{\alpha_j,\gamma_j} \in \left\{r_{\alpha_j,\gamma_j}, \psi_{\alpha_j,\gamma_j}, \omega_{\alpha_j,\gamma_j}\right\}$ for all $j \in \mathbb{N}$. Then we have for all $j \in \mathbb{N}$, $k \in \mathbb{N}_0$*

$$R_{\alpha_j,\gamma_j}(k) \le \lceil\alpha_j\rceil^{\lceil\alpha_j\rceil} r_{\alpha_j,\gamma_j}(k).$$

Particularly, we have for all $j \in \mathbb{N}$, $k \in \mathbb{N}_0$

$$R_{\alpha_j,\gamma_j}(k) \le \lceil\alpha_1\rceil^{\lceil\alpha_1\rceil} r_{\alpha_1,\gamma_j}(k).$$

Proof. On the one hand, it is obvious from Lemma 1 and Lemma 2 that

$$R_{\alpha_j,\gamma_j}(k) \le \lceil\alpha_j\rceil^{\lceil\alpha_j\rceil} r_{\alpha_j,\gamma_j}(k) \tag{4}$$

for all j, $k \in \mathbb{N}$. Since for all $j \in \mathbb{N}$

$$r_{\alpha_j,\gamma_j}(0) = \psi_{\alpha_j,\gamma_j}(0) = \omega_{\alpha_j,\gamma_j}(0) = 1,$$

we have

$$R_{\alpha_j,\gamma_j}(0) = 1 \le \lceil\alpha_j\rceil^{\lceil\alpha_j\rceil} = \lceil\alpha_j\rceil^{\lceil\alpha_j\rceil} r_{\alpha_j,\gamma_j}(0).$$

Thus we have for all $j \in \mathbb{N}$, $k \in \mathbb{N}_0$ that

$$R_{\alpha_j,\gamma_j}(k) \le \lceil\alpha_j\rceil^{\lceil\alpha_j\rceil} r_{\alpha_j,\gamma_j}(k).$$

On the other hand, noting for all j, $k \in \mathbb{N}$

$$r_{\alpha_j,\gamma_j}(k) \le r_{\alpha_1,\gamma_j}(k), \quad \psi_{\alpha_j,\gamma_j}(k) \le \psi_{\alpha_1,\gamma_j}(k), \quad \omega_{\alpha_j,\gamma_j}(k) \le \omega_{\alpha_1,\gamma_j}(k),$$

and for all $j \in \mathbb{N}$

$$r_{\alpha_j,\gamma_j}(0) = r_{\alpha_1,\gamma_j}(0) = 1, \quad \psi_{\alpha_j,\gamma_j}(0) = \psi_{\alpha_1,\gamma_j}(0) = 1, \quad \omega_{\alpha_j,\gamma_j}(0) = \omega_{\alpha_1,\gamma_j}(0) = 1,$$

we have for all $j \in \mathbb{N}$, $k \in \mathbb{N}_0$ that

$$R_{\alpha_j,\gamma_j}(k) \le R_{\alpha_1,\gamma_j}(k).$$

Hence, by (4) we further get for all $j \in \mathbb{N}$, $k \in \mathbb{N}_0$ that

$$R_{\alpha_j,\gamma_j}(k) \le R_{\alpha_1,\gamma_j}(k) \le \lceil\alpha_1\rceil^{\lceil\alpha_1\rceil} r_{\alpha_1,\gamma_j}(k).$$

□

Remark 4. *Let* $R_{\alpha_j,\gamma_j} \in \left\{r_{\alpha_j,\gamma_j}, \psi_{\alpha_j,\gamma_j}, \omega_{\alpha_j,\gamma_j}\right\}$ *for all* $j \in \mathbb{N}$. *Then we obtain*

$$R_{\alpha_j,\gamma_j}(0) = 1 \quad \text{and} \quad R_{\alpha_j,\gamma_j}(1) \geq \frac{\gamma_j}{2} \tag{5}$$

for all $j \in \mathbb{N}$. *Indeed, for all* $j \in \mathbb{N}$ *we have*

$$\psi_{\alpha_j,\gamma_j}(0) = r_{\alpha_j,\gamma_j}(0) = \omega_{\alpha_j,\gamma_j}(0) = 1,$$

which means $R_{\alpha_j,\gamma_j}(0) = 1$. *As a result of all* $j \in \mathbb{N}$, *we get*

$$\psi_{\alpha_j,\gamma_j}(1) = r_{\alpha_j,\gamma_j}(1) = \gamma_j \quad \text{and} \quad \omega_{\alpha_j,\gamma_j}(1) = \left(1 + \frac{1}{\gamma_j}\right)^{-1} \geq \frac{\gamma_j}{2},$$

which yields $R_{\alpha_j,\gamma_j}(1) \geq \frac{\gamma_j}{2}$.

3. L_2-Approximation in the Weighted Hilbert Spaces

In this paper we investigate the L_2-approximation $\mathrm{APP}_d : H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}) \to L_2([0,1]^d)$ given by

$$\mathrm{APP}_d(f) = f \quad \text{for all} \quad f \in H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$$

in weighted Hilbert space $H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ with weight $R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}} \in \{r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}, \psi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}, \omega_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}\}$. We note from Remark 1, Remark 2, Remark 3, and [15] that this L_2-approximation is compact for $1 < \alpha_1 \leq \alpha_2 \leq \cdots$.

We approximate APP_d by using the algorithm $A_{n,d}$ of the form

$$A_{n,d}(f) = \sum_{i=1}^{n} T_i(f) g_i, \text{ for } f \in H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}), \tag{6}$$

where $g_1, g_2, \ldots, g_n$ belong to $L_2([0,1]^d)$ and $T_1, T_2, \ldots, T_n$ are continuous linear functionals on $H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$.

We consider the worst case setting in which the error of the algorithm $A_{n,d}$ of the form (6) is defined as

$$e(A_{n,d}) := \sup_{||f||_{H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})} \leq 1} ||\mathrm{APP}_d(f) - A_{n,d}(f)||_{L_2}.$$

The error $e(A_{n,d})$ is also called the worst case error. The nth minimal worst case error is defined as

$$e(n, \mathrm{APP}_d) := \inf_{A_{n,d}} e(A_{n,d}) \quad \text{for} \quad n \geq 1,$$

which is the infimum error among all algorithms (6). For $n = 0$, we set $A_{0,d} = 0$. We call

$$e(0, \mathrm{APP}_d) = \sup_{||f||_{H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})} \leq 1} ||\mathrm{APP}_d(f)||_{L_2}$$

the initial error of the problem APP_d.

We are interested in how the worst case error for the algorithm $A_{n,d}$ depends on the numbers n and d. We define the information complexity as

$$n(\varepsilon, \mathrm{APP}_d) := \min\{n \in \mathbb{N}_0 : e(n, \mathrm{APP}_d) \leq \varepsilon\},$$

where $\varepsilon \in (0,1)$ and $d \in \mathbb{N}$. In this paper, we set $\mathbb{N}_0 = \{0, 1, \ldots\}$ and $\mathbb{N} = \{1, 2, \ldots\}$.

By the references [2,4] we know that the nth minimal worst case errors $e(n, \mathrm{APP}_d)$ and the information complexity $n(\varepsilon, \mathrm{APP}_d)$ are related to the eigenvalues of the continuously linear operator $W_d = \mathrm{APP}_d^* \mathrm{APP}_d : H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}) \to H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$, where APP_d^* is the operator dual to APP_d. The eigenvalues of W_d are denoted by $\{\lambda_{d,j}\}_{j\in\mathbb{N}}$ satisfying

$$\lambda_{d,1} \geq \lambda_{d,2} \geq \cdots \geq 0$$

and the corresponding orthogonal eigenvectors of $\{\lambda_{d,j}\}_{j\in\mathbb{N}}$ by $\{\eta_{d,j}\}_{j\in\mathbb{N}}$ satisfying

$$\langle \eta_{d,i}, \eta_{d,j} \rangle_{H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})} = \delta_{i,j}, \quad \text{for all } i,\, j \in \mathbb{N},$$

where

$$W_d \eta_{d,j} = \lambda_{d,j} \eta_{d,j}, \quad \text{for all } j \in \mathbb{N}.$$

Here $\delta_{i,j} = 1$ for $i = j$ and $\delta_{i,j} = 0$ for $i \neq j$. Then the nth minimal worst case error is attained for the algorithm

$$A_{n,d}^{\diamond} f = \sum_{i=1}^{n} \langle f, \eta_{d,i} \rangle_{H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})} \eta_{d,i}, \quad \text{for all } n \in \mathbb{N}$$

and

$$e(n, \mathrm{APP}_d) = e(A_{n,d}^{\diamond}) = \sqrt{\lambda_{d,n+1}}, \quad \text{for all } n \in \mathbb{N}.$$

The initial error $e(0, \mathrm{APP}_d) = \sqrt{\lambda_{d,1}}$. Hence, we have $e(n, \mathrm{APP}_d) = \sqrt{\lambda_{d,n+1}}$ for all $n \in \mathbb{N}_0$. This deduces that the information complexity is equal to

$$n(\varepsilon, \mathrm{APP}_d) = \min\Big\{ n \in \mathbb{N}_0 : \sqrt{\lambda_{d,n+1}} \leq \varepsilon \Big\} = \min\Big\{ n \in \mathbb{N}_0 : \lambda_{d,n+1} \leq \varepsilon^2 \Big\}. \tag{7}$$

Since the eigenvalues $\lambda_{d,j}$ with $j \in \mathbb{N}$ of the operator W_d are $R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{k})$ with $\boldsymbol{k} \in \mathbb{N}_0^d$ (see [4] p. 215), by (7) the information complexity of APP_d from the space $H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ is equal to

$$\begin{aligned} n(\varepsilon, \mathrm{APP}_d) &= \min\Big\{ n \in \mathbb{N}_0 : \lambda_{d,n+1} \leq \varepsilon^2 \Big\} = \Big| \Big\{ n \in \mathbb{N} : \lambda_{d,n} > \varepsilon^2 \Big\} \Big| \\ &= \Big| \Big\{ \boldsymbol{h} \in \mathbb{N}_0^d : R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{h}) > \varepsilon^2 \Big\} \Big| = \left| \left\{ \boldsymbol{h} \in \mathbb{N}_0^d : \prod_{j=1}^{d} R_{\alpha_j,\gamma_j}(h_j) > \varepsilon^2 \right\} \right|, \end{aligned} \tag{8}$$

with $\varepsilon \in (0,1)$ and $d \in \mathbb{N}$, where $|A|$ denotes the cardinality of set A.

Note that for the L_2-approximation APP_d from the space $H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ the absolute error criterion and the normalized error criterion are the same, since the initial error $e(0, \mathrm{APP}_d) = \sqrt{\lambda_{d,1}} = 1$.

4. Tractability in Weighted Hilbert Spaces and Main Results

In this paper we will study the classical tractability and the exponential convergence-tractability (EC-tractability) for the problem $\mathrm{APP} = \{\mathrm{APP}_d\}_{d\in\mathbb{N}}$ in the weighted Hilbert space $H_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}$.

4.1. Tractability and Main Results

We focus on the behaviours of the information complexity $n(\varepsilon, \mathrm{APP}_d)$ depending on the dimension d and the error threshold ε. Hence, we will study several notions about the classical tractability and the exponential convergence-tractability (EC-tractability) notions (see [4–9,11,12,16,22]).

Definition 1. *Let $APP = \{APP_d\}_{d\in\mathbb{N}}$. We say the following:*

- *Strong polynomial tractability (SPT) if there are positive numbers C and p such that*

$$n(\varepsilon, APP_d) \leq C(\varepsilon^{-1})^p \quad \text{for all } \; d \in \mathbb{N}, \varepsilon \in (0,1).$$

In this case we define the exponent p^{str} of SPT as

$$p^{str} := \inf\{p : \exists\, C > 0 \text{ such that } n(\varepsilon, APP_d) \le C(\varepsilon^{-1})^p, \forall\, d \in \mathbb{N}, \varepsilon \in (0,1)\}.$$

- *Polynomial tractability (PT) if there are positive numbers C, p, and q such that*

$$n(\varepsilon, APP_d) \le Cd^q(\varepsilon^{-1})^p \ \text{ for all } \ d \in \mathbb{N}, \varepsilon \in (0,1).$$

- *Quasi-polynomial tractability (QPT) if there are positive numbers C and t such that*

$$n(\varepsilon, APP_d) \le C\exp\big(t\big(1+\ln d\big)(1+\ln\varepsilon^{-1})\big) \ \text{ for all } \ d \in \mathbb{N}, \varepsilon \in (0,1).$$

- *Uniform weak tractability (UWT) if for all $t, s > 0$,*

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n(\varepsilon, APP_d)}{d^t + (\varepsilon^{-1})^s} = 0.$$

- *Weak tractability (WT) if*

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n(\varepsilon, APP_d)}{d + \varepsilon^{-1}} = 0.$$

- *(t,s)-weak tractability ((t,s)-WT) for fixed positive t and s if*

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n(\varepsilon, APP_d)}{d^t + (\varepsilon^{-1})^s} = 0.$$

We find that (1,1)-WT is the same as WT and

$$\text{SPT} \Longrightarrow \text{PT} \Longrightarrow \text{QPT} \Longrightarrow \text{UWT} \Longrightarrow \text{WT}.$$

In the above definitions regarding classical tractability, replacing ε^{-1} with $(1+\ln(\varepsilon^{-1}))$, we will have the following definitions about exponential convergence-tractability (EC-tractability).

Definition 2. *Let $APP = \{APP_d\}_{d\in\mathbb{N}}$. We say we have the following:*

- *Exponential convergence-strong polynomial tractability (EC-SPT) if there are positive numbers C and p such that*

$$n(\varepsilon, APP_d) \le C\big(1+\ln(\varepsilon^{-1})\big)^p \ \text{ for all } \ d \in \mathbb{N}, \varepsilon \in (0,1).$$

The exponent of EC-SPT is defined as

$$\inf\{p : \exists\, C > 0 \text{ such that } n(\varepsilon, APP_d) \le C\big(1+\ln(\varepsilon^{-1})\big)^p, \forall\, d \in \mathbb{N}, \varepsilon \in (0,1)\}.$$

- *Exponential convergence-polynomial tractability (EC-PT) if there are positive numbers C, p, and q such that*

$$n(\varepsilon, APP_d) \le Cd^q\big(1+\ln(\varepsilon^{-1})\big)^p \ \text{ for all } \ d \in \mathbb{N}, \varepsilon \in (0,1).$$

- *Exponential convergence-uniform weak tractability (EC-UWT) if for all $t, s > 0$*

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n(\varepsilon, APP_d)}{d^t + \big(1+\ln(\varepsilon^{-1})\big)^s} = 0.$$

- *Exponential convergence-weak tractability (EC-WT) if*

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n(\varepsilon, APP_d)}{d+\ln(\varepsilon^{-1})} = 0.$$

- *Exponential convergence-(t,s)-weak tractability (EC-(t,s)-WT) for fixed positive t and s if*

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n(\varepsilon, APP_d)}{d^t+\big(1+\ln(\varepsilon^{-1})\big)^s} = 0.$$

We note that EC-$(1,1)$-WT is the same as EC-WT, and

$$\text{EC-SPT} \Longrightarrow \text{EC-PT} \Longrightarrow \text{EC-QPT} \Longrightarrow \text{EC-UWT} \Longrightarrow \text{EC-WT}.$$

Obviously, if the problem APP has exponential convergence-tractability, then it has classical tractability and

$$\text{EC-}(t,s)\text{-WT} \Longrightarrow (t,s)\text{-WT}, \quad \text{EC-UWT} \Longrightarrow \text{UWT}, \quad \text{EC-WT} \Longrightarrow \text{WT}.$$

In the worst case setting the classical tractability and EC-tractability of the problem APP $= \{\mathrm{APP}_d\}_{d\in\mathbb{N}}$ in the weighted Hilbert space $H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ with $\boldsymbol{\gamma} = \{\gamma_j\}_{j\in\mathbb{N}}$ and $\boldsymbol{\alpha} = \{\alpha_j\}_{j\in\mathbb{N}}$ satisfying

$$1 \ge \gamma_1 \ge \gamma_2 \ge \cdots \ge 0, \text{ and } 1 < \alpha^* = \alpha_1 = \alpha_2 = \cdots$$

have been solved by [13,14,16,18] as follows:

- For $R_{d,\alpha^*,\gamma} \in \{r_{d,\alpha^*,\gamma}, \psi_{d,\alpha^*,\gamma}, \omega_{d,\alpha^*,\gamma}\}$, SPT holds iff PT holds iff

$$s_\gamma := \inf\left\{\kappa > 0 : \sum_{j=1}^{\infty} \gamma_j^\kappa < \infty\right\} < \infty$$

and the exponent of SPT is

$$p^{\mathrm{str}} = 2\max\Big(s_\gamma, \frac{1}{\alpha}\Big).$$

- For $R_{d,\alpha^*,\gamma} = r_{d,\alpha^*,\gamma}$, QPT, UWT, and WT are equivalent and hold iff

$$\gamma_I := \inf_{j\in\mathbb{N}} \gamma_j < 1.$$

For $R_{d,\alpha^*,\gamma} \in \{\psi_{d,\alpha^*,\gamma}, \omega_{d,\alpha^*,\gamma}\}$,

$$\gamma_I < \infty$$

implies QPT.
- For $R_{d,\alpha^*,\gamma} \in \{r_{d,\alpha^*,\gamma}, \psi_{d,\alpha^*,\gamma}, \omega_{d,\alpha^*,\gamma}\}$ and $t > 1$, (t,s)-WT holds for all $1 \ge \gamma_1 \ge \gamma_2 \ge \cdots \ge 0$.
- For $R_{d,\alpha^*,\gamma} = r_{d,\alpha^*,\gamma}$, EC-WT holds iff

$$\lim_{j\to\infty} \gamma_j = 0.$$

- For $R_{d,\alpha^*,\gamma} = r_{d,\alpha^*,\gamma}$ and $t < 1$, EC-$(t,1)$-WT holds iff

$$\lim_{j\to\infty} \frac{\ln j}{\ln(\gamma_j^{-1})} = 0.$$

In the worst case setting the classical tractability such as SPT, PT, and WT of the problem APP $= \{\mathrm{APP}_d\}_{d\in\mathbb{N}}$ in the weighted Hilbert space $H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ with $\boldsymbol{\gamma} = \{\gamma_j\}_{j\in\mathbb{N}}$ and $\boldsymbol{\alpha} = \{\alpha_j\}_{j\in\mathbb{N}}$ satisfying (1), i.e.,

$$1 \ge \gamma_1 \ge \gamma_2 \ge \cdots \ge 0, \text{ and } 1 < \alpha_1 \le \alpha_2 \le \cdots$$

has been solved by [18] as follows:

- For $R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}} \in \{r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}, \psi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}, \omega_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}\}$, SPT holds iff PT holds iff
$$\delta := \liminf_{j\to\infty} \frac{\ln \gamma_j^{-1}}{\ln j} > 0.$$
The exponent of SPT is
$$p^{\mathrm{str}} = 2\max\Big\{\frac{1}{\delta}, \frac{1}{\lceil \alpha_1 \rceil}\Big\}.$$
- For $R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}} = r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}$, WT holds iff
$$\lim_{j\to\infty} \gamma_j < 1.$$
- For $R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}} \in \{r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}, \psi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}, \omega_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}\}$ and $t > 1$, (t,s)-WT holds.

In this paper, we investigate the EC-tractability of the problem APP $= \{\mathrm{APP}_d\}_{d\in\mathbb{N}}$ in the weighted Hilbert space $H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ with $\boldsymbol{\gamma} = \{\gamma_j\}_{j\in\mathbb{N}}$ and $\boldsymbol{\alpha} = \{\alpha_j\}_{j\in\mathbb{N}}$ satisfying (1). We obtain sufficient and necessary conditions for EC-$(t,1)$-WT with $0 < t < 1$ and $t = 1$.

Theorem 1. *Let $\boldsymbol{\gamma} = \{\gamma_j\}_{j\in\mathbb{N}}$ and $\boldsymbol{\alpha} = \{\alpha_j\}_{j\in\mathbb{N}}$ satisfy (1). Then the problem APP $= \{APP_d\}_{d\in\mathbb{N}}$ in the weighted Hilbert spaces $H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ with $R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}} \in \{r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}, \psi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}, \omega_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}\}$*

(1) is EC-WT, if and only if
$$\lim_{j\to\infty} \gamma_j = 0.$$

(2) is EC-$(t,1)$-WT with $t < 1$, if and only if
$$\lim_{j\to\infty} \frac{\ln j}{\ln(\gamma_j^{-1})} = 0.$$

4.2. The Proof

In order to prove Theorem 1 we need the following Lemmas.

Lemma 4. *Let $\eta > 0$, $\varepsilon \in (0,1)$. We have for any $d \in \mathbb{N}$*

$$n(\varepsilon, APP_d) \le \varepsilon^{-2\eta} \prod_{j=1}^{d} \Big(1 + \lceil \alpha_1 \rceil^{\lceil \alpha_1 \rceil \eta} \zeta(\lceil \alpha_1 \rceil \eta) \gamma_j^{\eta}\Big).$$

Proof. By Lemma 3 we have

$$\begin{aligned}
\sum_{k=1}^{\infty} \lambda_{d,k}^{\eta} &= \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} (R_{d,\boldsymbol{\alpha},\gamma}(\boldsymbol{k}))^{\eta} = \prod_{j=1}^{d}\left(1+\sum_{k=1}^{\infty}\Big(R_{\alpha_j,\gamma_j}(k)\Big)^{\eta}\right) \\
&\le \prod_{j=1}^{d}\left(1+\sum_{k=1}^{\infty}\Big(\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil} r_{\alpha_1,\gamma_j}(k)\Big)^{\eta}\right) \\
&= \prod_{j=1}^{d}\left(1+\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\sum_{k=1}^{\infty}\Big(r_{\alpha_1,\gamma_j}(k)\Big)^{\eta}\right) \\
&= \prod_{j=1}^{d}\left(1+\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\sum_{k=1}^{\infty}\left(\frac{\gamma_j}{k^{\lceil\alpha_1\rceil}}\right)^{\eta}\right) \\
&= \prod_{j=1}^{d}\Big(1+\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta)\gamma_j^{\eta}\Big).
\end{aligned}$$

This yields

$$n\lambda_{d,n}^{\eta} \le \sum_{k=1}^{n}\lambda_{d,k}^{\eta} \le \sum_{k=1}^{\infty}\lambda_{d,k}^{\eta} \le \prod_{j=1}^{d}\Big(1+\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta)\gamma_j^{\eta}\Big),$$

which means

$$\lambda_{d,n} \le \frac{\prod_{j=1}^{d}\Big(1+\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta)\gamma_j^{\eta}\Big)^{1/\eta}}{n^{1/\eta}}.$$

It follows from the above inequality and (7)

$$n(\varepsilon, \mathrm{APP}_d) = \min\Big\{n\in\mathbb{N}_0 : \lambda_{d,n+1}\le\varepsilon^2\Big\},$$

that

$$n(\varepsilon, \mathrm{APP}_d) \le \varepsilon^{-2\eta}\prod_{j=1}^{d}\Big(1+\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta)\gamma_j^{\eta}\Big).$$

This proof is complete. □

Lemma 5. *Let $\varepsilon\in(0,1)$. We have for any $d\ge 2$*

$$n(\varepsilon, APP_d) \ge \left\lceil\Big(\varepsilon^{-2}\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}\gamma_d\Big)^{\frac{1}{\lceil\alpha_1\rceil}}\right\rceil.$$

Proof. Set

$$H = H(\varepsilon, d, \alpha) := \left\{h\in\mathbb{N}_0 : h\le\left\lceil\Big(\varepsilon^{-2}\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}\gamma_d\Big)^{\frac{1}{\lceil\alpha_1\rceil}}\right\rceil - 1\right\}.$$

If $h > \left\lceil\Big(\varepsilon^{-2}\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}\gamma_d\Big)^{\frac{1}{\lceil\alpha_1\rceil}}\right\rceil - 1$ and $d\ge 2$, by Lemma 3 we have

$$\prod_{j=1}^{d-1}R_{\alpha_j,\gamma_j}(h_j)R_{\alpha_d,\gamma_d}(h) \le R_{\alpha_d,\gamma_d}(h) \le \lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}r_{\alpha_1,\gamma_d}(h) = \lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}\frac{\gamma_d}{h^{\lceil\alpha_1\rceil}} \le \varepsilon^2$$

for any $\{h_1, \cdots, h_{d-1}\} \in \mathbb{N}_0^{d-1}$, which means

$$\left\{ \boldsymbol{h} \in \mathbb{N}_0^{d-1} : \prod_{j=1}^{d-1} R_{\alpha_j,\gamma_j}(h_j) R_{\alpha_d,\gamma_d}(h) > \varepsilon^2 \right\} = \varnothing \tag{9}$$

for all $h > \left\lceil \left(\varepsilon^{-2} \lceil \alpha_1 \rceil^{\lceil \alpha_1 \rceil} \gamma_d \right)^{\frac{1}{\lceil \alpha_1 \rceil}} \right\rceil - 1$. It follows from (8) and (9) that

$$\begin{aligned}
n(\varepsilon, \mathrm{APP}_d) &= \left| \left\{ \boldsymbol{h} \in \mathbb{N}_0^d : \prod_{j=1}^{d} R_{\alpha_j,\gamma_j}(h_j) > \varepsilon^2 \right\} \right| \\
&= \left| \left\{ \boldsymbol{h} \in \mathbb{N}_0^d : \prod_{j=1}^{d-1} R_{\alpha_j,\gamma_j}(h_j) R_{\alpha_d,\gamma_d}(h_d) > \varepsilon^2 \right\} \right| \\
&= \sum_{h \in \mathbb{N}_0} \left| \left\{ \boldsymbol{h} \in \mathbb{N}_0^{d-1} : \prod_{j=1}^{d-1} R_{\alpha_j,\gamma_j}(h_j) R_{\alpha_d,\gamma_d}(h) > \varepsilon^2 \right\} \right| \\
&= \sum_{h \in H} \left| \left\{ \boldsymbol{h} \in \mathbb{N}_0^{d-1} : \prod_{j=1}^{d-1} R_{\alpha_j,\gamma_j}(h_j) R_{\alpha_d,\gamma_d}(h) > \varepsilon^2 \right\} \right| \\
&= \sum_{h \in (H \setminus \{0\})} \left| \left\{ \boldsymbol{h} \in \mathbb{N}_0^{d-1} : \prod_{j=1}^{d-1} R_{\alpha_j,\gamma_j}(h_j) > \varepsilon^2 R^{-1}_{\alpha_d,\gamma_d}(h) \right\} \right| \\
&+ \left| \left\{ \boldsymbol{h} \in \mathbb{N}_0^{d-1} : \prod_{j=1}^{d-1} R_{\alpha_j,\gamma_j}(h_j) > \varepsilon^2 \right\} \right| \\
&= \sum_{h \in (H \setminus \{0\})} n(\varepsilon R^{-1/2}_{\alpha_d,\gamma_d}(h), \mathrm{APP}_{d-1}) + n(\varepsilon, \mathrm{APP}_{d-1}) \\
&= \sum_{h=1}^{\left\lceil \left(\varepsilon^{-2} \lceil \alpha_1 \rceil^{\lceil \alpha_1 \rceil} \gamma_d \right)^{\frac{1}{\lceil \alpha_1 \rceil}} \right\rceil - 1} n(\varepsilon R^{-1/2}_{\alpha_d,\gamma_d}(h), \mathrm{APP}_{d-1}) + n(\varepsilon, \mathrm{APP}_{d-1}) \\
&\geq \left\lceil \left(\varepsilon^{-2} \lceil \alpha_1 \rceil^{\lceil \alpha_1 \rceil} \gamma_d \right)^{\frac{1}{\lceil \alpha_1 \rceil}} \right\rceil.
\end{aligned}$$

This finishes the proof. □

Lemma 6. *For* $\prod_{j=1}^{d} \left(\frac{\gamma_j}{2} \right) > \varepsilon^2$ *and* $\varepsilon \in (0,1)$ *we have*

$$n(\varepsilon, APP_d) \geq 2^d.$$

Proof. Set

$$\mathcal{A}(\varepsilon, d) = \left\{ \boldsymbol{h} \in \mathbb{N}_0^d : \prod_{j=1}^{d} R_{\alpha_j,\gamma_j}(h_j) > \varepsilon^2 \right\}.$$

If $\boldsymbol{h} = \{h_1, h_2, \ldots, h_d\} \in \{0,1\}^d$, we have from (5) that

$$R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{h}) = \prod_{j=1}^{d} R_{\alpha_j,\gamma_j}(h_j) \geq \prod_{j=1}^{d} \left(\frac{\gamma_j}{2} \right).$$

Thus, we have $\{0,1\}^d \in \mathcal{A}(\varepsilon,d)$ for $\prod_{j=1}^{d}\left(\frac{\gamma_j}{2}\right) > \varepsilon^2$. Hence, it follows from (8) that

$$n(\varepsilon, \mathrm{APP}_d) = |\mathcal{A}(\varepsilon,d)| \geq \left|\left\{\boldsymbol{h} \in \{0,1\}^d : \prod_{j=1}^{d} R_{\alpha_j,\gamma_j}(h_j) > \varepsilon^2\right\}\right| = 2^d$$

for $\prod_{j=1}^{d}\left(\frac{\gamma_j}{2}\right) > \varepsilon^2$. This proof is complete. □

Proof of Theorem 1.

If there are infinitely many $\gamma_j = 0$ for $j \in \mathbb{N}$, the results are obviously true. Without loss of generality we discuss only that the γ_j are positive for $j \in \mathbb{N}$.

(1) Let $\delta > 0$ and take $\varepsilon = \prod_{j=1}^{d}\left(\frac{\gamma_j}{2}\right)^{\frac{1+\delta}{2}}$, then we have

$$\prod_{j=1}^{d}\left(\frac{\gamma_j}{2}\right) > \varepsilon^2.$$

It follows from Lemma 6 that

$$\begin{aligned}
\frac{\ln n(\varepsilon, \mathrm{APP}_d)}{d + \ln(\varepsilon^{-1})} &\geq \frac{d \ln 2}{d + \frac{1+\delta}{2} \cdot \ln\left(\prod_{j=1}^{d} (2\gamma_j^{-1})\right)} \\
&\geq \frac{d \ln 2}{d + \frac{1+\delta}{2} \cdot d \cdot \ln(2\gamma_d^{-1})} \\
&= \frac{\ln 2}{1 + \frac{1+\delta}{2} \cdot \left(\ln 2 + \ln(\gamma_d^{-1})\right)}. \qquad (10)
\end{aligned}$$

Assume that App is EC-WT, i.e., for the above fixed ε

$$\lim_{d\to\infty} \frac{\ln n(\varepsilon, \mathrm{APP}_d)}{d + \ln(\varepsilon^{-1})} = 0.$$

Combing (10) and the above equality we have

$$0 = \lim_{d\to\infty} \frac{\ln n(\varepsilon, \mathrm{APP}_d)}{d + \ln(\varepsilon^{-1})} \geq \lim_{d\to\infty} \frac{\ln 2}{1 + \frac{1+\delta}{2} \cdot \left(\ln 2 + \ln(\gamma_d^{-1})\right)}.$$

This implies $\lim_{d\to\infty} \gamma_d = 0$.

On the other hand, assume that we have $\lim_{d\to\infty} \gamma_d = 0$. For $\eta > 0$ we obtain from the upper bound in Lemma 4 that

$$\begin{aligned}
\limsup_{d+\varepsilon^{-1}\to\infty} \frac{\ln n(\varepsilon, \mathrm{APP}_d)}{d+\ln(\varepsilon^{-1})} &\le \limsup_{d+\varepsilon^{-1}\to\infty} \frac{2\eta\ln(\varepsilon^{-1}) + \sum_{j=1}^{d} \ln\Big(1 + \lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta)\gamma_j^{\eta}\Big)}{d+\ln(\varepsilon^{-1})} \\
&\le \limsup_{d+\varepsilon^{-1}\to\infty} \frac{2\eta\ln(\varepsilon^{-1}) + \sum_{j=1}^{d} \lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta)\gamma_j^{\eta}}{d+\ln(\varepsilon^{-1})} \\
&\le 2\eta + \limsup_{d\to\infty} \frac{\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta)\sum_{j=1}^{d}\gamma_j^{\eta}}{d} \\
&= 2\eta,
\end{aligned}$$

where we used $\ln(1+x) \le x$ for all $x \ge 0$ and $\limsup_{d\to\infty} \frac{\sum_{j=1}^{d}\gamma_j^{\eta}}{d} = 0$ if $\lim_{d\to\infty}\gamma_d = 0$. Setting $\eta \to 0$, we have

$$\limsup_{d+\varepsilon^{-1}\to\infty} \frac{\ln n(\varepsilon, \mathrm{APP}_d)}{d+\ln(\varepsilon^{-1})} = 0,$$

which yields that ET-WT holds.

(2) Assume that APP is EC-$(t,1)$-WT for $t < 1$. First, we note that $\lim_{d\to\infty}\gamma_d = 0$. Indeed, if $\lim_{d\to\infty}\gamma_d \ne 0$, we deduce from Theorem 1 (1) that EC-WT doesn't hold, i.e.,

$$0 < \limsup_{d+\varepsilon^{-1}\to\infty} \frac{\ln n(\varepsilon, \mathrm{APP}_d)}{d+\ln(\varepsilon^{-1})} \le \limsup_{d+\varepsilon^{-1}\to\infty} \frac{\ln n(\varepsilon, \mathrm{APP}_d)}{d^t+\ln(\varepsilon^{-1})}.$$

This deduces that EC-$(t,1)$-WT for $t < 1$ does not hold.
Next, we will prove $\lim_{j\to\infty} \frac{\ln j}{\ln(\gamma_j^{-1})} = 0$. Let $\varepsilon = \varepsilon_d \in (0,1)$ such that

$$\ln\Big(\varepsilon^{-2}\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}\gamma_d\Big)^{\frac{1}{\lceil\alpha_1\rceil}} = d^t$$

for large $d \in \mathbb{N}$. From the lower bound in Lemma 5 we obtain

$$\begin{aligned}
\frac{\ln n(\varepsilon, \mathrm{APP}_d)}{d^t+\ln(\varepsilon^{-1})} &\ge \frac{\ln\Big\lceil\Big(\varepsilon^{-2}\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}\gamma_d\Big)^{\frac{1}{\lceil\alpha_1\rceil}}\Big\rceil}{d^t+\ln(\varepsilon^{-1})} \ge \frac{\ln\Big(\varepsilon^{-2}\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}\gamma_d\Big)^{\frac{1}{\lceil\alpha_1\rceil}}}{d^t+\ln(\varepsilon^{-1})} \\
&= \frac{d^t}{d^t+\ln(\varepsilon^{-1})} = \frac{d^t}{d^t + \lceil\alpha_1\rceil d^t/2 + \ln(\gamma_d^{-1})/2 - \lceil\alpha_1\rceil(\ln\lceil\alpha_1\rceil)/2} \\
&= \frac{1}{1 + \lceil\alpha_1\rceil/2 + \ln(\gamma_d^{-1})/(2d^t) - \lceil\alpha_1\rceil(\ln\lceil\alpha_1\rceil)/(2d^t)}.
\end{aligned}$$

It follows from the assumption that

$$\begin{aligned}
0 = \limsup_{d+\varepsilon^{-1}\to\infty} \frac{\ln n(\varepsilon, \mathrm{APP}_d)}{d^t+\ln(\varepsilon^{-1})} &\ge \limsup_{d+\varepsilon^{-1}\to\infty} \frac{1}{1 + \lceil\alpha_1\rceil/2 + \ln(\gamma_d^{-1})/(2d^t) - \lceil\alpha_1\rceil(\ln\lceil\alpha_1\rceil)/(2d^t)} \\
&= \limsup_{d+\varepsilon^{-1}\to\infty} \frac{1}{1 + \lceil\alpha_1\rceil/2 + \ln(\gamma_d^{-1})/(2d^t)},
\end{aligned}$$

which implies

$$\lim_{d\to\infty} \frac{d^t}{\ln(\gamma_d^{-1})} = 0.$$

Using the fact that $d^t \geq \ln d^t = t \ln d \geq 0$ for large $d \in \mathbb{N}$, we have

$$0 \leq \lim_{d\to\infty} \frac{\ln d}{\ln(\gamma_d^{-1})} \leq \lim_{d\to\infty} \frac{d^t}{t\ln(\gamma_d^{-1})} = 0,$$

i.e.,

$$\lim_{d\to\infty} \frac{\ln d}{\ln(\gamma_d^{-1})} = 0.$$

On the other hand, assume that $\lim\limits_{j\to\infty} \frac{\ln j}{\ln(\gamma_j^{-1})} = 0$. Then we obtain that for all $\delta > 0$ there exists a positive number $N_\delta > 0$ such that

$$\gamma_j \leq j^{-\delta} \text{ for all } j \geq N_\delta. \tag{11}$$

Let $\eta > 0$. We get from Lemma 4 that

$$\begin{aligned}\ln n(\varepsilon, \mathrm{APP}_d) &\leq 2\eta \ln(\varepsilon^{-1}) + \sum_{j=1}^{d} \ln\Big(1 + \lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta)\gamma_j^\eta\Big) \\ &\leq 2\eta \ln(\varepsilon^{-1}) + \sum_{j=1}^{d} \lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta)\gamma_j^\eta, \end{aligned} \tag{12}$$

where we used $\ln(1+x) \leq x$ for all $x > 0$. Choose $\delta = \frac{2}{\eta}$. By (11) and (12) we get

$$\begin{aligned}\frac{\ln n(\varepsilon, \mathrm{APP}_d)}{d^t + \ln(\varepsilon^{-1})} &\leq \frac{2\eta\ln(\varepsilon^{-1}) + \sum\limits_{j=1}^{N_{2/\eta}-1} \lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta)\gamma_j^\eta + \sum\limits_{j=N_{2/\eta}}^{\max\{d,N_{2/\eta}\}} \lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta)\gamma_j^\eta}{d^t + \ln(\varepsilon^{-1})} \\ &\leq 2\eta + \frac{\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta)\gamma_1^\eta(N_{2/\eta}-1) + \lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta)\sum\limits_{j=N_{2/\eta}}^{\max\{d,N_{2/\eta}\}} j^{-2}}{d^t + \ln(\varepsilon^{-1})}.\end{aligned}$$

It follows that

$$\begin{aligned}\limsup_{d+\varepsilon^{-1}\to\infty} \frac{\ln n(\varepsilon, \mathrm{APP}_d)}{d^t + \ln(\varepsilon^{-1})} &\leq 2\eta + \lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta) \limsup_{d+\varepsilon^{-1}\to\infty} \frac{\sum\limits_{j=N_{2/\eta}}^{\max\{d,N_{2/\eta}\}} j^{-2}}{d^t + \ln(\varepsilon^{-1})} \\ &\leq 2\eta + \lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta) \limsup_{d+\varepsilon^{-1}\to\infty} \frac{\sum\limits_{j=1}^{\infty} j^{-2}}{d^t + \ln(\varepsilon^{-1})} \\ &= 2\eta.\end{aligned}$$

Setting $\eta \to 0$, we have

$$\limsup_{d+\varepsilon^{-1}\to\infty} \frac{\ln n(\varepsilon, \mathrm{APP}_d)}{d^t + \ln(\varepsilon^{-1})} = 0.$$

Therefore, Theorem 1 is proved.

□

Example 1. *An example for EC-WT.*

Assume that $\gamma_j = j^{-2}$ *and* $\alpha_j = j+1$ *for all* $j \in \mathbb{N}$*. Next, we will study EC-WT for the weighted Hilbert spaces* $H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ *with weight* $R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}} \in \{r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}, \psi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}, \omega_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}\}$.

Obviously, we have $\lim_{j\to\infty} \gamma_j = 0$*. By Lemma 4 we get*

$$\begin{aligned}
\limsup_{d+\varepsilon^{-1}\to\infty} \frac{\ln n(\varepsilon, APP_d)}{d+\ln(\varepsilon^{-1})} &\le \limsup_{d+\varepsilon^{-1}\to\infty} \frac{2\eta\ln(\varepsilon^{-1}) + \sum_{j=1}^{d} \ln\Big(1+\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta)\gamma_j^{\eta}\Big)}{d+\ln(\varepsilon^{-1})} \\
&\le \limsup_{d+\varepsilon^{-1}\to\infty} \frac{2\eta\ln(\varepsilon^{-1}) + \sum_{j=1}^{d} \lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta)\gamma_j^{\eta}}{d+\ln(\varepsilon^{-1})} \\
&\le 2\eta + \limsup_{d\to\infty} \frac{\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta)\sum_{j=1}^{d}\gamma_j^{\eta}}{d} \\
&= 2\eta + \limsup_{d\to\infty} \frac{\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta)\sum_{j=1}^{d} j^{-2\eta}}{d} \\
&= 2\eta,
\end{aligned}$$

where in the second inequality we used $\ln(1+x) \le x$ *for all* $x \ge 0$ *and* $\limsup_{d\to\infty} \frac{\sum_{j=1}^{d} j^{-2\eta}}{d} = 0$*. Setting* $\eta \to 0$*, we have*

$$\limsup_{d+\varepsilon^{-1}\to\infty} \frac{\ln n(\varepsilon, APP_d)}{d+\ln(\varepsilon^{-1})} = 0.$$

Hence, APP is EC-WT.

Example 2. *An example for EC-*$(t,1)$*-WT for* $t<1$*.*

Assume that $\gamma_j = 2^{-j}$ *and* $\alpha_j = 2j$ *for all* $j \in \mathbb{N}$*. Next, we will study EC-*$(t,1)$*-WT for* $t<1$ *for the weighted Hilbert spaces* $H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ *with weight* $R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}} \in \{r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}, \psi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}, \omega_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}\}$.

Note that $\lim_{j\to\infty} \frac{\ln j}{\ln(\gamma_j^{-1})} = \lim_{j\to\infty} \frac{\ln j}{j\ln 2} = 0$*. It follows from Lemma 4 that*

$$\begin{aligned}
\ln n(\varepsilon, APP_d) &\le 2\eta\ln(\varepsilon^{-1}) + \sum_{j=1}^{d} \ln\Big(1+\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta)\gamma_j^{\eta}\Big) \\
&= 2\eta\ln(\varepsilon^{-1}) + \sum_{j=1}^{d} \ln\Big(1+\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta)2^{-\eta j}\Big) \\
&\le 2\eta\ln(\varepsilon^{-1}) + \sum_{j=1}^{d} \lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta)2^{-\eta j},
\end{aligned}$$

where in the last inequality we used $\ln(1+x) \le x$ *for all* $x > 0$. *It yields that*

$$\limsup_{d+\varepsilon^{-1}\to\infty} \frac{\ln n(\varepsilon, APP_d)}{d^t + \ln(\varepsilon^{-1})} \le \limsup_{d+\varepsilon^{-1}\to\infty} \frac{2\eta \ln(\varepsilon^{-1}) + \sum_{j=1}^{d} \lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta)2^{-\eta j}}{d^t + \ln(\varepsilon^{-1})}$$
$$\le 2\eta + \limsup_{d+\varepsilon^{-1}\to\infty} \frac{\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta)\sum_{j=1}^{d} 2^{-\eta j}}{d^t + \ln(\varepsilon^{-1})}$$
$$\le 2\eta + \limsup_{d+\varepsilon^{-1}\to\infty} \frac{\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\eta}\zeta(\lceil\alpha_1\rceil\eta)\sum_{j=1}^{\infty} 2^{-\eta j}}{d^t + \ln(\varepsilon^{-1})}$$
$$= 2\eta.$$

Setting $\eta \to 0$, *we have*

$$\limsup_{d+\varepsilon^{-1}\to\infty} \frac{\ln n(\varepsilon, APP_d)}{d^t + \ln(\varepsilon^{-1})} = 0.$$

Hence, APP is EC-$(t, 1)$*-WT for* $t < 1$.

Remark 5. *We note that for Example 1 with* $\gamma_j = j^{-2}$ *and* $\alpha_j = j + 1$ *for all* $j \in \mathbb{N}$, *APP is EC-WT, but not EC-*$(t, 1)$*-WT for* $t < 1$. *Indeed, let* $\varepsilon = \varepsilon_d \in (0, 1)$ *such that*

$$\ln\Big(\varepsilon^{-2}\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}\gamma_d\Big)^{\frac{1}{\lceil\alpha_1\rceil}} = d, \quad i.e., \quad \varepsilon^{-1} = \frac{de^{d\lceil\alpha_1\rceil/2}}{\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil/2}}$$

for large $d \in \mathbb{N}$. *From Lemma 5 we have*

$$\frac{\ln n(\varepsilon, APP_d)}{d^t + \ln(\varepsilon^{-1})} \ge \frac{\ln\Big\lceil\Big(\varepsilon^{-2}\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}\gamma_d\Big)^{\frac{1}{\lceil\alpha_1\rceil}}\Big\rceil}{d^t + \ln(\varepsilon^{-1})} \ge \frac{\ln\Big(\varepsilon^{-2}\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}\gamma_d\Big)^{\frac{1}{\lceil\alpha_1\rceil}}}{d^t + \ln(\varepsilon^{-1})}$$
$$= \frac{d}{d^t + \ln(\varepsilon^{-1})} = \frac{d}{d^t + \lceil\alpha_1\rceil d/2 + \ln d - \lceil\alpha_1\rceil(\ln\lceil\alpha_1\rceil)/2}$$
$$= \frac{1}{d^{t-1} + \lceil\alpha_1\rceil/2 + \ln d/d - \lceil\alpha_1\rceil(\ln\lceil\alpha_1\rceil)/(2d)}.$$

For the above fixed ε *and* $t < 1$ *we obtain*

$$\lim_{d\to\infty} \frac{\ln n(\varepsilon, APP_d)}{d + \ln(\varepsilon^{-1})} \ge \lim_{d\to\infty} \frac{1}{d^{t-1} + \lceil\alpha_1\rceil/2 + \ln d/d - \lceil\alpha_1\rceil(\ln\lceil\alpha_1\rceil)/(2d)} = \frac{2}{\lceil\alpha_1\rceil}.$$

This means that APP is not EC-$(t, 1)$*-WT for* $t < 1$.

Remark 6. *Obviously, for Example 2 with* $\gamma_j = 2^{-j}$ *and* $\alpha_j = 2j$ *for all* $j \in \mathbb{N}$, *APP is also EC-WT. Indeed, if APP is EC-*$(t, 1)$*-WT for* $t < 1$, *then it is EC-WT. Assume that APP is EC-*$(t, 1)$*-WT for* $t < 1$, *then we have*

$$\lim_{d+\varepsilon^{-1}\to\infty} \frac{\ln n(\varepsilon, APP_d)}{d^t + \ln(\varepsilon^{-1})} = 0.$$

Since

$$0 \le \lim_{d+\varepsilon^{-1}\to\infty} \frac{\ln n(\varepsilon, APP_d)}{d + \ln(\varepsilon^{-1})} \le \lim_{d+\varepsilon^{-1}\to\infty} \frac{\ln n(\varepsilon, APP_d)}{d^t + \ln(\varepsilon^{-1})},$$

we further get

$$\lim_{d+\varepsilon^{-1}\to\infty} \frac{\ln n(\varepsilon, APP_d)}{d+\ln(\varepsilon^{-1})} = 0,$$

which means that APP is EC-WT.

5. Conclusions

In this paper we discuss the EC-WT and EC-$(t,1)$-WT with $t<1$ for the approximation problem APP in weighted Hilbert spaces $H_{R_{d,\alpha,\gamma}}$ for $R_{d,\alpha,\gamma} \in \{r_{d,\alpha,\gamma}, \psi_{d,\alpha,\gamma}, \omega_{d,\alpha,\gamma}\}$ with parameters $1 \geq \gamma_1 \geq \gamma_2 \geq \cdots \geq 0$ and $1 < \alpha_1 \leq \alpha_2 \leq \cdots$. We obtain the matching necessary and sufficient condition

$$\lim_{j\to\infty} \gamma_j = 0$$

on EC-WT, and the matching necessary and sufficient condition

$$\lim_{j\to\infty} \frac{\ln j}{\ln(\gamma_j^{-1})} = 0$$

on EC-$(t,1)$-WT with $t<1$. The weights are used to model the importance of the functions from the weighted Hilbert spaces, so we will further research the other EC-tractability notions such as EC-SPT, EC-PT, EC-QWT, and EC-UWT.

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