

About the Subgradient Method for Equilibrium Problems

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Abstract: Convergence results of the subgradient algorithm for equilibrium problems were mainly obtained using a Lipschitz continuity assumption on the given bifunctions. In this paper, we first provide a complexity result for monotone equilibrium problems without assuming Lipschitz continuity. Moreover, we give a convergence result of the value of the averaged sequence of iterates beyond Lipschitz continuity. Next, we derive a rate convergence in terms of the distance to the solution set relying on a growth condition. Applications to convex minimization and min–max problems are also stated. These ideas and results deserve to be developed and further refined.

Keywords: subgradient method; equilibrium problem; convex minimization; min–max problem

MSC: 49J53; 65K10

1. Introduction

The subgradient algorithm has recently enjoyed regained popularity, see for example [1–4]. However, the Lipschitz continuity assumption can be stringent even in a convex setting. It is well known that not only SVM (Support Vector Machine) but also Feed-Forward and Recurrent Neural Networks do not satisfy the Lipschitz continuity assumption, see [5]. Indeed, most l_2 -regularized convex learning problems lack this property, while l_2 regularization and weight decay are ubiquitous in the learning field. The proposed equilibrium approach will allow us not only to generalize some very recent results in convex minimization [1] and present them for min–max problems in a unified way but also to yield new insights into equilibrium problems. In order to go to the essential to share, we took the same paper outline as in [1] and we assume the reader has some basic knowledge of variational and convex analysis as can be found, for example, in [6,7].

The subgradient algorithm is a powerful tool for constructing algorithms to approximate solutions of optimization and equilibrium problems, the latter being the problem of finding $\bar{x} \in C$ such that

$$(\mathcal{P}) \quad \text{find } \bar{x} \in C, F(\bar{x}, y) \geq 0 \quad \forall y \in C,$$

where C is a given nonempty closed convex subset of $\mathbf{R}^d$ and $F : C \times C \rightarrow \mathbf{R}$ is a bifunction. The solution set of $(\mathcal{P})$ will be denoted by Γ . Such a problem is also known as the equilibrium problem in the sense of Blum and Oettli [8]. It is worth mentioning that variational inequalities, Nash equilibrium, the saddle-point problem, optimization, and many problems arising in applied nonlinear analysis are special cases of equilibrium problems. In this paper, we will be concerned with the convergence analysis of a subgradient method for solving problem $(\mathcal{P})$ in which F is not assumed to be Lipschitz continuous. For more simplicity and clarity, we suppose that F verifies the usual conditions:

- (A1) $F(x, x) = 0$ for all $x, y \in C$;
- (A2) F is monotone $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
- (A3) $\limsup_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x, y)$ for any $x, y, z \in C$;
- (A4) for each $x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous.



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2. The Main Results

Let us state the subgradient method which generates a sequence $(x_k)_{k \in N}$ by Algorithm 1:

Algorithm 1 (SGM)

Step 0: Choose $x_0 \in C$ and set $k = 0$.

Step 1: Let $x_k \in C$ and compute $\eta_k \in \partial_2 F(x_k, x_k)$, where $\partial_2 F$ stands for the convex subdifferential of F with respect to the second variable.

If $\eta_k = 0$, then stop.

Step 2: Update $x_{k+1} = P_C(x_k - \alpha_k \eta_k)$, where $\alpha_k = \frac{\beta_k}{\|\eta_k\|}$, for all $k \geq 0$ with $\beta_k > 0$.

Now we are in a position to provide a complexity result.

Theorem 1. Suppose F verifies conditions A_1 to A_4 and note by T the total number of iterations. Set $\bar{x}_T = \frac{\sum_{i=0}^T x_i}{T+1}$, $\beta_k = \frac{c}{\sqrt{T+1}}$, and $c > 0$ and let $\bar{x}$ be a solution of $(\mathcal{P})$. Then, $x_k \in \mathcal{B}(\bar{x}, r)$ for $k = 0, 1, \dots, T$ with $r = \sqrt{\|x_0 - \bar{x}\|^2 + c^2}$. Moreover, we have

$$F(\bar{x}, \bar{x}_T) \leq \frac{r^2 L_B}{2r\sqrt{T+1}}, \quad (1)$$

where L_B is the local Lipschitz continuity constant of $F(\bar{x}, \cdot)$ on the smallest open convex set $\mathcal{O}$ such that the closed ball $\mathcal{B}(\bar{x}, r) \subset \mathcal{O}$.

Proof. For all $x \in C$, we have

$$\begin{aligned} \|x_{k+1} - x\|^2 &\leq \|x_k - x - \alpha_k \eta_k\|^2 \\ &\leq \|x_k - x\|^2 - 2\alpha_k \langle \eta_k, x_k - x \rangle + \beta_k^2. \end{aligned}$$

On the other hand, $\eta_k \in \partial_2 F(x_k, x_k)$ implies that $F(x_k, x) \geq \langle \eta_k, x - x_k \rangle$, which combined with the monotonicity of F ensures that $\langle \eta_k, x_k - x \rangle \geq F(x, x_k)$. Consequently,

$$\|x_{k+1} - x\|^2 \leq \|x_k - x\|^2 - 2\frac{\beta_k}{\eta_k} F(x, x_k) + \beta_k^2. \quad (2)$$

Setting $x = \bar{x}$, we have

$$\|x_{k+1} - \bar{x}\|^2 \leq \|x_k - \bar{x}\|^2 + \beta_k^2, \quad (3)$$

because $\beta_k > 0$ together with $F(\bar{x}, x_k) \geq 0$. In other words, the sequence $(x_k)_{k \in N}$ is quasi-Fejér monotone to the solution set Γ .

From the latter inequality, we infer that

$$\|x_{k+1} - \bar{x}\|^2 \leq \|x_0 - \bar{x}\|^2 + \sum_{i=0}^k \beta_i^2 \leq \|x_0 - \bar{x}\|^2 + c^2, \text{ for } 0 \leq k \leq T. \quad (4)$$

Relation (4) assures that $x_k \in \mathcal{B}(\bar{x}, r)$ for $0 \leq k \leq T$.

In light of the consequence of [6]—Proposition 9.13, we have that $F(\bar{x}, \cdot)$ is locally Lipschitz continuous with constant L_B on $\mathcal{O}$ which, for $0 \leq k \leq T$, in turn gives that $\|\eta_k\| \leq L_B$. This combined with (2) leads to

$$\|x_{k+1} - \bar{x}\|^2 \leq \|x_k - \bar{x}\|^2 - 2\frac{\beta_k}{L_B} F(\bar{x}, x_k) + \beta_k^2. \quad (5)$$

From which we derive

$$\sum_{i=0}^T \beta_i F(\bar{x}, x_k) \leq \frac{L_B}{2} (\|x_0 - \bar{x}\|^2 + \sum_{i=0}^T \beta_i^2),$$

or equivalently

$$\sum_{i=0}^T \frac{c}{\sqrt{T+1}} F(\bar{x}, x_k) \leq \frac{L_B}{2} (\|x_0 - \bar{x}\|^2 + c^2).$$

Using the convexity of the function $F(\bar{x}, \cdot)$, we finally obtain

$$F(\bar{x}, \bar{x}_T) = F\left(\bar{x}, \sum_{i=0}^T \frac{1}{T+1} x_i\right) \leq \sum_{i=0}^T \frac{1}{T+1} F(\bar{x}, x_i) \leq \frac{r^2 L_B}{2r\sqrt{T+1}}.$$

This completes the proof. $\square$

A convergence result of the values of the averaged sequence of iterates generated by (SGM) is provided in the next Theorem.

Theorem 2. Suppose that the bifunction F verifies conditions A_1 to A_4 . Set $\bar{x}_k = \frac{\sum_{i=0}^k \beta_i x_i}{\sum_{i=0}^k \beta_i}$, $\sigma_k = \sum_{i=0}^k \beta_i$, and $(\beta_k)_{k \in \mathbb{N}} \in l^2/l^1$ and let $\bar{x}$ be a solution of $(\mathcal{P})$. Then, the sequence $(x_k)_{k \in \mathbb{N}} \subset \mathcal{B}(\bar{x}, \bar{r})$ where $\bar{r} = \sqrt{\|x_0 - \bar{x}\|^2 + \bar{c}}$ and $\bar{c} > 0$ is such that $\sum_{k=0}^{\infty} \beta_k^2 \leq \bar{c}$. Moreover,

$$\forall k \in \mathbb{N}, F(\bar{x}, \bar{x}_k) \leq \frac{r^2 L_B}{2\sigma_k}, \quad (6)$$

where L_B is the local Lipschitz continuity constant of $F(\bar{x}, \cdot)$ on the smallest open convex set $\mathcal{O}$ such that $\mathcal{B}(\bar{x}, r) \subset \mathcal{O}$.

Proof. We again have that

$$\|x_{k+1} - \bar{x}\|^2 \leq \|x_k - \bar{x}\|^2 - 2 \frac{\beta_k}{\eta_k} F(\bar{x}, x_k) + \beta_k^2. \quad (7)$$

Therefore,

$$\|x_{k+1} - \bar{x}\|^2 \leq \|x_0 - \bar{x}\|^2 + \sum_{i=0}^k \beta_i^2 \leq \|x_0 - \bar{x}\|^2 + \bar{c}. \quad (8)$$

This ensures that $(x_k)_{k \in \mathbb{N}} \subset \mathcal{B}(\bar{x}, \bar{r})$. Following the same lines as in the proof of Theorem 1, we obtain

$$\|x_{k+1} - \bar{x}\|^2 \leq \|x_k - \bar{x}\|^2 - 2 \frac{\beta_k}{L_B} F(\bar{x}, x_k) + \beta_k^2, \text{ for } k \in \mathbb{N}. \quad (9)$$

This together with the convexity of the function $F(\bar{x}, \cdot)$ yields to

$$F(\bar{x}, \bar{x}_k) = F\left(\bar{x}, \sum_{i=0}^k \frac{\beta_i}{\sigma_k} x_i\right) \leq \sum_{i=0}^k \frac{\beta_i}{\sigma_k} F(\bar{x}, x_i) \leq \frac{r^2 L_B}{2\sigma_k},$$

which leads to the announced result. $\square$

Remark 1. We can obtain more than the convergence of the averaged sequence of the iterates. Actually, we have

$$\liminf_{k \rightarrow +\infty} F(\bar{x}, x_k) = 0,$$

and the whole sequence $(x_k)_{k \in \mathbb{N}}$ converges to a solution of $(\mathcal{P})$.

Indeed, for all $k \geq 0$, the inequality

$$\|x_{k+1} - \bar{x}\|^2 \leq \|x_k - \bar{x}\|^2 - 2 \frac{\beta_k}{L_B} F(\bar{x}, x_k) + \beta_k^2,$$

leads to

$$\sum_{k=0}^{\infty} \beta_k F(\bar{x}, x_k) < +\infty.$$

Since $\sum_{k=0}^{\infty} \beta_k = +\infty$, this implies that $\liminf_{k \rightarrow +\infty} F(\bar{x}, x_k) = 0$.

Now, with the sequence $(x_k)_{k \in \mathbb{N}}$ being quasi-Fejér convergent to the set Γ , namely verifying (3) with $(\beta_k)_{k \in \mathbb{N}} \in l^2$, this implies its boundedness. Further, in view of (9), which is still valid for all $x \in C$ together with both lower semicontinuity and upper hemi-continuity assumptions, we obtain that any cluster point x^* of $(x_k)_{k \in \mathbb{N}}$ belongs to Γ . Consequently, the whole sequence converges to x^* , see for example [9], and we retrieve the main results in [10].

A convergence result based on a growth property.

Corollary 1. Suppose in addition to hypotheses A_1 to A_4 that F verifies the following growth property:

there exists $\gamma > 0$ such that $F(\bar{x}, x) \geq \gamma \text{dist}^2(x, \Gamma)$ for all $x \in C$ and some $\bar{x} \in \Gamma$.

Consider the sequence $(\beta_k)_{k \in \mathbb{N}}$ given by $\beta_k = \frac{L_B}{\gamma(k+1)}$ for all $k \in \mathbb{N}$. Then, for all $k \in \mathbb{N}$, $(x_k)_{k \in \mathbb{N}} \subset \mathcal{B}(\bar{x}, \bar{r})$, and we have

$$\text{dist}^2(x_{k+1}, \Gamma) \leq \frac{L_B^2}{\gamma^2(k+1)}, \quad (10)$$

where L_B is the local Lipschitz continuity constant of $F(\bar{x}, \cdot)$ on the smallest open convex set $\mathcal{O}$ such that $\mathcal{B}(\bar{x}, r) \subset \mathcal{O}$.

Proof. The beginning of Theorem 2 ensures that $(x_k)_{k \in \mathbb{N}}$ lies in $\mathcal{B}(\bar{x}, \bar{r})$ and assures again that $\|\eta_k\| \leq L_B$. Remember that (7) reads as

$$\|x_{k+1} - \bar{x}\|^2 \leq \|x_k - \bar{x}\|^2 - 2 \frac{\beta_k}{\eta_k} F(\bar{x}, x_k) + \beta_k^2. \quad (11)$$

By taking $\bar{x} = \text{proj}_{\Gamma}(x_k)$, we obtain

$$\text{dist}^2(x_{k+1}, \Gamma) \leq \|x_{k+1} - \bar{x}\|^2 \leq \text{dist}^2(x_k, \Gamma) - 2 \frac{\beta_k}{L_B} F(\bar{x}, x_k) + \beta_k^2.$$

This implies

$$\text{dist}^2(x_{k+1}, \Gamma) \leq \text{dist}^2(x_k, \Gamma) - \frac{2}{k+1} \text{dist}^2(x_k, \Gamma) + \frac{L_B^2}{\gamma^2(k+1)^2},$$

or equivalently

$$(k+1)^2 \text{dist}^2(x_{k+1}, \Gamma) \leq ((k+1)^2 - 2(k+1)) \text{dist}^2(x_k, \Gamma) + \frac{L_B^2}{\gamma^2}.$$

Following the same lines as in the proof of [1]—Corollary 1, for all $k \in \mathbb{N}$, we further have

$$(k+1)^2 \text{dist}^2(x_{k+1}, \Gamma) - k^2 \text{dist}^2(x_k, \Gamma) \leq \frac{L_B^2}{\gamma^2}.$$

Summing the last inequality for $i = 1$ to k leads to

$$(k+1)^2 \text{dist}^2(x_{k+1}, \Gamma) = \sum_{i=0}^k ((i+1)^2 \text{dist}^2(x_{i+1}, \Gamma) - i^2 \text{dist}^2(x_i, \Gamma)) \leq \frac{L_B^2}{\gamma^2} (k+1).$$

From which we derive

$$\text{dist}^2(x_{k+1}, \Gamma) \leq \frac{L_B^2}{\gamma^2(k+1)}.$$

□

3. Applications

In the convex minimization case, (SGM) coincides with the classical subgradient method, and we recover the results obtained in the convex setting in [1]. Indeed, just take $F(x, y) = f(y) - f(x)$, f being a proper convex lower semicontinuous convex function, clearly $\partial_2 F(x_k, x_k) = \partial f(x_k)$, and we retrieve

$$(SGM) \begin{cases} \text{compute } \eta_k \in \partial f(x_k) \\ \text{Update } x_{k+1} = P_C(x_k - \alpha_k \eta_k). \end{cases}$$

Theorem 1 reduces to

$$f(\bar{x}_k) - f(\bar{x}) \leq \frac{r^2 L_B}{2r\sqrt{T+1}}, \quad (12)$$

where $\bar{x}$ is a minimizer of f and L_B is the local Lipschitz continuity constant of f on the smallest open convex set $\mathcal{O}$ such that $\mathcal{B}(\bar{x}, r) \subset \mathcal{O}$. Theorem 2, in turn, leads to the fact that the sequence $(x_k)_{k \in \mathbb{N}} \subset \mathcal{B}(\bar{x}, \bar{r})$, where $\bar{r} = \sqrt{\|x_0 - \bar{x}\|^2 + \bar{c}}$ and $\bar{c} > 0$, is such that $\sum_{k=0}^{\infty} \beta_k^2 \leq \bar{c}$. Moreover, for all $k \in \mathbb{N}$, we obtain the following convergence result in terms of the suboptimality error:

$$f(\bar{x}_k) - f(\bar{x}) \leq \frac{r^2 L_B}{2\sigma_k}, \quad (13)$$

$\bar{x}_k, \sigma_k, \bar{r}, \bar{c}$ were defined in Theorem 2, and $(\beta_k)_{k \in \mathbb{N}} \in l^2/l^1$ and $\bar{x}$ are minimizers of f .

If we assume, in addition, for all $x \in C$ that

$$f(x) - f(\bar{x}) \geq \gamma \text{dist}^2(x, \Gamma), \quad (14)$$

then

$$\text{dist}^2(x_{k+1}, \Gamma) \leq \frac{L_B^2}{\gamma^2(k+1)}. \quad (15)$$

Likewise, by taking $F(v, w) = L(z, p) - L(x, y)$ with $w = (z, y)$, $v = (x, p)$ and L being a proper closed convex–concave function defined on $C \times C$, $C = Q \times P$ with Q, P closed convex sets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, then clearly

$$\partial_w F(v, w) = (\partial_1 L(\cdot, p), -\partial_2 L(x, \cdot)),$$

we recover the subgradient algorithm for the saddle function considered in [11] and more recently in [12] and we extend some results in [1] to the convex–concave case. More precisely, (SGM) reduces to

$$(SGM) \begin{cases} \text{compute } (\mu_k, \gamma_k) \in (\partial_1 L(x_k, y_k), \partial_2(-L)(x_k, y_k)) \\ \text{Update } (x_{k+1}, y_{k+1}) = P_C((x_k, y_k) - \alpha_k(\mu_k, \gamma_k)). \end{cases}$$

Theorem 1 reads, in this case, as

$$L(\bar{x}_T, \bar{y}) - L(\bar{x}, \bar{y}_T) \leq \frac{r^2 L_B}{2r\sqrt{T+1}}, \quad (16)$$

where $r = \sqrt{\|(x_0, y_0) - (\bar{x}, \bar{y})\|^2 + c^2}$ and $(\bar{x}, \bar{y})$ are a saddle point of L , namely $(\bar{x}, \bar{y})$ verifying $L(\bar{x}, y) \leq L(\bar{x}, \bar{y}) \leq L(x, \bar{y})$ for all $(x, y) \in P \times Q$ and L_B being the local Lipschitz continuity constant of $L(\cdot, \bar{x}) - L(\bar{x}, \cdot)$ on the smallest open convex set $\mathcal{O}$ such that $\mathcal{B}(\bar{x}, r) \subset \mathcal{O}$.

Theorem 2, in turn, leads to the sequence $(x_k, y_k)_{k \in \mathbb{N}} \subset \mathcal{B}((\bar{x}, \bar{y}), \bar{r})$, where $\bar{r} = \sqrt{\|(x_0, y_0) - (\bar{x}, \bar{y})\|^2 + \bar{c}}$ and $\bar{c} > 0$ is such that $\sum_{k=0}^{\infty} \beta_k^2 \leq \bar{c}$. Moreover, for all $k \in \mathbb{N}$, we obtain the following convergence result in terms of a merit function:

$$L(\bar{x}_k, \bar{y}) - L(\bar{x}, \bar{y}_k) \leq \frac{r^2 L_B}{2\sigma_k}, \quad (17)$$

$(\bar{x}_k, \bar{y}_k), \sigma_k, \bar{r}, \bar{c}$ were defined in Theorem 2 and $\beta_k \in l^2/l^1$ and $(\bar{x}, \bar{y})$ are saddle points of L . Now, if in addition, we assume for all $(x, y) \in C$ that

$$L(x, \bar{y}) - L(\bar{x}, y) \geq \gamma \text{dist}^2((x, y), \Gamma), \quad (18)$$

then

$$\text{dist}^2((x_{k+1}, y_{k+1}), \Gamma) \leq \frac{L_B^2}{\gamma^2(k+1)}. \quad (19)$$

4. Conclusions

To conclude, the equilibrium approach allowed us not only to generalize some very recent results obtained by X. Li, L. Zhao, D. Zhu, and A. M-Ch.; use them in convex minimization [3]; and present them for min–max problems in a unified way but also to yield new insights into equilibrium problems. More precisely, a complexity result for monotone equilibrium problems was provided without assuming Lipschitz continuity. Moreover, a convergence result of the value of the averaged sequence of iterates beyond Lipschitz continuity was also given. Finally, a rate convergence in terms of the distance to the solution set relying on a growth condition was derived. The proposed results were declined in the convex minimization context as well as in the convex–concave min–max setting.

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