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Multidimensional Hyperbolic Equations with Nonlocal Potentials: Families of Global Solutions

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Abstract: In spaces of arbitrary dimensions, hyperbolic differential–difference equations with potentials undergoing translations along arbitrary spatial directions are investigated. The fundamental novelty of this study is that, unlike previous investigations, no restrictions are imposed on the symbol of the differential–difference operator contained in the equation. For the investigated equation, we succeed to explicitly construct a multiparametric family of classical global solutions.

Keywords: differential–difference operators; multidimensional hyperbolic equations; nonlocal potentials; classical solutions

MSC: 35L10; 35R10



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1. Introduction

Differential–difference equations (equations containing translation operators apart from differential ones) of the hyperbolic type have been studied rather actively during the several last years (see, e.g., [1] and references therein). This interest (as well the unfailing worldwide interest in functional-differential equations and nonlocal problems as a whole, see [2–5] and references therein) is motivated by the following circumstances. First, it is already quite clear that efficient models of mathematical physics cannot be exhausted by *differential* equations anymore: apart from differential operators acting on the desired functions, we have to consider translations, convolutions, other integral operators, contractions and extensions of independent variables, fractional-differential operators, and so on. It is quite important that equations including operators of the above kinds (such equations are said to be *functional-differential* ones) link values of the desired function at different points because each model from the classical theory of *differential* equations is a simplification: using only differential equations, we ignore all *nonlocal* interactions in the modelled physics process. This fundamental enhancement is the meaning of differential–difference equations (and the nonlocal theory in general) from the viewpoint of applications. For instance, partial differential–difference equations arise in the theory of multi-layer plates and envelopes (see [2,6]), nonclassical diffusion theory including biomathematical applications (see [7–9]), and nonlinear optics (see [10–15]). From the purely theoretical viewpoint, the meaning of the nonlocal theory is as follows: since a lot of research methods and approaches profoundly developed for the classical theory become inapplicable in the nonlocal case, researchers have to invent a qualitatively new technique; this is a thrilling challenge and the achievements might be quite useful for other mathematical areas as well.

The present paper is devoted to the equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + au(x + h, t), \quad (1)$$

where a is a real parameter and $h := (h_1, \dots, h_n)$ is a vector parameter.

In [16], a family of global classical solutions is constructed for Equation (1) under the assumption that the following inequality is satisfied everywhere in $\mathbb{R}^n$:

$$|\xi|^2 - a \cos h \cdot \xi > 0. \quad (2)$$

This inequality means the negativity of the real part of the symbol of the differential–difference operator from the right-hand side of Equation (1), and restrictions of this kind, are imposed in almost all investigations of hyperbolic equations with nonlocal potentials (see [1] and references therein). As far as the author is aware, the only result for the case where the specified real part is allowed to change its sign is obtained in [17]: global solutions are obtained for the case where $n = 1$ and assumptions on the relations between the coefficient at the nonlocal term and the length of the translation are still imposed (instead of the sign-constancy of the real part of the symbol, they ensure its monotonicity now).

In the present paper, the dimension is arbitrary and no restrictions on a and h are imposed.

2. Results and Proofs

Assuming that $a \in (-\infty, \infty)$ and $(h_1, \dots, h_n) =: h \in \mathbb{R}^n$, consider Equation (1) in $\mathbb{R}^{n+1}$.

Introduce the following functions of the n -dimensional variable ξ :

$$\rho(\xi) = \left[\left(|\xi|^2 - a \cos h \cdot \xi \right)^2 + a^2 \sin^2 h \cdot \xi \right]^{\frac{1}{4}}, \quad (3)$$

$$\varphi(\xi) = \frac{1}{2} \begin{cases} \arctan \frac{a \sin h \cdot \xi}{|\xi|^2 - a \cos h \cdot \xi} & \text{for } |\xi|^2 - a \cos h \cdot \xi > 0, \\ \arctan \frac{a \sin h \cdot \xi}{a \cos h \cdot \xi - |\xi|^2} & \text{for } |\xi|^2 - a \cos h \cdot \xi < 0, \end{cases} \quad (4)$$

and

$$G_{\left\{ \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \right\}}(\xi) = \begin{cases} \rho(\xi) \left\{ \begin{smallmatrix} \sin \\ \cos \end{smallmatrix} \right\} \varphi(\xi) & \text{for } |\xi|^2 - a \cos h \cdot \xi > 0, \\ \rho(\xi) \left\{ \begin{smallmatrix} \cos \\ \sin \end{smallmatrix} \right\} \varphi(\xi) & \text{for } |\xi|^2 - a \cos h \cdot \xi < 0, \end{cases} \quad (5)$$

The following assertions are valid.

Lemma 1. *If $a \cos h \cdot \xi - |\xi|^2 \neq 0$, then the function*

$$F(x, t; \xi) = e^{tG_1(\xi)} \sin \left[tG_2(\xi) + \varphi(\xi) + x \cdot \xi \right] \quad (6)$$

depending on the n -dimensional parameter ξ , satisfies Equation (1) for each value of the $(n+1)$ -dimensional variable (x, t) .

Proof. Let $|\xi|^2 - a \cos h \cdot \xi > 0$. Then, $\varphi(\xi) = \frac{1}{2} \arctan \frac{a \sin h \cdot \xi}{|\xi|^2 - a \cos h \cdot \xi}$. Substitute function (6) in Equation (1):

$$\begin{aligned} \frac{\partial^2 F}{\partial t^2} &= G_1(\xi) e^{tG_1(\xi)} \sin \left[tG_2(\xi) + \varphi(\xi) + x \cdot \xi \right] + G_2(\xi) e^{tG_1(\xi)} \cos \left[tG_2(\xi) + \varphi(\xi) + x \cdot \xi \right], \\ &+ G_1(\xi) G_2(\xi) e^{tG_1(\xi)} \cos \left[tG_2(\xi) + \varphi(\xi) + x \cdot \xi \right] - G_2^2(\xi) e^{tG_1(\xi)} \sin \left[tG_2(\xi) + \varphi(\xi) + x \cdot \xi \right] \\ &= \left[G_1^2(\xi) - G_2^2(\xi) \right] e^{tG_1(\xi)} \sin \left[tG_2(\xi) + \varphi(\xi) + x \cdot \xi \right] + 2G_1(\xi) G_2(\xi) e^{tG_1(\xi)} \cos \left[tG_2(\xi) + \varphi(\xi) + x \cdot \xi \right], \end{aligned} \quad (7)$$

$$\frac{\partial^2 F}{\partial x_j^2} = -\xi_j^2 e^{tG_1(\xi)} \sin \left[tG_2(\xi) + \varphi(\xi) + x \cdot \xi \right], j = \overline{1, n},$$

and, therefore,

$$\Delta F = -|\xi|^2 e^{tG_1(\xi)} \sin \left[tG_2(\xi) + \varphi(\xi) + x \cdot \xi \right]. \quad (8)$$

Further,

$$2G_1(\xi)G_2(\xi) = \rho^2(\xi) \sin \varphi(\xi) \cos \varphi(\xi) = \rho^2(\xi) \sin 2\varphi(\xi) \quad (9)$$

and

$$G_1^2(\xi) - G_2^2(\xi) = \rho^2(\xi) \sin^2 \varphi(\xi) - \rho^2(\xi) \cos^2 \varphi(\xi) = -\rho^2(\xi) \cos 2\varphi(\xi). \quad (10)$$

Now, we note that $-\frac{\pi}{4} < \varphi(\xi) < \frac{\pi}{4}$ on $\mathbb{R}^1$ by definition. Then, $2\varphi(\xi) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ on $\mathbb{R}^1$ and, therefore, the function $\cos 2\varphi(\xi)$ is positive everywhere. Then,

$$\begin{aligned} \cos 2\varphi(\xi) &= \frac{1}{\sqrt{1 + \tan^2 2\varphi(\xi)}} \\ &= \left[1 + \left(\frac{a \sinh \cdot \xi}{|\xi|^2 - a \cosh \cdot \xi} \right)^2 \right]^{-\frac{1}{2}} = \sqrt{\frac{(|\xi|^2 - a \cosh \cdot \xi)^2}{(|\xi|^2 - a \cosh \cdot \xi)^2 + a^2 \sinh^2 \cdot \xi}}. \end{aligned}$$

By assumption, the numerator of the last fraction is positive. Hence, its denominator is positive a fortiori. Thus, the last relation has a sense. Further, the denominator of the last fraction is equal to $\rho^4(\xi)$. Therefore,

$$\cos 2\varphi(\xi) = \frac{\sqrt{(|\xi|^2 - a \cosh \cdot \xi)^2}}{\rho^2(\xi)} = \frac{|\xi|^2 - a \cosh \cdot \xi}{\rho^2(\xi)}$$

because $|\xi|^2 - a \cosh \cdot \xi$ is positive. Hence,

$$\sin 2\varphi(\xi) = \tan 2\varphi(\xi) \cos 2\varphi(\xi) = \frac{a \sinh \cdot \xi}{|\xi|^2 - a \cosh \cdot \xi} \frac{|\xi|^2 - a \cosh \cdot \xi}{\rho^2(\xi)} = \frac{a \sinh \cdot \xi}{\rho^2(\xi)}.$$

Thus,

$$\begin{aligned} \frac{\partial^2 F}{\partial t^2} - \Delta F &= -\rho^2(\xi) \cos 2\varphi(\xi) e^{tG_1(\xi)} \sin \left[tG_2(\xi) + \varphi(\xi) + x \cdot \xi \right] \\ &\quad + \rho^2(\xi) \sin 2\varphi(\xi) e^{tG_1(\xi)} \cos \left[tG_2(\xi) + \varphi(\xi) + x \cdot \xi \right] + |\xi|^2 e^{tG_1(\xi)} \sin \left[tG_2(\xi) + \varphi(\xi) + x \cdot \xi \right] \\ &= \left(a \cosh \cdot \xi - |\xi|^2 \right) e^{tG_1(\xi)} \sin \left[tG_2(\xi) + \varphi(\xi) + x \cdot \xi \right] \\ &\quad + a \sinh \cdot \xi e^{tG_1(\xi)} \cos \left[tG_2(\xi) + \varphi(\xi) + x \cdot \xi \right] + |\xi|^2 e^{tG_1(\xi)} \sin \left[tG_2(\xi) + \varphi(\xi) + x \cdot \xi \right] \\ &= a e^{tG_1(\xi)} \left(\cosh \cdot \xi \sin \left[tG_2(\xi) + \varphi(\xi) + x \cdot \xi \right] + \sinh \cdot \xi \cos \left[tG_2(\xi) + \varphi(\xi) + x \cdot \xi \right] \right) \\ &= a e^{tG_1(\xi)} \sin \left[tG_2(\xi) + \varphi(\xi) + x \cdot \xi + h \cdot \xi \right] = a e^{tG_1(\xi)} \sin \left[tG_2(\xi) + \varphi(\xi) + (x + h) \cdot \xi \right] = aF(x + h, t), \end{aligned} \quad (11)$$

i.e., function (6) satisfies (in the classical sense) Equation (1) for each ξ such that $|\xi|^2 - a \cosh \cdot \xi$ is positive.

Now, substitute function (6) in Equation (1) under the assumption that $|\xi|^2 - a \cosh \cdot \xi$ is negative.

The steps leading to relations (7)–(9) remain the same as above, but

$$G_1^2(\xi) - G_2^2(\xi) = \rho^2(\xi) \cos^2 \varphi(\xi) - \rho^2(\xi) \sin^2 \varphi(\xi) = \rho^2(\xi) \cos 2\varphi(\xi) \quad (12)$$

now.

Further, taking into account that $\varphi(\xi) = \frac{1}{2} \arctan \frac{a \sinh \cdot \xi}{a \cosh \cdot \xi - |\xi|^2}$ by assumption, we conclude (arguing as above) that $\cos 2\varphi(\xi)$ is positive everywhere and

$$\cos 2\varphi(\xi) = \frac{a \cosh \cdot \xi - |\xi|^2}{\rho^2(\xi)}$$

because $a \cosh \cdot \xi - |\xi|^2$ is positive. Hence,

$$\sin 2\varphi(\xi) = \tan 2\varphi(\xi) \cos 2\varphi(\xi) = \frac{a \sinh \cdot \xi}{a \cosh \cdot \xi - |\xi|^2} \frac{a \cosh \cdot \xi - |\xi|^2}{\rho^2(\xi)} = \frac{a \sinh \cdot \xi}{\rho^2(\xi)}.$$

Thus,

$$\begin{aligned} \frac{\partial^2 F}{\partial t^2} - \Delta F &= \rho^2(\xi) \cos 2\varphi(\xi) e^{tG_1(\xi)} \sin [tG_2(\xi) + \varphi(\xi) + x \cdot \xi] \\ &+ \rho^2(\xi) \sin 2\varphi(\xi) e^{tG_1(\xi)} \cos [tG_2(\xi) + \varphi(\xi) + x \cdot \xi] + |\xi|^2 e^{tG_1(\xi)} \sin [tG_2(\xi) + \varphi(\xi) + x \cdot \xi] \\ &= (a \cosh \cdot \xi - |\xi|^2) e^{tG_1(\xi)} \sin [tG_2(\xi) + \varphi(\xi) + x \cdot \xi] \\ &+ a \sinh \cdot \xi e^{tG_1(\xi)} \cos [tG_2(\xi) + \varphi(\xi) + x \cdot \xi] + |\xi|^2 e^{tG_1(\xi)} \sin [tG_2(\xi) + \varphi(\xi) + x \cdot \xi] \\ &= a e^{tG_1(\xi)} (\cosh \cdot \xi \sin [tG_2(\xi) + \varphi(\xi) + x \cdot \xi] + \sinh \cdot \xi \cos [tG_2(\xi) + \varphi(\xi) + x \cdot \xi]) \\ &= a e^{tG_1(\xi)} \sin [tG_2(\xi) + \varphi(\xi) + x \cdot \xi + h \cdot \xi] = a e^{tG_1(\xi)} \sin [tG_2(\xi) + \varphi(\xi) + (x+h) \cdot \xi] = aF(x+h, t), \end{aligned}$$

i.e., function (6) satisfies (in the classical sense) Equation (1) for each ξ such that $|\xi|^2 - a \cosh \cdot \xi$ is negative. $\square$

Lemma 2. If $a \cosh \cdot \xi - |\xi|^2 \neq 0$, then the function

$$H(x, t; \xi) = e^{-tG_1(\xi)} \sin [tG_2(\xi) - \varphi(\xi) - x \cdot \xi] \quad (13)$$

depending on the n -dimensional parameter ξ satisfies Equation (1) for each value of the $(n+1)$ -dimensional variable (x, t) .

Proof. Regardless of the sign of $a \cosh \cdot \xi - |\xi|^2$, we have (9) and the relations

$$\begin{aligned} \frac{\partial H}{\partial t} &= -G_1(\xi) e^{-tG_1(\xi)} \sin [tG_2(\xi) - \varphi(\xi) - x \cdot \xi] + G_2(\xi) e^{-tG_1(\xi)} \cos [tG_2(\xi) - \varphi(\xi) - x \cdot \xi], \\ \frac{\partial^2 H}{\partial t^2} &= G_1^2(\xi) e^{-tG_1(\xi)} \sin [tG_2(\xi) - \varphi(\xi) - x \cdot \xi] - G_1(\xi) G_2(\xi) e^{-tG_1(\xi)} \cos [tG_2(\xi) - \varphi(\xi) - x \cdot \xi] \\ &\quad - G_1(\xi) G_2(\xi) e^{-tG_1(\xi)} \cos [tG_2(\xi) - \varphi(\xi) - x \cdot \xi] - G_2^2(\xi) e^{-tG_1(\xi)} \sin [tG_2(\xi) - \varphi(\xi) - x \cdot \xi] \\ &= [G_1^2(\xi) - G_2^2(\xi)] e^{-tG_1(\xi)} \sin [tG_2(\xi) - \varphi(\xi) - x \cdot \xi] - 2G_1(\xi) G_2(\xi) e^{-tG_1(\xi)} \cos [tG_2(\xi) - \varphi(\xi) - x \cdot \xi] \end{aligned}$$

and

$$\Delta H = -|\xi|^2 e^{-tG_1(\xi)} \sin [tG_2(\xi) - \varphi(\xi) - x \cdot \xi].$$

If $|\xi|^2 - a \cosh \cdot \xi > 0$, then (10) is valid and $\cos 2\varphi(\xi) = \frac{|\xi|^2 - a \cosh \cdot \xi}{\rho^2(\xi)}$ (see the proof of Lemma 1 above). Then, $\sin 2\varphi(\xi) = \frac{a \sinh \cdot \xi}{\rho^2(\xi)}$ and, therefore,

$$\begin{aligned}
& \frac{\partial^2 H}{\partial t^2} - \Delta H = \left(a \cos h \cdot \xi - |\xi|^2 \right) e^{-t G_1(\xi)} \sin \left[t G_2(\xi) - \varphi(\xi) - x \cdot \xi \right] \\
& - a \sin h \cdot \xi e^{-t G_1(\xi)} \cos \left[t G_2(\xi) - \varphi(\xi) - x \cdot \xi \right] + |\xi|^2 e^{-t G_1(\xi)} \sin \left[t G_2(\xi) - \varphi(\xi) - x \cdot \xi \right] \\
& = a e^{-t G_1(\xi)} \left(\sin \left[t G_2(\xi) - \varphi(\xi) - x \cdot \xi \right] \cos h \cdot \xi - \cos \left[t G_2(\xi) - \varphi(\xi) - x \cdot \xi \right] \sin h \cdot \xi \right) \\
& = a e^{-t G_1(\xi)} \sin \left[t G_2(\xi) - \varphi(\xi) - x \cdot \xi - h \cdot \xi \right] = a e^{-t G_1(\xi)} \sin \left[t G_2(\xi) - \varphi(\xi) - (x + h) \cdot \xi \right] = a H(x + h, t),
\end{aligned}$$

i.e., function (13) satisfies (in the classical sense) Equation (1) provided that $|\xi|^2 - a \cos h \cdot \xi$ is positive.

If $|\xi|^2 - a \cos h \cdot \xi < 0$, then (12) is valid and $\cos 2\varphi(\xi) = \frac{a \cos h \cdot \xi - |\xi|^2}{\rho^2(\xi)}$, while $\sin 2\varphi(\xi)$ is still equal to $\frac{a \sin h \cdot \xi}{\rho^2(\xi)}$ (see the proof of Lemma 1 above). Therefore,

$$\begin{aligned}
& \frac{\partial^2 H}{\partial t^2} - \Delta H = \left(a \cos h \cdot \xi - |\xi|^2 \right) e^{-t G_1(\xi)} \sin \left[t G_2(\xi) - \varphi(\xi) - x \cdot \xi \right] \\
& - a \sin h \cdot \xi e^{-t G_1(\xi)} \cos \left[t G_2(\xi) - \varphi(\xi) - x \cdot \xi \right] + |\xi|^2 e^{-t G_1(\xi)} \sin \left[t G_2(\xi) - \varphi(\xi) - x \cdot \xi \right] \\
& = a e^{-t G_1(\xi)} \left(\sin \left[t G_2(\xi) - \varphi(\xi) - x \cdot \xi \right] \cos h \cdot \xi - \cos \left[t G_2(\xi) - \varphi(\xi) - x \cdot \xi \right] \sin h \cdot \xi \right) = a H(x + h, t),
\end{aligned}$$

i.e., function (13) satisfies (in the classical sense) Equation (1) provided that $|\xi|^2 - a \cos h \cdot \xi$ is negative. $\square$

Combining Lemmas 1 and 2, we obtain the following assertion:

Theorem 1. For each ξ , from $\mathbb{R}^n$ such that $|\xi|^2 - a \cos h \cdot \xi \neq 0$ and each real α and β , the function

$$\alpha F(x, t; \xi) + \beta H(x, t; \xi), \quad (14)$$

where $F(x, t; \xi)$ and $H(x, t; \xi)$ are introduced by relations (6) and (13), respectively, is an infinitely smooth solution of Equation (1) in $\mathbb{R}^{n+1}$.

3. Heuristic Considerations

As we see above, Theorem 1 is proved directly: function (14) is substituted in Equation (1). Such a proof is strict and clear, but it provides no explanation of how to find the solution. In this section, we explain how to use the well-known Gelfand–Shilov operational scheme (see, e.g., [18]) [Sec. 10] in the considered case.

First, we (formally) apply the Fourier transformation with respect to the variable x to Equation (1), which is a *partial differential–difference* equation. This yields the equation

$$\frac{d^2 \hat{u}}{dt^2} + \left[|\xi|^2 - a(\cos h \cdot \xi + i \sin h \cdot \xi) \right] \hat{u} = 0, \quad (15)$$

which is an *ordinary differential* equation depending on the parameter ξ .

Its general solution (up to arbitrary constants depending on the parameter ξ) is equal to

$$\frac{1}{\rho(\xi)} \left(e^{-t G_1(\xi)} e^{i[t G_2(\xi) - \varphi(\xi)]} - e^{t G_1(\xi)} e^{-i[t G_2(\xi) + \varphi(\xi)]} \right),$$

where the functions $\rho(\xi)$ and $\varphi(\xi)$ are defined by relations (3) and, respectively, (4), while the functions $G_{\left\{ \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \right\}}(\xi)$ are defined by relation (5) for each ξ such that $a \cos h \cdot \xi - |\xi|^2 \neq 0$.

Further steps of the classical Gelfand–Shilov scheme are as follows: (formally) apply the inverse Fourier transformation, discard terms with odd integrand functions, and choose the arbitrary constants depending on the parameter ξ such that purely imaginary terms are eliminated. However, one cannot perform the above actions in our case. The reason is that

one cannot ensure the convergence of the improper integral with respect to ξ , arising at the above remaining actions of the procedure. Instead, we truncate the explained scheme before integrating with respect to the dual variable ξ and show that if we treat that variable as a parameter, then the obtained function, which is represented by (14), satisfies Equation (1) for each value of that parameter such that $|\xi|^2 - a \cosh \cdot \xi \neq 0$. To verify this, we directly substitute (14) in Equation (1) in Section 2 above.

4. Discussion

As we note above, the fundamental novelty of the presented results is as follows: we do not insist anymore on the negativity of the real part of the symbol of the differential–difference operator acting with respect to spatial variables (denoted by $\Re R(\xi)$). Families of global solutions are explicitly constructed regardless of the sign of $\Re R(\xi)$. However, the exact form of the solutions depends on that sign. This leads to the necessity of studying the structure of the sets $\{\xi \in \mathbb{R}^n \mid |\xi|^2 > a \cosh \cdot \xi\} =: D_+$ and $\{\xi \in \mathbb{R}^n \mid |\xi|^2 < a \cosh \cdot \xi\} =: D_-$. They deserve a detailed further investigation, but it is worth to note certain properties right now:

1. Regardless of the sign of the coefficient a , the set D_+ cannot be empty.
2. If the coefficient a is positive, then the set D_- cannot be empty.
3. If the spatial variable is scalar, then there is a sufficient condition guaranteeing the following simple structure of those sets (see [17]): if $0 < a < \frac{2}{h^2}$ (here, h is a real constant), then the function $\xi^2 - a \cosh \xi$ has one and only one positive zero ξ_0 , $D_- = (-\xi_0, \xi_0)$, and $D_+ = (-\infty, -\xi_0) \cup (\xi_0, +\infty)$.

The assumption about the negativity of $\Re R(\xi)$, taken off in the present paper, cannot be treated as a technical restriction. To justify this, one has to clarify the classification problem for differential–difference (or, more generally, functional–differential) equations. When we pass from partial *differential* equations to partial *functional–differential* ones, the classical definition of types of equations does not fit anymore. A new definition is required for the theory of functional–differential equations. The notion of elliptic operators is naturally extended for the functional–differential case as follows: we say that L is an elliptic operator if the real part of its symbol is negative everywhere apart from the origin. For the case of differential operators, this ellipticity definition coincides with the classical one. Further, we can extend this new definition to other types of equations (at least, for second-order ones): we say that $\frac{\partial^2}{\partial t^2} - L$ is a hyperbolic operator and $\frac{\partial}{\partial t} - L$ is a parabolic operator if L is an elliptic operator. Within this approach, the following fundamentally new phenomenon arises: adding terms with first-order derivatives, we can change the type of the equation. This never occurs in the classical *differential* case: once we have the d’Alembert operator, then it remains to be hyperbolic after the adding of any (classical) potential. If the said potential is nonlocal (in particular, in the present paper, it undergoes a translation), then the hyperbolicity is not guaranteed to be preserved under such an operation. This is caused by the polynomial form of symbols of differential operators. Symbols of differential–difference operators are not polynomials anymore. This leads to a qualitatively greater diversity; in particular, the sign-constancy of the real part of the symbol of the operator L is not compulsory preserved.

Besides all, this illustrates a very general principle of the nonlocal theory: nonlocal terms of (formally) low orders can change the nature of the problem fundamentally. This is the reason why the term “low-order terms” is not applied in the theory of differential–difference equations: even though the translation operator acts on the desired function itself (i.e., on the zero-order term), the obtained nonlocal term cannot be treated as a subordinated term or as a small perturbation.

From this viewpoint, the novelty of the results provided in the present paper can be explained as follows. Earlier (apart from the prototype two-dimensional case considered in [17]), nonlocal potentials were added to the (differential) d’Alembert operator such that the sum (i.e., a *differential–difference* operator) remains to be hyperbolic. Now, the last

assumption about the hyperbolicity is taken off, but families of classical global solutions are explicitly constructed anyway.

Smoothness Preservation Issues

Since the pioneering works in [19,20], it has been known that, unlike the classical theory, the famous smoothing property of elliptic and parabolic operators takes no place in the nonlocal case: the smoothness of the solution might be broken regardless the smoothness of the right-hand side or (and) boundary value functions, while only the smoothness in subdomains is guaranteed. It turns out that this depends on the domain geometry very much, e.g., infinitely smooth solutions in half-spaces are constructed (see, e.g., [21,22] and references therein).

From this viewpoint, the hyperbolic case is especially interesting because classical (i.e., differential) hyperbolic operators do not possess the smoothing property. Hence, it is reasonable to expect that there are no smooth solutions in the nonlocal case *a fortiori*. However, families of infinitely smooth solutions are obtained in the present paper. There is no contradiction because global solutions (i.e., solutions in the whole space) of homogeneous equations are considered here. This can be illustrated by the following analog from the classical hyperbolic theory. Separating independent variables to solve a boundary value problem in a cylindrical domain, we obtain a series of functions. Each summand of this series is an infinitely smooth global solution of the equation, but it does not satisfy the investigated problem; it satisfies only the equation. The problem is solved by the whole series, and its smoothness is not infinite at all (though each of its summands is infinitely smooth).

Thus, the investigation of the above-mentioned phenomenon for equations studied in the present paper is a quite likely direction for the next step of the research; this can be undertaken once we find classes of well-posed (or, at least, natural) boundary value (nonlocal) problems for those equations. At the moment, this problem is completely open (to the knowledge of the author).

5. Conclusions

In this paper, we continue the investigation of differential–difference hyperbolic equations with nonlocal potentials, taking off restrictions for signs of real (or imaginary) parts of symbols of operators contained in the investigated equations. We pass from the prototype case of the unique independent spatial variable, investigated in [17], to the case of an arbitrary spatial dimension.

For Equation (1), we explicitly construct the following three-parameter family of smooth global solutions (both for positive and negative values of the real part of the symbol of the differential–difference operator from the right-hand side of the equation):

$$u(x, t) = \alpha F(x, t; \xi) + \beta H(x, t; \xi),$$

where

$$\begin{Bmatrix} F \\ H \end{Bmatrix}(x, t; \xi) = e^{\pm t G_1(\xi)} \sin \left[t G_2(\xi) \pm \varphi(\xi) \pm x \cdot \xi \right],$$

$$G_{\begin{Bmatrix} 1 \\ 2 \end{Bmatrix}}(\xi) = \begin{cases} \rho(\xi) \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} \varphi(\xi) & \text{for } |\xi|^2 > a \cosh \cdot \xi, \\ \rho(\xi) \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} \varphi(\xi) & \text{for } |\xi|^2 < a \cosh \cdot \xi, \end{cases}$$

$$\varphi(\xi) = \frac{1}{2} \arctan \frac{a \sinh \cdot \xi}{||\xi|^2 - a \cosh \cdot \xi|},$$

$$\rho(\xi) = \left[\left(|\xi|^2 - a \cosh \cdot \xi \right)^2 + a^2 \sinh^2 \cdot \xi \right]^{\frac{1}{4}},$$

$\alpha \in \mathbb{R}^1$, $\beta \in \mathbb{R}^1$, and $\xi \in \mathbb{R}^n$ and is such that $|\xi|^2 \neq a \cosh \cdot \xi$.

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