


On Finsler Surfaces with Isotropic Main Scalar

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Abstract: Let (M, F) be a Finsler surface with the isotropic main scalar $\mathcal{I} = \mathcal{I}(x)$. The well-known Berwald's theorem states that F is a Berwald metric if and only if it has a constant main scalar $\mathcal{I} = \text{constant}$. This ensures a kind of equality of two non-Riemannian quantities for Finsler surfaces. In this paper, we consider a positively curved Finsler surface and show that $\mathbf{H} = 0$ if and only if $\mathcal{I} = 0$. This provides an extension of Berwald's theorem. It follows that F has an isotropic scalar flag curvature if and only if it is Riemannian. Our results yield an infrastructural development of some equalities for two-dimensional Finsler manifolds.

Keywords: Finsler surface; main scalar; flag curvature; H-curvature

MSC: 53B40; 53C60

1. Introduction

By an investigation of the class of two-dimensional Finsler spaces, one can find that these spaces have special geometric properties that separate them from the spaces of higher dimensions [1–5]. For example, all two-dimensional Finsler metrics are C-reducible and do not need to be Randers or Kropina metrics. This fact means that the well-known conclusive theorem of Matsumoto-Hōjō does not work properly for two-dimensional Finsler spaces. Also, all two-dimensional Finsler spaces are relatively Landsberg spaces, implying that the relative rate of change in the Landsberg curvature of Finsler surfaces along geodesics is a scalar function on the tangent bundle. As another interesting subject, the study of the flag curvature of two-dimensional Finsler spaces, shows that these spaces inherently possess scalar flag curvature. Namely, the flag curvature is independent of the flag. It was Ludwig Berwald who systemically studied two-dimensional spaces and made very important pioneering works in this class of Finsler spaces [6].

Finsler surfaces have many geometric properties that are different from higher-dimensional Finsler spaces. In particular, for two-dimensional Finsler metrics, there exists a notion of the main scalar, denoted by $\mathcal{I} = \mathcal{I}(x, y)$, which is characterized by the following properties. The main scalar $\mathcal{I} = \mathcal{I}(x, y)$ of the two-dimensional Finsler manifold (M, F) vanishes identically if and only if the metric reduces to a Riemannian metric. Among the class of two-dimensional Finsler metrics, those with an isotropic main scalar $\mathcal{I} = \mathcal{I}(x)$ have interesting curvature properties. In [6], Berwald proved that a two-dimensional Finsler metric with an isotropic main scalar $\mathcal{I} = \mathcal{I}(x)$ is a Berwald metric or Landsberg metric if and only if $\mathcal{I}(x) = \text{constant}$. In [7], Matsumoto showed that a two-dimensional one-form metric is a Landsberg metric if and only if it is a Berwald metric. Berwald metrics are divided into two classes: either a T -Minkowskian metric or $\mathcal{I}(x) = \text{constant}$. Recently, Yang-Cheng proved that a Finsler surface with an isotropic main scalar is locally conformally flat if and only if $\mathcal{I}(x) = \text{constant}$ [8].

In [6], Berwald studied Finsler surfaces with the isotropic main scalar $\mathcal{I} = \mathcal{I}(x)$ and showed that all such Finsler metrics can be expressed as one of the following cases:



Citation: Tayebi, A.; Koh, W.S. On Finsler Surfaces with Isotropic Main Scalar. *Mathematics* **2024**, *12*, 2141. <https://doi.org/10.3390/math12132141>

Academic Editor: Mancho Manev

Received: 12 March 2024

Revised: 27 March 2024

Accepted: 30 March 2024

Published: 8 July 2024



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$$F = \theta^s \gamma^{1-s}; \quad (s = s(x) \neq 0, s(x) \neq 1); \tag{1}$$

$$F = \sqrt{(\theta^2 + \gamma^2)e^{2r \arctan(\frac{\theta}{\gamma})}}, \quad r = r(x), \tag{2}$$

$$F = \theta e^{\frac{\gamma}{\theta}}, \tag{3}$$

where $\theta = \theta_i(x)y^i$ and $\gamma = \gamma_i(x)y^i$ are two independent one-forms. Their main scalars are given, respectively, by

$$\mathcal{I}(x) = \pm \sqrt{\frac{(2s(x) - 1)^2}{\epsilon s(x)(s(x) - 1)}}, \tag{4}$$

$$\mathcal{I}(x) = \pm \sqrt{\frac{4r(x)^2}{1 + r(x)^2}}, \tag{5}$$

$$\mathcal{I} = \pm 2, \tag{6}$$

where ϵ in (4) is the index of F satisfying $\epsilon = 1$ if $s(x) > 1$ or $s(x) < 0$, and $\epsilon = -1$ if $0 < s(x) < 1$. If the main scalar satisfies $\mathcal{I} = 0$, which is equivalent to $s = 1/2$ in (1) or $r = 0$ in (2), then F reduces to a Riemannian metric. Based on the classification of Finsler surfaces with an isotropic main scalar, as above, Berwald classified projectively flat Finsler surfaces with an isotropic main scalar [6]. Moreover, Berwald proved the following rigidity result.

Theorem 1 ([6] Berwald Theorem). *Let (M, F) be a Finsler surface. Suppose that F has an isotropic main scalar. Then, F is a Berwald metric if and only if it has a constant main scalar.*

The E -curvature is obtained by taking a trace of the B -curvature. Also, the H -curvature is obtained by taking a horizontal derivation of the E -curvature along Finslerian geodesics.

Theorem 2. *Let (M, F) be a positively curved Finsler surface. Suppose that F has an isotropic main scalar. Then, $\mathbf{H} = 0$ if and only if F is Riemannian. Equivalently, $\mathbf{H} = 0$ if and only if F has a vanishing main scalar.*

Here, we remark upon two important subjects. First, every Berwald metric satisfies $\mathbf{H} = 0$. Then, Theorem 2 implies that every positively curved Berwald surface with an isotropic main scalar is Riemannian and then $\mathcal{I} = 0$. Obviously, Theorem 2 is an extension of the Berwald Theorem for positively curved surfaces. Second, the positive-definiteness of F in Theorem 2 is necessary. For an example, see the following.

Example 1. *Consider the manifold $M := \mathbb{R}^2$ with the local coordinates $(x^1, x^2) \in \mathbb{R}^2$. Then, the coordinate basis $(\partial/\partial x^i)$ induces the global coordinates $(y^1, y^2) \in T\mathbb{R}^2$. Let us put $a := x^1$, $b := x^2$, $p := y^1$, $q := y^2$, and $r := p/q$. The Berwald–Rund metric $F(x, y) := q(\epsilon + r)^2$ is a solution of the PDE: $\epsilon \partial \epsilon / \partial a = \partial \epsilon / \partial b$, where $\epsilon = \epsilon(a, b)$ is a given function [9]. One can see that if $q \leq 0$, then $F \leq 0$. Also, for $q > 0$, $F = 0$ along the ray $r = -\epsilon$. Thus, F is a y -local Finsler metric. This shows that F is only positive and strongly convex on the upper half (and with a ray excluded) of each tangent plane $T_x \mathbb{R}^2$. A simple calculation shows that F is a non-Riemannian Berwald metric. This implies that $\mathbf{H} = 0$ while $\mathcal{I} = 3/\sqrt{2}$.*

F has an isotropic mean Berwald curvature if its mean Berwald curvature satisfies

$$\mathbf{E} = \frac{n + 1}{2F} \kappa \mathbf{h},$$

where $\kappa = \kappa(x)$ is a scalar function on M and $\mathbf{h} := h_{ij} dx^i dx^j$ denotes the angular metric. For $y \in T_x M$, define the Landsberg curvature by $\mathbf{L}_y(u, v, w) := -1/2 \mathbf{g}_y(\mathbf{B}_y(u, v, w), y)$.

The quantity L is called the Landsberg curvature. The Finsler metric F is called a relatively isotropic Landsberg metric if it satisfies

$$L = vFC,$$

where $v = v(x)$ is a scalar function on M and C denotes the mean Cartan torsion of F . Also, the Finsler metric is called an isotropic Berwald metric if its Berwald curvature can be written as follows

$$B_y(u, v, w) = \frac{v}{F} \left\{ \mathbf{h}(u, v)(w - \mathbf{g}_y(w, \ell)\ell) + \mathbf{h}(v, w)(u - \mathbf{g}_y(u, \ell)\ell) + \mathbf{h}(w, u)(v - \mathbf{g}_y(v, \ell)\ell) + 2FC_y(u, v, w)\ell \right\},$$

where $v = v(x)$ is a scalar function on M . There is a generalization of isotropic Berwald metrics, namely, Douglas metrics.

Theorem 3. *Let (M, F) be Finsler surface with an isotropic main scalar. Then, the following holds:*

- (i) F has a relatively isotropic Landsberg curvature and an isotropic mean Berwald curvature;
- (ii) F is a Douglas metric if and only if it is an isotropic Berwald metric. In this case, F has an isotropic S -curvature.

A Finsler manifold (M, F) is called homogeneous if its group of isometries acts transitively on the manifold. For a homogeneous Douglas surface, we obtain the following.

Corollary 1. *Every homogeneous Douglas surface with an isotropic main scalar is Riemannian or locally Minkowskian.*

Finally, we consider a regular Finsler surface with a constant main scalar and prove the following.

Corollary 2. *Every Finsler surface with a constant main scalar is Riemannian or locally Minkowskian.*

2. Preliminaries

Let (M, F) be a Finsler metric and the following be the fundamental tensor of F :

$$g_y(u, v) := \frac{1}{2} \frac{\partial^2}{\partial s \partial t} \left[F^2(y + su + tv) \right]_{s=t=0}, \quad u, v \in T_x M.$$

The Cartan torsion of F is defined by

$$C_{ijk} := \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k}.$$

The tensor C_{ijk} is called the Cartan torsion.

For $y \in T_x M_0$, define $I_y : T_x M \rightarrow \mathbb{R}$ by

$$I_y(u) = \sum_{i=1}^n g^{ij}(y) C_y(u, \partial_i, \partial_j),$$

where $\{\partial_i\}$ is a basis for $T_x M$ at $x \in M$. The family $I := \{I_y\}_{y \in TM_0}$ is called the mean Cartan torsion. By Diecke’s theorem, a positive-definite Finsler metric F is Riemannian if and only if $I_y = 0$.

For a Finsler manifold (M, F) , its induced spray on TM is denoted by $\mathbf{G} = \mathbf{G}(x, y)$ which, in a standard coordinate (x^i, y^i) for TM_0 , is given by $\mathbf{G} = y^i \partial / \partial x^i - 2G^i(x, y) \partial / \partial y^i$, where

$$G^i := \frac{1}{4} g^{il} \left[\frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k - \frac{\partial F^2}{\partial x^l} \right],$$

where G^i represents the geodesic coefficients of F in the same local coordinate system.

The Busemann–Hausdorff volume form $dV_f := \sigma_F(x) dx^1 \cdots dx^n$ on an n -dimensional Finsler manifold (M, F) can be written as follows:

$$\sigma_F(x) := \frac{\text{Vol}(\mathbb{B}^n(1))}{\text{Vol}\left\{ (y^i) \in \mathbb{R}^n \mid F\left(y^i \frac{\partial}{\partial x^i} \Big|_x\right) < 1 \right\}}.$$

The S-curvature of F is given by

$$\mathbf{S}(x, y) := \frac{\partial G^i}{\partial y^i} - y^i \frac{\partial}{\partial x^i} \left[\ln \sigma_F \right]. \tag{7}$$

The distortion of F is given by

$$\tau(x, y) = \ln \frac{\sqrt{\det(g_{ij}(x, y))}}{\sigma(x)}.$$

By definition, the distortion τ is homogeneous of a degree of one with respect to y .

For a vector $y \in T_x M_0$, the Berwald curvature $\mathbf{B}_y : T_x M \times T_x M \times T_x M \rightarrow T_x M$ is defined by $\mathbf{B}_y(u, v, w) := B^i_{jkl}(y) u^j v^k w^l \partial / \partial x^i \Big|_x$, where

$$B^i_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.$$

F is called a Berwald metric if $\mathbf{B} = \mathbf{0}$. Every Berwald metric satisfies $\mathbf{S} = 0$ (see [10]).

For $y \in T_x M$, define the Landsberg curvature $\mathbf{L}_y : T_x M \times T_x M \times T_x M \rightarrow \mathbb{R}$ by

$$\mathbf{L}_y(u, v, w) := -\frac{1}{2} \mathbf{g}_y(\mathbf{B}_y(u, v, w), y).$$

A Finsler metric F is called a Landsberg metric if $\mathbf{L} = 0$.

The E -curvature of F is defined by

$$\mathbf{E}_y(u, v) := \frac{1}{2} \sum_{i=1}^n g^{ij}(y) \mathbf{g}_y(\mathbf{B}_y(u, v, \partial_i), \partial_j), \tag{8}$$

where $\{\partial_i\}$ is a basis for $T_x M$ at $x \in M$. In the local coordinates, $\mathbf{E}_y(u, v) := E_{ij}(y) u^i v^j$, where

$$E_{ij} := \frac{1}{2} B^m_{mij}.$$

Using a horizontal derivation of the mean of the Berwald curvature \mathbf{E} along Finslerian geodesics give us the H -curvature $\mathbf{H} = \mathbf{H}(x, y)$, which is defined by $\mathbf{H}_y = H_{ij} dx^i \otimes dx^j$, where

$$H_{ij} := E_{ij|_m} y^m.$$

The notation “|” denotes the horizontal derivation with respect to the Berwald connection.

The Douglas curvature is defined by

$$D^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l} - \frac{2}{n+1} \frac{\partial^3}{\partial y^j \partial y^k \partial y^l} \left[\frac{\partial G^m}{\partial y^m} y^i \right]. \tag{9}$$

If $\mathbf{D} = 0$, then F is a Douglas metric.

For a non-zero vector $y \in T_x M_0$, the Riemann curvature is a family of linear transformations $\mathbf{R}_y : T_x M \rightarrow T_x M$, which is defined by $\mathbf{R}_y(u) := R_k^i(y) u^k \partial / \partial x^i$, where

$$R_k^i(y) = 2 \frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}. \tag{10}$$

The family $\mathbf{R} := \{\mathbf{R}_y\}_{y \in TM_0}$ is called the Riemann curvature.

The flag curvature is denoted by $\mathbf{K} = \mathbf{K}(y, P)$ with the flag $P := \text{span}\{y, u\}$ and given by

$$\mathbf{K}(x, y, P) := \frac{\mathbf{g}_y(u, \mathbf{R}_y(u))}{\mathbf{g}_y(y, y)\mathbf{g}_y(u, u) - \mathbf{g}_y(y, u)^2}. \tag{11}$$

F is of the scalar flag curvature $\mathbf{K} = \mathbf{K}(x, y)$ if its Riemannian curvature satisfies the following

$$R_j^i = \mathbf{K} F^2 h_j^i.$$

Also, F is of isotropic and constant flag curvature if $\mathbf{K} = \mathbf{K}(x)$ and $\mathbf{K} = \text{constant}$, respectively.

Theorem 4 ([11] Akbar-Zadeh Theorem). *Let F be a Finsler metric of the scalar flag curvature $\mathbf{K} = \mathbf{K}(x, y)$ on a manifold M . Then, F has the isotropic flag curvature $\mathbf{K} = \mathbf{K}(x)$ if and only if $\mathbf{H} = 0$.*

In [12], Najafi, Shen, and Tayebi generalized the Akbar-Zadeh Theorem and proved the following.

Theorem 5 ([12] Najafi–Shen–Tayebi Theorem). *Let F be a Finsler metric of a scalar flag curvature on an n -dimensional manifold M . Let θ be an arbitrary one-form on M . Then*

$$\mathbf{H} = \frac{(n^2 - 1)\theta}{2F} \tag{12}$$

if and only if F is of a weakly isotropic flag curvature

$$\mathbf{K} = \frac{3\theta}{F} + \sigma, \tag{13}$$

where $\sigma = \sigma(x)$ is a scalar function on M .

Throughout this paper, we use the Berwald connection on Finsler manifolds. Let $\{e_j\}$ denote the local frame for $\pi^* TM$. Suppose that $\{\omega^i, \omega^{n+i}\}$ denotes the corresponding local co-frame for $T^*(TM_0)$. Let the set $\{\omega_j^i\}$ denote the local Berwald connection forms with respect to $\{e_j\}$. In this case, the connection forms of the Berwald connection satisfy the following structure equations:

- Torsion freeness

$$d\omega^i = \omega^j \wedge \omega_j^i; \tag{14}$$

- Almost metric compatibility

$$dg_{ij} - g_{kj}\omega^k_i - g_{ik}\omega^k_j = -2L_{ijk}\omega^k + 2C_{ijk}\omega^{n+k}, \tag{15}$$

where

$$\omega^i := dx^i, \quad \omega^{n+k} := dy^k + y^j\omega^k_j.$$

Then, the following holds

$$g_{ij|k} = -2L_{ijk}, \quad g_{ij,k} = 2C_{ijk}.$$

3. Proof of Theorem 2

For any Minkowskian plane (V, \mathcal{F}) and any vector $v \in V$ with $\mathcal{F}(v) \neq 0$, there is a non-zero vector $w \in V$ such that it is orthogonal to v with respect to the fundamental tensor raised by the Minkowski functional \mathcal{F} . The special and useful Berwald frame was founded and developed by L. Berwald to study Finsler surfaces [6]. It works under the assumption that the fundamental tensor is positive definite. Let (M, F) be a two-dimensional Finsler manifold. It is easy to see that for every $\mathbf{y} \in T_xM$, $x \in M$, there is a vector $\mathbf{y}^\perp \in T_xM_0$ such that

$$\mathbf{g}_y(\mathbf{y}, \mathbf{y}^\perp) = 0, \quad \mathbf{g}_y(\mathbf{y}^\perp, \mathbf{y}^\perp) = F(\mathbf{y}).$$

The pair $\{\mathbf{y}, \mathbf{y}^\perp\}$ is called the Berwald frame at \mathbf{y} .

Based on the Berwald frame, the Cartan torsion can be determined by a scalar function on the slit tangent bundle. Let us define

$$\mathcal{I}(\mathbf{y}) := \frac{\mathbf{C}_y(\mathbf{y}^\perp, \mathbf{y}^\perp, \mathbf{y}^\perp)}{F(\mathbf{y})}.$$

One can see that $\mathcal{I}(\lambda\mathbf{y}) = \mathcal{I}(\mathbf{y})$ holds for $\forall \lambda > 0$ and $\forall \mathbf{y} \in T_xM_0$. We call \mathcal{I} the main scalar of the Finsler metric F .

In most of the literature regarding Finsler geometry, the special notion (ℓ, m) was used instead of $\{\mathbf{y}, \mathbf{y}^\perp\}$. By considering this notion, for a scalar $T = T(x, y)$, we define the horizontal scalar derivatives $(T_{|1}, T_{|2})$ and vertical scalar derivatives $(T_{,1}, T_{,2})$ as follows

$$T_{|i} := T_{|1}\ell_i + T_{|2}m_i, \quad FT_{,i} := T_{,1}\ell_i + T_{,2}m_i, \tag{16}$$

where

$$T_{|i} := \frac{\partial T}{\partial x^i} - G^j_i \frac{\partial T}{\partial y^j}, \quad FT_{,i} := F \frac{\partial T}{\partial y^i}$$

denote the horizontal and vertical derivations with respect to the Berwald connection of F .

To prove Theorem 2, we need to the explicit construction of the flag curvature of a two-dimensional Finsler manifold. Then, we prove the following.

Lemma 1. *The flag curvature of every two-dimensional Finsler manifold (M, F) satisfies the following*

$$\mathbf{K}_{y^i} + T\tau_{y^i} = 0, \tag{17}$$

where

$$\mu := -\frac{2}{\mathcal{I}}\mathcal{I}_{|1}, \quad T(x, y) := \frac{1}{4}\left(4\mathbf{K} + \mu^2 - \frac{2}{F}\mu_{|k}y^k\right).$$

Proof. The curvature form of the Berwald connection is given by

$$\Omega^i_j = d\omega^i_j - \omega^k_j \wedge \omega^i_k = \frac{1}{2}R^i_{jkl}\omega^k \wedge \omega^l - B^i_{jkl}\omega^k \wedge \omega^{n+l}. \tag{18}$$

Differentiating (15) implies that

$$g_{pj}\Omega_i^p + g_{ip}\Omega_j^p = -2L_{ijk|s}\omega^k \wedge \omega^s - 2L_{ijk,s}\omega^k \wedge \omega^{n+s} - 2C_{ijs|k}\omega^k \wedge \omega^{n+s} - 2C_{ijs}\Omega^s - 2C_{ijs,k}\omega^{n+k} \wedge \omega^{n+s}. \tag{19}$$

Putting (18) into (19) gives us the following

$$g_{pj}B_{ikl}^p + g_{ip}B_{jkl}^p = 2C_{ijl|k} + 2L_{ijk,l}. \tag{20}$$

By (20), we have

$$g_{pk}B_{ijl}^p + g_{jp}B_{ikl}^p = 2C_{jkl|i} + 2L_{ijk,l}, \tag{21}$$

$$g_{pi}B_{jkl}^p + g_{kp}B_{ijl}^p = 2C_{ikl|j} + 2L_{ijk,l}. \tag{22}$$

From (20)+(21)–(22), we obtain

$$B_{jkl}^p = g^{ip} \{ C_{ijl|k} + C_{ikl|j} - C_{jkl|i} + L_{ijk,l} \}. \tag{23}$$

On the other hand, the Cartan torsion of a Finsler manifold (M, F) has no components in the direction ℓ^i , i.e., $C_{ijk}y^i = 0$. Then, it can be written in the Berwald frame (ℓ, m) as follows

$$FC_{ijk} = \mathcal{I}m_i m_j m_k. \tag{24}$$

Using a horizontal derivation of (24) implies that

$$FC_{ijk|s} = (\mathcal{I}_1 \ell_s + \mathcal{I}_2 m_s) m_i m_j m_k. \tag{25}$$

Contracting (25) with y^s yields

$$FL_{ijk} = \mathcal{I}_1 m_i m_j m_k. \tag{26}$$

By putting (25) and (26) in (23), we obtain

$$FB_{jkl}^i = \left\{ -2\mathcal{I}_1 \ell^i + (\mathcal{I}_{1,2} + \mathcal{I}_2) m^i \right\} m_j m_k m_l. \tag{27}$$

Let us put

$$\mathcal{I}_2 := \mathcal{I}_{1,2} + \mathcal{I}_2. \tag{28}$$

Thus, the Berwald curvature of Finsler surfaces is given by

$$B_{jkl}^i = \frac{1}{F} \left(\mathcal{I}_2 m^i - 2\mathcal{I}_1 \ell^i \right) m_j m_k m_l. \tag{29}$$

By (24) and (29), we have

$$B_{jkl}^i = -\frac{2\mathcal{I}_1}{F\mathcal{I}} C_{jkl} y^i + \frac{\mathcal{I}_2}{3F} \left\{ h_{jk} h_l^i + h_{kl} h_j^i + h_{lj} h_k^i \right\}, \tag{30}$$

where $\mathbf{h} = h_{ij} dx^i dx^j$ denotes the angular metric and is defined by the following

$$h_{ij} = g_{ij} - F^{-2} y_i y_j.$$

Then, for a Finsler surface, the Berwald curvature can be written as follows

$$B_{jkl}^i = -\frac{1}{F} \mu C_{jkl} y^i + \lambda \left\{ h_{jk} h_l^i + h_{kl} h_j^i + h_{lj} h_k^i \right\}, \tag{31}$$

where

$$\mu := -\frac{2}{\mathcal{I}}\mathcal{I}_1, \quad \lambda := \frac{1}{3}\mathcal{I}_2. \tag{32}$$

Taking a trace of (31) yields

$$E_{ij} = \frac{3}{2}\lambda F^{-1}h_{ij}. \tag{33}$$

Contracting (31) with y_i implies that

$$L_{ijk} + \frac{1}{2}\mu FC_{ijk} = 0. \tag{34}$$

For the Finsler metrics of the scalar flag curvature, the following holds

$$L_{ijk|m}y^m = -\frac{1}{3}F^2 \left\{ \mathbf{K}_{\cdot i}h_{jk} + \mathbf{K}_{\cdot j}h_{ik} + \mathbf{K}_{\cdot k}h_{ij} + 3\mathbf{K}C_{ijk} \right\} \tag{35}$$

and

$$J_{k|m}y^m = -\frac{1}{3}F^2 \left\{ (n+1)\mathbf{K}_{\cdot k} + 3\mathbf{K}I_k \right\}. \tag{36}$$

Taking a trace of (34) implies that

$$J_k = -\frac{1}{2}\mu FI_k. \tag{37}$$

Taking a horizontal derivation of (37) along Finslerian geodesics yields

$$J_{k|m}y^m = -\frac{1}{2}F(\mu_0 I_k + \mu J_k) = -\frac{1}{4}F(2\mu_0 - \mu^2 F)I_k. \tag{38}$$

By (36), (38), and $I_k = \tau_k$, we obtain (17). \square

Now, we are ready to present the key lemma that enables us to prove Theorem 2. More precisely, we prove the following.

Lemma 2. *Let (M, F) be a positively curved Finsler surface. Suppose that F has an isotropic main scalar. Then F has an isotropic Gaussian curvature if and only if it is Riemannian.*

Proof. Let $\mathcal{I} = \mathcal{I}(x)$. Then, by (28) and (32), we obtain

$$\lambda = \frac{1}{3}\mathcal{I}_2 = \frac{1}{3}\mathcal{I}|_2. \tag{39}$$

Equation (39) shows that if $\mathcal{I} = \mathcal{I}(x)$, then $\lambda = \lambda(x)$. Then, (33) and (39) imply that F has an isotropic mean Berwald curvature

$$E_{ij} = \frac{3}{2}\rho F^{-1}h_{ij}. \tag{40}$$

where $\rho := 1/3\mathcal{I}|_2$. Taking a horizontal derivation of (40) along Finslerian geodesics gives us

$$H_{ij} = \frac{3}{2F}\rho_{x^m}(x)y^m h_{ij}. \tag{41}$$

Every Finsler surface has a scalar flag curvature $\mathbf{K} = \mathbf{K}(x, y)$. According to Theorem 5, we find that F is of a weakly isotropic flag curvature

$$\mathbf{K} = \frac{3\rho_{x^m}(x)y^m}{F} + \sigma, \tag{42}$$

where $\sigma = \sigma(x)$ is a scalar function on M . On the other hand, by (17), we have

$$\mathbf{K}_{y^i} = \left(\frac{1}{2F} \mu_{x^k}(x) y^k - \mathbf{K} - \frac{1}{4} \mu^2(x) \right) \tau_{y^i}. \tag{43}$$

Putting (42) in (43) gives us

$$\mathbf{K}_{y^i} = \left(\frac{1}{2F} \mu_{x^k}(x) y^k - \frac{3}{F} \rho_{x^k}(x) y^k - \sigma - \frac{1}{4} \mu^2(x) \right) \tau_{y^i}. \tag{44}$$

If F is Riemannian, then $\tau_{y^i} = I_i = 0$, and (44) implies that $\mathbf{K}_{y^i} = 0$. It follows that $\mathbf{K} = \mathbf{K}(x)$.

Conversely, suppose that $\mathbf{K} = \mathbf{K}(x)$. We prove that F is Riemannian. First, we show that $\mathbf{K} = \sigma$. On the contrary, suppose that $\mathbf{K} \neq \sigma$. Then, by (42), we obtain

$$F = \frac{3}{\mathbf{K} - \sigma} \rho_{x^m}(x) y^m \tag{45}$$

which contradicts the positive-definiteness of the Finsler metric F . Then, $\rho_{x^m}(x) y^m = 0$, and we obtain

$$\mathbf{K} = \sigma. \tag{46}$$

Therefore, (44) reduces to the following

$$\left(\frac{1}{2F} \mu_{x^k}(x) y^k - \sigma - \frac{1}{4} \mu^2(x) \right) \tau_{y^i} = 0. \tag{47}$$

We claim that F is a Riemannian metric. On the contrary, suppose that F is not Riemannian. According to (47), we have two main cases as follows:

Case (i): $\mu^2(x) \neq -4\sigma$. Since F is not Riemannian, there is an open subset \mathcal{U} such that $\tau_{y^i}(x, y) \neq 0$ for any $x \in \mathcal{U}$. Then, (47) implies that

$$\mu_{x^k}(x) y^k - 2 \left(\sigma(x) + \frac{1}{4} \mu^2(x) \right) F = 0, \tag{48}$$

which is equal to

$$F = - \frac{2}{4\sigma(x) + \mu^2(x)} \mu_{x^m}(x) y^m. \tag{49}$$

Equation (49) contradicts the positive-definiteness of F . Then, this case does not hold.

Case (ii): Suppose that $\mu^2(x) = -4\sigma$. By assuming that $\mathbf{K} > 0$ and by (46), we have $\sigma > 0$. This is a contradiction. Thus, this case does not hold either. \square

Proof of Theorem 2. By Theorem 4, a Finsler metric of the scalar flag curvature $\mathbf{K} = \mathbf{K}(x, y)$ has an isotropic flag curvature $\mathbf{K} = \mathbf{K}(x)$ if and only if $\mathbf{H} = \mathbf{0}$. By assumption, we obtain $\mathbf{K} = \mathbf{K}(x)$. By Lemma 2, F reduces to a Riemannian metric. If F is Riemannian, then by (24), we obtain $\mathcal{I} = 0$. This completes the proof. \square

Proof of Theorem 3. Let F be a two-dimensional Finsler metric with an isotropic main scalar $\mathcal{I} = \mathcal{I}(x)$. We prove Theorem 3, part by part, as follows.

Proof of Part (i): By (32), we have

$$\mu := - \frac{2}{\mathcal{I}(x)} \mathcal{I}_{|1}(x), \quad \lambda := \frac{1}{3} \mathcal{I}_2 = \frac{1}{3} \mathcal{I}_{|2}(x). \tag{50}$$

In this case, (33) and (34) imply that F has an isotropic mean Berwald curvature and a relatively isotropic Landsberg curvature.

Proof of Part (ii): By part (i), F has an isotropic mean Berwald curvature. In [13], it is proved that every Douglas metric with an isotropic mean Berwald curvature is an isotropic Berwald metric. On the other hand, every isotropic Berwald metric is a Douglas metric. In [10], it is shown that every isotropic Berwald metric has an isotropic S-curvature. This completes the proof. \square

Proof of Corollary 1. By part (ii) of Theorem 3, F has an isotropic S-curvature, $\mathbf{S} = 3cF$, where $c = c(x)$ is a scalar function on M . In [14], it is proved that every homogeneous Finsler metric with an isotropic S-curvature satisfies $\mathbf{S} = 0$, i.e., $c = 0$. In this case, F reduces to a Berwald metric. By Szabó's Theorem in [15], F is Riemannian or locally Minkowskian. \square

Proof of Corollary 2. Let F be a two-dimensional Finsler metric with a constant main scalar $\mathcal{I} = \text{constant}$. In this case, (50) implies that $\lambda = \mu = 0$, and then, F satisfies $\mathbf{E} = 0$ and $\mathbf{L} = 0$. Then, F reduces to a Berwald metric and Szabó's Theorem completes the proof. \square

Author Contributions: Writing—original draft, A.T. and W.S.K. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data Sharing not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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