

# Fixed-Point Theorems for Fuzzy Mappings

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**Abstract:** Since the 1970s and 1980s, significant contributions have been made by Weiss, Butnariu, Heilpern, Chitra, Subrahmanyam, and others, extending fixed-point theorems to fuzzy mappings and topological spaces. This paper provides two generalizations of two important fixed-point theorems, one provided by Butnariu and the other provided by Chitra. The first generalization ensures that, under certain conditions, an acyclic fuzzy mapping has a fixed point. The second result ensures the existence of a point in the intersection of two or more fuzzy mappings considering contractible finite dimensional ANR spaces, which is a generalization of the statement provided by Chitra.

**Keywords:** fuzzy topology; fuzzy sets; fuzzy mapping; fixed point; coincidence point; multi-valued map

**MSC:** 54A40

**Citation:** de Andrade, A.E.R.; Wasques, V.F. Fixed-Point Theorems for Fuzzy Mappings. *Mathematics* **2024**, *12*, 2165. <https://doi.org/10.3390/math12142165>

Academic Editor: Francisco Gallego Lupiáñez

Received: 4 June 2024

Revised: 1 July 2024

Accepted: 7 July 2024

Published: 10 July 2024



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## 1. Introduction

The fixed-point theorem is a fundamental concept in mathematics with applications across several fields, such as functional analysis, game theory, economics, algebraic topology, and many others (see, for example, A. Antón-Sancho [1], A. E. R. Andrade et al. [2], C. B. Kim [3], and P. Dechboon et al. [4]). In this context, the classic Brouwer's fixed-point theorem (see [5]) states that any continuous function  $f : D \rightarrow D$  from a closed disk  $D$  to itself has at least one fixed point. Another fundamental result is Banach's fixed-point theorem (see [6]), also known as the Banach contraction principle, which guarantees the existence and uniqueness of fixed points for contractive mappings in complete metric spaces.

In 1965, the concept of fuzzy sets was introduced by Zadeh in [7]. The fuzzy sets theory was established in order to computationally incorporate uncertain parameters into mathematical models [8,9] and the necessity of extending the classical concept of the fixed point to this new framework became evident.

In the 1970s and 1980s, many mathematicians, including Weiss [10] and Butnariu [11], investigated fixed-point theorems in the context of fuzzy mappings and topological spaces. Subsequently, numerous fixed-point theorems for fuzzy sets and fuzzy mappings under suitable hypotheses were obtained by Heilpern [12], Chitra and Subrahmanyam [13], Azam and Arshad [14], Soo Lee and Jin Cho [15], Alansari et al. [16], Batul et al. [17], Shoaib et al. [18], and Sagheer et al. [19].

The theory of fixed points for fuzzy mappings and fuzzy metric space continues to be an active area of research, with applications in control theory, optimization, dynamic systems analysis, and machine learning (see, for example, [20–22]). Motivated by this interesting theory, the paper focuses on the study of fixed-point theorems in the context of fuzzy topological spaces, as well as extensions of some results provided in the literature. More precisely, this work is dedicated to generalizing the following results:

**Theorem 1. (Butnariu Theorem) ([11], Theorem 2.11)** *Let  $C$  be a non-empty convex and compact subset of  $\mathbb{R}^m$ . If  $R$  is a fuzzy function over  $C$  and  $R'$  is  $F$ -continuous, then  $R$  has a fixed point in  $C$  (the related definitions are shown in Section 2.1).*

**Theorem 2. (Chitra Theorem)** ([13], Theorem 3.6) Let  $K$  be a convex subset of the topological vector space  $E$ ,  $K_1$  be a compact and convex subset of the locally convex topological vector space  $F$ , and  $T, S : K \rightarrow L_1(K_1)$  be two fuzzy mappings, where

$$L_1(K_1) = \{A \in L(K_1) : \text{All the } \alpha\text{-level sets of } A \text{ are non-empty}\},$$

in which the following conditions are satisfied:

- (i) Every  $\alpha$ -level set of  $T(x)$ , for each  $x \in K$ , is a closed-convex subset of  $K_1$  and  $T$  is upper semicontinuous;
- (ii) For each  $u \in K$ ,  $S(u)$  is a  $\tau_1$ -open subset of  $L(K_1)$  (i.e.,  $S(u) : K_1 \rightarrow [0, 1]$  is lower semicontinuous) and for all  $z \in K$ ,  $\{y \in K_1 : S(z)(y) > 0\}$  is a non-empty convex subset of  $K_1$ . Thus, there exists  $u \in K$ , such that  $T(u) \cap S(u) \neq \emptyset$ .

The next section provides the preliminary concepts and results that are required for this paper.

## 2. Preliminary and Known Results

### 2.1. Fuzzy Mapping

A fuzzy subset  $A$  of a non-empty set  $X$  is characterized by a membership function  $\mu_A : X \rightarrow [0, 1]$ , where  $\mu_A(x)$  represents the membership degree of  $x \in X$  in  $A$ , in the following sense: the closer the value of  $\mu_A(x)$  comes to 1, the higher the membership of element  $x$  in the set  $A$  [7]. The  $\alpha$ -level of a fuzzy set  $A$  is defined by all elements  $x \in X$ , such that  $\mu_A(x) \geq \alpha$ , where  $\alpha \in (0, 1]$ . In particular, if  $X$  is a topological space, then the 0-level is defined by the closure of the support of  $A$ , that is,  $\{x \in X : \mu_A(x) > 0\}$ . The  $\alpha$ -levels of a fuzzy set  $A$  are denoted by  $[A]^\alpha$  and are depicted in Figure 1.

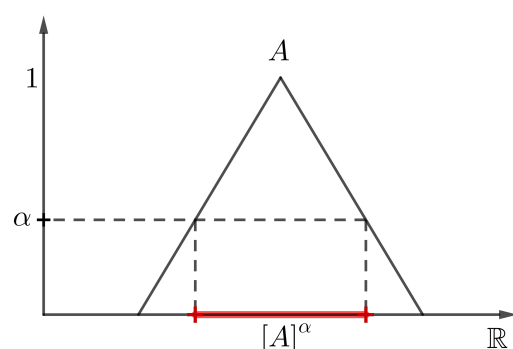


**Figure 1.** Representation of the  $\alpha$ -level of a fuzzy set  $A$ . The red interval stands for the  $[A]^\alpha$  and the black triangle stands for the fuzzy set called the triangular fuzzy number.

If  $A$  is a set in the ordinary sense of the term, its membership function assumes only two values—0 or 1, that is,  $\mu_A(x) = 1$  or 0, depending on whether  $x$  does or does not belong to  $A$ . Thus, in this case,  $\mu_A$  boils down to the characteristic function of a set  $A$ . This means that every real number and classical interval can be viewed as a particular example of fuzzy sets, where the membership function is given by the characteristic function. For notation convenience, the symbol  $A(x)$  will be considered instead of  $\mu_A(x)$ .

Now, set  $A = \emptyset$  is called an empty set if its characteristic function is given by  $A(x) = \chi_{\emptyset}(x) = 0$ , for all  $x \in X$ . Also, set  $X$  is universal if its characteristic function  $X(x) = \chi_X(x) = 1$ , for all  $x \in X$ . The union  $A \cup B$  and the intersection  $A \cap B$  of two fuzzy subsets,  $A$  and  $B$ , are the fuzzy subsets, whose membership functions are given by  $(A \cup B)(x) = A(x) \cup B(x) = \max\{A(x), B(x)\}$  and  $(A \cap B)(x) = A(x) \cap B(x) = \min\{A(x), B(x)\}$ , for all  $x \in X$  [23].

Let  $A$  and  $B$  be fuzzy subsets. The fuzzy subset  $B$  includes  $A$  ( $A \subseteq B$ ), if  $A(x) \leq B(x)$ , for all  $x \in X$  [8]. In this definition, it can also be said that  $A$  is contained in  $B$ .

Symbol  $L(X)$  stands for the class of fuzzy subsets of  $X$ . A fuzzy mapping over the set  $X$  is a function  $R$  from  $X$  to  $L(X)$ . If  $R$  is a fuzzy mapping over  $X$ , then  $R$  is also denoted by the fuzzy subset of  $X \times X$ , with the membership function  $R(x, y) = R(x)(y)$ , for any  $x, y \in X$ . A fixed point of the fuzzy mapping  $R$  over  $X$  is an element  $x_0$  of  $X$ , such that  $R(x_0, x_0) \geq R(x_0, x)$ , for any  $x \in X$ .

A fuzzy topology on  $X$  is a family  $\tau$  of fuzzy subsets in  $X$ , which satisfies the following conditions [24]:

- (a)  $\emptyset, X \in \tau$ ;
- (b) If  $A, B \in \tau$ , then  $A \cap B \in \tau$ ;
- (c) If  $A_i \in \tau$  for each  $i \in I$ , then  $\cup_{i \in I} A_i \in \tau$ .

The pair  $(X, \tau)$  is called a fuzzy topological space (fts) and every member of  $\tau$  is called a  $\tau$ -open fuzzy subset. It is known that the family of all lower semicontinuous functions on a topological space  $X$  into  $[0, 1]$  forms a fuzzy topology  $\tau_l$  for  $X$ .

A fuzzy subset  $V$  in a fts  $(X, \tau)$  is a neighborhood, of the fuzzy subset  $A$ , if and only if a  $\tau$ -open fuzzy subset  $O$  exists, such that  $A \subseteq O \subseteq V$ .

**Definition 1.** Let  $(X, \tau)$  be a fts, where  $X$  is a topological space. A fuzzy mapping  $R : X \rightarrow L(X)$  is said to be  $\tau$ -upper semicontinuous if for each  $x_0 \in X$  and for each neighborhood  $V$  of  $R(x_0)$ , a neighborhood  $U$  of  $x_0$  in  $X$  exists, such that  $R(U) \subseteq V$ .

In other words, Definition 1 establishes that if  $R$  is a  $\tau$ -upper semicontinuous fuzzy mapping, then  $R(u)(x) \leq V(x)$ , for all  $x \in X$  and for all  $u \in U$ .

Consider the following definition presented by D. Butnariu [11]: Let  $R$  be a fuzzy mapping over  $X$ ,  $C \subseteq X$  and  $A \in L(C)$ . The image of  $A$  by  $R$  is the fuzzy subset  $R[A] \in L(C)$  defined by

$$R[A](y) = \sup_{x \in C} \{R(x, y) \cap A(x)\}.$$

Fuzzy mapping  $R$  over  $X$  is said to be  $F$ -continuous if

$$R(x) \subseteq D \cap C \Rightarrow \exists V \in V(x) \text{ such that } R[V] \subseteq D \cap C,$$

for all  $x \in C$  and for all open subsets  $D \subseteq X$ , where  $V(x)$  denotes the set of all neighborhoods of  $x$  in  $X$ .

Fuzzy mapping  $R$  is said to be a *fuzzy function* if, for any  $x \in C$ ,

$$\hat{R}(x) = \left\{ y \in C : R(x)(y) = \max_{z \in C} R(x)(z) \right\},$$

contains only one point.

Let  $R$  be a fuzzy function. Thus, consider  $R'$  as the fuzzy mapping over  $C$  given by

$$R'(x)(y) = \begin{cases} R(x, y), & \text{if } y \in \hat{R}(x) \\ 0, & \text{otherwise} \end{cases}.$$

Space  $X$  is acyclic (with respect to the functor  $H^*(\cdot, \mathbb{Q})$ ) if  $H_0(X, \mathbb{Q}) = \mathbb{Q}$  and  $H_l(X, \mathbb{Q}) = 0$ , for all  $l > 0$ , where the symbol  $H_*$  stands for the Čech homology in the sense of [25]. Note that any contractible space is acyclic, in particular, every convex space is acyclic. Recall that a topological space  $X$  is contractible if the identity map on  $X$  is null-homotopic, that is, if it is homotopic to some constant map [26]. Intuitively, a contractible space is one that can be continuously shrunk to a point within that space.

Now consider the following class of fuzzy mappings: Let  $X, K$  be topological spaces.  $R : X \rightarrow L(K)$  is said to be an *acyclic fuzzy mapping* if, for any  $x \in X$ , it follows that  $\hat{R}(x) = \{y \in K : R(x)(y) = \max_{z \in C} R(x)(z)\}$  is acyclic. In particular, every fuzzy function is an acyclic fuzzy mapping. An example of an acyclic fuzzy mapping is the fuzzy

mapping  $R : \mathbb{R}^+ \rightarrow L(\mathbb{R})$  given by  $R(x) = [1, 2]e^x$ , where  $[1, 2]$  is an interval. The graphical representation of this fuzzy mapping is depicted in Figure 2.

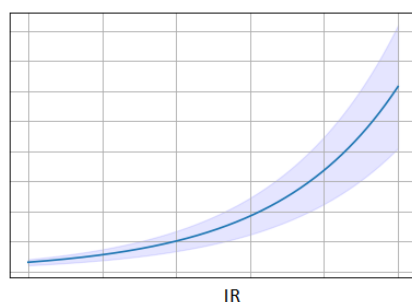


**Figure 2.** Representation of the fuzzy mapping  $R(x) = [1, 2]e^x$ . The blue region stands for the image of the fuzzy mapping.

## 2.2. Multi-Valued Maps

Let  $X$  and  $Y$  be two topological spaces, and assume that for each point  $x \in X$ , a non-empty closed subset  $F(x)$  of  $Y$  is given; in this case, we say that  $F$  is a *multi-valued map* from  $X$  into  $Y$ , and we write  $F : X \multimap Y$  [27]. A fixed point of  $F : X \multimap X$  is a point  $x \in X$ , such that  $x \in F(x)$ .

A multi-valued map  $F : X \multimap Y$  is called upper semicontinuous (u.s.c.) if

- (i)  $F(x)$  is compact for each  $x \in X$ ;
- (ii)  $F^{-1}(U) = \{x \in X \mid F(x) \subset U\}$  is an open subset of  $X$ , for each open subset  $U$  of  $Y$ .

An u.s.c. multi-valued map  $F : X \multimap Y$  is called acyclic, if set  $F(x)$  is an acyclic subset of  $Y$ , for every  $x \in X$ .

Let  $F : X \multimap Y$  be an u.s.c. multi-valued map. The graph of  $F$  is defined by

$$\Gamma_F = \{(x, y) \in X \times Y \mid y \in F(x)\}.$$

**Lemma 1.** Let  $X$  and  $Y$  be two topological spaces. If  $F : X \multimap Y$  is an u.s.c multi-valued map, then the graph  $\Gamma_F \subset X \times Y$  is a closed subset.

**Proof.** See [27], Proposition 1.1.  $\square$

Now, the last required result of the literature is presented, which was provided by Monis and Biazzi [28].

**Theorem 3.** Let  $(X, A)$  be a compact Hausdorff pair,  $K \subseteq \mathbb{R}^n$  be a compact and convex subset, and  $g : (X, A) \rightarrow (K, \partial K)$  be a continuous map, such that the induced homomorphism  $g_* : H_m(X, A) \rightarrow H_m(K, \partial K)$  is non-null, where  $m = \dim(K)$ . Thus, for every acyclic multi-valued map  $F : X \multimap K$ , there exists  $x \in X$ , such that  $g(x) \in F(x)$ .

## 3. Generalization of Butnariu and Chitra Theorems

This section presents the main contributions of this paper. First, the generalization of the Butnariu Theorem is provided.

### 3.1. Generalization of Butnariu Theorem

**Theorem 4.** Let  $(X, A)$  be a compact Hausdorff pair,  $K \subset \mathbb{R}^m$  be a compact and convex subset, and  $g : (X, A) \rightarrow (K, \partial(K))$  be a continuous map, such that  $g_* : H_m(X, A) \rightarrow H_m(K, \partial(K))$  is a non-null homomorphism. If  $R : X \rightarrow L(K)$  is an acyclic fuzzy mapping and  $R'$  is a  $F$ -continuous fuzzy mapping, then there exists  $x \in X$ , such that  $R(x, g(x)) \geq R(x, y)$ , for every  $y \in K$ . Moreover, if  $g : X \rightarrow K$  is an identity map, then  $x$  is a fixed point of  $R$ .

**Proof.** If  $R : X \rightarrow L(K)$  is an acyclic fuzzy mapping, then for any  $x \in X$ ,

$$\hat{R}(x) = \left\{ y \in K : R(x)(y) = \max_{z \in X} R(x)(z) \right\}$$

is acyclic.

If  $x_0 \in X$  exists, such that  $R(x_0) = \chi_{\emptyset}(x_0) = 0$ , then  $x_0$  is trivially a fixed point of  $R$ , as  $R(x_0, x_0) = R(x_0, x) = 0$ , for all  $x \in X$ . Hence, suppose  $R'(x)(y) = R(x)(y) > 0$  for every  $x \in X$  and  $y \in \hat{R}(x)$ . Consider now the multi-valued map  $\hat{R} : X \multimap K$ , given by

$$\hat{R}(x) = \left\{ y \in K : R(x)(y) = \max_{z \in C} R(x)(z) \right\} = \{ y \in K : R'(x)(y) > 0 \}.$$

As  $R'$  is  $F$ -continuous, for any open subset  $D \subseteq K$  with  $R'(x) \subseteq D$ , there exists a neighborhood  $V$  of  $x$ , such that  $R'[V] \subseteq D$ . Consequently, for every  $x \in V$  and  $y \in K$ , it follows that

$$R'(x)(y) \leq \sup_{z \in V} R'(z)(y) = R'[V](y) \leq D(y).$$

Therefore,  $R'(V) \subseteq D$ . Thus, for every  $x \in V$  and  $y \in \hat{R}(x)$ , it follows that

$$0 < R(x)(y) = R'(x)(y) \leq D(y),$$

which implies that  $D(y) = 1$ , as  $D$  is an ordinary set in  $K$ , that is,  $D(y) \in \{0, 1\}$ . Hence,  $\hat{R}(V) \subseteq D$ . Furthermore, hypothesis states that  $\hat{R}(x)$  is acyclic for every  $x \in X$ , so  $\hat{R}$  is an acyclic multi-valued map and from Theorem 3,  $x \in X$  exists, such that  $g(x) \in \hat{R}(x)$ , that is,

$$R(x, g(x)) \geq R(x, y), \quad \text{for all } y \in K.$$

□

### 3.2. Generalization of Chitra Theorem

Following the definition established in [27,29], this subsection considers the class of contractible ANR spaces, namely the space with the ANR metric. Recall that subset  $A \subseteq X$  is the retraction of  $X$  if a retraction  $r : X \rightarrow A$  exists, i.e.,  $r(x) = x$ , for every  $x \in A$ . The embedding of space  $X$  into  $Y$  can be viewed as a homeomorphism of  $h : X \rightarrow Y$  from  $X$  to  $Y$ , such that  $h(X)$  is a closed subset of  $Y$ . Now, the following definition can be established:

**Definition 2.** [27]  $X$  is an absolute neighbourhood retraction (ANR-space) if, and only if, for any space  $Y$  and for any embedding  $h : X \rightarrow Y$ , the set  $h(X)$  is a retraction of  $Y$ .

In view of [29], it follows that every convex subset of a Banach space is an ANR space (see Corollary III. 6.4; Proposition III.6.5 of [29]). Furthermore, every convex set in a locally convex topological vector space is a contractible ANR space (see Corollary II.14.2; Theorem III.3.1 and Proposition III.6.5. of [29]).

Now, consider the following result proved by J. F. McClendon [30]:

**Lemma 2** (Theorem 4.2 of [30]). Let  $X_1, \dots, X_n$  be topological Hausdorff spaces and  $X[i] = \prod_{j \neq i} X_j$ . If  $X_1, \dots, X_n$  are compact contractible finite dimensional ANR spaces,  $S_i \subset X_1 \times \dots \times X_n$  are either open or closed subsets, for  $i = 1, \dots, n$ , and

$$\{x_i \in X_i \mid (x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) \in S_i\}$$

is a contractible and non-empty set for any given  $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in X[i]$ , then the set  $S_1 \cap \dots \cap S_n$  is non-empty.

From now on, suppose that  $X_i$  is a compact contractible finite dimensional ANR topological spaces, for all  $i = 1, \dots, n$ . Now, let  $R_i : X[n] \rightarrow L(X_n)$  be fuzzy mappings, where  $X = X_1 \times \dots \times X_n$ , satisfying one of the following conditions:

- (i)  $R_i(x_1, \dots, x_{n-1})$  is a lower semicontinuous function from  $X_n$  into  $[0, 1]$  or  $R_i$  is  $F$ -continuous; and for each  $x = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in X[i]$ ;

$$B_x^i = \{x_i \in X_i : R_i(x_1, \dots, x_i, \dots, x_{n-1})(x_n) > 0\},$$

is a non-empty and contractible set.

- (ii)  $R_i(x) \in L(X_n)$  is an upper semicontinuous function for every  $x \in X[n]$ ,  $R_i$  is a  $\tau_l$ -upper semicontinuous fuzzy mapping, and

$$B_x^i = \{x_i \in X_i : R_i(x_1, \dots, x_i, \dots, x_{n-1})(x_n) = 1\},$$

is a non-empty and contractible set, for any given  $x = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in X[i]$ .

For case (i), consider  $S_i = \{(x_1, \dots, x_n) \in X \mid R_i(x_1, \dots, x_n) > 0\}$ , and for case (ii), consider  $S_i = \{(x_1, \dots, x_n) \in X \mid R_i(x_1, \dots, x_n) = 1\}$ . The next theorem provides a generalization of Chitra's Theorem:

**Theorem 5.** Let  $X_1, \dots, X_n$  be compact contractible finite dimensional ANR spaces,  $X[i] = \prod_{j \neq i} X_j$  and let  $R_i : X[n] \rightarrow L(X_n)$ ,  $i = 1, \dots, n-1$  be fuzzy mappings satisfying (i) or (ii); then, an element  $\bar{x} = (\bar{x}_1, \dots, \bar{x}_{n-1}) \in X[n]$  exists, such that  $\bigcap_{i=1}^n R_i(\bar{x}) \neq \emptyset$ . In this particular case, if  $n = 2$ ,  $X_1 = X_2 = X$  and  $R_2 : X \rightarrow L(X)$  is the trivial fuzzy mapping given by  $R_2(x) = \chi_x$ , then  $\bar{x}$  is a fixed point of  $R_1$ .

**Proof.** In the case of (i), if  $R_i(x_1, \dots, x_n)$  is a lower semicontinuous function from  $X$  into  $[0, 1]$ , then  $S_i = R_i^{-1}(0, 1]$  is an open subset of  $X$ .

On the other hand, if  $R_i$  is a  $F$ -continuous mapping, then it will be proven that  $S_i$  is a closed subset of  $X$ . Note that  $S_i$  is the graph of the multi-valued map  $\hat{S}_i : X[n] \multimap X_n$  given by

$$\hat{S}_i(x_1, \dots, x_{n-1}) = \{y \in X_n \mid R_i(x_1, \dots, x_{n-1})(y) > 0\}.$$

Furthermore,  $\hat{S}_i$  is an u.s.c multi-valued map. In fact, let  $D$  be an open set in  $X_n$ , such that  $\hat{S}_i(x_1, \dots, x_{n-1}) \subseteq D$ . Note that

$$R_i(x_1, \dots, x_{n-1})(y) > 0 \Rightarrow y \in \hat{S}_i(x_1, \dots, x_{n-1}) \subseteq D,$$

thus,  $R_i(x_1, \dots, x_{n-1}) \subseteq D$ , that is,  $R_i(x_1, \dots, x_{n-1})(y) \leq D(y)$ , for all  $y \in X_n$ . As  $R_i$  is a  $F$ -continuous mapping, a neighborhood  $V$  of  $x = (x_1, \dots, x_{n-1})$  exists, such that  $R_i(V) \subseteq D$ . Hence, for every  $y \in V$  and  $z \in \hat{S}_i(y)$ , it follows that  $0 < R_i(y)(z) \leq D(z)$ . Consequently,  $D(z) = 1$  for every  $z \in \hat{S}_i(y)$  and  $y \in V$ , that is,  $\hat{S}_i(V) \subseteq D$  and  $\hat{S}_i$  is an u.s.c multi-valued map. As  $S_i$  is a graph of  $\hat{S}_i$ , from Lemma 1, it follows that  $S_i$  is a closed set.

For the case (ii): Let  $\hat{S}_i : X[n] \multimap X_n$  be the multi-valued given by

$$\hat{S}_i(x_1, \dots, x_{n-1}) = \{y \in X_n \mid R_i(x_1, \dots, x_{n-1})(y) = 1\}.$$

Now, considering analogous arguments to those provided by Chitra's Theorem, it will be shown that  $\hat{S}_i$  is an upper semicontinuous (u.s.c) multi-valued map. Let  $U$  be an open set in  $X_n$ , such that  $\hat{S}_i(x_1, \dots, x_{n-1}) \subseteq U$ , and let  $g : X_n \rightarrow [0, 1]$  be the fuzzy set given by

$$g(z) = \begin{cases} 1 & \text{if } z \in U \\ \sup_{y \in X_n - U} R_i(x_1, \dots, x_{n-1})(y) & \text{if } z \in X_n - U. \end{cases}$$

Thus,  $g(z) < 1$  for all  $z \in X_n - U$ . Furthermore,  $g$  is a lower semicontinuous function with  $R_i(x_1, \dots, x_{n-1})(z) \leq g(z)$ , for all  $z \in X_n$ . Thus,  $g$  is an open neighborhood of

$R_i(x_1, \dots, x_{n-1})$ , and by the  $\tau_i$ -upper semicontinuity of  $R_i$ , there is a neighborhood  $V$  of  $(x_1, \dots, x_{n-1}) \in X[n]$ , such that  $R_i(V) \subseteq g$ . That is,  $R_i(y)z \leq g(z)$  for every  $z \in X_n$  and  $y \in V$ . Consequently,  $R_i(y)z = 1$  implies that  $g(z) = 1$ . Thus,  $\hat{S}_i(V) \subseteq U$ , and  $\hat{S}_i$  is an upper semicontinuous multi-valued map. As  $S_i$  is the graph of  $\hat{S}_i$ , from Lemma 1, it follows that  $S_i$  is a closed set.

As, in both cases,  $S_i \subseteq X$  are either open or closed subsets, and for each  $x = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in X[i]$ ,

$$\{x_i \in X_i : (x_1, \dots, x_n) \in S_i\}$$

is a non-empty and contractible set, for  $i = 1, \dots, n$ ; thus, all of the conditions of Lemma 2 are satisfied. Therefore,  $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n) \in \bigcap_{i=1}^n S_i$  exists, and it follows that

$$R_i(\bar{x}) = R_i(\bar{x}_1, \dots, \bar{x}_{n-1})(\bar{x}_n) > 0 \text{ for each } i = 1, \dots, n.$$

Consequently,  $\bigcap_{i=1}^n R_i(\bar{x}_1, \dots, \bar{x}_{n-1}) \neq \emptyset$ .  $\square$

#### 4. Conclusions

This paper extends two results about fixed-point theorems, presented in the literature. The first theorem was provided by Butnariu [11], which establishes certain conditions in order to guarantee that a fuzzy function has fixed points. Here, in this paper, a generalization of this result is provided by ensuring that under certain conditions, acyclic fuzzy mapping has a fixed point.

The second theorem, provided by Chitra [13], indicates some conditions under which two fuzzy mappings must be defined to ensure they intercept in at least one point. The second contribution of this paper is an extension of the result proposed by Chitra, ensuring the existence of a point in the intersection of two or more fuzzy mappings, considering contractible finite dimensional ANR spaces.

For further work, the purpose is to apply the generalizations theorems provided here in the area of Fuzzy Calculus, more specifically in the study of obtaining a solution for Fuzzy Differential Equations, in which the direction field is given by an acyclic fuzzy mapping.

**Author Contributions:** Conceptualization, A.E.R.d.A. and V.F.W.; Formal analysis, A.E.R.d.A. and V.F.W. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Data Availability Statement:** Data will be made available on request.

**Acknowledgments:** The authors thank the support of FAPESP n° 2023/03927-0 and PROPP/UFMS.

**Conflicts of Interest:** The authors declare no conflicts of interest.

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