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Marshall–Olkin Bivariate Weibull Model with Modified Singularity (MOBW-μ): A Study of Its Properties and Correlation Structure

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Abstract: We propose the "Marshall–Olkin Bivariate Weibull Model with Modified Singularity MOBW-μ", which focuses on bivariate distributions essential for reliability and survival analyses. Distributions such as the Marshall–Olkin bivariate exponential (MOBE) and the Marshall–Olkin bivariate Weibull (MOBW) are discussed. The MOBW-μ model is introduced, which incorporates a lag parameter μ in the singular part, and probabilistic properties such as the joint survival function, marginal density functions, and the bivariate hazard rate function are explored. In addition, aspects such as the correlation structure and survival copulation are addressed and we show that the correlation of the MOBW-μ is always lower than that of its copula, regardless of the parameters. The latter result implies that the MOBW-μ does not have the Lancaster's phenomenon that explains that any nonlinear transformation of variables decreases the correlation in absolute value. The article concludes by presenting a robust theoretical framework applicable to various disciplines.

Keywords: Marshall–Olkin bivariate Weibull; singularity; copula; survival bivariate; Lancaster's phenomenon; correlation structure

MSC: 60A05; 60B11

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1. Introduction

Bivariate and multivariate distributions with marginals set at $(0, \infty)$ play a crucial role in the modeling used in reliability and survival analyses. Although it is commonly assumed that the useful life of two random variables or a system with two components is independent, in many practical situations it is essential to consider some form of dependency between these components. Consequently, the bivariate distributions related to life expectancy emerge as essential components in the joint study of this phenomenon.

Several useful bivariate distributions have been proposed; see, for example, Refs. [1–3]. In this article we focus our attention on an important family of bivariate distributions introduced by [4] and called the Marshall–Olkin bivariate exponential (MOBE) distribution. This family arises from a context of reliability.

To understand the essence of the MOBE distribution, let us consider a system composed of two components that face three different types of impacts. These impacts, when occurring, result in the immediate destruction of the affected component. The first two types of impact exclusively affect the first and second components, separately. In contrast, the third type of impact involves the simultaneous destruction of both components. It is for this reason that the first two impacts are called "individual", while the third is characterized as "common". These impacts follow exponential waiting time distributions and occur independently. Representing the lifetimes of the two components as X_1 and X_2, the MOBE model arises from the stochastic representation.

$$(X_1, X_2) = (\min(T_1, T_3), \min(T_2, T_3)), \tag{1}$$

where T_i are independent and exponentially distributed with parameters λ_i, $i = 1, 2, 3$. The only bivariate distribution with exponential marginals generated by (1) is

$$S_{X_1,X_2}(x_1, x_2) = \exp\{-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 \max(x_1, x_2)\}, \tag{2}$$

which is called the MOBE distribution.

Due to the "common shock" identified by T_3 in (2), we have to $P(X_1 = X_2) = \frac{\lambda_3}{\lambda_1+\lambda_2+\lambda_3} > 0$, if $\lambda_3 > 0$, the distribution (2) has a singular component along the line $\mathrm{L} : \{x_1 = x_2\}$. Therefore, the MOBE is not absolutely continuous and lacks a probability density with respect to the two-dimensional Lebesgue measure.

The stochastic representation (1) and MOBE (2) have many applications in various fields, standing out in disciplines such as engineering, survival analysis, reliability, quality control, finance, insurance, and industry [5–9]. For example, Ref. [8] presents a review of different applications and recent developments in MOBE. In reliability theory, the Marshall–Olkin distribution has been applied in the context of dependent competing risks models, where the failure of a system can be attributed to multiple causes or failure modes. These models have been extended to accommodate different censoring schemes, such as type-I progressive interval censoring, dependent left censoring, and type-II censoring.

In survival analysis, ref. [9] discusses the application of semiparametric Marshall–Olkin models to the occurrence of metastases at multiple sites after breast cancer. The authors propose four models to compare the risk of developing a metastasis at different sites. The versatility and effectiveness of these tools make them fundamental resources for addressing a variety of problems and challenges in these areas, providing robust theoretical frameworks and models that can be adapted to different contexts and scenarios.

Numerous generalizations of bivariate Marshall–Olkin-type distributions have been proposed in the literature, as evidenced in previous studies [10–14]. Typically, authors assume several distributions for T_1, T_2, and T_3 in (2), but always maintaining their independence. Recently, many articles have been devoted to bivariate distributions of the Marshall–Olkin type: Ref. [15] proposed a bivariate Weibull distribution; Ref. [12] suggested a bivariate Kumaraswamy-exponential distribution; Ref. [13] introduced a bivariate Pareto distribution; and Ref. [14] introduced a new probabilistic model derived from the Marshall–Olkin shock model and applied it to data representing the failure times of a parallel system consisting of two identical motors in days.

In practice, we can find systems with two components that do not stop working immediately after a common collision. In this sense, Ref. [16] proposes an alternative to the model (2), called the modified Marshall–Olkin bivariate:

$$S_{X_1,X_2}(x_1, x_2) = \exp\{-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 \max(x_1, \mu x_2)\}, \tag{3}$$

generated by the stochastic representation

$$(X_1, X_2) = \left(\min(T_1, T_3), \min\left(T_2, \frac{T_3}{\mu}\right)\right), \tag{4}$$

where the random variables T_i are independent and exponentially distributed with parameters $\lambda_i > 0$, $i = 1, 2, 3$, and $\mu > 0$. That is, the failure times due to the common shock for the two devices will not necessarily be the same.

One can recognize that there is a positive mass concentrated on the line $\mathrm{L}_\mu : \{x_1 = \mu x_2\}$ and it can be verified that the contribution to the singularity is $P(X_1 = \mu X_2) = \frac{\mu\lambda_3}{\lambda_1+\lambda_2+\mu\lambda_3} > 0$. Clearly, X_1 and X_2 are independent if $\lambda_3 = 0$. The stochastic representation (4) has an added value with respect to the relationship (1), since it takes into account the asymmetric behavior of the two devices X_1 and X_2.

In practice, if in a bivariate data set in some cases two components take equal values, the MOBE distribution can be used quite effectively to analyze such data sets. Since the MOBE distribution has exponential marginals, if the bivariate data indicate a unimodal marginal probability density function or a non-constant hazard function, then the MOBE distribution may not be appropriate. Due to this restriction, Ref. [17] suggested a more flexible bivariate Weibull distribution, where the marginals are Weibull distributions and can be obtained along the same lines as the MOBE model.

The bivariate Weibull distribution proposed by [17] is

$$S_{X_1,X_2}(x_1,x_2) = \exp\left\{-\lambda_1 x_1^{\beta_1} - \lambda_2 x_2^{\beta_2} - \lambda_3 \max\left(x_1^{\beta_1}, x_2^{\beta_2}\right)\right\},\ \beta_1 > 0,\ \beta_2 > 0. \tag{5}$$

If $\beta_1 = \beta_2 = \beta$, the distribution (5) is called Marshall–Olkin Weibull bivariate (MOBW), if $\lambda_3 > 0$, this distribution has a singular component along the line $\mathrm{L} : \left\{x_1^\beta = x_2^\beta\right\}$. Several papers have been published with theoretical developments on the properties of the MOBW model, analysis of its correlation structure, and also inference on the model parameters, see [15,18–21].

In the context of two-component systems, it is common to find situations where the failure of one component due to a "common shock" can affect the operation of the other, either directly or indirectly [22]. However, in some cases, this effect may not be immediate and there may be a delay between the failure of one component and the failure of the other. This delay may be caused by a variety of factors, such as the time required to detect and respond to the failure, the duration of the repair or recovery processes, or the intrinsic characteristics of the system.

These situations can occur in different contexts. Think of a twin-engine aircraft; if it experiences a traumatic event, such as a collision, it is possible that both engines will be affected differently [23]. For example, one of the engines could be irreparably damaged while the other continues to function properly. In this situation, the aircraft can still be kept in flight, but it is now completely dependent on the one operational engine, which may eventually fail some time later. Also, in the case of certain eye diseases or injuries, a person may experience vision loss in both eyes. However, this loss of vision may not occur simultaneously. This situation is common in patients with diabetic retinopathy [24,25].

When the common shock does not cause the simultaneous failure of a system with two failure modes as described above, it is necessary to include a parameter that captures this difference between these failure times.

There is no statistical literature, theory, or applications related to the MOBW distribution with modified singularity such as that suggested by [16], in which, due to a common collision, in a system with two components, one of the components fails and the other component remains functional.

In this work, we propose an MOBW model incorporating a delay parameter μ in the singular part, which we call MOBW-μ. For the proposed model we study different probabilistic properties, we find the survival copula to model the dependence structure in the bivariate case, and we study its correlation structure; we show that the MOBW-μ does not have Lancaster's phenomenon.

The rest of the article is organized as follows. In Section 2, we introduce the new bivariate distribution MOBW-μ. In Section 3, we provide some probabilistic properties of the proposed model. In Section 4, we furnish and discuss the bivariate hazard rate function and hazard gradients. In Section 5, we find the survival copula that generates the proposed bivariate distribution. In Section 6, finally, we present the conclusions.

2. Bivariate Marshall–Olkin Weibull Model with Modified Singularity (MOBW-μ)

In this section, we present the survival function of the Bivariate Marshall–Olkin Weibull Model with Modified Singularity; we show that this distribution has an absolutely continuous component and a singular component. For this we start from the following:

If (X_1, X_2) has an MOBW distribution with modified singularity along the line $L_\mu : \{x_1 = \mu x_2,\ \mu > 0\}$ with parameters $(\lambda_1, \lambda_2, \lambda_3, \beta)$, which we will denote MOBW-$\mu(\lambda_1, \lambda_2, \lambda_3, \beta)$, then the joint survival function of (X_1, X_2) can be written as

$$S_{X_1,X_2}(x_1, x_2) = \exp\Big\{-\lambda_1 x_1^\beta - \lambda_2 x_2^\beta - \lambda_3 \max\Big(x_1^\beta, (\mu x_2)^\beta\Big)\Big\}, \tag{6}$$

for $x_1 > 0, x > 0$, and 0 elsewhere.

The marginal survival functions have a Weibull distribution:

$$S_{WE}(x_1; \beta, \lambda_1 + \lambda_3) = \exp\Big\{-(\lambda_1 + \lambda_3)x_1^\beta\Big\}, \tag{7}$$

$$S_{WE}\Big(x_2; \beta, \lambda_2 + \lambda_3\mu^\beta\Big) = \exp\Big\{-\Big(\lambda_2 + \lambda_3\mu^\beta\Big)x_2^\beta\Big\}, \tag{8}$$

with marginal density functions

$$f_{WE}(x_1; \beta, \lambda_1 + \lambda_3) = \beta(\lambda_1 + \lambda_3)x_1^{\beta-1}\exp\Big\{-(\lambda_1 + \lambda_3)x_1^\beta\Big\}, \tag{9}$$

$$f_{WE}\Big(x_2; \beta, \lambda_2 + \lambda_3\mu^\beta\Big) = \beta\Big(\lambda_2 + \lambda_3\mu^\beta\Big)x_2^{\beta-1}\exp\Big\{-\Big(\lambda_2 + \lambda_3\mu^\beta\Big)x_2^\beta\Big\}. \tag{10}$$

Model (6) can be written as

$$S_{X_1,X_2}(x_1, x_2) = \begin{cases} S_1(x_1, x_2), & \text{if } x_1 > \mu x_2 \\ S_2(x_1, x_2), & \text{if } x_1 < \mu x_2 \\ S_0(x), & \text{if } x_1 = \mu x_2 = x. \end{cases}$$

$$S_{X_1,X_2}(x_1, x_2) = \begin{cases} \exp\Big\{-(\lambda_1 + \lambda_3)x_1^\beta - \lambda_2 x_2^\beta\Big\}, & \text{if } x_1 > \mu x_2 \\ \exp\Big\{-\lambda_1 x_1^\beta - \big(\lambda_2 + \lambda_3\mu^\beta\big)x_2^\beta\Big\}, & \text{if } x_1 < \mu x_2 \\ \exp\Big\{-\Big(\lambda_1 + \frac{\lambda_2}{\mu^\beta} + \lambda_3\Big)x_1^\beta\Big\}, & \text{if } x_1 = \mu x_2 = x \end{cases}$$

$$= \begin{cases} \exp\Big\{-(\lambda_1 + \lambda_3)x_1^\beta\Big\}\exp\Big\{-\lambda_2 x_2^\beta\Big\}, & \text{if } x_1 > \mu x_2 \\ \exp\Big\{-\lambda_1 x_1^\beta\Big\}\exp\Big\{\big(\lambda_2 + \lambda_3\mu^\beta\big)x_2^\beta\Big\}, & \text{if } x_1 < \mu x_2 \\ \exp\Big\{-\Big(\lambda_1 + \frac{\lambda_2}{\mu^\beta} + \lambda_3\Big)x^\beta\Big\}, & \text{if } x_1 = \mu x_2 = x \end{cases}$$

$$= \begin{cases} S_{WE}(x_1; \beta, \lambda_1 + \lambda_3)S_{WE}(x_2; \beta, \lambda_2), & \text{if } x_1 > \mu x_2 \\ S_{WE}(x_1; \beta, \lambda_1)S_{WE}\big(x_2; \beta, \lambda_2 + \lambda_3\mu^\beta\big), & \text{if } x_1 < \mu x_2 \\ S_{WE}\Big(x; \beta, \lambda_1 + \frac{\lambda_2}{\mu^\beta} + \lambda_3\Big), & \text{if } x_1 = \mu x_2 = x. \end{cases}$$

3. Properties of the Proposed Model

In this section, we describe some probabilistic properties of the proposed MOBW-μ model, such as the definition of the bivariate density function necessary for the study of the bivariate hazard.

Theorem 1. *Given the joint survival function of (X_1, X_2) as in (6), then the joint density function of (X_1, X_2) is written as*

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} f_1(x_1, x_2), & \textit{if } x_1 > \mu x_2 \\ f_2(x_1, x_2), & \textit{if } x_1 < \mu x_2 \\ f_0(x), & \textit{if } x_1 = \mu x_2 = x, \end{cases} \tag{11}$$

where

$$f_1(x_1,x_2) = f_{WE}(x_1;\beta,\lambda_1+\lambda_3) f_{WE}(x_2;\beta,\lambda_2)$$

$$f_2(x_1,x_2) = f_{WE}(x_1;\beta,\lambda_1) f_{WE}\left(x_2;\beta,\lambda_2+\lambda_3\mu^\beta\right)$$

$$f_0(x) = \frac{\lambda_3\mu^\beta}{\lambda_1\mu^\beta+\lambda_2+\lambda_3\mu^\beta} f_{WE}\left(x;\beta,\lambda_1+\frac{\lambda_2}{\mu^\beta}+\lambda_3\right).$$

The proof of Theorem 1 is given in Appendix A.

It is clearly seen that the joint density function of (X_1, X_2) is expressed as the product of Weibull-distributed marginals for the absolutely continuous part and a Weibull-distributed marginal for the singular component. This joint density function can take different forms depending on its parameters. In addition, when $\mu = 1$, the MOBW-μ distribution reduces to the MOBW distribution.

The graphs in Figures 1 and 2 are generated for two values of the delay parameter, $\mu = 0.5$ and 1, and three values of the shape parameter, $\beta = 0.5$, 1, and 2. Figures 1 and 2 correspond to the graph of the bivariate density function and the contour lines for the bivariate density function of the proposed MOBW-μ model, respectively. We can see how the parameter μ controls the singularity and the parameter β the shape of the concentration of the bivariate data. In Figure 2, the colors represent different levels of the density function in two-dimensional space. The green-colored areas indicate lower values of density, and as the colors become warmer (towards yellow), they indicate higher values of density. The warmest areas are observed in the center of the graphs, indicating that this is the region with the highest probability of occurrence of the values of the variables X_1 and X_2.

When the parameter β is equal to 1, this corresponds to the graphs of the density function and contour lines of the MOBE model presented in [16].

Lemma 1. *If (X_1, X_2), it is distributed like the MOBW-μ with modified singularity along the line $L_\mu : \{x_1 = \mu x_2,\ \mu > 0\}$ of (11).*

1. $P(X_1 < \mu X_2) = \frac{\lambda_1\mu^\beta}{\lambda_1\mu^\beta+\lambda_2+\lambda_3\mu^\beta}$
2. $P(\mu X_2 < X_1) = \frac{\lambda_2}{\lambda_1\mu^\beta+\lambda_2+\lambda_3\mu^\beta}$
3. $P(X_1 = \mu X_2) = \frac{\lambda_3\mu^\beta}{\lambda_1\mu^\beta+\lambda_2+\lambda_3\mu^\beta}$
4. $P((\min(X_1, \mu X_2)) > t) = \exp\left\{-\left(\lambda_1 + \frac{\lambda_2}{\mu^\beta} + \lambda_3\right)t\right\}.$

The proof of Lemma 1 is given in Appendix B.

One of the important results of Lemma 1 is associated with property 3. Where it is easy to verify that the correlation between the random variables X_1 and X_2 is completely determined by the singularity, that is, $\rho = P(X_1 = \mu X_2)$.

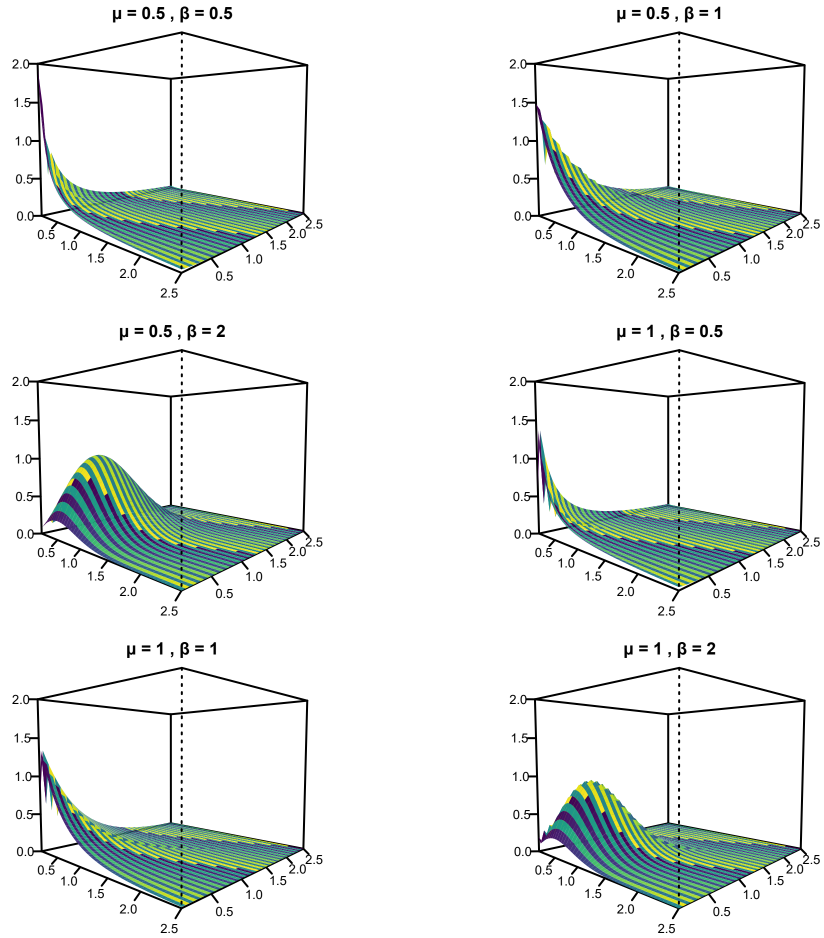

Figure 1. Bivariate density function for $f_{X_1,X_2}(x_1, x_2)$ in (11), with $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 1$ and different parameter values μ and β.

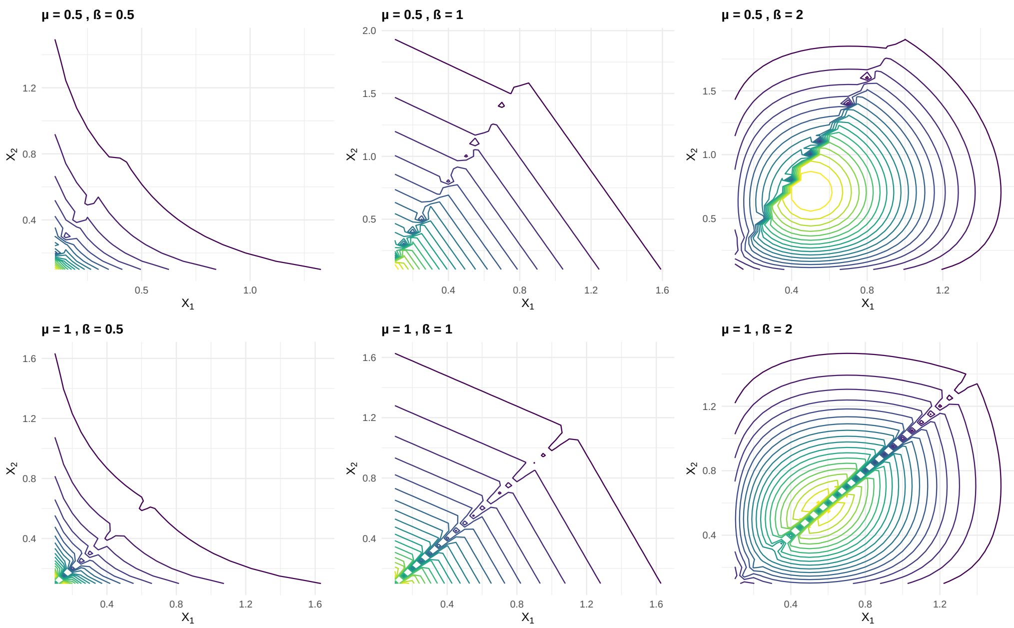

Figure 2. Contour lines for the function $f_{X_1,X_2}(x_1, x_2)$ in (11), with $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 1$ and different parameter values μ and β.

Theorem 2. *The conditional probability density functions of X_i, given $X_j = x_j$, $f_{X_i|X_j}(x_i|x_j)$, $i, j = 1, 2$; $i \neq j$, have the following formulas:*

$$f_{X_2|X_1}(x_2|x_1) = \begin{cases} f_{WE}(x_2; \beta, \lambda_2), & if \;\; x_1 > \mu x_2 \\ \frac{\lambda_1}{\lambda_1+\lambda_3} \frac{f_{WE}\left(x_2;\beta,\lambda_2+\lambda_3\mu^{\beta}\right)}{exp\left\{-\lambda_3 x_1^{\beta}\right\}}, & if \;\; x_1 < \mu x_2 \\ \frac{\lambda_3}{\lambda_1+\lambda_3} exp\left\{-\lambda_2\left(\frac{x}{\mu}\right)^{\beta}\right\}, & if \;\; x_1 <= \mu x_2 = x. \end{cases}$$

$$f_{X_2|X_1}(x_2|x_1) = \begin{cases} \frac{\lambda_2}{\lambda_2+\lambda_3\mu^{\beta}} \frac{f_{WE}(x_1;\beta,\lambda_1+\lambda_3)}{exp\left\{-\lambda_3(\mu x_2)^{\beta}\right\}}, & if \;\; x_1 > \mu x_2 \\ f_{WE}(x_1; \beta, \lambda_1), & if \;\; x_1 < \mu x_2 \\ \frac{\lambda_3\mu^{\beta}}{\lambda_2+\lambda_3\mu^{\beta}} exp\{-\lambda_1 x^{\beta}\}, & if \;\; x_1 <= \mu x_2 = x. \end{cases}$$

Proof. The proof follows immediately by substituting the joint pdf of (X_1, X_2) discussed in (11) and the marginal probability density function of X_j $(i = 1, 2)$ given in (9) and (10), using the following formula:

$$f_{X_i|X_j}(x_i|x_j) = \frac{f_{X_i,X_j}(x_i, x_j)}{f_{X_j}(x_j)}, \;\; i = 1, 2.$$

□

4. The Hazard Rate Function

In this section, we furnish and discuss the bivariate hazard rate function and hazard gradients of the hazard rate function for the MOBW-μ.

4.1. Hazard Rate Function

In the literature, the bivariate failure rate function is described in various methods. One of these ways was defined by [26] as

$$h_{X_1,X_2}(x_1, x_2) = \frac{f_{X_1,X_2}(x_1, x_2)}{S_{X_1,X_2}(x_1, x_2)}.$$

Theorem 3. *If $(X_1, X_2) \sim MOBW\text{-}\mu(\lambda_1, \lambda_2, \lambda_3, \beta)$, then the joint hazard rate function has the following form:*

$$h_{X_1,X_2}(x_1, x_2) = \begin{cases} h_1(x_1, x_2), \text{ if } \;\; x_1 > \mu x_2 \\ h_2(x_1, x_2), \text{ if } \;\; x_1 < \mu x_2 \\ h_0(x), \text{ if } \;\; x_1 = \mu x_2 = x \end{cases}$$

with

$$h_1(x_1, x_2) = \beta^2(\lambda_1 + \lambda_3)\lambda_2(x_1 x_2)^{\beta-1},$$
$$h_2(x_1, x_2) = \beta^2(\lambda_2 + \lambda_3\mu^{\beta})\lambda_1(x_1 x_2)^{\beta-1},$$
$$h_0(x) = \beta\lambda_3 x^{\beta-1}.$$

Proof. Using Equations (6) and (11), it is easy to prove the theorem. □

4.2. Hazard Gradients

The hazard rate function quantifies the failure rate in the univariate state, but in multivariate cases, the failure rate depends on the altered variable. As such, risk gradients for modeling bivariate and multivariate lifetime data are defined in [27,28].

The bivariate hazard gradient for continuous random variables X_1 and X_2 is given by

$$\begin{aligned}\gamma_{X_1,X_2}(x_1,x_2) &= (\gamma_{X_1}(x_1,x_2), \gamma_{X_2}(x_1,x_2)) \\ &= \left(-\frac{\partial}{\partial x_1}\log S_{X_1,X_2}(x_1,x_2), -\frac{\partial}{\partial x_2}\log S_{X_1,X_2}(x_1,x_2)\right).\end{aligned}$$

For $(X_1, X_2) \sim$ MOBW-$\mu(\lambda_1,\lambda_2,\lambda_3,\beta)$, the hazard gradients are given by

$$\gamma_{X_1}(x_1,x_2) = \begin{cases} \beta(\lambda_1+\lambda_3)x_1^{\beta-1},\ x_1 > \mu x_2 \\ \lambda_1 x_1^{\beta-1},\ x_1 < \mu x_2 \\ \beta\left(\lambda_1 + \frac{\lambda_2}{\mu^\beta} + \lambda_3\right)x^{\beta-1},\ x_1 = \mu x_2 = x \end{cases} \tag{12}$$

and

$$\gamma_{X_2}(x_1,x_2) = \begin{cases} \lambda_1 x_2^{\beta-1},\ x_1 > \mu x_2 \\ \beta(\lambda_2+\lambda_3\mu^\beta)x_2^{\beta-1},\ x_1 < \mu x_2 \\ \beta\left(\lambda_1 + \frac{\lambda_2}{\mu^\beta} + \lambda_3\right)x^{\beta-1},\ x_1 = \mu x_2 = x. \end{cases} \tag{13}$$

From (12) and (13) we can conclude that for fixed x_2, $\gamma_{X_1}(x_1,x_2)$ assumes the same shape as a Weibull distribution according to the value of β. This hazard function can be increasing, decreasing, or constant according to whether the parameter β is greater than, less than, or equal to one. Similarly, the hazard function $\gamma_{X_2}(x_1,x_2)$ holds for a fixed x_1.

5. Copula and Dependency Measures

The dependence between two random variables X_1 and X_2 is completely determined by their joint distribution $F_{X_1,X_2}(x_1,x_2)$. The copula is a powerful tool to study the dependence between variables. Any distribution function can be expressed in copula form, in which dependence and marginals can be studied separately [29].

The most central theorem within the theory of copulas is Sklar's theorem. This theorem states that every multivariate distribution function can be expressed through its univariate marginals and a copula that describes the dependence between the random variables [29]. Copulas are of interest primarily because they act as a link between random variables and their joint distribution function, and this is what we will focus our attention on in this section.

Definition 1. *Let $C : \mathbb{I}^2 \to \mathbb{I}$ be a copula. The complement of the union of all open sets in $\mathbb{I}^2$ that have C-measure zero is called the support of C, and is denoted by supp(C). If supp(C) = $\mathbb{I}^2$, we say that C has full support.*

Again, let (U_1, U_2) be a random vector with standard uniform marginals and joint distribution function C. The support of C is the subset of $\mathbb{I}^2$ in which (U_1, U_2) can take values. Conversely, given a subset $A \subset \mathbb{I}^2$ which satisfies $A \cup \text{supp}(C) = \emptyset$, then $P[(U_1,U_2) \in A] = 0$.

Definition 2. *Let $C : \mathbb{I}^2 \to \mathbb{I}$ be a copula. Then, C can be split into an absolutely continuous component A_C and a singular component S_C, where*

$$A_C(u,v) = \int_0^u \int_0^v \frac{\partial^2}{\partial s \partial t} C(s,t) ds dt \tag{14}$$

and $S_C(u,v) = C(u,v) - A_C(u,v)$.

Note that C is singular if and only if the Lebesgue measure of its support is zero. Additionally, the C-measures of the absolutely continuous component and singular component are given by $A_C(1,1)$ and $S_C(1,1)$, respectively.

Lemma 2. *If (X_1, X_2) is distributed as the MOBW-μ with a modified singularity along the line $L_\mu : \{x_1 = \mu x_2\}$ like in (6), the survival copula for (X_1, X_2) is*

$$C_{\alpha_1,\alpha_2(\mu)}(u,v) = \begin{cases} u^{1-\alpha_1}v, & \text{if } u^{\alpha_1} \geq v^{\alpha_2(\mu)} \\ uv^{1-\alpha_2(\mu)}, & \text{if } u^{\alpha_1} \leq v^{\alpha_2(\mu)} \end{cases} \tag{15}$$

with

$$\alpha_1 = \frac{\lambda_3}{\lambda_1 + \lambda_3}; \quad \alpha_2(\mu) = \frac{\lambda_3 \mu^\beta}{\lambda_2 + \lambda_3 \mu^\beta}.$$

The proof of Theorem 1 is given in Appendix C.

Remark 1. *When $\mu = 1$ and $\beta = 1$, the copula in (15) belongs to the two-parameter Marshall–Olkin family. The range of variability in the parameters in (15) is $0 < \alpha_1, \alpha_2(\mu) < 1$. If we extend the range of variability to $0 \leq \alpha_1, \alpha_2(\mu) \leq 1$, so $C_{\alpha_1,0} = \Pi$ and $C_{1,1} = M$, these are the copulas of independence and the upper Fréchet–Hoeffding, respectively [30].*

The copulas in this family have full support, but they are neither absolutely continuous nor singular. We compute the absolutely continuous component by first finding the partial derivatives:

$$\frac{\partial^2}{\partial u \partial v} C_{\alpha_1,\alpha_2(\mu)}(u,v) = \begin{cases} (1-\alpha_1)u^{\alpha_1}, & \text{if } u^{\alpha_1} \geq v^{\alpha_2(\mu)} \\ (1-\alpha_2(\mu))v^{-\alpha_2(\mu)}, & \text{if } u^{\alpha_1} \leq v^{\alpha_2(\mu)} \end{cases}$$

and then, evaluating the double integral in (14), we obtain

$$A_{\alpha_1,\alpha_2(\mu)}(u,v) = C_{\alpha_1,\alpha_2(\mu)}(u,v) - \frac{\alpha_1 \alpha_2(\mu)}{\alpha_1 + \alpha_2(\mu) - \alpha_1\alpha_2(\mu)} \left[\min\left(u^{\alpha_1}, v^{\alpha_2(\mu)}\right)\right]^{\frac{\alpha_1+\alpha_2(\mu)-\alpha_1\alpha_2(\mu)}{\alpha_1\alpha_2(\mu)}}$$

and is supported on the line $u^{\alpha_1} = v^{\alpha_2(\mu)}$. The $C_{\alpha_1,\alpha_2(\mu)}$ − measure of the singular component is given by

$$S_{\alpha_1,\alpha_2(\mu)}(1,1) = \frac{\alpha_1\alpha_2(\mu)}{\alpha_1 + \alpha_2(\mu) - \alpha_1\alpha_2(\mu)}.$$

Returning to the initial configuration, the singular component corresponds to the case in which, due to the common collision, component 1 fails and sometime later component 2 fails, which is when $X_2 = \mu X_1$, $(\mu < 1)$. When $\mu = 1$, this corresponds to the standard case in which, due to the common collision, the two components fail simultaneously.

$$S(x_1, x_2) = C(S(x_1), S(x_2)) = A_C(S(x_1), S(x_2)) + S_C(S(x_1), S(x_2)).$$

The singular component, S_C, has its support on the line $v = u^{\frac{\alpha_1}{\alpha_2(\mu)}}$. Remembering what we defined in Equation (A2), we obtain that $v = u^{\frac{\alpha_1}{\alpha_2(\mu)}}$ evaluated at $u = S(x_1)$ and $v = S(x_2)$ corresponds to

$$S(x_2) = [S(x_1)]^{\frac{\alpha_1}{\alpha_2(\mu)}} = [S(x_1)]^{\frac{\lambda_2+\lambda_3\mu^\beta}{(\lambda_1+\lambda_3)\mu^\beta}},$$

by making easy developments we arrive at $\mu x_2 = x_1$, and to conclude, we have

$$P[T_3 < \min(T_1, T_2)] = \frac{\alpha_1\alpha_2(\mu)}{\alpha_1 + \alpha_2(\mu) - \alpha_1\alpha_2(\mu)} = \frac{\lambda_3\mu^\beta}{\lambda_2 + (\lambda_1 + \lambda_3)\mu^\beta},$$

which is coherent with what we would obtain from manually computing this probability for the survival function of the proposed MOBW-μ model.

Figure 3 shows a series of graphs representing bivariate data simulated with the copula $C_{\alpha_1,\alpha_2(\mu)}(u,v)$. The parameters μ and β affect the shape and orientation of the copula-simulated data as follows:

- Parameter μ: Controls the location or center of the data, this location is associated with the uniqueness of the data. As the value of μ decreases, the concentration of the data shifts toward the lower left corner of the graph. This suggests that a higher μ places the concentration of the data towards the upper right, while a lower μ shifts it towards the lower left. This can be seen as a lag parameter in bivariate data.
- Parameter β: This parameter seems to influence the dispersion and correlation between the variables u and v. When β is 0.5, the data are more spread out and the contour lines show a less pronounced positive correlation. As β increases to 1 and then 2, the data show a stronger positive correlation, evidenced by the steeper slope of the blue contour lines becoming tighter around a curve passing from the lower left corner to the upper right corner of the graph.

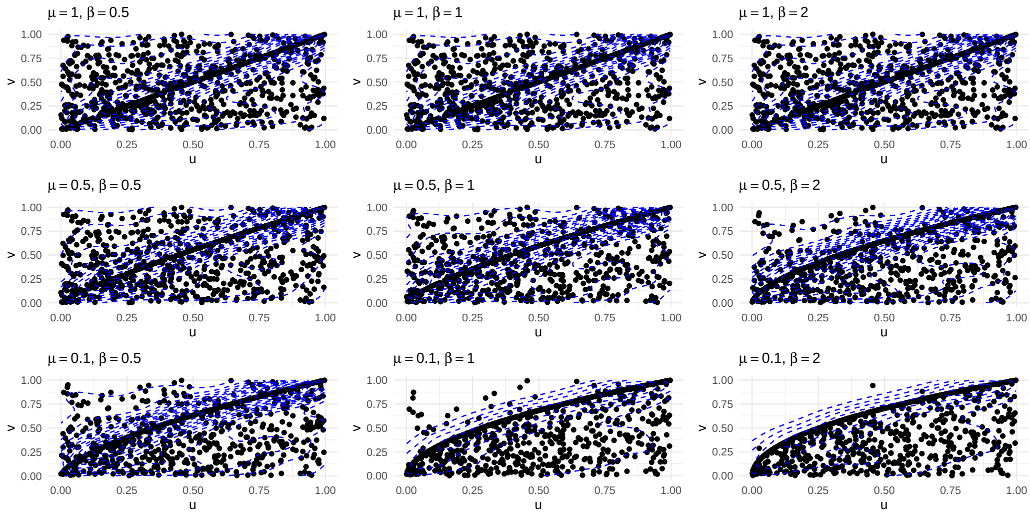

Figure 3. Bivariate data simulated with copula $C_{\alpha_1,\alpha_2(\mu)}(u,v)$ with $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 1$, and different parameter values μ and β.

Measures of Association

In the context of Marshall–Olkin copula models, various measures of association are used to quantify the relationship of random variables. For the copula associated with the model we propose, we present the correlation coefficient and compare it with the correlation coefficient of the random variables X_1 and X_2.

- *Correlation of the Marshall–Olkin copula*

  For the Marshall–Olkin survival copula given in (15), the correlation coefficient is

$$\rho_C = \frac{3\alpha_1\alpha_2(\mu)}{2\alpha_1 + 2\alpha_2 - \alpha_1\alpha_2(\mu)} = \frac{\lambda_3\mu^\beta}{\frac{2}{3}\lambda_1\mu^\beta + \frac{2}{3}\lambda_2 + \lambda_3\mu^\beta}. \tag{16}$$

We note from (16) that the copula correlation is still affected by the scale and singularity parameters of the original marginals. As a result of Lemma 1 we have that the correlation coefficient of the random variables is $\rho = P(X_1 = \mu X_2) = \frac{\lambda_3\mu^\beta}{\lambda_1\mu^\beta+\lambda_2+\lambda_3\mu^\beta}$, showing that the transformation of the copula MOBW-μ increases the correlation; this behavior can be observed in Figure 4. This implies that MOBW-μ does not have the Lancaster's phenomenon. This result is similar to that found by [31] in the study of the correlation structure of the MOBE distribution.

Lancaster's phenomenon holds significant relevance within the realm of neural networks, as expounded upon in [30]. This principle, which posits that any nonlinear transformation of variables leads to a reduction in correlation, has been extensively explored in the academic literature concerning various bivariate distributions. Examples include distributions with Poisson, negative binomial, and gamma marginals, as detailed in references such as [1,31].

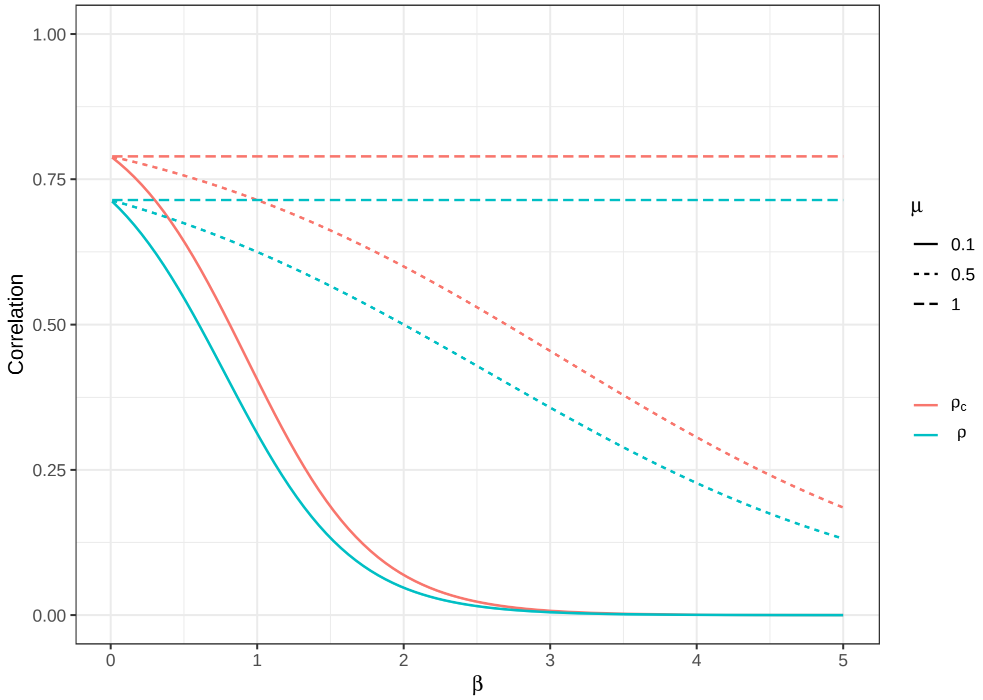

Figure 4. Correlation behavior (ρ) of MOBW-μ and the correlation (ρ_C) of the survival copula $C_{\alpha_1,\alpha_2(\mu)}(u,v)$ for different values of μ and β, with $\lambda_1 = \lambda_2 = 1$, and $\lambda_3 = 5$.

6. Conclusions

In this work, the MOBW-μ model was introduced, which incorporates a delay parameter in the singular part, which allows systems with two components that do not stop working immediately after a common collision to be addressed. This approach extends the versatility of the MOBE model by considering asymmetric behavior between devices X_1 and X_2. Furthermore, the importance of this model in practice is highlighted, since it can be adapted to situations where two components do not fail simultaneously after a common event, which makes it relevant for applications in various areas such as engineering, survival analysis, quality control, and insurance.

Secondly, the contribution of the article is proposing the MOBW-μ model as an extension of the MOBE model, providing a robust theoretical framework to address reliability and survivability problems in bivariate systems. The introduction of a delay parameter μ in the singular part of the model allows the asymmetric dynamics between the components to be better captured, resulting in a more flexible tool applicable to a variety of contexts. The correlation structure investigated for the proposed model and its copula indicate that the correlation of the MOBW-μ is always lower than that of its copula, regardless of the parameters, thus showing that MOBW-μ does not have Lancaster's phenomenon.

These conclusions underline the relevance and potential of the MOBW-μ model to improve the analysis and understanding of systems with two components in various disciplines, offering a more realistic and adaptable approach to practical situations where the dependency between components is crucial.

Future Work

As potential future work, the following topics can be identified:

- Explore the problem of parameter identification.
- Estimation and inference of the parameters of the proposed model with different methods, following the approach of [32–34].
- Incorporation of a Bayesian approach for parameter estimation, following the approach of [35].
- Carry out a simulation study comparing the estimates of the proposed model with modified singularity with the Marshall–Olkin models when the singularity is assumed to be on the straight line $X_1 = X_2$.
- Present applications of Marshall–Olkin models with modified singularity in dependency problems in different areas.
- Incorporate covariates to study bivariate regression models in the context of the MOBW-μ models.

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Abbreviations

The following abbreviations are used in this manuscript:

MOBE	Marshall–Olkin bivariate exponential
MOBW	Marshall–Olkin Weibull bivariate
MOBW-μ	Marshall–Olkin Weibull Bivariate Modified

Appendix A

Proof of Theorem 1. The expressions of $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ can be obtained simply by taking $\frac{\partial^2}{\partial x_1 \partial x_2} S_{X_1,X_2}(x_1, x_2)$ for $x_1 > \mu x_2$ and $x_1 < \mu x_2$ respectively, but naturally, $f_0(.)$ cannot be obtained similarly. To find $f_0(.)$, let us use the fact that

$$\int_0^\infty \int_{\mu x_2}^\infty f_1(x_1, x_2) dx_1 dx_2 + \int_0^\infty \int_{\frac{x_1}{\mu}}^\infty f_2(x_1, x_2) dx_2 dx_1 + \int_0^\infty f_0(x) dx = 1,$$

where

$$\begin{aligned}
I_1 &= \int_0^\infty \int_{\mu x_2}^\infty f_1(x_1, x_2) dx_1 dx_2 \\
&= \int_0^\infty \int_{\mu x_2}^\infty f_{WE}(x_1; \beta, \lambda_1 + \lambda_3) f_{WE}(x_2; \beta, \lambda_2) dx_1 dx_2 \\
&= \int_0^\infty \left(f_{WE}(x_2; \beta, \lambda_2) \int_{\mu x_2}^\infty f_{WE}(x_1; \beta, \lambda_1 + \lambda_3) dx_1 \right) dx_2 \\
&= \int_0^\infty \left(\beta \lambda_2 x_2^{\beta-1} \exp\left\{-\lambda_2 x_2^\beta\right\} \int_{\mu x_2}^\infty \beta(\lambda_1 + \lambda_3) x_1^{\beta-1} \exp\left\{-(\lambda_1 + \lambda_3) x_1^\beta\right\} dx_1 \right) dx_2 \\
&= \int_0^\infty \beta \lambda_2 x_2^{\beta-1} \exp\left\{-\lambda_2 x_2^\beta\right\} \exp\left\{-(\lambda_1 + \lambda_3) \mu^\beta x_2^\beta\right\} dx_2
\end{aligned}$$

$$
\begin{aligned}
I_2 &= \textstyle\int_0^\infty \int_{\frac{x_1}{\mu}}^\infty f_2(x_1,x_2)dx_2dx_1 \\
&= f_{WE}(x_1;\beta,\lambda_1)f_{WE}\left(x_2;\beta,\lambda_2+\lambda_3\mu^\beta\right)dx_2dx_1 \\
&= \int_0^\infty \left(f_{WE}(x_1;\beta,\lambda_1)\int_{\frac{x_1}{\mu}}^\infty f_{WE}\left(x_2;\beta,\lambda_2+\lambda_3\mu^\beta\right)dx_2\right)dx_1 \\
&= \int_0^\infty \left(\beta\lambda_1 x_1^{\beta-1}\exp\left\{-\lambda_1 x_1^\beta\right\}\int_{\frac{x_1}{\mu}}^\infty \beta\left(\lambda_2+\lambda_3\mu^\beta\right)x_2^{\beta-1}\exp\left\{-\left(\lambda_2+\lambda_3\mu^\beta\right)x_2^\beta\right\}dx_2\right)dx_1 \\
&= \int_0^\infty \beta\lambda_1 x_1^{\beta-1}\exp\left\{-\lambda_1 x_1^\beta\right\}\exp\left\{-\left(\lambda_2+\lambda_3\mu^\beta\right)\left(\frac{x_1}{\mu}\right)^\beta\right\}dx_1.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\int_0^\infty f_0(x)dx =& 1-I_1-I_2 \\
=& 1-\int_0^\infty\left[\beta\lambda_2\left(\frac{x}{\mu}\right)^{\beta-1}\exp\left\{-\left(\lambda_1\mu^\beta+\lambda_2+\lambda_3\mu^\beta\right)\left(\frac{x}{\mu}\right)^\beta\right\}\frac{1}{\mu}+\right. \\
&\left.\beta\lambda_1 x^{\beta-1}\exp\left\{-\left(\lambda_1+\frac{\lambda_2}{\mu^\beta}+\lambda_3\right)x^\beta\right\}\right]dx \\
=& 1-\int_0^\infty \beta\left(\lambda_1+\frac{\lambda_2}{\mu^\beta}\right)x^{\beta-1}\exp\left\{-\left(\lambda_1+\frac{\lambda_2}{\mu^\beta}+\lambda_3\right)x^\beta\right\}dx \\
=& \int_0^\infty\left[\beta\left(\lambda_1+\frac{\lambda_2}{\mu^\beta}+\lambda_3\right)x^{\beta-1}\exp\left\{-\left(\lambda_1+\frac{\lambda_2}{\mu^\beta}+\lambda_3\right)x^\beta\right\}-\right. \\
&\left.\beta\left(\lambda_1+\frac{\lambda_2}{\mu^\beta}\right)x^{\beta-1}\exp\left\{-\left(\lambda_1+\frac{\lambda_2}{\mu^\beta}+\lambda_3\right)x^\beta\right\}\right]dx \\
=& \int_0^\infty \beta\lambda_3 x^{\beta-1}\exp\left\{-\left(\lambda_1+\frac{\lambda_2}{\mu^\beta}+\lambda_3\right)x^\beta\right\}dx
\end{aligned}
$$

Thus,

$$
\begin{aligned}
f_0(x) &= \beta\lambda_3 x^{\beta-1}\exp\left\{-\left(\lambda_1+\frac{\lambda_2}{\mu^\beta}+\lambda_3\right)x^\beta\right\} \\
&= \frac{\lambda_3}{\lambda_1+\frac{\lambda_2}{\mu^\beta}+\lambda_3}f_{WE}\left(x;\beta,\lambda_1+\frac{\lambda_2}{\mu^\beta}+\lambda_3\right) \\
&= \frac{\lambda_3\mu^\beta}{\lambda_1\mu^\beta+\lambda_2+\lambda_3\mu^\beta}f_{WE}\left(x;\beta,\lambda_1+\frac{\lambda_2}{\mu^\beta}+\lambda_3\right).
\end{aligned}
$$

□

Note that the function $f_{X_1,X_2}(x_1,x_2)$ may be considered to be a density function for the MOBW-μ distribution if it is understood that the first two terms are the densities with respect to the two-dimensional Lebesgue measure and the third term is a density function with respect to the one-dimensional Lebesgue measure; see for example [36]. It is clear that the joint cumulative distribution function of X_1 and X_2 can be written as a mixture of an absolute continuous part and a singular part, as follows.

Appendix B

Proof of Lemma 1. In an experiment, the three results $X_1 < \mu X_2$, $X_1 > \mu X_2$, and $X_1 = \mu X_2$ are mutually exclusive. Below is the proof of Lemma 1 for each item.

For item 1,

$$\begin{aligned}
P(X_1 < \mu X_2) &= \int_0^{\infty}\int_0^{\mu x_2} f_{WE}(x_1;\beta,\lambda_1) f_{WE}\left(x_2;\beta,\lambda_2+\lambda_3\mu^{\beta}\right)dx_1dx_2 \\
&= \int_0^{\infty} f_{WE}\left(x_2;\beta,\lambda_2+\lambda_3\mu^{\beta}\right)\left[\int_0^{\mu x_2} f_{WE}(x_1;\beta,\lambda_1)dx_1\right]dx_2 \\
&= \int_0^{\infty} \beta\left(\lambda_2+\lambda_3\mu^{\beta}\right)x_2^{\beta-1}\exp\left\{-\left(\lambda_2+\lambda_3\mu^{\beta}\right)x_2^{\beta}\right\}\left(1-\exp\left\{-\lambda_1\mu^{\beta}x_2^{\beta}\right\}\right)dx_2 \\
&= 1-\frac{\lambda_2+\lambda_3\mu^{\beta}}{\lambda_2+(\lambda_1+\lambda_3)\mu^{\beta}} = \frac{\lambda_1\mu^{\beta}}{\lambda_1\mu^{\beta}+\lambda_2+\lambda_3\mu^{\beta}}.
\end{aligned}$$

For item 2,

$$\begin{aligned}
P(\mu X_2 < X_1) &= \int_0^{\infty}\int_0^{\frac{x_1}{\mu}} f_{WE}(x_1;\beta,\lambda_1+\lambda_3) f_{WE}(x_2;\beta,\lambda_2)dx_2dx_1 \\
&= \int_0^{\infty} f_{WE}(x_1;\beta,\lambda_1+\lambda_3)\left[\int_0^{\frac{x_1}{\mu}} f_{WE}(x_2;\beta,\lambda_2)dx_2\right]dx_1 \\
&= \int_0^{\infty} \beta(\lambda_1+\lambda_3)x_1^{\beta-1}\exp\left\{-(\lambda_1+\lambda_3)x_1^{\beta}\right\}\exp\left\{1-\exp\left(-\lambda_2\frac{x_1^{\beta}}{\mu^{\beta}}\right)\right\}dx_1 \\
&= 1-\frac{\lambda_1+\lambda_3}{\lambda_1+\frac{\lambda_2}{\mu^{\beta}}+\lambda_3} = \frac{\lambda_2}{\lambda_1\mu^{\beta}+\lambda_2+\lambda_3\mu^{\beta}}.
\end{aligned}$$

For item 3,

$$\begin{aligned}
P(X_1 = \mu X_2) &= 1-P(X_1<\mu X_2)-P(\mu X_2<X_1) \\
&= 1-\frac{\lambda_1\mu^{\beta}}{\lambda_1\mu^{\beta}+\lambda_2+\lambda_3\mu^{\beta}}-\frac{\lambda_2}{\lambda_1\mu^{\beta}+\lambda_2+\lambda_3\mu^{\beta}} \\
&= \frac{\lambda_3\mu^{\beta}}{\lambda_1\mu^{\beta}+\lambda_2+\lambda_3\mu^{\beta}}.
\end{aligned}$$

For item 4,

$$\begin{aligned}
P(\min(X_1,\mu X_2)>t) &= P\left(X_1>t, X_2>\frac{t}{\mu}\right) = S_{X_1,X_2}\left(t,\frac{t}{\mu}\right) \\
&= \exp\left\{-\lambda_1 t^{\beta}-\lambda_2\left(\frac{t}{\mu}\right)^{\beta}-\lambda_3\max\left(t^{\beta},\left(\mu\frac{t}{\mu}\right)^{\beta}\right)\right\} \\
&= \exp\left\{-\left(\lambda_1+\frac{\lambda_2}{\mu^{\beta}}+\lambda_3\right)t\right\}.
\end{aligned}$$

□

Appendix C

Proof of Lemma 2. We know that

$$\max\left(x_1^{\beta},(\mu x_2)^{\beta}\right) = x_1^{\beta}+(\mu x_2)^{\beta}-\min\left(x_1^{\beta},(\mu x_2)^{\beta}\right). \tag{A1}$$

Substituting (A1) into (6):

$$
\begin{aligned}
S_{X_1,X_2}(x_1,x_2) &= \exp\Big\{-\lambda_1 x_1^\beta - \lambda_2 x_2^\beta - \lambda_3 x_1^\beta - \lambda_3(\mu x_2)^\beta + \lambda_3 \min\Big(x_1^\beta, (\mu x_2)^\beta\Big)\Big\} \\
&= \exp\Big\{-(\lambda_1+\lambda_3)x_1^\beta - \Big(\lambda_2 + \lambda_3\mu^\beta\Big)x_2^\beta + \lambda_3 \min\Big(x_1^\beta, (\mu x_2)^\beta\Big)\Big\} \\
&= S_{X_1}(x_1) S_{X_2}(x_2) \exp\Big\{\lambda_3 \min\Big(x_1^\beta, (\mu x_2)^\beta\Big)\Big\}.
\end{aligned}
$$

Let $u = S_{X_1}(x_1)$, the $u = \exp\Big\{-(\lambda_1+\lambda_3)x_1^\beta\Big\}$ and we arrive at the following: $\exp\Big(\lambda_3 x_1^\beta\Big) = u^{-\frac{\lambda_3}{\lambda_1+\lambda_3}}$ and for $v = S_{X_2}(x_2)$ we have $\exp\Big\{\lambda_3(\mu x_2)^\beta\Big\} = v^{-\frac{\lambda_3\mu^\beta}{\lambda_2+\lambda_3\mu^\beta}}$. Therefore, the survival copula of (X_1, X_2) is given by

$$
\begin{aligned}
C_{\alpha_1,\alpha_2(\mu)}(u,v) &= S_{X_1,X_2}\Big(S_{X_1}^{-1}(u), S_{X_2}^{-1}(v)\Big) = uv\min\left(u^{-\frac{\lambda_3}{\lambda_1+\lambda_3}}, v^{-\frac{\lambda_3\mu^\beta}{\lambda_2+\lambda_3\mu^\beta}}\right) \\
&= uv\min\Big(u^{-\alpha_1}, v^{-\alpha_2(\mu)}\Big) = \min\Big(u^{1-\alpha_1}v, uv^{1-\alpha_2(\mu)}\Big) \\
&= \begin{cases} u^{1-\alpha_1}v, & \text{if} \quad u^{\alpha_1} \geq v^{\alpha_2(\mu)} \\ uv^{1-\alpha_2(\mu)}, & \text{if} \quad u^{\alpha_1} \leq v^{\alpha_2(\mu)} \end{cases}
\end{aligned}
$$

with

$$
\alpha_1 = \frac{\lambda_3}{\lambda_1+\lambda_3}; \quad \alpha_2(\mu) = \frac{\lambda_3\mu^\beta}{\lambda_2+\lambda_3\mu^\beta}. \tag{A2}
$$

□

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