

Article

One-Bit Function Perturbation Impact on Robust Set Stability of Boolean Networks with Disturbances

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Abstract: This paper investigates the one-bit function perturbation (OBFP) impact on the robust set stability of Boolean networks with disturbances (DBNs). Firstly, the dynamics of these networks are converted into the algebraic forms utilizing the semi-tensor product (STP) method. Secondly, OBFP's impact on the robust set stability of DBNs is divided into two situations. Then, by constructing a state set and defining an index vector, several necessary and sufficient conditions to guarantee that a DBN under OBFP can stay robust set stable unchanged are provided. Finally, a biological example is proposed to demonstrate the effectiveness of the obtained theoretical results.

Keywords: boolean networks; disturbances; one-bit function perturbation; robust set stability; semi-tensor product of matrices

MSC: 93D09



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1. Introduction

A Boolean network is a typical binary logical dynamic system, which was first introduced by Kauffman for modeling and analyzing gene regulatory networks (GRNs) [1]. Now, it has attracted extensive attention and has become an effective model of many other complex networks, including wireless sensor networks, neural networks and networked evolutionary games, etc. [2–4]. With the development of practical research problems, BNs have been generalized in many aspects [5,6]. It turns out that when modeling BNs, the undesired signals (or, equivalently, the disturbances) are unavoidable due to internal or external environmental perturbations. For instance, in gene regulatory networks, the disturbances may be gene mutation, the duplication or deletion of fragments in genetic recombination, and external environmental stimuli, etc. [7]. Based on the H_∞ complementary sensitivity function, the design method of a decentralized control scheme was provided in reference [8], which can improve the performance of the system in terms of disturbance rejection and tracking. Considering the model subject to disturbance, the concept of BNs with disturbances (DBNs) was presented. In fact, DBNs serve as models of complex networks, which describe complex biological phenomena and have been widely investigated [9–11].

In classical control theory, there is a fundamental and meaningful issue, that is, the robust set stability of system [12]. The robust set stability of system means that all the initial states can eventually converge to an attractor subset of a given set under arbitrary disturbance sequence. Recently, the semi-tensor product (STP) method has been introduced, which establishes an algebraic state space representation to analyze the robust set stability of DBNs [13]. Specifically, Zhong et al. presented a robust reachable sets method for investigating the robust set stability of DBNs [14]. The robust set stability of probabilistic DBNs was studied in [15], and some efficient criteria for robust set stability were proposed based on the largest robust invariant set. Moreover, STP was also used to study various control problems of DBNs, such as stabilization [16], controllability [17], detectability [18], synchronization [19] and output tracking [20], etc. In addition to DBNs, the STP method

has been also successfully applied to many other logical networks, including switched BNs [21], mix-valued logical networks [22], game theory [23], etc.

It is noted that gene mutation is universal in GRNs. These mutations have a significant impact on organisms. For instance, sickle cell anemia is caused by gene mutation in hemoglobin [24]. When modeling GRNs with BNs, function perturbation is introduced to describe gene mutation [25]. Recently, the effect of function perturbation on the dynamics of logical networks has been widely investigated. Under STP framework, reference [16] studied OBFP impact on the stability of BNs, while references [5,26] first migrated this issue to probabilistic BNs and switched BNs, respectively. Additionally, the effect of OBFP on the robust stability of DBNs was discussed in [27]. The effect of function perturbation on other fundamental problems of logical networks, such as optimal control [28], observability [29], detectability [30], has been investigated. It should be pointed out that function perturbation may change the attractors of the original DBNs. In fact, the robust set stability of DBNs is closely related to the attractors of DBNs, which may also be influenced due to function perturbation. There is a question that comes up naturally: how will the OBFP influence the robust set stability of DBNs? From the perspective of network models, most existing works do not consider disturbances. When the system contains disturbances, the evolution trajectory of each state is not unique, which will make the considered problem more complicated. In terms of the research issue, many references studied the stability problem of the system under OBFP. If the target set is a set rather than a fixed point, the robust set stability must consider the relationship between the trajectories of perturbed state and the target set. However, robust stability does not need to consider this aspect. To our best knowledge, there exist few results about the robust set stability for DBNs subject to function perturbation at present.

In this paper, we aim to propose some criteria to detect the robust set stability for DBNs affected by OBFP. The main contributions are as follows. (i) The concept of OBFP in DBNs is first proposed. Based on the possible relationship between the perturbed state and the largest invariant set, the robustness analyses for the set stability of DBNs are divided into two situations. For the case in which the perturbed state does not belong to the largest invariant set, a necessary and sufficient condition that DBNs stay robust set stable unchanged is derived. The condition to verify whether the DBN is still robust set stable is established, when the perturbed state belongs to the largest invariant set. (ii) Our results can be seen as an extension of reference [27]. When the given set becomes a single point, our results will degenerate into the results of robust stability of DBNs. However, the methods proposed in this article are easier to understand and detect than previous techniques.

The rest of this paper is organized as follows. Section 2 shows some notations and definitions about semi-tensor product, and presents the algebraic state space representation of a BN with disturbances. Section 3 presents the main results. Section 4 provides a biological example to describe the validity of the proposed method, and Section 5 is a brief conclusion.

2. Preliminaries

This section gives some necessary symbols and definitions, as well as algebraic representations of BNs with disturbances.

2.1. Semi-Tensor Product of Matrices

We first give some notations in Table 1.

Some other notations can be explained as follows. If $\text{Col}(A) \subset \Delta_m$, then $A \in \mathcal{M}_{m \times n}$ is called a logical matrix. If $M = [\delta_n^{i_1}, \delta_n^{i_2}, \dots, \delta_n^{i_m}]$, then briefly denote it as $M = \delta_n[i_1 \ i_2 \ \dots \ i_m]$. A matrix $A \in \mathcal{M}_{m \times n}$ is called a Boolean matrix, if all its entries are either 0 or 1. Assume $A = (a_{ij}), B = (b_{ij}) \in \mathcal{B}_{m \times n}$, then $A +_{\mathcal{B}} B = (a_{ij} \vee b_{ij})$, where “ $\vee$ ” represents the logical operator “or”. $(\mathcal{B}) \sum_{i=1}^n L_i := L_1 +_{\mathcal{B}} L_2 +_{\mathcal{B}} \dots +_{\mathcal{B}} L_n$. $A^{(k)} = A^{(k-1)} \times_{\mathcal{B}} A$, where k is a positive integer.

Table 1. Notations.

Notations	Definitions
$\mathbb{Z}_+$	the set of positive integers
$\mathcal{M}_{m \times n}$	the set of $m \times n$ real matrices
A^T	the transpose of matrix A
$\mathcal{D}$	$\{0, 1\}$
$\mathcal{D}^n$	$\underbrace{\mathcal{D} \times \mathcal{D} \times \cdots \times \mathcal{D}}_n$
$\text{Col}_i(A)$	the i -th column of matrix A
$\text{Col}(A)$	the set of columns of matrix A
δ_n^i	$\text{Col}_i(I_n)$
Δ_n	$\{\delta_n^1, \delta_n^2, \dots, \delta_n^n\}$
$\mathbf{0}_n$	$\underbrace{(0, 0, \dots, 0)}_n^T$
$\mathbf{1}_n$	$\underbrace{(1, 1, \dots, 1)}_n^T$
$\mathcal{L}_{m \times n}$	the set of $m \times n$ logical matrices
$\mathcal{B}_{m \times n}$	the set of $m \times n$ Boolean matrices
$[A]_{i,j}$	(i, j) -element of matrix A

Definition 1 ([13]). The semi-tensor product of two matrices $A \in \mathcal{M}_{m \times n}$ and $B \in \mathcal{M}_{p \times t}$ is defined as

$$A \ltimes B = (A \otimes I_{\frac{\alpha}{n}})(B \otimes I_{\frac{\alpha}{p}}),$$

where $\alpha = \text{lcm}(n, p)$ is the least common multiple of n and p , and $\otimes$ is the Kronecker product.

Semi-tensor product is a generalization of ordinary matrix product, and it retains almost all the basic properties of ordinary matrix product. Throughout this paper, we omit the symbol “ $\ltimes$ ”.

Next, we give an example to illustrate how to calculate the semi-tensor product of two matrices by Definition 1.

Example 1. Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 1 & 2 \\ 3 & 2 & 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix},$$

then

$$A \ltimes B = A(B \otimes I_2) = \begin{bmatrix} 3 & 4 & -3 & -5 \\ 4 & 7 & -5 & -8 \\ 5 & 2 & -7 & -4 \end{bmatrix}.$$

For a logical variable $x \in \mathcal{D}$, we define its vector form as $(x, 1 - x)^T$, then there is an equivalence relationship between $\mathcal{D}$ and Δ . It is easy to see that if x_i is the vector form of logical variable X_i , then there is a one-to-one correspondence between $X = (X_1, X_2, \dots, X_n)^T \in \mathcal{D}^n$ and $x = \ltimes_{i=1}^n x_i \in \Delta_{2^n}$. We call x the vector form of X .

Lemma 1 ([13]). Consider a logical mapping $f : \mathcal{D}^n \rightarrow \mathcal{D}$. There exists a unique matrix $M_f \in \mathcal{L}_{2 \times 2^n}$, called the structure matrix of f such that

$$f(X_1, X_2, \dots, X_n) = M_f \ltimes_{i=1}^n x_i,$$

where $x_i \in \Delta$ is the vector form of $X_i \in \mathcal{D}$, $i = 1, 2, \dots, n$.

2.2. Algebraic Representations of BNs with Disturbances

A BN with finite disturbance inputs can be described as

$$X_i(t+1) = f_i(X(t), \xi(t)), \quad i = 1, 2, \dots, n, \quad (1)$$

where $X(t) = (X_1(t), X_2(t), \dots, X_n(t)) \in \mathcal{D}^n$, $\xi(t) = (\xi_1(t), \xi_2(t), \dots, \xi_q(t)) \in \mathcal{D}^q$ are the state and disturbance, respectively, $f_i : \mathcal{D}^{n+q} \rightarrow \mathcal{D}$ are Boolean functions.

Since the number of variables in (1) is finite, and each variable just takes values from a finite set $\mathcal{D}$, we can see that (1) is a standard finite-valued discrete dynamical system.

Denote the vector forms of X_i and ξ_i by x_i and ε_i , respectively, that is, $x_i = (X_i, 1 - X_i)^T$, $\varepsilon_i = (\xi_i, 1 - \xi_i)^T$. By Lemma 1, system (1) can be expressed as

$$x_i(t+1) = M_{f_i} \varepsilon(t) x(t), \quad i = 1, 2, \dots, n, \quad (2)$$

where $x(t) = \times_{i=1}^n x_i \in \Delta_{2^n}$, $\varepsilon(t) = \times_{i=1}^q \varepsilon_i \in \Delta_{2^q}$, and $M_{f_i} \in \mathcal{L}_{2 \times 2^{q+n}}$.

Multiplying the n equations in (2) together yields

$$x(t+1) = L \varepsilon(t) x(t), \quad (3)$$

where $L = M_{f_1} * M_{f_2} * \dots * M_{f_n} \in \mathcal{L}_{2^n \times 2^{q+n}}$, and $*$ is the Khatri–Rao product of matrices. System (3) is called the algebraic form of BN (1), and the matrix $L := [L_1 \ L_2 \ \dots \ L_{2^q}] \in \mathcal{L}_{2^n \times 2^{q+n}}$ is called the state transition matrix of system (3).

For initial state $x(0) \in \Delta_{2^n}$, under disturbance sequence $\varepsilon := \{\varepsilon(t), t \in [0, \tau]\} \subseteq \Delta_{2^q}$, the state of system (3) at time $\tau + 1$ is indicated as $x(\tau + 1; x(0), \varepsilon)$. According to the algebraic form (3), the concept for robust set stability of DBNs is reviewed below [14].

Definition 2 ([14]). System (3) is said to be robustly stable to the nonempty set $\mathcal{W} \subseteq \Delta_{2^n}$, if for any initial state $x(0) \in \Delta_{2^n}$, there exists $\tau \in \mathbb{Z}_+$ such that $x(t; x(0), \varepsilon) \in \mathcal{W}$ for any $t \geq \tau$ and arbitrary disturbance sequence ε .

In the following, a numerical example is provided to check whether the system is robustly stable to set $\mathcal{W}$ via Definition 2.

Example 2. Consider a DBN as follows:

$$x(t+1) = L \varepsilon(t) x(t), \quad (4)$$

where the state transition matrix of system (4) is $L = [L_1 \ L_2]$, $L_1 = \delta_4[2 \ 1 \ 1 \ 3]$ and $L_2 = \delta_4[1 \ 1 \ 2 \ 2]$. In the following, we check whether system (4) will robustly stable to set $\mathcal{W} = \{\delta_4^1, \delta_4^2\}$. For $x(0) = \delta_4^1$, through simple calculations, it holds that

$$\begin{aligned} x(1; x(0) = \delta_4^1, \varepsilon(0) = \delta_2^1) &= L \varepsilon(0) x(0) = L \delta_2^1 \delta_4^1 = L_1 \delta_4^1 = \delta_4^2, \\ x(1; x(0) = \delta_4^1, \varepsilon(0) = \delta_2^2) &= L \varepsilon(0) x(0) = L \delta_2^2 \delta_4^1 = L_2 \delta_4^1 = \delta_4^1, \end{aligned} \quad (5)$$

which means that state δ_4^1 can reach set $\mathcal{W}$ in one step under arbitrary disturbance input ε . Similarly, for $x(0) = \delta_4^2$, we have

$$\begin{aligned} x(1; x(0) = \delta_4^2, \varepsilon(0) = \delta_2^1) &= L \varepsilon(0) x(0) = L \delta_2^1 \delta_4^2 = L_1 \delta_4^2 = \delta_4^1, \\ x(1; x(0) = \delta_4^2, \varepsilon(0) = \delta_2^2) &= L \varepsilon(0) x(0) = L \delta_2^2 \delta_4^2 = L_2 \delta_4^2 = \delta_4^1. \end{aligned} \quad (6)$$

Combining with Equations (5) and (6), we can obtain that the states in set $\mathcal{W}$ will always remain in set $\mathcal{W}$ under arbitrary disturbance sequence ε . Thus, $\mathcal{W}$ is a robust invariant set. For $x(0) = \delta_4^3$, skipping some redundant calculations, it can be obtained that $x(1; x(0) = \delta_4^3, \varepsilon) \in \mathcal{W}$ under arbitrary disturbance input ε .

Next, we will judge whether $x(0) = \delta_4^4$ can be robustly stabilized to the set $\mathcal{W}$. By simple calculations, we obtain $x(1; x(0) = \delta_4^4, \varepsilon(0) = \delta_2^1) = \delta_4^3$ and $x(1; x(0) = \delta_4^4, \varepsilon(0) = \delta_2^2) = \delta_4^2$, which means that state δ_4^4 can reach set $\{\delta_4^2, \delta_4^3\}$ in one step. According to the above discussion, we obtain that both state δ_4^2 and state δ_4^3 can robustly reach set $\mathcal{W}$ in the next step, which means that state δ_4^4 can reach set $\mathcal{W}$ at the second step under arbitrary disturbance sequence ε . It follows that the trajectory of the system (4) starting from any initial state $x(0) \in \Delta_4$ will robustly reach set $\mathcal{W}$ and then stay there forever under arbitrary disturbance sequence ε . Based on Definition 2, system (4) is robustly stable to set $\mathcal{W}$.

As is well known, the largest robust invariant subset of a given set plays an important role in robust set stability. The following criterion is reviewed for the robust set stability of DBNs.

Lemma 2 ([22]). *System (3) is robustly stable to $\mathcal{W}$ if and only if it can be robustly stable to the largest invariant subset of $\mathcal{W}$.*

There are many methods to calculate the largest robust invariant subset; thus, we will not elaborate on them further. For more details, please refer to [10,22]. This paper assumes that the largest robust invariant subset contained in $\mathcal{W}$ is $I(\mathcal{W})$.

When using DBNs to model gene regulatory networks, gene mutation can be regarded as the function perturbation. The definition of one-bit function perturbation is proposed as below.

Definition 3. *The one-bit function perturbation on system (3) is that only one column in the state transition matrix L is changed.*

In order to study the robust set stability of DBNs subject to OBFP, we give a natural assumption as follows.

Assumption 1. *Before OBFP occurs, system (3) is robustly stable to set $\mathcal{W}$.*

3. Main Results

In this section, we devote to discussing the impact of OBFP on the robust set stability of DBNs. First, we analyze the change in state transition matrix when the OBFP occur.

Given $l \in \{1, \dots, 2^{n+q}\}$. Assume that after OBFP occurs, $\text{Col}_l(L)$ is perturbed from $\delta_{2^n}^\gamma$ to $\delta_{2^n}^{\gamma^*}$, where $\gamma \neq \gamma^*$. The state $\delta_{2^n}^{\gamma^*}$ is called the perturbed state. Obviously, the state transition matrix L of (3) is changed into a new matrix $\hat{L} \in \mathcal{L}_{2^n \times 2^{n+q}}$, where

$$\text{Col}_i(\hat{L}) = \begin{cases} \delta_{2^n}^{\gamma^*}, & \text{if } i = l; \\ \text{Col}_i(L), & \text{otherwise.} \end{cases} \quad (7)$$

Hence, system (3) under OBFP becomes the following form:

$$x(t+1) = \hat{L}\varepsilon(t)x(t). \quad (8)$$

Similarly, for initial state $x(0) \in \Delta_{2^n}$, under disturbance sequence $\varepsilon := \{\varepsilon(t), t \in [0, \tau]\} \subseteq \Delta_{2^q}$, the state of system (8) at time $\tau+1$ is indicated as $\hat{x}(\tau+1; x(0), \varepsilon)$.

Lemma 3 ([13]). *For any integer $1 \leq i \leq 2^{n+q}$, there exist unique positive integers $i_1 \in [1, 2^q]$ and $i_2 \in [1, 2^n]$ such that*

$$\delta_{2^{n+q}}^i = \delta_{2^q}^{i_1} \times \delta_{2^n}^{i_2}, \quad (9)$$

where $i = (i_1 - 1)2^n + i_2$.

Set $l = (k^* - 1)2^n + \varphi^*$. It follows from Definition 3 that OBFP only affects state $\delta_{2^n}^{\varphi^*}$ when $\varepsilon = \delta_{2^n}^{k^*}$. Therefore, for any $x = \delta_{2^n}^{\varphi}$ and any $\varepsilon = \delta_{2^n}^k$, if $\varphi \neq \varphi^*$, it follows that

$$L\varepsilon x = \delta_{2^n}^{\alpha_{\varphi}^k} = \hat{L}\varepsilon x. \quad (10)$$

If $\varphi = \varphi^*$ and $k = k^*$, one has

$$L\varepsilon x = \delta_{2^n}^{\gamma} \neq \delta_{2^n}^{\gamma^*} = \hat{L}\varepsilon x. \quad (11)$$

The possible relationship between the perturbed state $\delta_{2^n}^{\gamma^*}$ and the largest robust invariant set $I(\mathcal{W})$ can be divided into two cases: Case 1: $\delta_{2^n}^{\gamma^*} \notin I(\mathcal{W})$; Case 2: $\delta_{2^n}^{\gamma^*} \in I(\mathcal{W})$. Next, we analyze how OBFP affects the robust set stability of system (3) based on the above two cases.

Firstly, let $P = (\mathcal{B}) \sum_{k=1}^{2^q} L_k$ and construct a state set:

$$\Omega_{\varphi^*} = \{\delta_{2^n}^{\theta} : [Q]_{\varphi^*, \theta} > 0\} \cup \{\delta_{2^n}^{\varphi^*}\}, \quad (12)$$

where $Q = (\mathcal{B}) \sum_{i=1}^{2^n} P^{(i)}$. Thus the set Ω_{φ^*} contains all the states that can reach state $\delta_{2^n}^{\varphi^*}$ under some disturbance sequence before OBFP, and includes $\delta_{2^n}^{\varphi^*}$ itself. Then, construct a matrix $\hat{Q} = (\mathcal{B}) \sum_{i=1}^{2^{n+1}} \hat{P}^{(i)}$, where $\hat{P} = (\mathcal{B}) \sum_{k=1}^{2^q} \hat{L}_k$.

Denote the index vector of a given set $\mathcal{W}$ as $J_{\mathcal{W}} \in \mathcal{B}_{2^n \times 1}$, where

$$(J_{\mathcal{W}})_i = \begin{cases} 1, & \text{if } \delta_{2^n}^i \notin \mathcal{W}, \\ 0, & \text{if } \delta_{2^n}^i \in \mathcal{W}, \end{cases}$$

and $(J_{\mathcal{W}})_i$ is the i -th element of $J_{\mathcal{W}}$.

In the light of the set Ω_{φ^*} , matrix $\hat{Q}$ and index vector $J_{\mathcal{W}}$, we provide several criteria to detect whether a BN with arbitrary disturbance is still robustly stable to the set $\mathcal{W}$ after OBFP.

Theorem 1. Suppose Assumption 1 holds and $\delta_{2^n}^{\gamma^*} \notin I(\mathcal{W})$. System (3) under OBFP is still robustly stable to the set $\mathcal{W}$, if and only if one of the following two conditions holds

- (i) $\delta_{2^n}^{\gamma^*} \notin \Omega_{\varphi^*}$,
- (ii) $\delta_{2^n}^{\gamma^*} \in \Omega_{\varphi^*}$, $J_{\Delta_{2^n} \setminus \mathcal{W}}^T \text{Col}_{\varphi^*}(\hat{Q}) = 0$.

Proof. (Necessity) We prove the necessity by contradiction. Assume that $\delta_{2^n}^{\gamma^*} \in \Omega_{\varphi^*}$ and $J_{\Delta_{2^n} \setminus \mathcal{W}}^T \text{Col}_{\varphi^*}(\hat{Q}) \neq 0$. It follows from Assumption 1 that for any initial state $x(0) \in \Delta_{2^n}$, it can robustly reach set $I(\mathcal{W})$ in finite steps before OBFP occurs. This together with $\delta_{2^n}^{\gamma^*} \in \Omega_{\varphi^*}$ means that there exists at least one path from $\delta_{2^n}^{\gamma^*}$ to $I(\mathcal{W})$ include $\delta_{2^n}^{\varphi^*}$ before OBFP. Without loss of generality, we suppose that $\delta_{2^n}^{\gamma^*}$ can be steered to $\delta_{2^n}^{\varphi^*}$ at the s_1 th step and $\delta_{2^n}^{\gamma^*}$ can be steered to $I(\mathcal{W})$ at the s th step, where $s_1 < s$. Then, the path from $\delta_{2^n}^{\gamma^*}$ to $I(\mathcal{W})$ can be described as

$$\delta_{2^n}^{\gamma^*} \xrightarrow{\varepsilon(0)} \dots \xrightarrow{\varepsilon(s_1-1)} \delta_{2^n}^{\varphi^*} \rightarrow \dots \xrightarrow{\varepsilon(s-1)} I(\mathcal{W}), \quad (13)$$

where the corresponding disturbance sequence is $\delta_{2^q}^{k_0}, \dots, \delta_{2^q}^{k_{s_1-1}}, \dots, \delta_{2^q}^{k_{s-1}}$.

Based on Equation (11), one has $\delta_{2^n}^{\gamma^*} = \hat{L}\delta_{2^q}^{k^*}\delta_{2^n}^{\varphi^*}$, that is, $\delta_{2^n}^{\varphi^*}$ can reach $\delta_{2^n}^{\gamma^*}$ in one step under disturbance input $\varepsilon = \delta_{2^q}^{k^*}$ after OBFP. By selecting disturbance sequence $\varepsilon_1 =: \{\delta_{2^q}^{k^*}, \delta_{2^q}^{k_0}, \dots, \delta_{2^q}^{k_{s_1-1}}\}$, we can obtain the following path

$$\delta_{2^n}^{\varphi^*} \rightarrow \delta_{2^n}^{\gamma^*} \rightarrow \dots \rightarrow \delta_{2^n}^{\varphi^*}. \quad (14)$$

There forms a new cycle (14) for system (3) after OBFP.

Since $J_{\Delta_{2^n} \setminus \mathcal{W}}^T \text{Col}_{\varphi^*}(\hat{Q}) \neq 0$, we know that there is a positive integer $\tau \geq 2^n$ satisfying $x(\tau; \delta_{2^n}^{\varphi^*}, \varepsilon) \notin \mathcal{W}$ in the above cycle (14). That is, the state $\delta_{2^n}^{\varphi^*}$ could not be stabilized set $\mathcal{W}$ in finite time after OBFP, which is a contradiction to the fact that system (3) is still robustly stable to set $\mathcal{W}$.

(Sufficiency) Suppose condition (i) holds firstly. By following Assumption 1, for any initial state $x(0) \in \Delta_{2^n}$, it can robustly reach set $I(\mathcal{W})$ in finite steps before OBFP occurs. Hence, for initial state $x(0) = \delta_{2^n}^{\theta}$, the paths from $\delta_{2^n}^{\theta}$ to set $I(\mathcal{W})$ have the following two situations.

- (i) $\delta_{2^n}^{\theta} \notin \Omega_{\varphi^*}$, which implies that set $I(\mathcal{W})$ is reachable from $\delta_{2^n}^{\theta}$ and there exists no path from $\delta_{2^n}^{\theta}$ to set $I(\mathcal{W})$ containing $\delta_{2^n}^{\varphi^*}$, simultaneously.
- (ii) $\delta_{2^n}^{\theta} \in \Omega_{\varphi^*}$, which implies that set $I(\mathcal{W})$ is reachable from $\delta_{2^n}^{\theta}$ and there exists at least one path from $\delta_{2^n}^{\theta}$ to set $I(\mathcal{W})$ containing $\delta_{2^n}^{\varphi^*}$, simultaneously.

First, we discuss $\delta_{2^n}^{\theta} \notin \Omega_{\varphi^*}$. Without loss of generality, we suppose that $I(\mathcal{W})$ is reachable from $\delta_{2^n}^{\theta}$ at the λ th step. The path from $\delta_{2^n}^{\theta}$ to set $I(\mathcal{W})$ is described as

$$\delta_{2^n}^{\theta} \rightarrow \dots \rightarrow x(t) \rightarrow \dots \rightarrow I(\mathcal{W}), \quad (15)$$

where $\varepsilon := \{\varepsilon(t) = \delta_{2^q}^{k_t}, t \in [0, \lambda - 1]\} \subseteq \Delta_{2^q}$, and the $\{x(1), \dots, x(\lambda - 1)\}$ is a sequence of states in the path from $\delta_{2^n}^{\theta}$ to $I(\mathcal{W})$. Apparently, $x(t) \neq \delta_{2^n}^{\varphi^*}, t \in [1, \lambda - 1]$.

After OBFP, it follows from (10) that

$$\begin{aligned} \hat{x}(\lambda; \delta_{2^n}^{\theta}, \varepsilon) &= \hat{L}\varepsilon(\lambda - 1)x(\lambda - 1) = \hat{L}\varepsilon(\lambda - 1)\hat{L}\varepsilon(\lambda - 2)x(\lambda - 2) \\ &= \dots \\ &= \times_{t=\lambda-1}^0 (\hat{L}\varepsilon(t))\delta_{2^n}^{\theta} = L\varepsilon(\lambda - 1)x(\lambda - 1) \\ &= L\varepsilon(\lambda - 1)L\varepsilon(\lambda - 2)x(\lambda - 2) \\ &= \dots \\ &= \times_{t=\lambda-1}^0 (L\varepsilon(t))\delta_{2^n}^{\theta} = x(\lambda; \delta_{2^n}^{\theta}, \varepsilon) \\ &\in I(\mathcal{W}). \end{aligned}$$

Thus, OBFP has no effect on the path (15). Due to the arbitrariness of path (15), it holds $I(\mathcal{W})$ is robustly reachable from any state $\delta_{2^n}^{\theta} \in \Delta_{2^n}$.

In the following, we consider $\delta_{2^n}^{\theta} \in \Omega_{\varphi^*}$. Select an arbitrary path from $\delta_{2^n}^{\theta}$ to $I(\mathcal{W})$ as

$$\begin{aligned} \delta_{2^n}^{\theta} &\rightarrow \dots \rightarrow x(t_1) \rightarrow \dots \rightarrow \delta_{2^n}^{\varphi^*} \rightarrow \delta_{2^n}^{\eta} \rightarrow \\ &\dots \rightarrow x(t_2) \rightarrow \dots \rightarrow I(\mathcal{W}), \end{aligned} \quad (16)$$

where $\varepsilon := \{\varepsilon(t) = \delta_{2^q}^{i_t} : t \in [0, \tau_1 + \tau_2 - 1]\} \subseteq \Delta_{2^q}$, $\tau_1 + \tau_2$ indicates the number of steps from $\delta_{2^n}^{\theta}$ to $I(\mathcal{W})$. Here, the corresponding state sequence in the path from $\delta_{2^n}^{\theta}$ to $\delta_{2^n}^{\eta}$ is $\{x(t_1) = \delta_{2^n}^{i_{t_1}} : t_1 \in [1, \tau_1 - 1]\}$, and the corresponding state sequence from $\delta_{2^n}^{\eta}$ to $I(\mathcal{W})$ is $\{x(t_2) = \delta_{2^n}^{i_{t_2}} : t_2 \in [\tau_1 + 1, \tau_1 + \tau_2 - 1]\}$. Clearly, $x(\tau_1 - 1) = \delta_{2^n}^{\varphi^*}, x(\tau_1) = \delta_{2^n}^{\eta}$ and $x(t) \neq \delta_{2^n}^{\varphi^*}, t \in \{1, \dots, \tau_1 + \tau_2 - 1\} \setminus \{\tau_1 - 1\}$.

In path (16), the relationship between the state $\delta_{2^n}^\eta$ and perturbed state $\delta_{2^n}^{\gamma^*}$ can be divided into two situations: (i) $\delta_{2^n}^\eta \neq \delta_{2^n}^{\gamma^*}$; (ii) $\delta_{2^n}^\eta = \delta_{2^n}^{\gamma^*}$. For $\delta_{2^n}^\eta \neq \delta_{2^n}^{\gamma^*}$, similar to the analysis of path (15), OBFP has no effect on the path (16), and every state $\delta_{2^n}^\eta \in \Delta_{2^n}$ can still robustly reach to $I(\mathcal{W})$ after OBFP. For $\delta_{2^n}^\eta = \delta_{2^n}^{\gamma^*}$, when OBFP occurs, it holds that

$$\begin{cases} \hat{x}(\tau_1 - 1; \delta_{2^n}^\theta, \varepsilon) = \times_{t=\tau_1-2}^0 (\hat{L}\delta_{2q}^{i_t}) \delta_{2^n}^\theta \\ \quad = \times_{t=\tau_1-2}^0 (L\delta_{2q}^{i_t}) \delta_{2^n}^\theta = \delta_{2^n}^{\varphi^*}, \\ \hat{x}(1; \delta_{2^n}^{\varphi^*}, \varepsilon) = \hat{L}\delta_{2q}^{k^*} \delta_{2^n}^{\varphi^*} = \delta_{2^n}^{\gamma^*}. \end{cases} \quad (17)$$

Due to $\delta_{2^n}^{\gamma^*} \notin \Omega_{\varphi^*}$, we know that $\delta_{2^n}^{\varphi^*}$ is not reachable from $\delta_{2^n}^{\gamma^*}$. Combined with Assumption 1, we derive that

$$\begin{aligned} \hat{x}(\tau_3 - 1; \delta_{2^n}^{\gamma^*}, \varepsilon) &= \times_{t=\tau_3-1}^0 (\hat{L}\delta_{2q}^{j_t}) \delta_{2^n}^{\gamma^*} \\ &= \times_{t=\tau_3-1}^0 (L\delta_{2q}^{j_t}) \delta_{2^n}^{\gamma^*} \\ &\in I(\mathcal{W}), \end{aligned} \quad (18)$$

where $\varepsilon := \{\varepsilon(t) = \delta_{2q}^{j_t} : t \in [0, \tau_3 - 1]\} \subseteq \Delta_{2q}$ and τ_3 indicates the number of steps from $\delta_{2^n}^{\gamma^*}$ to $I_s(\mathcal{W})$. Combining with (17) and (18), it can be concluded that

$$\begin{aligned} &\hat{x}(\tau_1 + \tau_3; \delta_{2^n}^\theta, \varepsilon) \\ &= \times_{t=\tau_3-1}^0 (\hat{L}\delta_{2q}^{j_t}) \hat{L}\delta_{2q}^{k^*} \times_{t=\tau_1-2}^0 (\hat{L}\delta_{2q}^{i_t}) \delta_{2^n}^\theta \\ &= \times_{t=\tau_1+\tau_3-1}^0 (\hat{L}\varepsilon(t)) \delta_{2^n}^\theta \\ &\in I(\mathcal{W}), \end{aligned}$$

which shows that path (16) can be rewritten as

$$\begin{aligned} &\delta_{2^n}^\theta \rightarrow \cdots \rightarrow x(t_1) \rightarrow \cdots \\ &\rightarrow \delta_{2^n}^{\varphi^*} \rightarrow \delta_{2^n}^{\gamma^*} \rightarrow \cdots \rightarrow x(t_2) \rightarrow \cdots \rightarrow I(\mathcal{W}) \\ &\quad \downarrow \\ &\delta_{2^n}^{\gamma^*} \rightarrow \cdots \rightarrow x(t_3) \rightarrow \cdots \rightarrow I(\mathcal{W}), \end{aligned} \quad (19)$$

where $\varepsilon := \{\varepsilon(t) = \delta_{2q}^{i_t} : t \in [0, \tau_1 - 2]\} \cup \{\varepsilon(t) = \delta_{2q}^{k^*} : t = \tau_1 - 1\} \cup \{\varepsilon(t) = \delta_{2q}^{j_{t-\tau_1}} : t \in [\tau_1, \tau_1 + \tau_3 - 1]\}$, $\tau_1 + \tau_3$ denotes the number of steps from $\delta_{2^n}^\theta$ to set $I(\mathcal{W})$. Here, $\{x(t_3) : t_3 \in [\tau_1 + 1, \tau_1 + \tau_3 - 1]\}$ is a sequence of states in the path from $\delta_{2^n}^{\gamma^*}$ to set $I(\mathcal{W})$. Due to the arbitrariness of path (19), we obtain that $\delta_{2^n}^\theta$ can robustly reach to $I(\mathcal{W})$ if $\delta_{2^n}^\eta = \delta_{2^n}^{\gamma^*}$. On the basis of the above analysis, for $\delta_{2^n}^\eta \in \Omega_{\varphi^*}$, it holds that set $I(\mathcal{W})$ is still robustly reachable from every state $\delta_{2^n}^\eta \in \Delta_{2^n}$ after OBFP occurs.

To sum up, if condition (i) holds, system (3) is still robustly stable to the set $I(\mathcal{W})$.

Next, we suppose that condition (ii) holds. For any state $\delta_{2^n}^\eta \in \Delta_{2^n}$, we just discuss the situation: $\delta_{2^n}^\theta \in \Omega_{\varphi^*}$ and $\delta_{2^n}^\eta = \delta_{2^n}^{\gamma^*}$. The analysis of other situations is similar to the proof in condition (i). If $\delta_{2^n}^\theta \in \Omega_{\varphi^*}$ and $\delta_{2^n}^\eta = \delta_{2^n}^{\gamma^*}$, without loss of generality, the path from $\delta_{2^n}^\theta$ to $\delta_{2^n}^{\gamma^*}$ can be described as

$$\delta_{2^n}^\theta \rightarrow \cdots \rightarrow x(t_1) \rightarrow \cdots \rightarrow \delta_{2^n}^{\varphi^*} \rightarrow \delta_{2^n}^{\gamma^*}, \quad (20)$$

where $\varepsilon := \{\varepsilon(t) = \delta_{2q}^{l_t} : t \in [0, l_1 - 2]\} \cup \{\varepsilon(t) = \delta_{2q}^{k^*} : t = l_1 - 1\}$, l_1 denotes the number of steps from δ_{2n}^θ to $\delta_{2n}^{\gamma^*}$. This together with $\delta_{2n}^{\gamma^*} \in \Omega_{\varphi^*}$ shows that a new cycle as (14) is formed for system (3) after OBFP. Denote the cycle (14) by $\Lambda = \{\delta_{2n}^{\varphi^*}, \delta_{2n}^{\gamma^*}, \delta_{2n}^{\gamma_1^*} \cdots \delta_{2n}^{\gamma_s^*}\}$.

Since $\int_{\Delta_{2n} \setminus \mathcal{W}} \text{Col}_{\varphi^*}(\hat{Q}) = 0$, we have $\Lambda \subseteq \mathcal{W}$, which means state $\delta_{2n}^\theta \in \Delta_{2n}$ can robustly reach to set $\mathcal{W}$ after OBFP occurs. Due to the arbitrariness of δ_{2n}^θ , by Definition 2, system (3) is still robustly stable to the set $\mathcal{W}$. $\square$

Below we discuss when OBFP is the case 2, that is, $\delta_{2n}^{\gamma^*} \in I(\mathcal{W})$. The following theorem can be drawn.

Theorem 2. Suppose Assumption 1 holds and $\delta_{2n}^{\gamma^*} \in I(\mathcal{W})$. Then, system (3) under OBFP is still robustly stable to the set $\mathcal{W}$.

Proof. The relationship between the perturbed state $\delta_{2n}^{\gamma^*}$ and the set Ω_{φ^*} can be divided into two situations: (1) $\delta_{2n}^{\gamma^*} \notin \Omega_{\varphi^*}$ and (2) $\delta_{2n}^{\gamma^*} \in \Omega_{\varphi^*}$. Below, we demonstrate that OBFP has no effect on the robust set stability of system (3) in the above two situations.

If $\delta_{2n}^{\gamma^*} \notin \Omega_{\varphi^*}$, we can obtain from the proof of the sufficiency of Theorem 1 that the path from δ_{2n}^θ to set $I(\mathcal{W})$ can be described as (15). Moreover, OBFP does not affect the path (15). Combining with Assumption 1, we can conclude that every $\delta_{2n}^\theta \in \Delta_{2n}$ can be stabilized to set $\mathcal{W}$ under arbitrary disturbance sequence.

If $\delta_{2n}^{\gamma^*} \in \Omega_{\varphi^*}$, the path from δ_{2n}^θ to set $I(\mathcal{W})$ can be described as (16). When $\varepsilon(\tau_1 - 1) \neq \delta_{2q}^{k^*}$, the path (16) is not affected by OBFP. When $\varepsilon(\tau_1 - 1) = \delta_{2q}^{k^*}$, after OBFP occurs, the path (16) changes to be

$$\delta_{2n}^\theta \rightarrow \cdots \rightarrow x(t_1) \rightarrow \cdots \delta_{2n}^{\varphi^*} \rightarrow \delta_{2n}^{\gamma^*}. \quad (21)$$

Since $\delta_{2n}^{\gamma^*} \in I(\mathcal{W})$, it follows from the property of the largest robust invariant set that δ_{2n}^θ can be stabilized to set $I(\mathcal{W})$ and stay in set $I(\mathcal{W})$ forever under arbitrary disturbance sequence. Thus, every $\delta_{2n}^\theta \in \Delta_{2n}$ can robustly reach to $I(\mathcal{W})$ after OBFP occurs.

Due to the arbitrariness of δ_{2n}^θ , one has that system (3) is still robustly stable to the set $\mathcal{W}$ under Assumption 1 and Case 2 of OBFP. $\square$

Remark 1. In [27], OBFP impact is only considered for the robust stability problem, where the target set is a single point set. The system is robust stable only when the trajectories of perturbed state cannot form a cycle. If the trajectory of perturbed state does not form a cycle, then the system can maintain robust set stability. However, even if a new cycle is formed, the system may still be robust set stable. As long as the new cycle is in the target set, the robust set stability of the system can be achieved. It is a difficult work to determine whether the new cycle is in the target set, which is the main technical difficulty [16] (see Theorem 1). In addition, this article needs to consider whether the perturbation state belongs to the target set, which is also different from References [16,27] (see Theorem 2).

Since both the robust synchronization and the robust output tracking can be transformed into the robust set stability problems, the results of this paper can be extended to the OBFP impact on the robust synchronization and robust output tracking of DBNs, which indicates that the research content of this work is more general than [16,27].

4. Illustrative Example

In this section, an illustrative example is given to show the impact of OBFP on robust set stability for DBNs.

Example 3. Consider the biological example: a reduced *E. coli* lactose operon network [21]. The six genes, termed *lac* mRNA, the high-concentration lactose, medium-concentration lactose, the extracellular glucose, the extracellular glucose, the high extracellular glucose and the medium extracellular glucose are denoted by state variable X_1 , state variable X_2 , state variable X_3 , input variable U_1 ,

input variable U_2 and input variable U_3 , respectively. An environmental signal is denoted by disturbance input ξ . The network model is

$$\begin{cases} X_1(t+1) = \neg U_1(t) \wedge (X_2(t) \vee X_3(t)), \\ X_2(t+1) = \neg U_1(t) \wedge U_2(t) \wedge X_1(t) \wedge \xi(t), \\ X_3(t+1) = \neg U_1(t) \wedge (U_2(t) \vee (U_3(t) \wedge X_1(t))). \end{cases} \quad (22)$$

Denote $1 \sim \delta_2^1$ and $0 \sim \delta_2^2$, the algebraic form of (22) is represented as

$$x(t+1) = Mu(t)\varepsilon(t)x(t), \quad (23)$$

where $x(t)$, $u(t)$, $\varepsilon(t)$ are the vector forms of $(X_1(t), X_2(t), X_3(t))$, $(U_1(t), U_2(t), U_3(t))$, $\xi(t)$ respectively, and $M = \delta_8[8\ 8\ 8\ 8\ \dots\ 4\ 4\ 4\ 8]$.

A feasible state feedback controller is pre-given as

$$u(t) = \delta_8[5\ 8\ 5\ 5\ 5\ 8\ 5\ 5]x(t). \quad (24)$$

By adding the control (24) into (23), we derive the following closed-loop system:

$$x(t+1) = L\varepsilon(t)x(t), \quad (25)$$

where $L = [L_1, L_2]$ with $L_1 = \delta_8[1\ 4\ 1\ 5\ 3\ 4\ 3\ 7]$ and $L_2 = \delta_8[3\ 4\ 3\ 7\ 3\ 4\ 3\ 7]$.

Given a set $\mathcal{W} = \{\delta_8^1, \delta_8^2, \delta_8^3\}$. By simple calculation, it can be obtained that $I(\mathcal{W}) = \{\delta_8^1, \delta_8^3\}$,

$$P = L_1 + {}_{\mathcal{B}}L_2 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (26)$$

and

$$Q = ({}_{\mathcal{B}}) \sum_{i=1}^8 P^{(i)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (27)$$

Apparently, DBN (25) is robustly stable to set $\{\delta_8^1, \delta_8^3\}$. The state trajectory graph of DBN (25) before OBFP is shown as Figure 1.

In the following, the impact of OBFP on DBNs will be discussed.

(1) After OBFP, $\text{Col}_4(L)$ is perturbed from δ_8^5 to δ_8^8 . Thus, the perturbed state is δ_8^8 . The state transition matrix of system (25) is changed to be $\hat{L} = \delta_8[1\ 4\ 1\ 8\ 3\ 4\ 3\ 7\ 3\ 4\ 3\ 7\ 3\ 4\ 3\ 7]$. By following Lemma 3, we obtain $k^* = 1$ and $\varphi^* = 4$.

According to Equation (27), one has $\Omega_4 = \{\delta_8^2, \delta_8^4, \delta_8^6\}$. Since $\delta_8^8 \notin I(\mathcal{W})$ and $\delta_8^8 \notin \Omega_4$, it follows from Theorem 1 that DBN (25) is still robustly stable to the set $\mathcal{W}$ after OBFP. The corresponding state trajectory graph of dynamics can be described by Figure 2.

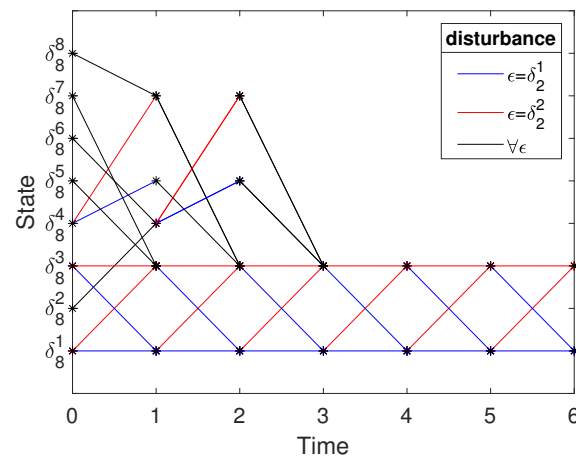


Figure 1. The state trajectory of DBN (25) before function perturbation.

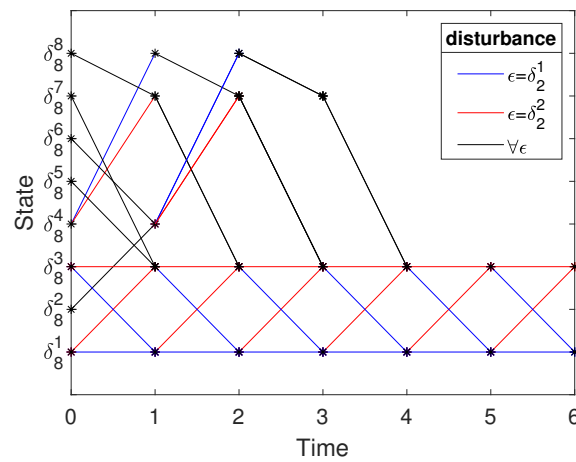


Figure 2. State trajectory graph of dynamics after function perturbation in Case 1.

(2) After OBFP, $\text{Col}_{14}(L)$ is perturbed from δ_8^4 to δ_8^3 , that is the perturbed state is δ_8^3 . From Lemma 3, we obtain $k^* = 2$ and $\varphi^* = 6$. The state transition matrix of system (25) is changed to be $\hat{L} = \delta_8[1 \ 4 \ 1 \ 5 \ 3 \ 4 \ 3 \ 7 \ 3 \ 4 \ 3 \ 7 \ 3 \ 3 \ 7]$.

Since $\delta_8^3 \in I(\mathcal{W})$, we know that OBFP is the Case 2. Therefore, we can conclude from Theorem 2 that DBN (25) under OBFP is still robustly stable to set $\mathcal{W}$. The state trajectory graph of dynamics is described in Figure 3.

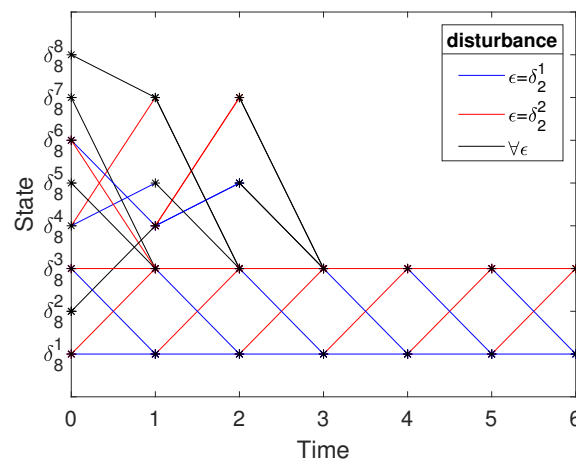


Figure 3. State trajectory graph of dynamics after function perturbation in Case 2.

5. Conclusions

We have investigated robust set stability about DBNs affected by OBFP. Based on the algebraic representation of a DBN, we have provided several necessary and sufficient conditions to guarantee that a DBN under OBFP can stay robust set stable unchanged. In the actual GRNs, gene mutations often occur in a stochastic manner and occur at multiple bits simultaneously. Hence, future work can study multi-bit stochastic function perturbations impact on the behavior of DBNs.

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References

1. Kauffman, S.A. Metabolic stability and epigenesis in randomly constructed genetic nets. *J. Theor. Biol.* **1969**, *22*, 437–467. [[CrossRef](#)] [[PubMed](#)]
2. Nazi, A.; Raj, M.; Di Francesco, M.; Ghosh, P.; Das, S.K. Deployment of robust wireless sensor networks using gene regulatory networks: An isomorphism-based approach. *Pervasive Mob. Comput.* **2014**, *13*, 246–257. [[CrossRef](#)]
3. Wang, L.; Pichler, E.E.; Ross, J. Oscillations and chaos in neural networks: An exactly solvable model. *Proc. Natl. Acad. Sci. USA* **1990**, *87*, 9467–9471. [[CrossRef](#)] [[PubMed](#)]
4. Guo, P.; Wang, Y.; Li, H. Algebraic formulation and strategy optimization for a class of evolutionary networked games via semi-tensor product method. *Automatica* **2013**, *49*, 3384–3389. [[CrossRef](#)]
5. Li, H.; Wang, S.; Li, X.; Zhao, G. Perturbation analysis for controllability of logical control networks. *SIAM J. Control Optim.* **2020**, *58*, 3632–3657. [[CrossRef](#)]
6. Zhong, J.; Yu, Z.; Li, Y.; Lu, J. State estimation for probabilistic Boolean networks via outputs observation. *IEEE Trans. Neural Netw. Learn. Syst.* **2021**, *33*, 4699–4711. [[CrossRef](#)] [[PubMed](#)]
7. Sun, Y.; Feng, G.; Cao, J. Robust stochastic stability analysis of genetic regulatory networks with disturbance attenuation. *Neurocomputing* **2012**, *79*, 39–49. [[CrossRef](#)]
8. KR, A.G.; Mahapatra, S.; Ranjan Mahapatro, S. Design of a decentralized control law for variable area coupled tank systems using H_∞ complimentary sensitivity function. *Asian J. Control* **2024**, *26*, 1540–1552.
9. Sun, Q.; Li, H.; Gao, H. Lebesgue sampling approach to robust stabilization of Boolean control networks with external disturbances. *J. Frankl. Inst.* **2023**, *360*, 2794–2810. [[CrossRef](#)]
10. Li, H.; Xie, L.; Wang, Y. On robust control invariance of Boolean control networks. *Automatica* **2016**, *68*, 392–396. [[CrossRef](#)]
11. Achu Govind, K.; Mahapatra, S.; Mahapatro, S.R. Enhanced decentralised fractional-order control using tree seed optimisation and singular value analysis. *Int. J. Syst. Sci.* **2024**, *55*, 1437–1464. [[CrossRef](#)]
12. Li, L.; Zhang, A.; Lu, J. Robust set stability of probabilistic Boolean networks under general stochastic function perturbation. *Inf. Sci.* **2022**, *582*, 833–849. [[CrossRef](#)]
13. Cheng, D.; Qi, H.; Li, Z. *Analysis and Control of Boolean Networks*; Springer: London, UK, 2011.
14. Zhong, J.; Ho, D.W.; Lu, J.; Xu, W. Global robust stability and stabilization of Boolean network with disturbances. *Automatica* **2017**, *84*, 142–148. [[CrossRef](#)]
15. Wang, J.; Liu, W.; Fu, S.; Xia, J. On robust set stability and set stabilization of probabilistic Boolean control networks. *Appl. Math. Comput.* **2022**, *422*, 126992. [[CrossRef](#)]
16. Li, H.; Yang, X.; Wang, S. Robustness for stability and stabilization of Boolean networks with stochastic function perturbations. *IEEE Trans. Autom. Control* **2020**, *66*, 1231–1237. [[CrossRef](#)]
17. Li, F.; Tang, Y. Pinning controllability for a Boolean network with arbitrary disturbance inputs. *IEEE Trans. Cybern.* **2019**, *51*, 3338–3347. [[CrossRef](#)] [[PubMed](#)]
18. Wang, B.; Feng, J. Detectability of Boolean networks with disturbance inputs. *Syst. Control Lett.* **2020**, *145*, 104783. [[CrossRef](#)]
19. Chen, H.; Liang, J. Local synchronization of interconnected Boolean networks with stochastic disturbances. *IEEE Trans. Neural Netw. Learn. Syst.* **2019**, *31*, 452–463. [[CrossRef](#)]
20. Zhao, Y.; Zhao, X.; Fu, S.; Xia, J. Robust Output Tracking of Boolean Control Networks over Finite Time. *Mathematics* **2022**, *10*, 4078. [[CrossRef](#)]

21. Li, F.; Tang, Y. Set stabilization for switched Boolean control networks. *Automatica* **2017**, *78*, 223–230. [[CrossRef](#)]
22. Wang, J.; Fu, S.; De Leone, R.; Xia, J.; Qiao, L. On robust control invariance and robust set stabilization of mix-valued logical control networks. *Int. J. Robust Nonlinear Control* **2022**, *32*, 10347–10357. [[CrossRef](#)]
23. Fu, S.; Pan, Y.; Feng, J.; Zhao, J. Strategy optimisation for coupled evolutionary public good games with threshold. *Int. J. Control* **2022**, *95*, 562–571. [[CrossRef](#)]
24. Chen, R.; Eshleman, J.; Brodsky, R.; Medof, M. Glycophosphatidylinositol-anchored protein deficiency as a marker of Mutator phenotypes in cancer. *Cancer Res.* **2001**, *61*, 654–658. [[PubMed](#)]
25. Xiao, Y.; Dougherty, E.R. The impact of function perturbations in Boolean networks. *Bioinformatics* **2007**, *23*, 1265–1273. [[CrossRef](#)] [[PubMed](#)]
26. Wu, J.; Liu, Y.; Ruan, Q.; Lou, J. Robust stability of Switched Boolean Networks with function perturbation. *Nonlinear Anal. Hybrid Syst.* **2022**, *46*, 101216. [[CrossRef](#)]
27. Deng, L.; Fu, S.; Wang, J.; Zhang, F. Function Perturbation Impact on Robust Stability and Stabilization of Boolean Networks with Disturbances. *IEEE Access* **2023**, *11*, 84514–84521. [[CrossRef](#)]
28. Li, H.; Yang, X. Robust optimal control of logical control networks with function perturbation. *Automatica* **2023**, *152*, 110970. [[CrossRef](#)]
29. Wang, S.; Li, H. Graph-based function perturbation analysis for observability of multivalued logical networks. *IEEE Trans. Neural Netw. Learn. Syst.* **2020**, *32*, 4839–4848. [[CrossRef](#)]
30. Chen, Q.; Li, H. Robust weak detectability analysis of Boolean networks subject to function perturbation. *IEEE Trans. Circuits Syst. II Express Briefs* **2022**, *69*, 5004–5008. [[CrossRef](#)]

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