Cohomology and Crossed Modules of Modified Rota–Baxter Pre-Lie Algebras

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Abstract: The goal of the present paper is to provide a cohomology theory and crossed modules of modified Rota–Baxter pre-Lie algebras. We introduce the notion of a modified Rota–Baxter pre-Lie algebra and its bimodule. We define a cohomology of modified Rota–Baxter pre-Lie algebras with coefficients in a suitable bimodule. Furthermore, we study the infinitesimal deformations and abelian extensions of modified Rota–Baxter pre-Lie algebras and relate them with the second cohomology groups. Finally, we investigate skeletal and strict modified Rota–Baxter pre-Lie 2-algebras. We show that skeletal modified Rota–Baxter pre-Lie 2-algebras can be classified into the third cohomology group, and strict modified Rota–Baxter pre-Lie 2-algebras are equivalent to the crossed modules of modified Rota–Baxter pre-Lie algebras.

Keywords: pre-Lie algebra; modified Rota–Baxter operator; cohomology; deformation; abelian extension; pre-Lie 2-algebra; crossed module

MSC: 17A01; 17B30; 17B10; 17B38; 17B56

1. Introduction

Cayley [1] first introduced pre-Lie algebras (also called left-symmetric algebras) in the context of rooted tree algebras. Independently, Gerstenhaber [2] also introduced pre-Lie algebras in the deformation theory of rings and algebras. Pre-Lie algebras arose from the study of affine manifolds, affine structures on Lie groups and convex homogeneous cones [3], and then appeared in geometry and physics, such as integrable systems, classical and quantum Yang–Baxter equations [4,5], quantum field theory, Poisson brackets, operands, complex and symplectic structures on Lie groups and Lie algebras [6]. Also see [7–18] for some interesting related studies about pre-Lie algebras.

Rota–Baxter operators on associative algebras were first introduced by Baxter [19] in his study of probability fluctuation theory, and then it was further developed by Rota [20]. The Rota–Baxter operator has been widely used in many fields of mathematics and physics, including combinatorics, number theory, operands and quantum field theory [21]. The cohomology and deformation theory of Rota–Baxter operators of weight zero have been studied on various algebraic structures; see [22–26]. Recently, Wang and Zhou [27] and Das [28] studied Rota–Baxter associative algebras of any weight using different methods. Inspired by Wang and Zhou’s work, Das [29] considered the cohomology and deformations of weighted Rota–Baxter Lie algebras. The authors in [30,31] developed the cohomology, extensions and deformations of Rota–Baxter 3-Lie algebras with any weight. In [32], Chen, Lou and Sun studied the cohomology and extensions of Rota–Baxter Lie triple systems. See also [33] for weighted Rota–Baxter Lie supertriple systems.

The term modified Rota–Baxter operator stemmed from the notion of the modified classical Yang–Baxter equation, which was also introduced in the work of Semenov-Tian-
Shansky [34] as a modification of the operator form of the classical Yang–Baxter equation. Recently, Jiang and Sheng the established cohomology and deformation theory of modified \( r \)-matrices in [35]. Inspired by the modified \( r \) matrix [34,35], due to the importance of pre-Lie algebras, we naturally study modified Rota–Baxter pre-Lie algebras. More precisely, we introduce the notion of a modified Rota–Baxter pre-Lie algebra and its bimodule. We define a cochain map, \( \Upsilon \), and then the cohomology of modified Rota–Baxter pre-Lie algebras with coefficients in a bimodule is constructed. Finally, as applications of our proposed cohomology theory, we consider the infinitesimal deformations and abelian extensions of a modified Rota–Baxter pre-Lie algebra in terms of second cohomology groups. In addition, we further classify skeletal modified Rota–Baxter pre-Lie 2-algebras using the third cohomology group of a modified Rota–Baxter pre-Lie algebra, and show that strict modified Rota–Baxter pre-Lie 2-algebras are equivalent to crossed modules of modified Rota–Baxter pre-Lie algebras.

This paper is organized as follows. In Section 2, we introduce the concept of modified Rota–Baxter pre-Lie algebras, and give its bimodules. In Section 3, we establish the cohomology theory of modified Rota–Baxter pre-Lie algebras with coefficients in a bimodule, and apply it to the study of infinitesimal deformation. In Section 4, we discuss an abelian extension of the modified Rota–Baxter pre-Lie algebras in terms of our second cohomology groups. Finally, in Section 5, we classify skeletal modified Rota–Baxter pre-Lie 2-algebras using the third cohomology group. Then, we introduce the notion of crossed modules of modified Rota–Baxter pre-Lie algebras, and show that strict modified Rota–Baxter pre-Lie 2-algebras are equivalent to crossed modules of modified Rota–Baxter pre-Lie algebras.

Throughout this paper, \( \mathbb{K} \) denotes a field of characteristic zero. All the vector spaces and (multi)linear maps are taken over \( \mathbb{K} \).

2. Bimodules of Modified Rota–Baxter Pre-Lie Algebras

In this section, we introduce the notion of modified Rota–Baxter pre-Lie algebras and give some examples. Next, we propose the bimodule of modified Rota–Baxter pre-Lie algebras. Finally, we establish a new modified Rota–Baxter pre-Lie algebra and give its bimodule.

First, let us recall some definitions and results of pre-Lie algebra and its bimodules from [2,8].

**Definition 1** ([2]). A pre-Lie algebra is a pair \((\mathcal{P}, \bullet)\) consisting of a vector space, \(\mathcal{P}\), and a binary operation, \(\bullet : \mathcal{P} \times \mathcal{P} \to \mathcal{P}\), such that for all \(a, b, c \in \mathcal{P}\), the associator:

\[
(a, b, c) = (a \bullet b) \bullet c - a \bullet (b \bullet c),
\]

is symmetric in \(a, b\), i.e.,

\[
(a, b, c) = (b, a, c), \quad \text{or equivalently,} \quad (a \bullet b) \bullet c - a \bullet (b \bullet c) = (b \bullet a) \bullet c - b \bullet (a \bullet c). \tag{1}
\]

Given a pre-Lie algebra \((\mathcal{P}, \bullet)\), the commutator, \([a, b]^c = a \bullet b - b \bullet a\), defines a Lie algebra structure on \(\mathcal{P}\), which is called the sub-adjacent Lie algebra of \((\mathcal{P}, \bullet)\), and we denote it by \(\mathcal{P}^c\).

Inspired by the modified \( r \)-matrix [34,35], we propose the notion of a modified Rota–Baxter operator on pre-Lie algebras.

**Definition 2.** (i) Let \((\mathcal{P}, \bullet)\) be a pre-Lie algebra. A modified Rota–Baxter operator on \(\mathcal{P}\) is a linear map, \(M : \mathcal{P} \to \mathcal{P}\), subject to the following:

\[
Ma \bullet Mb = M(Ma \bullet b + a \bullet Mb) - a \bullet b \quad \text{for all} \quad a, b \in \mathcal{P}. \tag{2}
\]

Furthermore, the triple \((\mathcal{P}, \bullet, M)\) is called a modified Rota–Baxter pre-Lie algebra, simply denoted by \((\mathcal{P}, M)\).
Then, the triple $(\mathcal{P}, \bullet, \text{id}_\mathcal{P})$ is a modified Rota–Baxter pre-Lie algebra, where $\text{id}_\mathcal{P}: \mathcal{P} \to \mathcal{P}$ is an identity mapping.

**Example 2.** Let $(\mathcal{P}, \bullet)$ be a two-dimensional pre-Lie algebra and $\{e_1, e_2\}$ be a basis, whose nonzero products are given as follows:

$$e_1 \bullet e_2 = e_1, \quad e_2 \bullet e_2 = e_2.$$

Then, the triple $(\mathcal{P}, \bullet, M)$ is a two-dimensional modified Rota–Baxter pre-Lie algebra, where $M = \begin{pmatrix} 1 & k \\ 0 & -1 \end{pmatrix}$, for $k \in \mathbb{K}$.

**Example 3.** Let $(\mathcal{P}, \bullet)$ be a pre-Lie algebra. If a linear map, $M: \mathcal{P} \to \mathcal{P}$, is a modified Rota–Baxter operator, then $-M$ is also a modified Rota–Baxter operator.

**Definition 3 ([16]).** Let $(\mathcal{P}, \bullet)$ be a pre-Lie algebra. A Rota–Baxter operator of weight-1 on $\mathcal{P}$ is a linear map, $R : \mathcal{P} \to \mathcal{P}$, subject to the following:

$$Ra \bullet Rb = R(Ra \bullet b + a \bullet Rb - a \bullet b) \text{ for all } a, b \in \mathcal{P}.$$  

Then, the triple $(\mathcal{P}, \bullet, R)$ is called a Rota–Baxter pre-Lie algebra of weight-1.

**Proposition 1.** Let $(\mathcal{P}, \bullet)$ be a pre-Lie algebra. If a linear map, $R : \mathcal{P} \to \mathcal{P}$, is a Rota–Baxter operator of weight -1, then the map, $2R - \text{id}_\mathcal{P}$, is a modified Rota–Baxter operator on $\mathcal{P}$.

**Proof.** For any $a, b \in \mathcal{P}$, we have the following:

$$
(2R - \text{id}_\mathcal{P})a \bullet (2R - \text{id}_\mathcal{P})b = (2Ra - a) \bullet (2Rb - b) = 4Ra \bullet Rb - 2Ra \bullet b - 2a \bullet Rb + a \bullet b = 4R(Ra \bullet b + a \bullet Rb - a \bullet b) - 2Ra \bullet b - 2a \bullet Rb + a \bullet b = (2R - \text{id}_\mathcal{P})(2(2R - \text{id}_\mathcal{P})a \bullet b + a \bullet (2R - \text{id}_\mathcal{P})b) - a \bullet b.
$$

The proposition follows. \qed

Recall from [16] that a Nijenhuis operator on a pre-Lie algebra $(\mathcal{P}, \bullet)$ is a linear map, $N : \mathcal{P} \to \mathcal{P}$, that satisfies the following,

$$Na \bullet Nb = N(Na \bullet b + a \bullet Nb - N(a \bullet b)),$$

for all $a, b \in \mathcal{P}$. The relationship between the modified Rota–Baxter operator and Nijenhuis operator is as follows, which proves to be obvious.

**Proposition 2.** Let $(\mathcal{P}, \bullet)$ be a pre-Lie algebra and $N : \mathcal{P} \to \mathcal{P}$ be a linear map. If $N^2 = \text{id}_\mathcal{P}$, then $N$ is a Nijenhuis operator if, and only if, $N$ is a modified Rota–Baxter operator.

**Definition 4 ([8]).** Let $(\mathcal{P}, \bullet)$ be a pre-Lie algebra and $V$ a vector space. A bimodule of $\mathcal{P}$ on $V$ consists of a pair $(\bullet_1, \bullet_2)$, where $\bullet_1 : \mathcal{P} \times V \to V$ and $\bullet_2 : V \times \mathcal{P} \to V$ are two linear maps satisfying the following:

$$a \bullet_1 (b \bullet_1 u) - (a \bullet_2 b) \bullet_1 u = b \bullet_1 (a \bullet_1 u) - (b \bullet_1 a) \bullet_1 u,$$

$$a \bullet_2 (u \bullet_2 b) - (a \bullet_1 u) \bullet_2 b = u \bullet_2 (a \bullet_2 b) - (u \bullet_2 a) \bullet_2 b \text{ for all } a, b \in \mathcal{P}, u \in V.$$
Definition 5. A bimodule of the modified Rota–Baxter pre-Lie algebra \((P, \bullet, M)\) is a quadruple \((V; \bullet_r, \bullet_r, M_V)\) such that the following conditions are satisfied:

(i) \((V; \bullet_r, \bullet_r)\) is a bimodule of the pre-Lie algebra \((P, \bullet)\);

(ii) \(M_V : V \to V\) is a linear map satisfying the following equations,

\[
Ma \bullet_r M_V u = M_V (Ma \bullet_l u + a \bullet_l M_V u) - a \bullet_l u, \tag{3}
\]

\[
M_V u \bullet_r Ma = M_V (M_V u \bullet_r a + u \bullet_r Ma) - u \bullet_r a, \tag{4}
\]

for \(a \in P\) and \(u \in V\). In this case, the quadruple \((V; \bullet_r, \bullet_r, M_V)\) is also called a representation over \((P, \bullet, M)\).

Example 4. \((P; \bullet_l = \bullet_r = \bullet_r, M)\) is an adjoint bimodule of the modified Rota–Baxter pre-Lie algebra \((P, \bullet, M)\).

Next, we construct the semidirect product of the modified Rota–Baxter pre-Lie algebra.

Proposition 3. The quadruple \((V; \bullet_r, \bullet_r, M_V)\) is a bimodule of a modified Rota–Baxter pre-Lie algebra \((P, \bullet, M)\) if, and only if, \(P \oplus V\) is a modified Rota–Baxter pre-Lie algebra with the following maps,

\[
(a + u) \bullet_r (b + v) := a \bullet b + a \bullet_v u \bullet_l b,
\]

\[
M \oplus M_V (a + u) = Ma + M_V u,
\]

for \(a \in P\) and \(u \in V\). In the case, the modified Rota–Baxter pre-Lie algebra \(P \oplus V\) is called a semidirect product of \(P\) and \(V\), denoted by \(P \ltimes V = (P \oplus V, \bullet_r, M \oplus M_V)\).

Proof. Firstly, it is easy to verify that \((P \oplus V, \bullet_r)\) is a pre-Lie algebra. In addition, for any \(a, b \in P\) and \(u, v \in V\), via Equations (2)–(4), we have

\[
M \oplus M_V (a + u) \bullet_r M \oplus M_V (b + v) = (Ma + M_V u) \bullet_r (Mb + M_V v)
\]

\[
= Ma \bullet Mb + Ma \bullet_r M_V v + M_V u \bullet_r Mb
\]

\[
= M(Ma \bullet b + a \bullet Mb) - a \bullet b + M_V (Ma \bullet_l u + a \bullet_l M_V u) - a \bullet_l u
\]

\[
+ M_V (M_V u \bullet_r b + u \bullet_r Mb) - u \bullet_r b
\]

\[
= M \oplus M_V ((a + u) \bullet_r M \oplus M_V (b + v) + M \oplus M_V (a + u) \bullet_r (b + v)) - (a + u) \bullet_r (b + v),
\]

which means that \((P \oplus V, \bullet_r, M \oplus M_V)\) is a modified Rota–Baxter pre-Lie algebra. \(\square\)

Proposition 4. Let \((P, \bullet, M)\) be a modified Rota–Baxter pre-Lie algebra. Define a new operation as follows:

\[
a \bullet_M b = Ma \bullet b + a \bullet Mb \text{ for all } a, b \in P.
\]  \(\tag{5}\)

Then, (i) \((P, \bullet_M)\) is a pre-Lie algebra. We denote this pre-Lie algebra as \(P_M\).

(ii) \((P_M, M)\) is a modified Rota–Baxter pre-Lie algebra.
Proof. (i) For any $a, b, c \in \mathcal{P}$, according to Equations (1) and (2), we have the following:

\[
(a \bullet_M b) \bullet_M c - a \bullet_M (b \bullet_M c) = M(Ma \bullet (b + a) - Mb) \bullet c + (Mb \bullet (a + b) \bullet c - Mc) - a \bullet M(Mb \bullet c + b \bullet Mc) = M Mb \bullet a \bullet (Mb \bullet c + a \bullet Mc) - b \bullet M(Ma \bullet c + a \bullet Mc) = (b \bullet_M a) \bullet_M c - b \bullet_M (a \bullet_M c).
\]

Thus, $(\mathcal{P}, \bullet_M)$ is a pre-Lie algebra.

(ii) For any $a, b \in \mathcal{P}$, according to Equation (2), we have

\[
Ma \bullet_M Mb = M^2 a \bullet Mb + Ma \bullet M^2 b \quad \text{and} \quad Mb \bullet_M Ma = M^2 b \bullet Ma + Mb \bullet M^2 a.
\]

Hence, $(\mathcal{P}, \bullet_M)$ is a modified Rota–Baxter pre-Lie algebra.

\[
\text{Proposition 5. Let } (V; \bullet^M, \bullet^l, M_V) \text{ be a bimodule of the modified Rota–Baxter pre-Lie algebra, } (\mathcal{P}, \bullet_M, M). \text{ Define two bilinear maps, } \bullet^M : \mathcal{P} \times V \rightarrow V \text{ and } \bullet^l : V \times \mathcal{P} \rightarrow V, \text{ via the following:}
\]

\[
a \bullet^M u := Ma \bullet u - M_V(a \bullet u), \quad \text{(6)}
\]

\[
u \bullet^l a := u \bullet ma - M_V(u \bullet r a) \text{ for all } a \in \mathcal{P}, u \in V. \quad \text{(7)}
\]

Then, $(V; \bullet^M, \bullet^l, M_V)$ is a bimodule of a pre-Lie algebra $\mathcal{P}_M$. Moreover, $(V; \bullet^M, \bullet^l, M_V)$ is a bimodule of a modified Rota–Baxter pre-Lie algebra $(\mathcal{P}_M, M)$.

\[
\text{Proof. First, by direct verification, we determine that } (V; \bullet^M, \bullet^l, M_V) \text{ is a bimodule of the pre-Lie algebra } \mathcal{P}_M. \text{ Further, for any } a \in \mathcal{P} \text{ and } u \in V, \text{ according to Equation (3), we have the following:}
\]

\[
Ma \bullet^M_M M_V u = M^2 a \bullet r_M v - M (Ma \bullet l_M M_V u) = M_V(M^2 a \bullet u + Ma \bullet M_V u) - Ma \bullet u - M^2 (Ma \bullet l_M M_V u) + M_V(a \bullet u) = M (Ma \bullet u) - M_V (Ma \bullet u - M_V (Ma \bullet u). \quad \text{Similarly, according to Equation (4), there is also } M^2 u \bullet^M_M Ma = M_V(M^2 u \bullet^M_M a + u \bullet^M_M Ma) - u \bullet^M_M a. \text{ Hence, } (V; \bullet^M, \bullet^l, M_V) \text{ is a bimodule of } (\mathcal{P}_M, M). \]

\[
\text{Example 5. } (\mathcal{P}; \bullet^M, \bullet^l, M, M) \text{ is an adjoint bimodule of the modified Rota–Baxter pre-Lie algebra } (\mathcal{P}_M, M), \text{ where}
\]

\[
a \bullet^M_M b := Ma \bullet b - M(a \bullet b), \quad a \bullet^M_M b := a \bullet Mb - M(a \bullet b),
\]

for any $a, b \in \mathcal{P}$.

3. Cohomology of Modified Rota–Baxter Pre-Lie Algebras

In this section, we develop the cohomology of a modified Rota–Baxter pre-Lie algebra with coefficients in its bimodule.
Let us recall the cohomology theory of pre-Lie algebras in [17]. Let \((\mathcal{P}, \cdot)\) be a pre-Lie algebra and \((V; \cdot_j, \bullet_r)\) be a bimodule of it. Denote the \(n\)–cochains of \(\mathcal{P}\) with coefficients in representation \(V\) via the following:

\[
C^n_{\text{PLie}}(\mathcal{P}, V) := \text{Hom}(\mathcal{P}^\otimes n, V).
\]

The coboundary operator \(\delta : C^n_{\text{PLie}}(\mathcal{P}, V) \to C^{n+1}_{\text{PLie}}(\mathcal{P}, V)\), for \(a_1, \ldots, a_{n+1} \in \mathcal{P}\) and \(g \in C^n_{\text{PLie}}(\mathcal{P}, V)\), as follows:

\[
\delta g(a_1, \ldots, a_{n+1}) = \sum_{i=1}^n (-1)^{i+1} a_i \cdot_j g(a_1, \ldots, \hat{a}_i, \ldots, a_{n+1}) + \sum_{i=1}^n (-1)^{i+1} g(a_1, \ldots, a_i, a_{n+1}) \bullet_r a_i
\]

\[
- \sum_{i=1}^n (-1)^{i+1} g(a_1, \ldots, a_i, a_{n+1}) \bullet_r a_i + \sum_{1 \leq i < j \leq n} (-1)^{i+j} g([a_i, a_j]^r, a_1, \ldots, \hat{a}_i, \ldots, \hat{a}_j, a_{n+1}).
\]

Then, it is proven in [17] that \(\delta^2 = 0\). Let us denote, via \(H^n_{\text{PLie}}(\mathcal{P}, V)\), the cohomology group associated to the cochain complex \((C^n_{\text{PLie}}(\mathcal{P}, V), \delta)\).

We first study the cohomology of the modified Rota–Baxter operator.

Let \((\mathcal{P}, \cdot, M)\) be a modified Rota–Baxter pre-Lie algebra and \((V; \cdot_j, \bullet_r, M_V)\) be a bimodule of it. Recall that Proposition 4 and Proposition 5 give a new pre-Lie algebra, \(\mathcal{P}_M\), and a new bimodule, \(V_M = (V; \cdot_M, \bullet_M)\), over \(\mathcal{P}_M\). Consider the cochain complex of \(\mathcal{P}_M\) with coefficients in \(V_M\):

\[
(C^n_{\text{PLie}}(\mathcal{P}_M, V_M), \delta_M) = (\oplus_{n=1}^\infty C^n_{\text{PLie}}(\mathcal{P}_M, V_M), \delta_M).
\]

More precisely, \(C^n_{\text{PLie}}(\mathcal{P}_M, V_M) = \text{Hom}(\mathcal{P}_M^\otimes n, V_M)\) and its coboundary map, \(\delta_M : C^n_{\text{PLie}}(\mathcal{P}_M, V_M) \to C^{n+1}_{\text{PLie}}(\mathcal{P}_M, V_M)\), for \(a_1, \ldots, a_{n+1} \in \mathcal{P}_R\) and \(f \in C^n_{\text{PLie}}(\mathcal{P}_M, V_M)\), are given as follows:

\[
\delta_M f(a_1, \ldots, a_{n+1}) = \sum_{i=1}^n (-1)^{i+1} \left(M a_i \cdot [f(a_1, \ldots, \hat{a}_i, \ldots, a_{n+1}) - M_V (f(a_1, \ldots, \hat{a}_i, \ldots, a_{n+1}))]\right)
\]

\[
+ \sum_{i=1}^n (-1)^{i+1} \left(f(a_1, \ldots, a_i, a_{n+1}) \bullet_r M a_{n+1} - M_V (f(a_1, \ldots, a_i, a_{n+1}) \bullet_r a_{n+1})\right)
\]

\[
- \sum_{i=1}^n (-1)^{i+1} f(a_1, \ldots, \hat{a}_i, \ldots, a_{n+1}, a_i, a_{n+1}) + a_i \cdot a_{n+1}
\]

\[
+ \sum_{1 \leq i < j \leq n} (-1)^{i+j} f(M a_i \cdot a_j + a_i \cdot M a_j, a_{i+1}, \ldots, \hat{a}_i, \ldots, \hat{a}_j, a_{n+1}).
\]

In particular, for \(n = 1\),

\[
\delta_M f(a_1, a_2) = M a_1 \cdot f(a_2) - M_V (f(a_1) \cdot f(a_2)) + f(a_1) \cdot M a_2 - M_V (f(a_1) \cdot f(a_2)) - f(M a_1 \cdot a_2 + a_1 \cdot M a_2).
\]

**Definition 6.** Let \((\mathcal{P}, \cdot, M)\) be a modified Rota–Baxter pre-Lie algebra and \((V; \cdot_j, \bullet_r, M_V)\) be a bimodule of it. Then, the cochain complex \((C^n_{\text{PLie}}(\mathcal{P}_M, V_M), \delta_M)\) is called the cochain complex of the modified Rota–Baxter operator, \(M\), with coefficients in \(V_M\), denoted by \((C^n_{\text{MRBO}}(\mathcal{P}, V), \delta_M)\).

The cohomology of \((C^n_{\text{MRBO}}(\mathcal{P}, V), \delta_M)\), denoted by \(H^n_{\text{MRBO}}(\mathcal{P}, V)\), is called the cohomology of modified Rota–Baxter operator \(M\) with coefficients in \(V_M\).

In particular, when \((\mathcal{P}; \cdot_M = \cdot^M, \bullet^M, M)\) is the adjoint bimodule of \((\mathcal{P}_M, M)\), we denote \((C^n_{\text{MRBO}}(\mathcal{P}, \mathcal{P}), \delta_M)\) as \((C^n_{\text{MRBO}}(\mathcal{P}), \delta_M)\) and call it the cochain complex of modified
Rota–Baxter operator $M$, denote $\mathcal{H}_M^{\ast}(P, P)$ as $\mathcal{H}_M^{\ast}(P)$ and call it the cohomology of modified Rota–Baxter operator $M$.

Next, we will combine the cohomology of pre-Lie algebras and the cohomology of modified Rota–Baxter operators to construct a cohomology theory for modified Rota–Baxter pre-Lie algebras.

Let us construct the following cochain map. For any $n \geq 1$, we define a linear map, $Y : C^n_{\text{PLie}}(P, V) \to C^n_{\text{MRBO}}(P, V)$, via the following:

\[(Yf)(a_1, \ldots, a_n) = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor+1} \sum_{1 \leq j_1 < \cdots < j_{2i-2} \leq n} f(a_1, \ldots, Ma_{j_1}, \ldots, Ma_{j_{2i-2}}, \ldots, a_n)
- \sum_{1 \leq j_1 < \cdots < j_{2i-2} \leq n} M_Mf(a_1, \ldots, Ma_{j_1}, \ldots, Ma_{j_{2i-2}}, \ldots, a_n), \text{if } n \text{ is an even,}\]

\[(Yf)(a_1, \ldots, a_n) = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor+1} \sum_{1 \leq j_1 < \cdots < j_{2i-1} \leq n} f(a_1, \ldots, Ma_{j_1}, \ldots, Ma_{j_{2i-1}}, \ldots, a_n)
- \sum_{1 \leq j_1 < \cdots < j_{2i-1} \leq n} M_Mf(a_1, \ldots, Ma_{j_1}, \ldots, Ma_{j_{2i-1}}, \ldots, a_n), \text{if } n \text{ is an odd.}\]

Among them, when the subscript of $j_{2i-3}$ is negative, $f$ is a zero map. For example, when $n = 1$, according to Equation (11), the map $Y : C^1_{\text{PLie}}(P, V) \to C^1_{\text{MRBO}}(P, V)$ is as follows:

\[(Yf)(a_1) = f(Ma_1) - M_Mf(a_1).\]

**Lemma 1.** Map $Y$ is a cochain map, i.e., $Y \circ \delta = \delta_M \circ Y$. In other words, the following diagram is commutative:

\[
\begin{array}{ccc}
C^1_{\text{PLie}}(P, V) & \overset{\delta}{\longrightarrow} & C^2_{\text{PLie}}(P, V) \\
Y & \downarrow & Y \\
C^1_{\text{MRBO}}(P, V) & \overset{\delta_M}{\longrightarrow} & C^2_{\text{MRBO}}(P, V)
\end{array}
\]

**Proof.** It can be proven by using similar arguments to those in Appendix A in [31]. Here, we only prove the case of $n = 1$. For any $f \in C^1_{\text{PLie}}(P, V)$ and $a, b \in P$, according to Equations (2)–(10) and (12), we have the following:

\[(Y\delta f)(a, b) = (\delta f)(Ma, Mb) - M_V((\delta f)(Ma, b)) = (\delta f)(a, b)\]

\[(\delta_M Y f)(a, b) = Ma \bullet f(Mb) + f(Ma) \bullet Mb - f(Ma \bullet Mb) - M_V(Ma \bullet f(a) + f(Ma) \bullet b - f(Ma \bullet b))\]

and

\[(\delta_M Y f)(a, b) = Ma \bullet f(Mb) + f(Ma) \bullet Mb - f(Ma \bullet Mb) + f(a) \bullet Mb - f(a \bullet Mb)\]

Further comparing Equations (13) and (14), we have (13) = (14). Therefore, $Y \circ \delta = \delta_M \circ Y$. □
Definition 7. Let \((\mathcal{P}, \bullet, M)\) be a modified Rota–Baxter pre-Lie algebra and \((V; \bullet_l, \bullet_r, M_V)\) be a bimodule of \(V\). We attribute the cochain complex \((C^*_\text{MRBPLie}(\mathcal{P}, V), \partial)\) of a modified Rota–Baxter pre-Lie algebra \((\mathcal{P}, \bullet, M)\) with coefficients in \((V; \bullet_l, \bullet_r, M_V)\) to the negative shift in the mapping cone of \(Y\), that is, let
\[
C^1_{\text{MRBPLie}}(\mathcal{P}, V) = C^1_{\text{PLie}}(\mathcal{P}, V) \quad \text{and} \quad C^n_{\text{MRBPLie}}(\mathcal{P}, V) := C^n_{\text{PLie}}(\mathcal{P}, V) \oplus C^{n-1}_{\text{MRBO}}(\mathcal{P}, V) \quad \text{for} \quad n \geq 2.
\]

The coboundary map \(\partial : C^1_{\text{MRBPLie}}(\mathcal{P}, V) \to C^2_{\text{MRBPLie}}(\mathcal{P}, V)\) is given by the following:
\[
\partial(f) = (\delta f, -Yf) \quad \text{for all} \quad f \in C^1_{\text{MRBPLie}}(\mathcal{P}, V);
\]

For \(n \geq 2\), the coboundary map \(\partial : C^n_{\text{MRBPLie}}(\mathcal{P}, V) \to C^{n+1}_{\text{MRBPLie}}(\mathcal{P}, V)\) is given by the following:
\[
\partial(f, g) = (\delta f, -\delta M g - Y f) \quad \text{for all} \quad (f, g) \in C^n_{\text{MRBPLie}}(\mathcal{P}, V).
\]

The cohomology of \((C^*_{\text{MRBPLie}}(\mathcal{P}, V), \partial)\), denoted by \(H^*_{\text{MRBPLie}}(\mathcal{P}, V)\), is called the cohomology of the modified Rota–Baxter pre-Lie algebra \((\mathcal{P}, \bullet, M)\) with coefficients in \((V; \bullet_l, \bullet_r, M_V)\). In particular, when \((V; \bullet_l, \bullet_r, M_V) = (\mathcal{P}; \bullet_l = \bullet_r = \bullet, M_V)\), we just denote \((C^*_{\text{MRBPLie}}(\mathcal{P}, V), \partial)\) and \(H^*_{\text{MRBPLie}}(\mathcal{P}, V)\) by \((C^*_{\text{MRBPLie}}(\mathcal{P}), \partial)\), \(H^*_{\text{MRBPLie}}(\mathcal{P})\), and call them the cochain complex and the cohomology of the modified Rota–Baxter pre-Lie algebra \((\mathcal{P}, \bullet, M)\), respectively.

It is obvious that there is a short exact sequence of cochain complexes:
\[
0 \to C^*_{\text{MRBO}}(\mathcal{P}, V) \longrightarrow C^*_{\text{MRBPLie}}(\mathcal{P}, V) \longrightarrow C^*_{\text{PLie}}(\mathcal{P}, V) \to 0.
\]

This induces a long exact sequence of cohomology groups:
\[
\cdots \to H^n_{\text{MRBPLie}}(\mathcal{P}, V) \to H^n_{\text{PLie}}(\mathcal{P}, V) \to H^n_{\text{MRBO}}(\mathcal{P}, V) \to H^{n+1}_{\text{MRBPLie}}(\mathcal{P}, V) \to H^{n+1}_{\text{PLie}}(\mathcal{P}, V) \to \cdots.
\]

At the end of this section, we use the established cohomology theory to characterize infinitesimal deformations of modified Rota–Baxter pre-Lie algebras.

Definition 8. An infinitesimal deformation of the modified Rota–Baxter pre-Lie algebra \((\mathcal{P}, \bullet, M)\) is a pair \((\bullet_t, M_t)\) of the following forms,
\[
\bullet_t = \bullet + \bullet_t t, \quad M_t = M + M_t t,
\]

such that the following conditions are satisfied:
(i) \((\bullet_t, M_1) \in C^2_{\text{MRBPLie}}(\mathcal{P})\);
(ii) \((\mathcal{P}[\![t]\!], \bullet_t, M_t)\) is a modified Rota–Baxter pre-Lie algebra over \(\mathbb{K}[\![t]\!]\).

Proposition 6. Let \((\mathcal{P}[\![t]\!], \bullet_t, M_t)\) be an infinitesimal deformation of modified Rota–Baxter pre-Lie algebra \((\mathcal{P}, \bullet, M)\). Then, \((\bullet_t, M_1)\) is a 2-cocycle in the cochain complex \((C^*_{\text{MRBPLie}}(\mathcal{P}), \partial)\).

Proof. Suppose \((\mathcal{P}[\![t]\!], \bullet_t, M_t)\) is a modified Rota–Baxter pre-Lie algebra. Then, for any \(a, b, c \in \mathcal{P}\), we have
\[
(a \bullet_t b) \bullet_t c - a \bullet_t (b \bullet_t c) = (b \bullet_t a) \bullet_t c - b \bullet_t (a \bullet_t c),
\]
\[
M_t a \bullet_t M_t b = M_t (M_t a \bullet_t (b + a) \bullet_t M_t b) - a \bullet_t b.
\]

Comparing coefficients of \(t^1\) on both sides of the above equations, we have
\[
(a \bullet_t b) \cdot c + (a \bullet b) \bullet_t c - a \bullet (b \bullet_1 c) - a \bullet_t (b \bullet_1 c) = (b \bullet_t a) \cdot c + (b \bullet a) \bullet_t c - b \bullet (a \bullet_1 c) - b \bullet (a \bullet_1 c),
\]
\[
M_t a \cdot M_t b + M_a \cdot M_t b + M_t b + M_a \cdot M_1 b = M(M_t a \cdot b + M_a \cdot b + a \cdot M_1 b + a \cdot M_1 M_t b) + M_t (M_a \cdot b + a \cdot M_t b) - a \bullet_1 b.
\]
Therefore, $\partial(\bullet_1, M_1) = (\delta \bullet_1, -\delta M_1 - Y \bullet_1) = 0$, that is, $(\bullet_1, M_1)$ is a 2-cocycle. \hfill \Box

**Definition 9.** The 2-cocycle $(\bullet_1, M_1)$ is called the infinitesimal of the infinitesimal deformation $(\mathcal{P}[[t]], \bullet_1, M_1)$ of the modified Rota–Baxter pre-Lie algebra $(\mathcal{P}, \bullet, M)$.

**Definition 10.** Let $(\mathcal{P}[[t]], \bullet_1, M_1)$ and $(\mathcal{P}[[t]], \bullet_1', M_1')$ be two infinitesimal deformations of a modified Rota–Baxter pre-Lie algebra $(\mathcal{P}, \bullet, M)$. An isomorphism from $(\mathcal{P}[[t]], \bullet_1, M_1)$ to $(\mathcal{P}[[t]], \bullet_1', M_1')$ is a linear map, $\varphi_1 = \text{id} + t\varphi_1$, where $\varphi_1 : \mathcal{P} \rightarrow \mathcal{P}$ is a linear map, such that:

\begin{align}
\varphi_1 \circ \bullet_1' &= \bullet_1 \circ (\varphi_1 \otimes \varphi_1), \\
\varphi_1 \circ M_1' &= M_1 \circ \varphi_1.
\end{align}

In this case, we say that the two infinitesimal deformations $(\mathcal{P}[[t]], \bullet_1, M_1)$ and $(\mathcal{P}[[t]], \bullet_1', M_1')$ are equivalent.

**Proposition 7.** The infinitesimals of two equivalent infinitesimal deformations of $(\mathcal{P}, \bullet, M)$ are in the same cohomology class in $H^2_{\text{MRBPLie}}(\mathcal{P})$.

**Proof.** Let $\varphi_1 : (\mathcal{P}[[t]], \bullet_1, M_1) \rightarrow (\mathcal{P}[[t]], \bullet_1', M_1')$ be an isomorphism. By expanding Equations (15) and (16) and comparing the coefficients of $t^i$ on both sides, we have:

$$
\bullet_1' - \bullet_1 = \bullet \circ (\varphi_1 \otimes \text{id}) + \bullet \circ (\text{id} \otimes \varphi_1) - \varphi_1 \circ \bullet = \delta \varphi_1,
$$

$$
M_1' - M_1 = M \circ \varphi_1 - \varphi_1 \circ M = -Y \varphi_1,
$$

that is, we have the following:

$$(\bullet_1', M_1') - (\bullet_1, M_1) = (\delta \varphi_1, -Y \varphi_1) = \partial(\varphi_1) \in B^2_{\text{MRBPLie}}(\mathcal{P}).$$

Therefore, $(\bullet_1', M_1')$ and $(\bullet_1, M_1)$ are cohomologous and belong to the same cohomology class in $H^2_{\text{MRBPLie}}(\mathcal{P})$. \hfill \Box

**4. Abelian Extensions of Modified Rota–Baxter Pre-Lie Algebras**

In this section, we prove that any abelian extension of a modified Rota–Baxter pre-Lie algebra has a bimodule and a 2-cocycle. It is further proven that they are classified by the second cohomology, as one would expect of a good cohomology theory.

**Definition 11.** Let $(\mathcal{P}, \bullet, M)$ be a modified Rota–Baxter pre-Lie algebra and $(V, \bullet_V, M_V)$ be an abelian modified Rota–Baxter pre-Lie algebra with the trivial product $\bullet_V$. An abelian extension $(\hat{\mathcal{P}}, \hat{\bullet}, \hat{M})$ of $(\mathcal{P}, \bullet, M)$ by $(V, \bullet_V, M_V)$ is a short exact sequence of morphisms of modified Rota–Baxter pre-Lie algebras,

$$
0 \longrightarrow (V, \bullet_V, M_V) \xrightarrow{i} (\hat{\mathcal{P}}, \hat{\bullet}, \hat{M}) \xrightarrow{p} (\mathcal{P}, \bullet, M) \longrightarrow 0
$$

such that $\hat{M}u = M_Vu$ and $u \bullet_V v = 0$, for $u, v \in V$, i.e., $V$ is an abelian ideal of $\hat{\mathcal{P}}$.

**Definition 12.** A section of an abelian extension $(\hat{\mathcal{P}}, \hat{\bullet}, \hat{M})$ of $(\mathcal{P}, \bullet, M)$ by $(V, \bullet_V, M_V)$ is a linear map, $s : \mathcal{P} \rightarrow \hat{\mathcal{P}}$, such that $p \circ s = \text{id}_\mathcal{P}$ and $s \circ M = \hat{M} \circ s$. 
Definition 13. Let \((\hat{\mathcal{P}}_1, \hat{\ast}_1, \hat{M}_1)\) and \((\hat{\mathcal{P}}_2, \hat{\ast}_2, \hat{M}_2)\) be two abelian extensions of \((\mathcal{P}, \ast, M)\) by \((V, \ast_V, M_V)\). They are said to be equivalent if there is an isomorphism of modified Rota–Baxter pre-Lie algebras, \(F : (\hat{\mathcal{P}}_1, \hat{\ast}_1, \hat{M}_1) \to (\hat{\mathcal{P}}_2, \hat{\ast}_2, \hat{M}_2)\) such that the following diagram is commutative:

\[
\begin{array}{cccc}
0 & \to & (V, \ast_V, M_V) & \xrightarrow{i_1} & (\hat{\mathcal{P}}_1, \hat{\ast}_1, \hat{M}_1) & \xrightarrow{p_1} & (\mathcal{P}, \ast, M) & \to & 0 \\
\| & & \| & & \| & & \| & & \\
0 & \to & (V, \ast_V, M_V) & \xrightarrow{i_2} & (\hat{\mathcal{P}}_2, \hat{\ast}_2, \hat{M}_2) & \xrightarrow{p_2} & (\mathcal{P}, \ast, M) & \to & 0.
\end{array}
\]  

(17)

Now for an abelian extension \((\hat{\mathcal{P}}, \hat{\ast}, \hat{M})\) of \((\mathcal{P}, \ast, M)\) by \((V, \ast_V, M_V)\) with a section, \(s : \mathcal{P} \to \hat{\mathcal{P}}\), we define two bimodule maps, \(s_1 : \mathcal{P} \times V \to V, s_r : V \times \mathcal{P} \to V\), by

\[a \ast_1 u = s(a) \hat{\ast} u, u \ast_2 a = u \hat{\ast} s(a)\quad\text{for all } a \in \mathcal{P}, u \in V.\]

Proposition 8. With the above notations, \((V; \ast_1, \ast_r, M_V)\) is a bimodule of the modified Rota–Baxter pre-Lie algebra \((\mathcal{P}, \ast, M)\) and does not depend on the choice of \(s\).

Proof. First, for any other section, \(s' : \mathcal{P} \to \hat{\mathcal{P}}\), for any \(a \in \mathcal{P}\), we have the following:

\[p(s(a) - s'(a)) = p(s(a)) - p(s'(a)) = a - a = 0.\]

Thus, there exists an element, \(u \in V\), such that \(s'(a) = s(a) + u\). Note that \(V\) is an abelian ideal of \(\hat{\mathcal{P}}\); this yields the following:

\[s'(x) \ast u = (s(x) + v) \ast u = s(x) \ast u, \quad u \hat{\ast} s'(x) = u \hat{\ast} (s(x) + v) = u \hat{\ast} s(x).\]

This means that \(\ast_1, \ast_r\) does not depend on the choice of \(s\).

Next, for any \(a, b \in \mathcal{P}\) and \(u \in V\), \(V\) is an abelian ideal of \(\hat{\mathcal{P}}\) and \(s(a) \hat{\ast} s(b) - s(a \ast b) \in V\); we have the following:

\[a \ast_1 (b \ast_1 u) - (a \ast b) \ast_1 u = s(a) \hat{\ast} (s(b) \hat{\ast} u) - s(a \ast b) \hat{\ast} u\]

\[= s(a) \hat{\ast} (s(b) \hat{\ast} u) - (s(a) \hat{\ast} s(b)) \hat{\ast} u\]

\[= s(b) \hat{\ast} (s(a) \hat{\ast} u) - (s(b) \hat{\ast} s(a)) \hat{\ast} u\]

\[= b \ast_1 (a \ast_1 u) - (b \ast a) \ast_1 u.\]

By the same token, there is also \(a \ast_1 (u \ast b) - (a \ast_1 u) \ast b\), \(a \ast_1 (a \ast b) - (u \ast a) \ast b\). This shows that \((V; \ast_1, \ast_r)\) is a bimodule of the pre-Lie algebra \((\mathcal{P}, \ast)\).

On the other hand, according to \(\hat{M}s(a) - s(Ma) \in V\), we have the following:

\[Ma \ast_1 M_V u = s(Ma) \ast_M M_V u = \hat{M}s(a) \ast_M M_V u = \hat{M}s(a) \ast_M u = \hat{M} (\hat{M}s(a) \ast_M u + s(a) \ast_M u) - s(a) \ast_M u = M_V (s(Ma) \ast_M u + s(a) \ast_M u) - s(a) \ast_M u = M_V (Ma \ast_1 u + a \ast_1 M_V u) - a \ast_1 u.\]

In the same way, there is also \(M_V u \ast_1 Ma = M_V (M_V u \ast_1 a + u \ast_1 Ma) - u \ast_1 a\). Hence, \((V; \ast_1, \ast_r, M_V)\) is a bimodule of \((\mathcal{P}, \ast, M)\). \(\square\)

Let \((\hat{\mathcal{P}}, \hat{\ast}, \hat{M})\) be an abelian extension of \((\mathcal{P}, \ast, M)\) by \((V, \ast_V, M_V)\) and \(s : \mathcal{P} \to \hat{\mathcal{P}}\) be a section of it. Define the maps \(\omega : \mathcal{P} \times \mathcal{P} \to V\) and \(\chi : \mathcal{P} \to V\) by the following, respectively:

\[\omega(a, b) = s(a) \hat{\ast} s(b) - s(a \ast b),\]

\[\chi(a) = \hat{M}s(a) - s(Ma)\quad\text{for all } a, b \in \mathcal{P}.\]
We transfer the modified Rota–Baxter pre-Lie algebra structure on \( \hat{\mathcal{P}} \) to \( \mathcal{P} \oplus V \) by endowing \( \mathcal{P} \oplus V \) with a multiplication, \( \cdot, \omega \), and a modified Rota–Baxter operator, \( M_\chi \), defined by the following:

\[
(a + u) \cdot_\omega (b + v) = a \cdot b + a \cdot v + u \cdot b + \omega(a, b),
\]

\[
M_\chi(a + u) = Ma + \chi(a) + M_V u \quad \text{for all } a, b \in \mathcal{P}, u, v \in V.
\]

**Proposition 9.** The triple \( (\mathcal{P} \oplus V, \cdot_\omega, M_\chi) \) is a modified Rota–Baxter pre-Lie algebra if, and only if, \( (\omega, \chi) \) is a 2-cocycle of the modified Rota–Baxter pre-Lie algebra \( (\mathcal{P}, \cdot, M) \) with the coefficient in \( (V, \cdot_V, M_V) \). In this case,

\[
0 \xrightarrow{i} (V, \cdot_V, M_V) \xrightarrow{\iota} (\mathcal{P} \oplus V, \cdot_\omega, M_\chi) \xrightarrow{p} (\mathcal{P}, \cdot, M) \xrightarrow{\pi} 0
\]

is an abelian extension.

**Proof.** The triple \( (\mathcal{P} \oplus V, \cdot_\omega, M_\chi) \) is a modified Rota–Baxter pre-Lie algebra if, and only if, for any \( a, b, c \in \mathcal{P} \) and \( u, v, w \in V \), the following equations hold true:

\[
\omega(a, b) \cdot c + \omega (a \cdot b, c) - a \cdot \omega (b, c) - \omega (a, b \cdot c) = \omega (b, a) \cdot c + \omega (b \cdot a, c) - b \cdot \omega (a, c) - \omega (b, a \cdot c),
\]

\[
Ma \cdot_\chi(b) + \chi(a) \cdot Mb + \omega (Ma, Mb) = \chi (Ma \cdot b + a \cdot Mb) + M_V (\chi(a) \cdot b + a \cdot \chi(b) + \omega (Ma, b) + \omega (a, Mb)) - \omega (a, b).
\]

Using Equations (22) and (23), we have \( \partial \omega = 0 \) and \( -\delta M_\chi - Y \omega = 0 \), respectively. Therefore, \( \partial (\omega, \chi) = (\partial \omega, -\delta M_\chi - Y \omega) = 0 \), that is, \( (\omega, \chi) \) is a 2-cocycle.

Conversely, if \( (\omega, \chi) \) is a 2-cocycle of \( (\mathcal{P}, \cdot, M) \) with the coefficient in \( (V, \cdot_V, M_V) \), then we have \( \partial (\omega, \chi) = (\partial \omega, -\delta M_\chi - Y \omega) = 0 \), in which case Equations (20) and (21) hold true. Hence, \( (\mathcal{P} \oplus V, \cdot_\omega, M_\chi) \) is a modified Rota–Baxter pre-Lie algebra. \( \square \)

**Proposition 10.** Let \( (\hat{\mathcal{P}}, \cdot, \hat{M}) \) be an abelian extension of \( (\mathcal{P}, \cdot, M) \) by \( (V, \cdot_V, M_V) \) and \( s \) be a section of it. If the pair \( (\omega, \chi) \) is a 2-cocycle of \( (\mathcal{P}, \cdot, M) \) with the coefficient in \( (V, \cdot_V, M_V) \) constructed using the section \( s \), then its cohomology class does not depend on the choice of \( s \).

**Proof.** Let \( s_1, s_2 : \mathcal{P} \to \hat{\mathcal{P}} \) be two distinct sections; according to Proposition 9, we have two corresponding 2-cocycles, \( (\omega_1, \chi_1) \) and \( (\omega_2, \chi_2) \), respectively. Define a linear map, \( \gamma : \mathcal{P} \to V \), by \( \gamma(a) = s_1(a) - s_2(a) \). Then,

\[
\omega_1(a, b) = s_1(a) \cdot s_1(b) - s_1(a \cdot b)
\]

\[
= (s_2(a) + \gamma(a)) \cdot s_2(b) + \gamma(b)) - (s_2(a \cdot b) + \gamma(a \cdot b))
\]

\[
= s_2(a) \cdot_2 s_2(b) - s_2(a \cdot b) + s_2(a) \cdot_2 \gamma(b) + \gamma(a) \cdot_2 s_2(b) + \gamma(a) \cdot_2 \gamma(b) - \gamma(a \cdot b)
\]

\[
= s_2(a) \cdot_2 s_2(b) - s_2(a \cdot b) + a \cdot_1 \gamma(b) + \gamma(a) \cdot_1 b - \gamma(a \cdot b)
\]

\[
= \omega_2(a, b) + \delta \gamma(a, b)
\]
Theorem 1. Abelian extensions of a modified Rota–Baxter pre-Lie algebra \((P, \bullet, M)\) by \((V, \bullet, \gamma, M_\gamma)\) are classified by the second cohomology group, \(H^2_{\text{MRBPLie}}(P, V)\).

Proof. Assume that \((\hat{P}_1, \hat{\bullet}_1, \hat{M}_1)\) and \((\hat{P}_2, \hat{\bullet}_2, \hat{M}_2)\) are equivalent abelian extensions of \((P, \bullet, M)\) by \((V, \bullet, \gamma, M_\gamma)\) with the associated isomorphism \(F : (\hat{P}_1, \hat{\bullet}_1, \hat{M}_1) \to (\hat{P}_2, \hat{\bullet}_2, \hat{M}_2)\) such that the diagram in (17) is commutative. Let \(s_1\) be a section of \((\hat{P}_1, \hat{\bullet}_1, \hat{M}_1)\). As \(p_2 \circ F = p_1\), we have the following:

\[ p_2 \circ (F \circ s_1) = p_1 \circ s_1 = \text{id}_P. \]

That is, \(F \circ s_1\) is a section of \((\hat{P}_2, \hat{\bullet}_2, \hat{M}_2)\). Denote \(s_2 := F \circ s_1\). Since \(F\) is an isomorphism of modified Rota–Baxter pre-Lie algebras such that \(F|_V = \text{id}_V\), we have the following:

\[
\omega_2(a, b) = s_2(a) \hat{\bullet}_2 s_2(b) - s_2(a \bullet b)
= F \circ s_1(a) \hat{\bullet}_2 F \circ s_1(b) - F \circ s_1(a \bullet b)
= F(s_1(a) \hat{\bullet}_1 s_1(b) - s_1(a \bullet b))
= F(\omega_1(a, b))
= \omega_1(a, b)
\]

and

\[
\chi_2(a) = \hat{M}_2 s_2(a) - s_2(Ma)
= \hat{M}(F \circ s_1(a)) - F \circ s_1(Ma)
= \hat{M}(s_1(a)) - s_1(M(a))
= \chi_1(a).
\]

Thus, two isomorphic abelian extensions give rise to the same element in \(H^2_{\text{MRBPLie}}(P, V)\).

Conversely, given two 2-cocycles \((\omega_1, \chi_1)\) and \((\omega_2, \chi_2)\), we can construct two abelian extensions, \((P \oplus V, \bullet_{\omega_1}, M_{\chi_1})\) and \((P \oplus V, \bullet_{\omega_2}, M_{\chi_2})\), via Proposition 9. If they represent the same cohomological class in \(H^2_{\text{MRBPLie}}(P, V)\), then there is a linear map, \(\iota : P \to V\), such that

\[
(\omega_1, \chi_1) - (\omega_2, \chi_2) = \partial(\iota).
\]

Define a linear map, \(F_i : P \oplus V \to P \oplus V\), by \(F_i(a + u) := a + \iota(a) + u\), \(a \in P, u \in V\). Then, it is easy to verify that \(F_i\) is an isomorphism of the two abelian extensions \((P \oplus V, \bullet_{\omega_1}, M_{\chi_1})\) and \((P \oplus V, \bullet_{\omega_2}, M_{\chi_2})\). \(\square\)
5. Modified Rota–Baxter Pre-Lie 2-Algebras and Crossed Modules

In this section, we introduce the notion of modified Rota–Baxter pre-Lie 2-algebras and show that skeletal modified Rota–Baxter pre-Lie 2-algebras are classified by 3-cocycles of modified Rota–Baxter pre-Lie algebras.

We first recall the notion of pre-Lie 2-algebras from [18], which is a categorification of a pre-Lie algebra.

A pre-Lie 2-algebra is a quintuple, \((\mathcal{P}_0, \mathcal{P}_1, h, l_2, l_3)\), where \(h : \mathcal{P}_1 \to \mathcal{P}_0\) is a linear map, \(l_2 : \mathcal{P}_1 \times \mathcal{P}_1 \to \mathcal{P}_{1+j}\) are bilinear maps and \(l_3 : \mathcal{P}_0 \times \mathcal{P}_0 \times \mathcal{P}_0 \to \mathcal{P}_1\) is a trilinear map, such that for any \(a, b, c, x \in \mathcal{P}_0\) and \(u, v \in \mathcal{P}_1\), the following equations are satisfied:

\[
\begin{align*}
hl_2(a, u) &= l_2(a, h(u)), \\
l_2(u, a) &= l_2(h(u), a), \\
l_2(h(u), v) &= l_2(u, h(v)), \\
l_3(a, b, c) &= l_2(a, l_2(b, c)) - l_2(l_2(a, b), c) - l_2(b, l_2(a, c)) + l_2(l_2(b, a), c), \\
l_3(a, b, h(u)) &= l_2(a, l_2(b, u)) - l_2(l_2(a, b), u) - l_2(b, l_2(a, u)) + l_2(l_2(b, a), u), \\
l_3(h(u), b, c) &= l_2(u, l_2(b, c)) - l_2(l_2(u, b), c) - l_2(b, l_2(u, c)) + l_2(l_2(b, u), c), \\
l_2(x, l_3(a, b, c)) &= l_2(a, l_3(x, b, c)) + l_2(b, l_3(x, a, c)) - l_2(l_3(x, b, a), c)
+ l_2(l_3(x, a, b), c) - l_3(a, l_2(b, x)) + l_3(x, b, l_2(a, c)) - l_3(x, a, l_2(b, c)) - l_3(l_2(x, a) - l_2(a, x), b, c)
+ l_3(l_2(x, b) - l_2(b, x), a, c) - l_3(l_2(a, b) - l_2(b, a), x, c) = 0.
\end{align*}
\]

Motivated by [18] and [26], we propose the notion of a modified Rota–Baxter pre-Lie 2-algebra.

**Definition 14.** A modified Rota–Baxter pre-Lie 2-algebra consists of a pre-Lie 2-algebra, \(\mathcal{P} = (\mathcal{P}_0, \mathcal{P}_1, h, l_2, l_3)\) and a modified Rota–Baxter 2-operator \(\mathfrak{M} = (M_0, M_1, M_2)\) on \(\mathcal{P}\), where \(M_0 : \mathcal{P}_0 \to \mathcal{P}_0\), \(M_1 : \mathcal{P}_1 \to \mathcal{P}_1\) and \(M_2 : \mathcal{P}_0 \times \mathcal{P}_0 \to \mathcal{P}_1\), for any \(a, b, c \in \mathcal{P}_0, u \in \mathcal{P}_1\), satisfying the following equations:

\[
\begin{align*}
M_0 \circ h &= h \circ M_1, \\
h M_2(a, b) + l_2(M_0 a, M_0 b) &= M_0 (l_2(M_0 a, b) + l_2(a, M_0 b)) - l_2(a, b), \\
M_2(h(u), b) + l_2(M_1 u, M_0 b) &= M_1 (l_2(M_1 u, b) + l_2(u, M_0 b)) - l_2(u, b), \\
M_2(a, h(u)) + l_2(M_0 a, M_1 u) &= M_1 (l_2(M_0 a, u) + l_2(a, M_1 u)) - l_2(a, u), \\
M_1 l_2(a, M_2(b, c)) &= M_2(M_0 a, M_2(b, c)) + l_2(M_0 b, M_2(a, c)) - M_1 l_2(b, M_2(a, c)) - l_2(M_2(b, a), M_0 c) + M_1 l_2(M_2(b, a), c) + l_2(M_2(a, b), M_0 c) - M_1 l_2(M_2(a, b), c) - M_2(b, l_2(M_0 a, c) + l_2(a, M_0 c)) - M_2(a, l_2(M_0 b, c) + l_2(b, M_0 c)) + M_1 l_3(M_0 a, b, M_0 c) + l_2(M_0 a, M_0 b, M_0 c) + l_2(M_0 a, M_0 b, M_0 c) - l_3(M_0 a, M_0 b, c) + M_1 l_3(a, M_0 b, M_0 c) + l_3(M_0 a, M_0 b, M_0 c) - l_3(a, M_0 b, M_0 c) + M_1 l_3(a, b, c) = 0.
\end{align*}
\]

We denote a modified Rota–Baxter pre-Lie 2-algebra by \((\mathcal{P}, \mathfrak{M})\).

A modified Rota–Baxter pre-Lie 2-algebra is said to be skeletal (resp. strict) if \(h = 0\) (resp. \(l_3 = 0, M_2 = 0\)).

**Example 6.** For any modified Rota–Baxter pre-Lie algebra, \((\mathcal{P}, \bullet, M), (\mathcal{P}_0 = \mathcal{P}_1 = \mathcal{P}, h = 0, l_2 = \bullet, M_0 = M_1 = M)\) is a strict modified Rota–Baxter pre-Lie 2-algebra.

**Proposition 11.** Let \((\mathcal{P}, \mathfrak{M})\) be a modified Rota–Baxter pre-Lie 2-algebra.
(i) If \((\Psi, \mathfrak{M})\) is skeletal or strict, then \((P_0, \bullet_0, M_0)\) is a modified Rota–Baxter pre-Lie algebra, where \(a \bullet_0 b = l_2(a, b)\) for any \(a, b \in P_0\).

(ii) If \((\Psi, \mathfrak{M})\) is strict, then \((P_1, \bullet_1, M_1)\) is a modified Rota–Baxter pre-Lie algebra, where \(u \bullet_1 v = l_2(b(u), v) = l_2(u, b(v))\) for any \(u, v \in P_1\).

(iii) If \((\Psi, \mathfrak{M})\) is skeletal or strict, then \((P_1; \bullet_1, \bullet_1, M_1)\) is a bimodule of \((P_0, \bullet_0, M_0)\) where \(a \bullet_1 u = l_2(a, u)\) and \(u \bullet_1 a = l_2(u, a)\) for \(a \in P_0, u \in P_1\).

**Proof.** Then, (i), (ii) and (iii) can be directly verified by Equations (24)–(29) and (31)–(34).

**Theorem 2.** There is a one-to-one correspondence between skeletal modified Rota–Baxter pre-Lie 2-algebras and 3-cocycles of modified Rota–Baxter pre-Lie algebras.

**Proof.** Let \((\Psi, \mathfrak{M})\) be a skeletal modified Rota–Baxter pre-Lie 2-algebra. According to Proposition 11, we can consider the cohomology of modified Rota–Baxter pre-Lie algebra to be \((P_0, \bullet_0, M_0)\) with coefficients in the bimodule \((P_1; \bullet_1, \bullet_1, M_1)\). For any \(a, b, c, x \in P_0\), combining Equations (8) and (30), we have the following:

\[
\delta l_3(x, a, b, c) = x \bullet_1 l_3(a, b, c) - a \bullet_1 l_3(x, b, c) + b \bullet_1 l_3(x, a, c) + l_3(a, b, x) \bullet_1 c - l_3(x, b, a) \bullet_1 c + l_3(x, a, b) \bullet_1 c - l_3(a, x, b, c) - l_3(x, a, b, 0) c - l_3(x, 0, a, b) c + l_3(x, 0, 0, b, c) - l_3(0, a, 0, b, c)
\]

\[
=l_2(x, l_3(a, b, c)) - l_2(a, l_3(x, b, c)) + l_2(b, l_3(x, a, c)) + l_2(l_3(a, b, x), c) - l_2(l_3(x, b, a), c) + l_2(l_3(x, a, b), c) - l_3(a, x, b, c) - l_3(l_3(a, b) - l_2(a, b), x, c) - l_3(x, l_2(b, x), a, c) - l_3(l_2(a, b) - l_2(a, b), x, c) = 0.
\]

According to Equations (9) and (35), the following hold true:

\[
(- \delta_M M_2 - Y l_3)(a, b, c) = - \delta_M M_2(a, b, c) - Y l_3(a, b, c)
\]

\[
= - M_0 a \bullet_1 M_2(b, c) + M_1(a \bullet_1 M_2(b, c)) + M_0 b \bullet_1 M_2(a, c) - M_1(b \bullet_1 M_2(a, c)) - M_2(b, a) \bullet_1 M_0 c + M_1(M_2(b, a) \bullet_1 c) + M_2(a, b) \bullet_1 M_0 c - M_1(M_2(a, b) \bullet_1 c) + M_2(b, M_0 a) \bullet_1 c + a \bullet_1 M_0 c - M_2(a, M_0 b) \bullet_1 c + b \bullet_1 M_0 c - l_3(M_0 a, M_0 b, M_0 c)
\]

\[
= - l_2(M_0 a, M_2(b, c)) + M_1 l_2(a, M_2(b, c)) + l_2(M_0 b, M_2(a, c)) - M_1 l_2(b, M_2(a, c)) - l_2(M_2(b, a), M_0 c) + M_1 l_2(M_2(b, a), c) + l_2(M_2(a, b), M_0 c) - M_1 l_2(M_2(a, b), c) - M_2(b, l_2(M_0 a, c) + l_2(a, l_2(c, 0), c) - M_2(a, l_2(M_0 b, c) + l_2(b, l_2(c, 0), c))
\]

\[
= - l_3(M_0 a, b, c) - l_3(a, M_0 b, c) - l_3(a, b, M_0 c) + M_1 l_3(a, b, c) = 0.
\]

Thus, \(\partial l_3, M_2 = (\delta l_3, - \delta_M M_2 - Y l_3) = 0\), that is \((l_3, M_2) \in C^3_{\text{RBPLie}}(P_0, P_1)\) is a 3-cocycle of a modified Rota–Baxter pre-Lie algebra \((P_0, \bullet_0, M_0)\) with coefficients in the bimodule \((P_1; \bullet_1, \bullet_1, M_1)\).

Conversely, suppose that \((l_3, M_2) \in C^3_{\text{RBPLie}}(P, V)\) is a 3-cocycle of a modified Rota–Baxter pre-Lie algebra \((P, \bullet, M)\) with coefficients in the bimodule \((V; \bullet_1, \bullet_1, M_V)\). Then, \((\Psi, \mathfrak{M})\) is a skeletal modified Rota–Baxter pre-Lie 2-algebra, where \(\Psi = (P_0 = P, P_1 = V)\).
According to Equation (26), we have

Theorem 3.

Baxter pre-Lie algebras, $h((l_{\text{pre-Lie algebras}})).$

Define the following two operations on $\mathcal{P}$. 

\[a \cdot b, l_2(a, u) = a \cdot u, l_2(u, a) = u \cdot a \text{ for any } a, b \in \mathcal{P}_0, u \in \mathcal{P}_1. \]

Whitehead [36] introduced the notion of crossed modules in the context of homotopy theory. At the end of this section, we introduce the notion of crossed modules of modified Rota–Baxter pre-Lie algebras and show that they are equivalent to strict modified Rota–Baxter pre-Lie 2-algebras.

**Definition 15.** A crossed module of modified Rota–Baxter pre-Lie algebras is a quadruple $((\mathcal{P}_0, \bullet_0, M_0), (\mathcal{P}_1, \bullet_1, M_1), h, (l_1, l_2))$, where $(\mathcal{P}_0, \bullet_0, M_0)$ and $(\mathcal{P}_1, \bullet_1, M_1)$ are modified Rota–Baxter pre-Lie algebras, $h: \mathcal{P}_1 \rightarrow \mathcal{P}_0$ is a homomorphism of modified Rota–Baxter pre-Lie algebras and $(\mathcal{P}_1; \bullet_1, \bullet_2, M_1)$ is a bimodule of $(\mathcal{P}_0, \bullet_0, M_0)$, for any $a \in \mathcal{P}_0, u, v \in \mathcal{P}_1$, satisfying the following equations:

\[
h(a \cdot u) = a \cdot_0 h(u), h(u \cdot_1 a) = h(u) \cdot_0 a, \quad (36)
\]

\[
h(u) \cdot_1 v = u \cdot_1 h(v) = u \cdot_1 v. \quad (37)
\]

**Example 7.** Let $(\mathcal{P}, \bullet, M)$ be a modified Rota–Baxter pre-Lie algebra, $F$ be its two-sided ideal—that is, $F$ satisfies $\mathcal{F} \bullet \mathcal{F} \subseteq \mathcal{F}, \mathcal{F} \bullet \mathcal{F} \subseteq \mathcal{F}$ and $M(\mathcal{F}) \subseteq \mathcal{F}$—and $\ker: \mathcal{F} \rightarrow \mathcal{P}$ be the inclusion. Then, $(\mathcal{F}, \bullet, F, M, \ker$, in, $(\bullet_1 = \bullet_2 = \bullet_3))$ is a crossed module of modified Rota–Baxter pre-Lie algebras. In particular, $((\mathcal{P}, \bullet, M), (\mathcal{P}, \bullet, M), \id_{\mathcal{P}}, (\bullet_1 = \bullet_2 = \bullet_3))$ is a crossed module of modified Rota–Baxter pre-Lie algebras.

**Example 8.** Let $F: (\mathcal{P}_1, \bullet_1, M_1) \rightarrow (\mathcal{P}_0, \bullet_0, M_0)$ be a homomorphism of modified Rota–Baxter pre-Lie algebras. Then, $\ker(F)$ is a two-sided ideal of $(\mathcal{P}_1, \bullet_1, M_1)$. Thus, according to Example 7, $((\mathcal{P}_1, \bullet_1, M_1), (\ker(F), \bullet_2, M_1|_{\ker(F)}), \in, (\bullet_1 = \bullet_2 = \bullet_3))$ is a crossed module of modified Rota–Baxter pre-Lie algebras.

**Example 9.** Let $(V, \bullet_1, \bullet_2, \bullet_3)$ be a bimodule over a modified Rota–Baxter pre-Lie algebra $(\mathcal{P}, \bullet, M)$. Endow $V$ with the trivial pre-Lie algebra structure, $\bullet_3 = 0$; in this case, $((\mathcal{P}, \bullet, M), (V, \bullet_1, \bullet_2, \bullet_3), 0, (\bullet_1, \bullet_2))$ is a crossed module of modified Rota–Baxter pre-Lie algebras.

**Theorem 3.** There is a one-to-one correspondence between strict modified Rota–Baxter pre-Lie 2-algebras and crossed modules of modified Rota–Baxter pre-Lie algebras.

**Proof.** Let $((\mathcal{P}_0, \mathcal{P}_1, h, l_2, l_3 = 0), (M_0, M_1, M_2 = 0))$ be a strict modified Rota–Baxter pre-Lie 2-algebra. Define the following two operations on $\mathcal{P}_0$ and $\mathcal{P}_1$:

\[
a \cdot_0 b = l_2(a, b),
\]

\[
u \cdot_1 v = l_2(h(u), v) = l_2(u, h(v)) \quad \text{for all } a, b \in \mathcal{P}_0, u, v \in \mathcal{P}_1.
\]

It is straightforward to see that both $(\mathcal{P}_0, \bullet_0, M_0)$ and $(\mathcal{P}_1, \bullet_1, M_1)$ are modified Rota–Baxter pre-Lie algebras. $l_2$ also gives rise to two maps: $\bullet: \mathcal{P}_0 \times \mathcal{P}_1 \rightarrow \mathcal{P}_1, \bullet: \mathcal{P}_1 \times \mathcal{P}_0 \rightarrow \mathcal{P}_1$ by

\[
a \bullet_0 u = l_2(a, u), u \bullet_1 a = l_2(u, a) \quad \text{for all } a \in \mathcal{P}_0, u \in \mathcal{P}_1.
\]

According to (33) and (34), we deduce that $(\mathcal{P}_1, \bullet_0, \bullet_1, M_1)$ is a bimodule of $(\mathcal{P}_0, \bullet_0, M_0)$. According to Equation (26), we have

\[
h(u \bullet_1 v) = h(l_2(u), v) = h(l_2(h(u), v)) = h(u) \cdot_0 h(v),
\]
which implies that \( h \) is a homomorphism of modified Rota–Baxter pre-Lie algebras. Furthermore, we have
\[
\begin{align*}
    h(a \bullet_1 u) &= h l_2(a, u) = l_2(a, h(u)) = a \bullet_0 h(u), \\
    h(u \bullet_2 a) &= h l_2(u, a) = l_2(h(u), a) = h(u) \bullet_0 a, \\
    h(u) \bullet_1 v &= l_2(h(u), v) = l_2(u, h(v)) = u \bullet_0 h(v) = u \bullet_1 v.
\end{align*}
\]
Thus, we obtain a crossed module of modified Rota–Baxter pre-Lie algebras.

Conversely, a crossed module of modified Rota–Baxter pre-Lie algebras \( ((\mathcal{P}_0, \bullet_0, M_0), (\mathcal{P}_1, \bullet_1, M_1), h, (\bullet_2, \bullet)) \) gives rise to a strict modified Rota–Baxter pre-Lie 2-algebra \( ((\mathcal{P}_0, \mathcal{P}_1, h, l_2, l_3 = 0), (M_0, M_1, M_2 = 0)) \), in which \( l_2 : g_i \times g_j \to g_{i+j} \) are given by
\[
\begin{align*}
    l_2(a, b) &= a \bullet_0 b, \\
    l_2(u, v) &= u \bullet_1 v, \\
    l_2(a, u) &= a \bullet_1 u, \\
    l_2(u, a) &= u \bullet_0 a,
\end{align*}
\]
for all \( a \in \mathcal{P}_0, u, v \in \mathcal{P}_1 \). The crossed module equations give various equations for strict modified Rota–Baxter pre-Lie 2-algebras. The proof is completed. \( \square \)

6. Conclusions

In the current research, we mainly study a modified Rota–Baxter pre-Lie algebra, which includes a modified Rota–Baxter operator and a pre-Lie algebra. More precisely, we introduce the bimodule of a modified Rota–Baxter pre-Lie algebra. We show that a modified Rota–Baxter pre-Lie algebra induces a pre-Lie algebra, and the bimodule of a modified Rota–Baxter pre-Lie algebra induces the bimodule of a pre-Lie algebra. Considering this fact, we define the cohomology of a modified Rota–Baxter operator on a pre-Lie algebra. Using the cohomology of pre-Lie algebras, we construct a cochain map, and the cohomology of modified Rota–Baxter pre-Lie algebras is defined. We study infinitesimal deformations of modified Rota–Baxter pre-Lie algebras and show that equivalent infinitesimal deformations are in the same second cohomology group. We investigate abelian extensions of modified Rota–Baxter pre-Lie algebras by using the second cohomology group. Additionally, the notion of modified Rota–Baxter pre-Lie 2-algebra is introduced, which is the categorization of a modified Rota–Baxter pre-Lie algebra. We study the skeletal modified Rota–Baxter pre-Lie 2-algebras using the third cohomology group. Finally, we introduce the notion of crossed modules of modified Rota–Baxter pre-Lie algebras, give some examples, and prove that strict modified Rota–Baxter pre-Lie 2-algebras are equivalent to crossed modules of modified Rota–Baxter pre-Lie algebras.

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References

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