

## Article

# Coefficient Functionals of Sakaguchi-Type Starlike Functions Involving Caputo-Type Fractional Derivatives Subordinated to the Three-Leaf Function

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**Abstract:** A challenging part of studying geometric function theory is figuring out the sharp boundaries for coefficient-related problems that crop up in the Taylor–Maclaurin series of univalent functions. Using Caputo-type fractional derivatives to define the families of Sakaguchi-type starlike functions with respect to symmetric points, this article aims to investigate the first three initial coefficient estimates, the bounds for various problems such as Fekete–Szegő inequality, and the Zalcman inequalities, by subordinating to the function of the three leaves domain. Fekete–Szegő-type inequalities and initial coefficients for functions of the form  $\mathcal{H}^{-1}$  and  $\frac{\zeta}{\mathcal{H}(\zeta)}$  and  $\frac{1}{2} \log\left(\frac{\mathcal{H}(\zeta)}{\zeta}\right)$  connected to the three leaves functions are also discussed.

**Keywords:** analytic functions; subordination; three-leaf function; Caputo-type fractional derivatives; convolution; coefficient inequalities; Fekete–Szegő functional; Krushkal inequality

**MSC:** 30C45; 33C50

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## 1. Introduction, Definitions, and Preliminaries

Let  $\mathbb{U}_d = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$  be an open unit disc of the complex plane  $\mathbb{C}$ , and let  $\mathcal{A}$  be the space of all analytical functions in  $\mathbb{U}_d$ . An analytic function  $\mathcal{H}$  is called univalent (a conformal map) if it is injective. The class  $\mathcal{S}$  consists of univalent functions  $\mathcal{H}$  in  $\mathbb{U}_d$ , and is normalized by  $\mathcal{H}'(0) - 1 = 0 = \mathcal{H}(0)$ . Thus, the power series representation of  $\mathcal{H} \in \mathcal{S}$  is given by

$$\mathcal{H}(\zeta) = \zeta + \sum_{t \geq 2} d_t \zeta^t \quad (\zeta \in \mathbb{U}_d) \quad (1)$$

Due to the Riemann Mapping theorem [1], we know that to study the univalent functions in simply connected domains, it is enough to look at these functions at  $\mathbb{U}_d$ . Moreover, they are unique if they satisfy  $\mathcal{H}'(0) - 1 = 0 = \mathcal{H}(0)$ . Thus, we shall pay special attention to the class  $\mathcal{S}$  of univalent and analytic functions. Geometrically speaking, studying  $g$  rather than  $\mathcal{H}$  corresponds to first translating univalent functions theory the image domain by the vector  $\mathcal{H}(0)$ , dilating by the factor  $|\mathcal{H}'(0)|$ , and rotating through the angle  $\arg(\mathcal{H}'(0))$ , which is reversible. Moreover,  $\mathcal{S}$  is preserved under a number of elementary transformations, as conjugation, rotation, disk automorphism, etc.

In  $\mathbb{U}_d$ , let  $\mathcal{P}$  represent the family of regular, positive-real part functions  $p(\zeta)$  that are assumed as

$$p(\zeta) = 1 + \sum_{t=1}^{\infty} c_t \zeta^t \quad (\zeta \in \mathbb{U}_d). \quad (2)$$

We say that  $g_1$  is subordinated to  $g_2$  for two functions  $g_1, g_2 \in \mathcal{A}$ , and that  $g_1 \prec g_2$  symbolically, if there exists an analytic function  $w$  with the properties  $|w(\zeta)| \leq |\zeta|$  and  $w(0) = 0$  such that, for  $\zeta \in \mathbb{U}_d$ ,  $g_1(\zeta) = g_2(w(\zeta))$ . Additionally, in the event where  $g_2 \in \mathcal{S}$ , the condition becomes

$$g_1 \prec g_2 \Leftrightarrow g_1(0) = g_2(0) \text{ and } g_1(\mathbb{U}_d) \subset g_2(\mathbb{U}_d).$$

Now, for brevity, we recall the following definitions of fractional derives.

**Definition 1.** Assume that, in a simply connected region of the  $\zeta$ -plane including the origin, the function  $\mathcal{H}$  is analytic. For order  $\tau$ , the fractional integral of  $\mathcal{H}$  is defined as

$$\mathcal{D}_{\zeta}^{-\tau} \mathcal{H}(\zeta) = \frac{1}{\Gamma(\tau)} \int_0^{\zeta} \frac{\mathcal{H}(t)}{(\zeta - t)^{1-\tau}} dt, \quad \tau > 0, \quad (3)$$

and the fractional derivatives of  $\mathcal{Y}$  order  $\tau$  is

$$\mathcal{D}_{\zeta}^{\tau} \mathcal{H}(\zeta) = \frac{1}{\Gamma(1-\tau)} \frac{d}{d\zeta} \int_0^{\zeta} \frac{\mathcal{H}(t)}{(\zeta - t)^{\tau}} dt, \quad 0 \leq \tau < 1, \quad (4)$$

where the multiplicity of  $(\zeta - t)^{1-\tau}$  and  $(\zeta - t)^{-\tau}$  is removed by requiring  $\log(\zeta - t)$  to be real when  $\zeta - t > 0$ .

**Definition 2.** The fractional derivative of  $\mathcal{H}$  of order  $n + \tau$  is

$$\mathcal{D}_{\zeta}^{n+\tau} \mathcal{H}(\zeta) = \frac{d^n}{d\zeta^n} \mathcal{D}_{\zeta}^{\tau} \mathcal{H}(\zeta), \quad 0 \leq \tau < 1; n \in \mathbb{N}_0. \quad (5)$$

With the aid of the above definitions, and their known extensions involving fractional derivative and fractional integrals, Srivastava and Owa [2] introduced the operator

$$\Theta^{\varrho} : \mathcal{A} \rightarrow \mathcal{A}$$

defined by

$$\Theta^{\varrho} \mathcal{H}(\zeta) = \Gamma(2 - \varrho) \zeta^{\varrho} \mathcal{D}_{\zeta}^{\varrho} \mathcal{H}(\zeta) = \zeta + \sum_{t=2}^{\infty} \Phi(t, \varrho) d_t \zeta^t,$$

where

$$\Phi(t, \varrho) = \frac{\Gamma(t+1)\Gamma(2-\varrho)}{\Gamma(t+1-\varrho)}$$

and  $\varrho \in \mathbb{R}$ ;  $\varrho \neq 2, 3, 4, \dots$ . For  $\mathcal{H} \in \mathcal{A}$ , and various choices of  $\varrho$ , we obtain different operators

$$\Theta^0 \mathcal{H}(\zeta) = \mathcal{H}(\zeta) = \zeta + \sum_{t=2}^{\infty} d_t \zeta^t \quad (6)$$

$$\Theta^1 \mathcal{H}(\zeta) = \zeta \mathcal{H}'(\zeta) = \zeta + \sum_{t=2}^{\infty} t d_t \zeta^t \quad (7)$$

$$\Theta^j \mathcal{H}(\zeta) = \Theta(\Theta^{j-1} \mathcal{H}(\zeta)) = \zeta + \sum_{t=2}^{\infty} t^j d_t \zeta^t, \quad j = 1, 2, 3, \dots \quad (8)$$

which is known as the Salagean operator (see the article by Salagean, [3]). Also note that

$$\Theta^{-1}\mathcal{H}(\zeta) = \frac{2}{\zeta} \int_0^{\zeta} \mathcal{H}(t) dt = \zeta + \sum_{t=2}^{\infty} \left( \frac{2}{t+1} \right) d_t \zeta^t$$

and

$$\Theta^{-j}\mathcal{Y}(\zeta) = \Theta^{-1}(\Theta^{-j+1}\mathcal{H}(\zeta)) = \zeta + \sum_{t \geq 2} \left( \frac{2}{t+1} \right)^j d_t \zeta^t, (j = 1, 2, 3, \dots) \quad (9)$$

which is called the Libera integral operator; generalized by Bernardi [4], it is given by

$$\frac{1+\nu}{\zeta^\nu} \int_0^{\zeta} t^{\nu-1} \mathcal{H}(t) dt = \zeta + \sum_{t=2}^{\infty} \left( \frac{1+\nu}{t+1} \right) d_t \zeta^t, \quad \nu = 1, 2, 3, \dots$$

which is commonly known as Bernardi integral operator. Throughout the article, we look at Caputo's definition (see the article by Caplinger and Causey, [5]) of the fractional-order derivative, assumed as

$$\mathcal{D}^\alpha \mathcal{H}(t) = \frac{1}{\Gamma(t-\alpha)} \int_a^t \frac{\mathcal{H}^{(t)}(\tau)}{(n-\tau)^{\alpha+1-t}} d\tau, \quad (10)$$

where  $t-1 < \Re(\alpha) \leq t$ ,  $t \in \mathbb{N}$ , and the parameter  $\alpha$  is allowed to be real or even complex, where  $\alpha$  is the initial value of the function  $\mathcal{H}$ . In 2010, Salah and Darus, in [6], defined the following operator:

$$\mathcal{C}_\tau^\vartheta \mathcal{H}(\zeta) = \frac{\Gamma(2+\vartheta-\tau)}{\Gamma(\vartheta-\tau)} \zeta^{\tau-\vartheta} \int_0^{\zeta} \frac{\Theta^\vartheta \mathcal{H}(t)}{(\zeta-t)^{\tau+1-\vartheta}} dt \quad (11)$$

where  $\vartheta$  (real number) and  $(\vartheta-1 < \tau < \vartheta < 2)$ . With simple straightforward computations for  $\mathcal{H} \in \mathcal{A}$ , we obtain (see also [7])

$$\begin{aligned} \mathcal{C}_\tau^\vartheta \mathcal{H}(\zeta) &= \zeta + \sum_{t=2}^{\infty} \frac{\Gamma(2+\vartheta-\tau)\Gamma(2-\vartheta)(\Gamma(t+1))^2}{\Gamma(t-\vartheta+1)\Gamma(t+\vartheta-\tau+1)} d_t \zeta^t, \quad \zeta \in \mathbb{U}_d \\ &= \zeta + \sum_{t=2}^{\infty} \mathfrak{M}_t d_t \zeta^t, \quad \zeta \in \mathbb{U}_d \end{aligned} \quad (12)$$

where

$$\mathfrak{M}_t = \frac{\Gamma(2+\vartheta-\tau)\Gamma(2-\vartheta)(\Gamma(t+1))^2}{\Gamma(t-\vartheta+1)\Gamma(t+\vartheta-\tau+1)}. \quad (13)$$

Further, note that  $\mathcal{C}_0^0 \mathcal{H}(\zeta) = \mathcal{H}(\zeta)$  and  $\mathcal{C}_1^1 \mathcal{H}(\zeta) = \zeta \mathcal{H}'(\zeta)$ .

Pommerenke [8] introduced the Hankel determinant  $H_{q,t}(\mathcal{H})$ , where the parameters  $q, t \in \mathbb{N} = \{1, 2, 3, \dots\}$  for function  $\mathcal{H} \in \mathcal{S}$  of the form (1) are as follows:

$$H_{q,t}(\mathcal{H}) = \begin{vmatrix} d_n & d_{n+1} & \dots & d_{n+q-1} \\ d_{n+1} & d_{n+2} & \dots & d_{n+q} \\ \vdots & \vdots & \dots & \vdots \\ d_{n+q-1} & d_{n+q} & \dots & d_{n+2q-2} \end{vmatrix}. \quad (14)$$

By fixing the values for  $q$  and  $n$ , the Hankel determinants for various orders can be obtained. For instance, if  $q = 2$  if  $n = 1$ , then

$$|H_{2,1}(\mathcal{H})| = |d_3 - d_2^2|, \text{ where } d_1 = 1.$$

Note that  $H_{2,1}(\mathcal{H}) = d_3 - d_2^2$  is the classical Fekete–Szegő functional. Furthermore, when  $q = 2$  and  $n = 2$ , the second Hankel determinant is

$$|H_{2,2}(\mathcal{H})| = |d_2 d_4 - d_3^2|.$$

For various subclasses of class  $\mathcal{A}$ , several authors investigated the optimal value of the upper bound for  $|H_{2,1}(\mathcal{H})|$  and  $|H_{2,2}(\mathcal{H})|$  (see [9–14] for details). Remember that the Hankel determinant has a wide range of applications, such as linear filtering theory [15], discrete inverse scattering, and discretization of specific integral equations from mathematical physics. In the recent past, a number of researchers have investigated the subclasses of starlike functions by defining as follows: If  $\mathcal{H} \in \mathcal{A}$  holds the following criteria, it is considered starlike:

$$\mathcal{S}^*(Y) = \left\{ \mathcal{H} \in \mathcal{A} : \frac{\zeta \mathcal{H}'(\zeta)}{\mathcal{H}(\zeta)} \prec Y(\zeta) \right\}, \quad (15)$$

where  $Y(\zeta) = (1 + \zeta)/(1 - \zeta)$ . Lately, by varying  $Y$  in (15), some subclasses of  $\mathcal{S}$  whose image domains have some interesting geometrical configurations have been extensively studied for initial coefficient bounds, and we list a few Hankel inequalities from the literature below:

1. Cho et al. [16] fixed  $Y(\zeta) = 1 + \sin \zeta$  and Mendiratta et al. [17] considered  $Y(\zeta) = e^\zeta$ , and discussed the class  $\mathcal{S}$  on certain geometric properties and radii problems.
2. Sharma et al. [18] developed  $Y(\zeta) = 1 + \frac{4}{3}\zeta + \frac{2}{3}\zeta^2$ , a pedal shaped domain, and a Wani and Swaminathan [19] fixed  $Y(\zeta) = 1 + \zeta - \frac{1}{3}\zeta^3$ , which maps  $\mathbb{U}_d$  onto the interior of the 2-cusped kidney-shaped region, and examined applicability for certain subclasses  $\mathcal{S}$  in the general coefficient problem.
3. Raina and Sokół [20,21] developed  $Y(\zeta) = \zeta + \sqrt{1 + \zeta^2}$ , which maps  $\mathbb{U}_d$  to a crescent shaped region, also assuming  $Y(\zeta) = \sqrt{1 + \zeta}$ , which is bounded by lemniscate of Bernoulli in right half plan they found the initial Taylor coefficients for subclasses  $\mathcal{S}$ , and discussed various geometrical inequalities.

Various subclasses of starlike functions were introduced and discussed (see [22–25]) by choosing a particular  $Y$  in (15), such as those related to Bell numbers, the shell-like curve associated with Fibonacci numbers, conic domain functions, van der Pol Numbers (VPN) and, rational functions. Gandhi [26] recently defined the class of starlike functions linked to three Leaf functions, i.e.,

$$\mathcal{S}_{3\mathcal{L}}^* = \left\{ \mathcal{H} \in \mathcal{A} : \frac{\zeta \mathcal{H}'(\zeta)}{\mathcal{H}(\zeta)} \prec 1 + \frac{4}{5}\zeta + \frac{1}{5}\zeta^4 \right\} \quad (\zeta \in \mathbb{U}_d).$$

Sakaguchi [27] investigated the family of star functions in 1959,  $\mathcal{S}_s^*$  of starlike functions with respect to other points, which helped to popularize the  $\mathcal{S}^*$  family of star functions. Later, Das and Singh [28] developed a family of convex functions  $\mathcal{K}_s$  with symmetry elements in 1977. In two articles, they provided the following description:

$$\begin{aligned} \mathcal{S}_s^* &= \left\{ \mathcal{H} \in \mathcal{A} : \Re \frac{2\zeta \mathcal{H}'(\zeta)}{\mathcal{H}(\zeta) - \mathcal{H}(-\zeta)} > 0, \quad (\zeta \in \mathbb{U}_d) \right\}, \\ \mathcal{K}_s &= \left\{ \mathcal{H} \in \mathcal{A} : \Re \frac{2(\zeta \mathcal{H}'(\zeta))'}{(\mathcal{H}(\zeta) - \mathcal{H}(-\zeta))'} > 0, \quad (\zeta \in \mathbb{U}_d) \right\}. \end{aligned}$$

Sakaguchi further stated in the same study that the families of convex and odd starlike functions are contained in the class  $\mathcal{S}_s^*$ , which is a subfamily of the set  $\mathcal{C}$  of close-to-convex functions that perform in relation to the source. After that, a number of mathematicians presented a plethora of novel subfamilies of univalent functions with regard to symmetric points, and examined issues of the coefficient kind, a few of which are shown in [29–33], etc. Influenced by the books and references mentioned above, and the reference therein, in this article, we present a new class  $\mathcal{SST}_{3\mathcal{L}}$ , making use of Caputo's defini-

tion (see Caplinger and Causey, [5]) of the fractional-order derivative operator (CFD), as defined below,

$$\mathcal{SST}_{3\mathcal{L}}^{\theta,\tau} = \left\{ \mathcal{H} \in \mathcal{A} : \frac{2\zeta (\mathcal{C}_{\tau}^{\theta} \mathcal{H}(\zeta))'}{\mathcal{C}_{\tau}^{\theta} \mathcal{H}(\zeta) - \mathcal{C}_{\tau}^{\theta} \mathcal{H}(-\zeta)} \prec 1 + \frac{4}{5}\zeta + \frac{1}{5}\zeta^4 \right\} \quad (\zeta \in \mathbb{U}_d) \quad (16)$$

and examine, for coefficient bounds, the Fekete–Szegő inequality and Zalcman conjecture for  $\mathcal{H} \in \mathcal{SST}_{3\mathcal{L}}^{\theta,\tau}$ .

## 2. Initial Bounds for $\mathcal{H} \in \mathcal{SST}_{3\mathcal{L}}^{\theta,\tau}$

We recall the following lemmas, which are required for the proofs of our main findings.

**Lemma 1** ([34]). *Let  $p \in \mathcal{P}$  and be given as in (2). Then, for  $x$  and  $\sigma$  with  $|x| \leq 1, |\sigma| \leq 1$ , such that*

**Lemma 2.**

$$2c_2 = c_1^2 + x(4 - c_1^2), \quad (17)$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)\sigma, \quad (18)$$

**Lemma 3.** *If  $p \in \mathcal{P}$ , and is assumed to be as in (2), then*

$$|c_{t+k} - \varphi c_t c_k| \leq 2, \text{ for } 0 \leq \varphi \leq 1 \quad (19)$$

$$|c_t| \leq 2 \text{ for } t \geq 1, \quad (20)$$

$$|c_2 - \nu c_1^2| \leq 2 \max\{1, |2\nu - 1|\}, \quad \zeta \in \mathbb{C} \quad (21)$$

$$|\mathcal{J}c_1^3 - \mathcal{K}c_1c_2 + \mathcal{L}c_3| \leq 2|\mathcal{J}| + 2|\mathcal{K} - 2\mathcal{J}| + 2|\mathcal{J} - \mathcal{K} + \mathcal{L}|, \quad (22)$$

We note that Inequalities (19), (20) and (22) in the above can be found in [8], that (21) is from [10], and also that (22) was evaluated in [35] (see also [36]).

**Lemma 4** ([37]). *Let  $m, t, l$  and  $r$  with  $0 < m < 1, 0 < r < 1$ , and*

$$\begin{aligned} & 8r(1-r) \left[ (mt - 2l)^2 + (m(r+m) - t)^2 \right] + m(1-m)(t - 2rm)^2 \\ & \leq 4m^2(1-m)^2r(1-r). \end{aligned}$$

*If  $p \in \mathcal{P}$ , and is given as in (2), then*

$$\left| lc_1^4 + rc_2^2 + 2mc_1c_3 - \frac{3}{2}tc_1^2c_2 - c_4 \right| \leq 2.$$

**Theorem 1.** *Let  $\mathcal{H} \in \mathcal{SST}_{3\mathcal{L}}^{\theta,\tau}$  and be given by (1). Then,*

$$|d_2| \leq \frac{2}{5\mathfrak{M}_2}, \quad (23)$$

$$|d_3| \leq \frac{2}{5\mathfrak{M}_3}, \quad (24)$$

$$|d_4| \leq \frac{1}{5\mathfrak{M}_4}, \quad (25)$$

and

$$|d_5| \leq \frac{1}{5\mathfrak{M}_5}, \quad (26)$$

where  $\mathfrak{M}_t$  ( $t \in \{2, 3, 4, 5\}$ ) is defined by (13).

**Proof.** Let  $\mathcal{H} \in \mathcal{SST}_{3\mathcal{L}}^{\theta, \tau}$ . Then, (16) can be expressed in Schwarz function  $w$  as

$$\frac{2\zeta (\mathcal{C}_\tau^\theta \mathcal{H}(\zeta))'}{\mathcal{C}_\tau^\theta \mathcal{H}(\zeta) - \mathcal{C}_\tau^\theta \mathcal{H}(-\zeta)} = 1 + \frac{4}{5}w(\zeta) + \frac{1}{5}(w(\zeta))^4 \quad (\zeta \in \mathbb{U}_d). \quad (27)$$

Also, if  $p \in \mathcal{P}$ , then it can be stated using the Schwarz function  $w$  as

$$p(\zeta) = 1 + c_1\zeta + c_2\zeta^2 + c_3\zeta^3 \cdots = \frac{1 + w(\zeta)}{1 - w(\zeta)},$$

equivalently,

$$w(\zeta) = \frac{p(\zeta) - 1}{p(\zeta) + 1} = \frac{c_1\zeta + c_2\zeta^2 + c_3\zeta^3 + \cdots}{2 + c_1\zeta + c_2\zeta^2 + c_3\zeta^3 + \cdots}. \quad (28)$$

From (27), by simple computation,

$$\frac{2\zeta (\mathcal{C}_\tau^\theta \mathcal{H}(\zeta))'}{\mathcal{C}_\tau^\theta \mathcal{H}(\zeta) - \mathcal{C}_\tau^\theta \mathcal{H}(-\zeta)} = 1 + 2\mathfrak{M}_2 d_2 \zeta + 2\Xi_3 d_3 \zeta^2 + (4\mathfrak{M}_4 d_4 - 2\mathfrak{M}_2 \mathfrak{M}_3 d_2 d_3) \zeta^3 + (4\mathfrak{M}_5 d_5 - 2d_3^2 \mathfrak{M}_3^2) \zeta^4 + \cdots. \quad (29)$$

By using the series expansion (28), we have

$$\begin{aligned} 1 + \frac{4}{5}w(\zeta) + \frac{1}{5}(w(\zeta))^4 &= 1 + \frac{2}{5}c_1\zeta + \left(\frac{2}{5}c_2 - \frac{1}{5}c_1^2\right)\zeta^2 + \left(\frac{1}{10}c_1^3 - \frac{2}{5}c_2c_1 + \frac{2}{5}c_3\right)\zeta^3 \\ &+ \left(-\frac{3}{80}c_1^4 + \frac{3}{10}c_1^2c_2 - \frac{2}{5}c_3c_1 - \frac{1}{5}c_2^2 + \frac{2}{5}c_4\right)\zeta^4 + \cdots. \end{aligned} \quad (30)$$

Comparing (29) and (30), we obtain

$$d_2 = \frac{1}{5\mathfrak{M}_2}c_1, \quad (31)$$

$$d_3 = \frac{1}{5\mathfrak{M}_3}c_2 - \frac{1}{10\mathfrak{M}_3}c_1^2, \quad (32)$$

$$d_4 = \frac{1}{20\mathfrak{M}_4} \left[ \frac{3}{10}c_1^3 - \frac{8}{5}c_1c_2 + 2c_3 \right] \quad (33)$$

and

$$d_5 = -\frac{1}{10\mathfrak{M}_5} \left( \frac{7}{160}c_1^4 + \frac{3}{10}c_2^2 + c_3c_1 - \frac{3}{20}c_1^2c_2 - c_4 \right). \quad (34)$$

Using (20) in (31), we have

$$|d_2| \leq \frac{2}{5\mathfrak{M}_2}.$$

By (32), we obtain

$$d_3 = \frac{1}{5\mathfrak{M}_3} \left( c_2 - \frac{1}{2}c_1^2 \right),$$

now, by relating (19), we have

$$|d_3| \leq \frac{2}{5\mathfrak{M}_3}.$$

Expanding (33), we attain

$$|d_4| = \frac{1}{20\mathfrak{M}_4} \left| \frac{3}{10}c_1^3 - \frac{8}{5}c_1c_2 + 2c_3 \right|.$$

By applying (22), we obtain

$$|d_4| \leq \frac{1}{20\mathfrak{M}_4} \left[ 2 \left| \frac{3}{10} \right| + 2 \left| \frac{8}{5} - 2 \left( \frac{3}{10} \right) \right| + 2 \left| \frac{3}{10} - \frac{8}{5} + 2 \right| \right] = \frac{1}{5\Xi_4}.$$

Using Lemma 4 in (34), we obtain

$$|d_5| \leq \frac{1}{5\mathfrak{M}_5}.$$

□

Next, we find Fekete–Szegő inequality for  $\mathcal{H} \in \mathcal{SST}_{3\mathcal{L}}^{\theta, \tau}$ .

**Theorem 2.** If  $\mathcal{H} \in \mathcal{SST}_{3\mathcal{L}}^{\theta, \tau}$ , and is given as in (1), then

$$|d_3 - \aleph d_2^2| \leq \frac{2}{5\mathfrak{M}_3} \max \left\{ 1, \frac{2|\aleph|\mathfrak{M}_3}{5\mathfrak{M}_2^2} \right\}, \quad (\aleph \in \mathbb{C}) \quad (35)$$

where  $\mathfrak{M}_t$  ( $t \in \{2, 3\}$ ) is defined by (13).

**Proof.** Using (31) and (32), we can write

$$|d_3 - \aleph d_2^2| = \left| \frac{1}{5\mathfrak{M}_3} c_2 - \frac{1}{10\Xi_3} c_1^2 - \aleph \frac{c_1^2}{25\mathfrak{M}_2^2} \right|.$$

By rearranging, we have

$$|d_3 - \aleph d_2^2| = \frac{1}{5\mathfrak{M}_3} \left| c_2 - \left( \frac{2\aleph\mathfrak{M}_3 + 5\mathfrak{M}_2^2}{10\mathfrak{M}_2^2} \right) c_1^2 \right|. \quad (36)$$

Applying (21), we obtain

$$|d_3 - \aleph d_2^2| \leq \frac{2}{5\mathfrak{M}_3} \max \left\{ 1, \frac{2|\aleph|\mathfrak{M}_3}{5\mathfrak{M}_2^2} \right\}.$$

□

Fixing  $\aleph = 1$ , then we have the following result.

**Corollary 1.** If  $\mathcal{H} \in \mathcal{SST}_{3\mathcal{L}}^{\theta, \tau}$ , and is of the form (1), then

$$|d_3 - d_2^2| \leq \frac{2}{5\mathfrak{M}_3} \max \left\{ 1, \frac{2\mathfrak{M}_3}{5\mathfrak{M}_2^2} \right\}. \quad (37)$$

where  $\mathfrak{M}_t$  ( $t \in \{2, 3\}$ ) is defined by (13).

### 3. Coefficient Inequalities for the Function $\mathcal{H}^{-1}$

**Theorem 3.**  $\mathcal{H} \in \mathcal{SST}_{3\mathcal{L}}^{\theta, \tau}$ , and is of the form (1). If the analytic continuation to  $\mathbb{U}_d$  of the inverse function of  $\mathcal{H}$  with  $|w| < r_0$ , ( $r_0 \geq \frac{1}{4}$ ) the radius of the Koebe domain, and  $\mathcal{H}^{-1}(w) = w + \sum_{t=2} \hbar_t w^t$ , then

**Theorem 4.**

$$|\hbar_2| \leq \frac{2}{5\mathfrak{M}_2}, \quad (38)$$

$$|\hbar_3| \leq \frac{2}{5\mathfrak{M}_3} \max \left\{ 1, \frac{4\mathfrak{M}_3}{5\mathfrak{M}_2^2} \right\} \quad (39)$$

and then, for  $\rho \in \mathbb{C}$ ,

$$|\hbar_3 - \rho \hbar_2^2| \leq \frac{2}{5\mathfrak{M}_3} \max \left\{ 1, \left| \frac{4\mathfrak{M}_3 + 2\rho\mathfrak{M}_3}{5\mathfrak{M}_2^2} \right| \right\}, \quad (40)$$

where  $\mathfrak{M}_t$  ( $t \in \{2, 3\}$ ) is defined by (13).

**Proof.** If

$$\mathcal{H}^{-1}(w) = w + \sum_{t \geq 2} \hbar_t w^t \quad (41)$$

it can be seen that

$$\mathcal{H}^{-1}(\mathcal{H}(\zeta)) = \mathcal{H}(\mathcal{H}^{-1}(\zeta)) = \zeta. \quad (42)$$

From (42), we obtain

$$\mathcal{H}^{-1}(\zeta + \sum_{t \geq 2} d_t \zeta^t) = \zeta. \quad (43)$$

Thus, (42) and (43) yield

$$\zeta + (d_2 + \hbar_2)\zeta^2 + (d_3 + 2d_2\hbar_2 + \hbar_3)\zeta^3 + \dots = \zeta. \quad (44)$$

Thus, it is evident from equating the corresponding coefficients of  $\zeta$  that

$$\hbar_2 = -d_2, \quad (45)$$

$$\hbar_3 = 2d_2^2 - d_3 = -(d_3 - 2d_2^2). \quad (46)$$

The estimate for  $|\hbar_2| = |d_2|$  follows immediately from (23). Letting  $\aleph = 2$  in (35), we obtain the estimate  $|\hbar_3|$ . To find the Fekete–Szegő inequality for the inverse function, consider

$$|\hbar_3 - \rho \hbar_2^2| = |2d_2^2 - d_3 - \rho d_2^2| = |-d_3 - (\rho - 2)d_2^2| = |d_3 - (2 - \rho)d_2^2|.$$

Thus, by fixing  $\aleph = (2 - \rho)$  in (35), we obtain the desired result.  $\square$

#### 4. Coefficient Associated with $\frac{\zeta}{\mathcal{H}(\zeta)}$

The coefficient bounds and Fekete–Szegő results for the function  $H(\zeta)$  are given by

$$H(\zeta) = \frac{\zeta}{\mathcal{H}(\zeta)} = 1 + \sum_{t=1}^{\infty} u_t \zeta^t \quad (\zeta \in \mathbb{U}_d). \quad (47)$$

where  $\mathcal{H} \in \mathcal{SST}_{3\mathcal{L}}^{\theta, \tau}$  are found in this section.

**Theorem 5.** If  $\mathcal{H}$  of the form (1) belongs to  $\mathcal{SST}_{h, \delta, 3\mathcal{L}}^{\alpha, q, n}$  and  $H(\zeta)$  is given by (47), then

$$|u_1| \leq \frac{2}{5\mathfrak{M}_2}, \quad (48)$$

$$|u_2| \leq \frac{2}{5\mathfrak{M}_3} \max \left\{ 1, \frac{2\mathfrak{M}_3}{5\mathfrak{M}_2^2} \right\} \quad (49)$$

$$|u_2 - \wp u_1^2| \leq \frac{2}{5\mathfrak{M}_3} \max \left\{ 1, \left| \frac{2(1 - \wp)\mathfrak{M}_3}{5\mathfrak{M}_2^2} \right| \right\}, \text{ for } \wp \in \mathbb{C}, \quad (50)$$

where  $\mathfrak{M}_t$  ( $t \in \{2, 3\}$ ) is defined by (13).

**Proof.** By routine calculation, we obtain

$$H(\zeta) = \frac{\zeta}{\mathcal{H}(\zeta)} = 1 - d_2\zeta + (d_2^2 - d_3)\zeta^2 + (d_2^3 + 2d_2d_3 - d_4)\zeta^3 + \dots \quad (51)$$

Comparing the coefficients of  $\zeta$ ,  $\zeta^2$  and  $\zeta^3$  on both sides of (47) and (51), we obtain

$$u_1 = -d_2, \quad (52)$$

and

$$u_2 = d_2^2 - d_3 = -(d_3 - d_2^2) \quad (53)$$

$$u_3 = c_2^3 + 2c_2c_3 - c_4 = -(c_4 - 2c_2c_3 - c_2^3). \quad (54)$$

The estimate for  $|u_1| = |d_2|$  follows immediately from (23). The bound for  $u_2$  is followed by Corollary 1. For  $\wp \in \mathbb{C}$ , we obtain

$$\begin{aligned} |u_2 - \wp u_1^2| &= |-d_3 + d_2^2 - \wp d_2^2| \\ &= |d_3 - (1 - \wp)d_2^2|, \end{aligned}$$

Thus, by taking  $\aleph = (1 - \wp)$  in (35), we obtain the desired result.

□

## 5. Initial Logarithmic Coefficient Bounds for $\mathcal{H} \in \mathcal{SST}_{3\mathcal{L}}^{\theta, \tau}$

The logarithmic coefficients of a given function  $\mathcal{H}$ , denoted by  $\gamma_t := \gamma_t(\mathcal{H})$ , are defined as

$$\frac{1}{2} \log \left( \frac{\mathcal{H}(\zeta)}{\zeta} \right) = \sum_{t=1}^{\infty} \gamma_t \zeta^t. \quad (55)$$

The theory of Schlicht functions is significantly impacted by these coefficients in various estimations. In 1985, de-Branges [38] determined that

$$\sum_{k=1}^t k(t-k+1) |\gamma_k|^2 \leq \sum_{k=1}^t \frac{t-k+1}{k}, \text{ for } t \geq 1,$$

and for the particular function  $\mathcal{H}(\zeta) = \zeta / (1 - e^{i\theta}\zeta)$  with  $\theta \in \mathbb{R}$ , equality is attained. This inequality is the source of the most general version of the famous Bieberbach–Robertson–Milin conjectures concerning the Taylor coefficients of  $\mathcal{H} \in \mathcal{A}$ . To learn more about the explanation of de-Brange’s claim, one refer to [39–41]. Kayumov [42] in 2005 answered Brennan’s conjecture in terms of conformal mappings by converting the logarithmic coefficients. Several studies on logarithmic coefficients have made significant progress [43–45]. From the above definition, in the following section, the logarithmic coefficients for  $\mathcal{H} \in \mathcal{A}$  are as given below:

$$\gamma_1 = \frac{1}{2}d_2 \quad (56)$$

$$\gamma_2 = \frac{1}{2} \left( d_3 - \frac{1}{2}d_2^2 \right) \quad (57)$$

$$\gamma_3 = \frac{1}{2} \left( d_4 - d_2d_3 + \frac{1}{3}d_2^3 \right). \quad (58)$$

**Theorem 6.** Let  $\mathcal{H} \in \mathcal{SST}_{3\mathcal{L}}^{\emptyset, \tau}$ , and be as in (1), then

$$\begin{aligned} |\gamma_1| &\leq \frac{1}{5\mathfrak{M}_2}, \\ |\gamma_2| &\leq \frac{1}{5\mathfrak{M}_3} \max \left\{ 1, \left| \frac{\mathfrak{M}_3}{5\mathfrak{M}_2^2} \right| \right\}, \\ |\gamma_3| &\leq \frac{1}{10\mathfrak{M}_4}, \end{aligned}$$

where  $\mathfrak{M}_t$  ( $t \in \{2, 3, 4\}$ ) are defined by (13).

**Proof.** Applying (20) and (33) in (56), (57), and (58), we obtain

$$\gamma_1 = \frac{1}{10\mathfrak{M}_2} c_1, \quad (59)$$

$$\begin{aligned} \gamma_2 &= \frac{1}{2} \left( \frac{1}{5\mathfrak{M}_3} c_2 - \frac{1}{10\mathfrak{M}_3} c_1^2 - \frac{c_1^2}{50\mathfrak{M}_2^2} \right), \\ &= \frac{1}{10\mathfrak{M}_3} \left( c_2 - \frac{5\mathfrak{M}_2^2 + \mathfrak{M}_3}{10\mathfrak{M}_2^2} c_1^2 \right). \end{aligned} \quad (60)$$

The bounds of  $\gamma_1$  and  $\gamma_2$  are clear. Now, to find  $\gamma_3$  using (58), we obtain

$$\gamma_3 = \frac{1}{40\mathfrak{M}_4} \left[ 2c_3 - \frac{8}{5} \left( 1 + \frac{\mathfrak{M}_4}{2\mathfrak{M}_2\mathfrak{M}_3} \right) c_1 c_2 + \left( \frac{3}{10} + \frac{2\mathfrak{M}_4}{5\mathfrak{M}_2\mathfrak{M}_3} + \frac{4\mathfrak{M}_4}{25\mathfrak{M}_2^2} \right) c_1^3 \right],$$

then

$$|\gamma_3| = \frac{1}{40\mathfrak{M}_4} \left| 2c_3 - \frac{8}{5} \left( 1 + \frac{\mathfrak{M}_4}{2\mathfrak{M}_2\mathfrak{M}_3} \right) c_1 c_2 + \left( \frac{3}{10} + \frac{2\mathfrak{M}_4}{5\mathfrak{M}_2\mathfrak{M}_3} + \frac{4\mathfrak{M}_4}{25\mathfrak{M}_2^2} \right) c_1^3 \right|.$$

Using (22) and a triangle inequality, we obtain

$$|\gamma_3| \leq \frac{1}{10\mathfrak{M}_4}.$$

□

**Theorem 7.** If  $\mathcal{H} \in \mathcal{SST}_{3\mathcal{L}}^{\emptyset, \tau}$ , and is as in (1), then for  $\aleph \in \mathbb{C}$ ,

$$|\gamma_2 - \aleph \gamma_1^2| \leq \frac{1}{5\mathfrak{M}_3} \max \left\{ 1, \left| \frac{(1 + \aleph)\mathfrak{M}_3}{5\mathfrak{M}_2^2} \right| \right\},$$

where  $\mathfrak{M}_t$  ( $t \in \{2, 3\}$ ) are defined by (13).

**Proof.** From (59) and (60), we have

$$\begin{aligned} |\gamma_2 - \aleph \gamma_1^2| &= \left| \frac{1}{10\mathfrak{M}_3} \left( c_2 - \frac{5\mathfrak{M}_2^2 + \mathfrak{M}_3}{10\mathfrak{M}_2^2} c_1^2 \right) - \frac{\aleph}{100\mathfrak{M}_2^2} c_1^2 \right|, \\ &= \frac{1}{10\mathfrak{M}_3} \left| \left( c_2 - \frac{5\mathfrak{M}_2^2 + \mathfrak{M}_3}{10\mathfrak{M}_2^2} c_1^2 \right) - \frac{\aleph \mathfrak{M}_3}{10\mathfrak{M}_2^2} c_1^2 \right|, \\ &= \frac{1}{10\mathfrak{M}_3} \left| \left( c_2 - \frac{5\mathfrak{M}_2^2 + (1 + \aleph)\mathfrak{M}_3}{10\mathfrak{M}_2^2} c_1^2 \right) \right| \end{aligned}$$

Using (21) and the triangle inequality, we have

$$|\gamma_2 - \aleph \gamma_1^2| \leq \frac{1}{5\mathfrak{M}_3} \max \left\{ 1, \left| \frac{(1 + \aleph)\mathfrak{M}_3}{5\mathfrak{M}_2^2} \right| \right\}.$$

□

Fixing  $\aleph = 1$ , we obtain the following consequence.

**Corollary 2.** Let  $\mathcal{H}$  of the form (1), and belong to  $\mathcal{ST}_{3\mathcal{L}}^{\theta, \tau}$ . Then,

$$|\gamma_2 - \gamma_1^2| \leq \frac{1}{5\mathfrak{M}_3} \max \left\{ 1, \left| \frac{2\mathfrak{M}_3}{5\mathfrak{M}_2^2} \right| \right\}.$$

## 6. Zalcman Functional

One of the main theories of geometric function theory, put forth by Lawrence Zalcman in 1960, holds that the inequality held by the coefficients of class  $\mathcal{S}$  is

$$|d_t^2 - d_{2t-1}| \leq (t-1)^2. \quad (61)$$

The Koebe function  $k(\zeta) = \frac{\zeta}{(1-\zeta)^2}$  is widely recognized, and the above expression represents the equality of rotations. When  $t = 2$ , the widely known Fekete–Szegő inequality holds true. Many researchers have examined the Zalcman functional (see [46–48]).

**Theorem 8.** If  $\mathcal{H} \in \mathcal{A}$  and is in  $\mathcal{ST}_{3\mathcal{L}}^{\theta, \tau}$ , then

$$|d_3^2 - d_5| \leq \frac{1}{5\mathfrak{M}_5}, \quad (62)$$

where  $\mathfrak{M}_5$  is defined by (13).

**Proof.** Using (32) and (34) to obtain Zalcman functional,

$$|d_3^2 - d_5| = \frac{1}{10\mathfrak{M}_5} \left| \left( \frac{\mathfrak{M}_5}{10\mathfrak{M}_3^2} + \frac{7}{160} \right) c_1^4 - \left( \frac{\mathfrak{M}_5}{\mathfrak{M}_3^2} + \frac{3}{20} \right) c_1^2 c_2 + d_3 c_1 - \left( \frac{2\mathfrak{M}_5}{5\mathfrak{M}_3^2} + \frac{3}{10} \right) c_2^2 - c_4 \right|.$$

Using Lemma 4, we can obtain the desired result (62). □

## 7. Krushkal Inequality for the Class $\mathcal{ST}_{3\mathcal{L}}^{\theta, \tau}$

For a choice of  $t = 4$ ,  $p = 1$ , and for  $t = 5$ ,  $p = 1$ , Krushkal introduced and proved the inequality

$$|d_t^p - d_2^{p(t-1)}| \leq 2^{p(t-1)} - t^p \quad (63)$$

for the whole class of univalent functions in [49].

Now, in the below theorem for  $\mathcal{H} \in \mathcal{ST}_{3\mathcal{L}}^{\theta, \tau}$ , we will give direct proof of (63).

**Theorem 9.** If  $\mathcal{H} \in \mathcal{ST}_{3\mathcal{L}}^{\theta, \tau}$ , then

$$|d_4 - d_2^3| \leq \frac{1}{5\mathfrak{M}_4},$$

where  $\mathfrak{M}_4$  is defined by (13).

**Proof.** From Equations (31) and (33), we obtain

$$|d_4 - d_2^3| = \frac{1}{20\mathfrak{M}_4} \left| \left( \frac{3}{10} - \frac{4\mathfrak{M}_4}{25\mathfrak{M}_2^2} \right) c_1^3 - \frac{8}{5} c_2 c_1 + 2c_3 \right|.$$

By using (22) with the above condition, we obtain the desired result.  $\square$

**Theorem 10.** Let  $\mathcal{H}$  of the form (1) belong to  $\mathcal{SST}_{3\mathcal{L}}^{\theta, \tau}$ . Then,

$$|d_5 - d_2^4| \leq \frac{1}{5\mathfrak{M}_5},$$

where  $\mathfrak{M}_5$  is defined by (13).

**Proof.** From Equations (31) and (33), we obtain

$$|d_5 - d_2^4| = -\frac{1}{10\mathfrak{M}_5} \left[ \left( \frac{7}{160} + \frac{10\mathfrak{M}_5}{625\mathfrak{M}_2^4} \right) c_1^4 + \frac{3}{10} c_2^2 + c_3 c_1 - \frac{3}{20} c_1^2 c_2 - c_4 \right]$$

Using Lemma 4, we can obtain the necessary result for the last expression.  $\square$

## 8. Conclusions

This study focuses on using the analytic coefficients of a subclass of Sakaguchi-type functions that are subordinated to a domain with a three-leaf domain. In function theory, one of the main challenges is estimating the coefficients that arise in analytic univalent functions. The general principle underlying determining the boundaries of the coefficients in various the expression of univalent function families' coefficients into Carathéodory function coefficients. Inequalities known for the class of Carathéodory functions can be used to explore coefficient functionals. We used a novel approach in the current study to find the bounds for a number of coefficient-related problems, such as the second-order Hankel determinant, the Zalcman inequalities, and the Fekete–Szegő inequalities, and also for  $\mathcal{H}^{-1}$  and  $\frac{\zeta}{\mathcal{H}(\zeta)}$  and  $\frac{1}{2} \log \left( \frac{\mathcal{H}(\zeta)}{\zeta} \right)$  linked with the function of the three leaves. The study can also be extended to bi-univalent function classes (see [7,50]). Since power law transformation and analytic coefficients derived from complex functions are being used more and more in digital image processing (DIP) these days, a solid foundation for the examination of phase information in pictures and the improvement of particular aspects of the picture are needed. Complex image processing tasks, including contrast enhancement, spatial filtering, and picture segmentation, can be accomplished by combining power law transformation with analytical coefficients. The value of analytical coefficients and power law transformation will be useful tools for processing digital images in the future; thus, we hope the coefficients derived in this article may find more applications in DIP (see [51]).

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