

Characterizations of Commutativity of Prime Ring with Involution by Generalized Derivations

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Abstract: In the paper, we investigate the commutativity of a two-torsion free prime ring $\mathcal{R}$ provided with generalized derivations, and some well-known results that characterize the commutativity of prime rings through generalized derivations have been generalized. Moreover, we provide some examples to testify that the assumed restriction in our theorems cannot be omitted.

Keywords: generalized derivations; prime ring; involution; commutativity

MSC: 13C12

1. Introduction

Throughout this article, $\mathcal{R}$ denotes an associative ring with center $Z(\mathcal{R})$. $\mathcal{R}$ is called a prime ring if $a\mathcal{R}b=0$ implies $a=0$ or $b=0$ and a semi-prime ring if $a\mathcal{R}a=0$ implies $a=0$ for all $a, b \in \mathcal{R}$. A ring is said to be two-torsion free if $2a=0$ (where $a \in \mathcal{R}$) implies $a=0$. In [1], the authors provide a more detailed definition. For all $a, b \in \mathcal{R}$, $[a, b]$ denotes the Lie product $ab - ba$ and $a \circ b$ denotes the Jordan product $ab + ba$.

An additive mapping $*$: $\mathcal{R} \rightarrow \mathcal{R}$ is called an involution if $*$ is an anti-automorphism of order 2 such that $(ab)^* = b^*a^*$ and $(a^*)^* = a$ for all $a, b \in \mathcal{R}$. If an element r in a ring with involution $(\mathcal{R}, *)$ satisfies the condition $r^* = r$, then it will be termed as Hermitian. Similarly, r is termed skew-Hermitian if r satisfies $r^* = -r$. The sets of all Hermitian and skew-Hermitian elements of $\mathcal{R}$ will be denoted by $H(\mathcal{R})$ and $S(\mathcal{R})$, respectively. The involution is classified as the first kind if $Z(\mathcal{R}) \subseteq H(\mathcal{R})$; otherwise, it is classified as the second kind. In the latter case, $S(\mathcal{R}) \cap Z(\mathcal{R}) \neq \{0\}$. It is clear that any prime ring with an involution $*$ is also semi-prime (i.e., $a\mathcal{R}b = a\mathcal{R}b^* = \{0\}$ yields that $a=0$ or $b=0$), but the converse is in general not true; an example is given to illustrate this in [2].

An additive mapping d : $\mathcal{R} \rightarrow \mathcal{R}$ is called a derivation if $d(ab) = d(a)b + ad(b)$ for all $a, b \in \mathcal{R}$. An additive mapping $\mathcal{F}$: $\mathcal{R} \rightarrow \mathcal{R}$ is called a generalized derivation if there exists a derivation d such that $\mathcal{F}(ab) = \mathcal{F}(a)b + ad(b)$ holds for all $a, b \in \mathcal{R}$, and d is called the associated derivation of $\mathcal{F}$.

In ring theory, the commutativity of a prime ring $\mathcal{R}$ is a classic topic. The relation between the commutativity of the prime ring $\mathcal{R}$ and some special maps in $\mathcal{R}$ was surveyed by numerous scholars. In [3], the authors proved if an Artinian ring has a nontrivial commutative self-isomorphism, then the ring is commutative. In [4], Posner proved that a prime ring $\mathcal{R}$ is commutative if it has a nonzero centralizable derivation (Posner's second theorem). To this day, there are still many scholars exploring the topic (see [5–10]). In [11], the author investigates the commutativity theory of two-torsion free prime ring $\mathcal{R}$ with involution of the second kind under one of the following conditions:

- (i) $\mathcal{F}([a, a^*]) \pm [\mathcal{F}(a), a^*] = 0$;
- (ii) $\mathcal{F}(a \circ a^*) \pm \mathcal{F}(a) \circ a^* = 0$;
- (iii) $\mathcal{F}([a, a^*]) \pm [\mathcal{F}(a), \mathcal{F}(a^*)] = 0$;
- (iv) $[\mathcal{F}(a), a^*] \pm [a, \mathcal{F}(a^*)] = 0$ for all $a \in \mathcal{R}$.



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In the paper, our purpose is to continue this line of investigation by studying the commutativity criteria for rings with involution that admit two generalized derivations satisfying certain algebraic identities. We consider two generalized derivations $\mathcal{F}$ and $\mathcal{G}$ on a two-torsion free prime ring $\mathcal{R}$ with involution of the second kind, which satisfy any of the following properties:

- (i) $\mathcal{F}([a, a^*]) \pm [\mathcal{G}(a), a^*] = 0$ for all $a \in \mathcal{R}$;
- (ii) $\mathcal{F}(a \circ a^*) \pm \mathcal{G}(a) \circ a^* = 0$ for all $a \in \mathcal{R}$;
- (iii) $[\mathcal{F}(a), a^*] \pm [a, \mathcal{G}(a^*)] = 0$ for all $a \in \mathcal{R}$.

Moreover, we will generalize some theorem of [11] from another perspective, and we will prove that on a two-torsion free prime ring $\mathcal{R}$ with involution of the second kind, a generalized derivation $\mathcal{F}$ satisfies one of the following equations:

- (i) $\mathcal{F}([a, a^*]) \pm [\mathcal{F}(a), a^*] \in Z(\mathcal{R})$ for all $a \in \mathcal{R}$;
- (ii) $\mathcal{F}(a \circ a^*) \pm \mathcal{F}(a) \circ a^* \in Z(\mathcal{R})$ for all $a \in \mathcal{R}$.

Then, $\mathcal{R}$ is commutative.

2. Some Preliminaries

We will introduce some of the basic identities in [11], which will be used frequently in the following proofs.

$$\begin{aligned} [ab, c] &= a[b, c] + [a, c]b \text{ and } [a, bc] = b[a, c] + [a, b]c, \\ a \circ (bc) &= (a \circ b)c - b[a, c] = b(a \circ c) + [a, b]c, \\ (ab) \circ c &= a(b \circ c) - [a, c]b = (a \circ c)b + a[b, c]. \end{aligned}$$

Subsequently, we provide some results concerning prime and semi-prime rings, which will be frequently used in the demonstration of theorems.

Lemma 1 ([7], Fact 1). *Let $(\mathcal{R}, *)$ be a two-torsion free prime ring with involution provided with a derivation d . Then, $d(h) = 0$ for all $h \in Z(\mathcal{R}) \cap H(\mathcal{R})$ implies that $d(z) = 0$ for all $z \in Z(\mathcal{R})$.*

Lemma 2 ([7], Fact 2). *Let $(\mathcal{R}, *)$ be a two-torsion free prime ring with involution. If “*” is of the second kind, then $H(\mathcal{R}) \cap Z(\mathcal{R}) \neq \{0\}$.*

Lemma 3 ([5], Remark 1). *Let $\mathcal{R}$ be a prime ring with center $Z(\mathcal{R})$. If $a, ab \in Z(\mathcal{R})$ for some $a, b \in \mathcal{R}$, then $a = 0$ or $b \in Z(\mathcal{R})$.*

Lemma 4 ([6], Lemma 2.3). *If $\mathcal{R}$ is a prime ring in which $[a, b] \in Z(\mathcal{R})$ for all $a, b \in \mathcal{R}$, then $\mathcal{R}$ is commutative.*

Lemma 5 ([12], Theorem 2). *Let $\mathcal{R}$ be a two-torsion free prime ring and $\mathcal{F}$ a generalized derivation associated with a nonzero derivation d of $\mathcal{R}$ such that $[\mathcal{F}(a), a] \in Z(\mathcal{R})$ for all $a \in J$, where J is nonzero Jordan ideal. Then, $\mathcal{R}$ is commutative.*

Lemma 6 ([4], Theorem 2). *Let $\mathcal{R}$ be a prime ring and d a nonzero derivation of $\mathcal{R}$ such that $[d(a), a] = 0$ for all $a \in \mathcal{R}$. Then, $\mathcal{R}$ is commutative.*

Lemma 7 ([8], Theorem 2.1). *Let $(\mathcal{R}, *)$ be a two-torsion free prime ring with involution of the second kind. If $\mathcal{R}$ admits a generalized derivation $\mathcal{F}$ associated with a nonzero derivation d such that $\mathcal{F}([a, a^*]) = 0$ for all $a \in \mathcal{R}$, then $\mathcal{R}$ is commutative.*

Lemma 8 ([8], Theorem 2.2). Let $(\mathcal{R}, *)$ be a two-torsion free prime ring with involution of the second kind. If $\mathcal{R}$ admits a generalized derivation $\mathcal{F}$ associated with a derivation d such that $\mathcal{F}(a \circ a^*) = 0$ for all $a \in \mathcal{R}$, then $\mathcal{F} = 0$.

Lemma 9 ([10], Theorem 2). Let $(\mathcal{R}, *)$ be a two-torsion free prime ring with involution of the second kind. If $\mathcal{R}$ admits a nonzero generalized derivation $\mathcal{F}$ associated with a derivation d such that $[\mathcal{F}(a), a^*] \in Z(\mathcal{R})$ for all $a \in \mathcal{R}$, then $\mathcal{R}$ is commutative.

3. Commutativity of Prime Ring with Involution by Generalized Derivations

Theorem 1. Let $(\mathcal{R}, *)$ be a two-torsion free prime ring with involution of the second kind, and let $(\mathcal{F}, f)$ and $(\mathcal{G}, g)$ be generalized derivations associated with derivations f and g of $\mathcal{R}$, respectively, where $g \neq 0$ or $f \neq 0$. Then, the following assertions are equivalent:

- (1) $\mathcal{F}([a, a^*]) + [\mathcal{G}(a), a^*] = 0$ for all $a \in \mathcal{R}$;
- (2) $\mathcal{R}$ is commutative.

Proof. If $\mathcal{F} = 0$ or $\mathcal{G} = 0$, it follows that $\mathcal{R}$ is commutative by Lemma 9 and Lemma 7.

Now, we only consider the case $\mathcal{F} \neq 0$ and $\mathcal{G} \neq 0$. By the hypothesis, we have

$$\mathcal{F}([a, a^*]) + [\mathcal{G}(a), a^*] = 0 \text{ for all } a \in \mathcal{R}. \quad (1)$$

The linearization of (1) indicates that

$$\mathcal{F}([a, b^*]) + \mathcal{F}([b, a^*]) + [\mathcal{G}(a), b^*] + [\mathcal{G}(b), a^*] = 0 \text{ for all } a, b \in \mathcal{R}. \quad (2)$$

By replacing b with b^* in (2), we can derive that

$$\mathcal{F}([a, b]) + \mathcal{F}([b^*, a^*]) + [\mathcal{G}(a), b] + [\mathcal{G}(b^*), a^*] = 0 \text{ for all } a, b \in \mathcal{R}. \quad (3)$$

Since $*$ is the second kind and Lemma 2, we can obtain $H(\mathcal{R}) \cap Z(\mathcal{R}) \neq \{0\}$. Substituting bh for b in (3), where $h \in H(\mathcal{R}) \cap Z(\mathcal{R}) \neq \{0\}$, we obtain

$$([a, b] + [b^*, a^*])f(h) + ([b^*, a^*])g(h) = 0 \text{ for all } a, b \in \mathcal{R}. \quad (4)$$

Case 1: $f(h) \neq 0$.

If $g(h) = 0$, by Equation (4), we have

$$([a, b] + [b^*, a^*])f(h) = 0 \text{ for all } a, b \in \mathcal{R}. \quad (5)$$

Since $f(h) \neq 0$, we can obtain $[a, b] + [b^*, a^*] = 0$ for all $a, b \in \mathcal{R}$ by Equation (5). Substituting bs for b in above equation where $s \in S(\mathcal{R}) \cap Z(\mathcal{R}) \neq \{0\}$, we obtain that

$$2([a, b])s = 0 \text{ for all } a, b \in \mathcal{R}. \quad (6)$$

Since $\mathcal{R}$ is two-torsion free prime and $s \neq 0$, we arrive at $[a, b] = 0$ for all $a, b \in \mathcal{R}$. Hence, $\mathcal{R}$ is commutative.

If $g(h) \neq 0$, replacing b by bs in (4), we find that

$$([a, b] - [b^*, a^*])f(h) - ([b^*, a^*])g(h)s = 0 \text{ for all } a, b \in \mathcal{R}. \quad (7)$$

Comparing (4) and (7), we find that

$$[a, b]f(h) = 0 \text{ for all } a, b \in \mathcal{R}. \quad (8)$$

Because $f(h) \neq 0$, we obtain that $[a, b] = 0$ for all $a, b \in \mathcal{R}$ and $\mathcal{R}$ is commutative.

Case 2: $f(h) = 0$.

By Equation (4), we have

$$([b^*, a^*])g(h) = 0 \text{ for all } a, b \in \mathcal{R}. \quad (9)$$

By the primeness of $\mathcal{R}$, it follows that $([b^*, a^*]) = 0$ for all $a, b \in \mathcal{R}$ or $g(h) = 0$. If $[b^*, a^*] = 0$ for all $a, b \in \mathcal{R}$, then $\mathcal{R}$ is commutative. If $g(h) = 0$, then $g(z) = 0$ for all $z \in Z(\mathcal{R})$ by Lemma 1. For the same reason, we can obtain that $f(z) = 0$ for all $z \in \mathcal{R}$. Substituting bs for b in (3), where $s \in S(\mathcal{R}) \cap Z(\mathcal{R}) \neq \{0\}$, we have

$$\mathcal{F}([a, b]) + [\mathcal{G}(a), b] = 0 \text{ for all } a, b \in \mathcal{R}. \quad (10)$$

By replacing b with ba in (10), we obtain that

$$[a, b]f(a) + b[\mathcal{G}(a), a] = 0 \text{ for all } a, b \in \mathcal{R}. \quad (11)$$

Substituting cb for b in (11), where $c \in \mathcal{R}$, we obtain

$$[a, c]bf(a) = 0 \text{ for all } a, b, c \in \mathcal{R}. \quad (12)$$

Since $\mathcal{R}$ is prime, by Equation (12), we have $[a, c]\mathcal{R}f(a) = 0$ for all $a, b, c \in \mathcal{R}$. Once again, by the primeness of $\mathcal{R}$, we arrive at $[a, c] = 0$ for all $a, c \in \mathcal{R}$ or $f(a) = 0$ for all $a \in \mathcal{R}$. If $f \neq 0$, then $[a, c] = 0$ for all $a, c \in \mathcal{R}$, and we can conclude that $\mathcal{R}$ is commutative. If $f = 0$ and $g \neq 0$, by Equation (11), we can obtain $b[\mathcal{G}(a), a] = 0$ for all $a, b \in \mathcal{R}$. This means $[\mathcal{G}(a), a] = 0$, and it follows that $\mathcal{R}$ is commutative by Lemma 5.

This completes the proof. $\square$

Use similar techniques, we can obtain the following theorem.

Theorem 2. Let $(\mathcal{R}, *)$ be a two-torsion free prime ring with involution of the second kind, and let $(\mathcal{F}, f)$ and $(\mathcal{G}, g)$ be generalized derivations associated with derivations f and g of $\mathcal{R}$, respectively, where $g \neq 0$ or $f \neq 0$; then, the following assertions are equivalent:

- (1) $\mathcal{F}([a, a^*]) - [\mathcal{G}(a), a^*] = 0$ for all $a \in \mathcal{R}$;
- (2) $\mathcal{R}$ is commutative.

Corollary 1 ([11] Theorem 3.1). Let $(\mathcal{R}, *)$ be a two-torsion free prime ring with involution of the second kind, $\mathcal{F}$ be a generalized derivation associated with nonzero derivation d of $\mathcal{R}$ such that $\mathcal{F}([a, a^*]) + [\mathcal{F}(a), a^*] = 0$ for all $a \in \mathcal{R}$; then, $\mathcal{R}$ is commutative.

Corollary 2. Let $(\mathcal{R}, *)$ be a two-torsion free prime ring with involution of the second kind, and let $(\mathcal{F}, f), (\mathcal{G}, g)$ be generalized derivations associated with derivations f and g of $\mathcal{R}$, respectively, where $g \neq 0$ or $f \neq 0$; then, the following assertions are equivalent:

- (1) $\mathcal{F}([a, b]) \pm [\mathcal{G}(a), b] = 0$ for all $a, b \in \mathcal{R}$;

(2) $\mathcal{R}$ is commutative.

The following example demonstrates that the condition “ $*$ is of second kind” is necessary in Theorems 1 and 2.

Example 1. Let S be the ring of integers and $\mathcal{R} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}, a, b, c, d \in S \right\}$. Define maps $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. It is straightforward to check that $\mathcal{R}$ is a two-torsion free prime ring and $*$ is an involution of the first kind. Let us define a derivation f on $\mathcal{R}$ by $f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix}$. Then, $\mathcal{F} = f$ is a generalized derivation and $f \neq 0$. It is easy to see that $\mathcal{F}$ and $\mathcal{G} = I$ satisfy the condition of Theorems 1 and 2. But $\mathcal{R}$ is not commutative.

The subsequent example demonstrates that Theorems 1 and 2 are not applicable to semi-prime rings.

Example 2. If we consider $\mathbb{C}$ to be a field of complex numbers with the conjugation involution and define $\mathcal{R} = (\mathcal{R}, \mathbb{C})$, then it is easy to see that $\mathcal{R}$ is a semi-prime ring with involution of the second kind, where $(\mathcal{R}, \mathbb{C})^* = (\mathcal{R}^*, \overline{\mathbb{C}})$. If we define the maps $\mathcal{F} : \mathcal{R} \rightarrow \mathcal{R}$ by $\mathcal{F}(a, b) = (\mathcal{F}(a), 0)$, $\mathcal{G} : \mathcal{R} \rightarrow \mathcal{R}$ by $\mathcal{G}(a, b) = (a, 0)$, for all $(a, b) \in \mathcal{R}$, it is clear that $\mathcal{F}$ and $\mathcal{G}$ create a generalized derivation, which satisfies the conditions of Theorems 1 and 2, but $\mathcal{R}$ is not commutative.

Theorem 3. Let $(\mathcal{R}, *)$ be a two-torsion free prime ring with involution of the second kind, while $(\mathcal{F}, f)$ and $(\mathcal{G}, g)$ are generalized derivations associated with derivations f and g of $\mathcal{R}$, respectively, where $g \neq 0$ or $f \neq 0$, such that $\mathcal{F}(a \circ a^*) + \mathcal{G}(a) \circ a^* = 0$ for all $a \in \mathcal{R}$; then, $\mathcal{R}$ is commutative.

Proof. First, we claim that $\mathcal{F} \neq 0$ and $\mathcal{G} \neq 0$.

If $\mathcal{G} = 0$, we can obtain $\mathcal{F} = 0$ by Lemma 8, which is a contradiction.

Now, we assume that $\mathcal{F} = 0$. By the hypothesis, we have

$$\mathcal{G}(a) \circ a^* = 0 \text{ for all } a \in \mathcal{R}. \quad (13)$$

The linearization of (13) gives that

$$\mathcal{G}(a) \circ b^* + \mathcal{G}(b) \circ a^* = 0 \text{ for all } a, b \in \mathcal{R}. \quad (14)$$

Replacing b^* by b in (14), we obtain that

$$\mathcal{G}(a) \circ b + \mathcal{G}(b^*) \circ a^* = 0 \text{ for all } a, b \in \mathcal{R}. \quad (15)$$

Replacing b by bh in (15), for all $h \in H(\mathcal{R}) \cap Z(\mathcal{R}) \neq \{0\}$ and $b \in \mathcal{R}$, we have

$$(b^* \circ a^*)g(h) = 0 \text{ for all } a, b \in \mathcal{R}. \quad (16)$$

By virtue of the primeness hypothesis, we can derive that either $g(h) = 0$ or $b \circ a = 0$ for all $a, b \in \mathcal{R}$. In the latter case, it is obvious that $\mathcal{R} = 0$, which is impossible. Consequently, we only need to consider the case $g(h) = 0$ for all $h \in H(\mathcal{R}) \cap Z(\mathcal{R}) \neq \{0\}$. By Lemma 2,

we have $g(Z(\mathcal{R})) = 0$. Replace b by bs in (15), where $s \in S(\mathcal{R}) \cap Z(\mathcal{R}) \neq \{0\}$, and we obtain that

$$(\mathcal{G}(a) \circ b - \mathcal{G}(b^*) \circ a^*)s = 0 \text{ for all } a, b \in \mathcal{R}. \quad (17)$$

Multiplying by s in the right side of (15), we obtain

$$(\mathcal{G}(a) \circ b + \mathcal{G}(b^*) \circ a^*)s = 0 \text{ for all } a, b \in \mathcal{R}. \quad (18)$$

Combining (17) with (18) yields that $\mathcal{G}(a) \circ b = 0$. Substituting $h \in Z(\mathcal{R}) \neq \{0\}$ for b , we obtain $2\mathcal{G}(a)h = 0$ for all $a \in \mathcal{R}$. Since $\mathcal{R}$ is two-torsion free prime and $h \neq 0$, which implies $\mathcal{G} = 0$, a contradiction. Accordingly, we conclude that $\mathcal{F} \neq 0$.

By the hypothesis, we have

$$\mathcal{F}(a \circ a^*) + \mathcal{G}(a) \circ a^* = 0 \text{ for all } a, b \in \mathcal{R}. \quad (19)$$

Using the linearization of (19), we find that

$$\mathcal{F}(a \circ b^*) + \mathcal{F}(b \circ a^*) + \mathcal{G}(b) \circ a^* + \mathcal{G}(a) \circ b^* = 0 \text{ for all } a, b \in \mathcal{R}. \quad (20)$$

Replacing b^* by b in (20), we obtain that

$$\mathcal{F}(a \circ b) + \mathcal{F}(b^* \circ a^*) + \mathcal{G}(b^*) \circ a^* + \mathcal{G}(a) \circ b = 0 \text{ for all } a, b \in \mathcal{R}. \quad (21)$$

Substituting bh for b in (21), where $h \in H(\mathcal{R}) \cap Z(\mathcal{R}) \neq \{0\}$, we have

$$(a \circ b + b^* \circ a^*)f(h) + (b^* \circ a^*)g(h) = 0 \text{ for all } a, b \in \mathcal{R}. \quad (22)$$

We divide the proof of the theorem into the following two steps.

Steps 1. We claim that $g(h) = 0$ and $f(h) = 0$ for all $h \in H(\mathcal{R}) \cap Z(\mathcal{R})$.

If $g(h) \neq 0$ and $f(h) = 0$ for all $h \in H(\mathcal{R}) \cap Z(\mathcal{R}) \neq \{0\}$, by Equation (22), we can obtain Equation (16). By the above proof, we know this is contradictory.

If $g(h) \neq 0$ and $f(h) \neq 0$, substituting bs for b in (22), where $s \in S(\mathcal{R}) \cap Z(\mathcal{R}) \neq \{0\}$, we obtain

$$2(a \circ b)f(h)s = 0 \text{ for all } a, b \in \mathcal{R}. \quad (23)$$

Since $\mathcal{R}$ is two-torsion free prime and $f(h) \neq 0$, this implies that $a \circ b = 0$; thus, it is easy to see that $\mathcal{R} = \{0\}$, which is impossible.

If $g(h) = 0$ and $f(h) \neq 0$, by Equation (22), we obtain $(a \circ b + b^* \circ a^*)f(h) = 0$. Since $\mathcal{R}$ is a prime ring and $f(h) \neq 0$, we obtain that $a \circ b + b^* \circ a^* = 0$. Replacing b by $h \in H(\mathcal{R}) \cap Z(\mathcal{R}) \neq \{0\}$, we arrive at $a + a^* = 0$ by Lemma 3. Similarly, for $s \in H(\mathcal{R}) \cap Z(\mathcal{R}) \neq \{0\}$, we can obtain that $a - a^* = 0$ for all $a \in \mathcal{R}$. Accordingly, we find that $a = 0$ for all $a \in \mathcal{R}$, which is a contradiction. Hence, $g(h) = 0$ and $f(h) = 0$.

Step 2. We claim that $f(h) = 0$ and $g(h) = 0$ for all $h \in H(\mathcal{R}) \cap Z(\mathcal{R})$ and $\mathcal{R}$ is commutative.

By Lemma 1, we have $f(Z(\mathcal{R})) = 0$ and $g(Z(\mathcal{R})) = 0$. Replacing b by bs in (21) for all $s \in S(\mathcal{R}) \cap Z(\mathcal{R}) \neq \{0\}$ and $b \in \mathcal{R}$, we can obtain

$$(\mathcal{F}(a \circ b) - \mathcal{F}(b^* \circ a^*) - \mathcal{G}(b^*) \circ a^* + \mathcal{G}(a) \circ b)s = 0 \text{ for all } a, b \in \mathcal{R}. \quad (24)$$

Multiplying by s in the right side of (21), and combining with (24) yields that

$$\mathcal{F}(a \circ b) + \mathcal{G}(a) \circ b = 0 \text{ for all } a, b \in \mathcal{R}. \quad (25)$$

Replacing b by ba in (25), we have

$$(a \circ b)f(a) - b[\mathcal{G}(a), a] = 0 \text{ for all } a, b \in \mathcal{R}. \quad (26)$$

Substituting tb for b in (26), where $t \in \mathcal{R}$, we arrive at

$$[a, t]bf(a) = 0 \text{ for all } a, b \in \mathcal{R}. \quad (27)$$

By Equation (27), we have $[a, t]\mathcal{R}f(a) = 0$ for all $a, t \in \mathcal{R}$, which implies either $[a, t] = 0$ for all $a, t \in \mathcal{R}$ or $f = 0$. If $f \neq 0$, we can obtain $[a, t] = 0$ and $\mathcal{R}$ is commutative.

If $f = 0$, by Equation (26), we have

$$b[\mathcal{G}(a), a] = 0 \text{ for all } a, b \in \mathcal{R}. \quad (28)$$

Once again invoking the primeness, it follows that $[\mathcal{G}(a), a] = 0$; the application of Lemma 5 yields that $\mathcal{R}$ is commutative.

This completes the proof. $\square$

By using the same techniques as previously mentioned, with appropriate adjustments, we are able to obtain the following theorem.

Theorem 4. Let $(\mathcal{R}, *)$ be a two-torsion free prime ring with involution of the second kind, and let $(\mathcal{F}, f), (\mathcal{G}, g)$ be generalized derivations associated with derivations f and g of $\mathcal{R}$, respectively, where $g \neq 0$ or $f \neq 0$, such that $\mathcal{F}(a \circ a^*) - \mathcal{G}(a) \circ a^* = 0$ for all $a \in \mathcal{R}$; then, $\mathcal{R}$ is commutative.

Corollary 3. Let $(\mathcal{R}, *)$ be a two-torsion free prime ring with involution of the second kind, and let $(\mathcal{F}, f), (\mathcal{G}, g)$ be generalized derivations associated with derivations f and g of $\mathcal{R}$, respectively, where $g \neq 0$ or $f \neq 0$, such that $\mathcal{F}(a \circ b) \pm \mathcal{G}(a) \circ b = 0$ for all $a, b \in \mathcal{R}$; then, $\mathcal{R}$ is commutative.

Theorem 5. Let $(\mathcal{R}, *)$ be a two-torsion free prime ring with involution of the second kind, and let $(\mathcal{F}, f)$ and $(\mathcal{G}, g)$ be generalized derivations associated with nonzero derivations f and g of $\mathcal{R}$, respectively; then, the following assertions are equivalent:

- (1) $[\mathcal{F}(a), a^*] + [a, \mathcal{G}(a^*)] = 0$ for all $a \in \mathcal{R}$;
- (2) $\mathcal{R}$ is commutative.

Proof. By the hypothesis, we have

$$[\mathcal{F}(a), a^*] + [a, \mathcal{G}(a^*)] = 0 \text{ for all } a \in \mathcal{R}. \quad (29)$$

The linearization of (29) gives that

$$[\mathcal{F}(a), b^*] + [\mathcal{F}(b), a^*] + [a, \mathcal{G}(b^*)] + [b, \mathcal{G}(a^*)] = 0 \text{ for all } a, b \in \mathcal{R}. \quad (30)$$

Replacing b^* by b in (30), we have

$$[\mathcal{F}(a), b] + [\mathcal{F}(b^*), a^*] + [a, \mathcal{G}(b)] + [b^*, \mathcal{G}(a^*)] = 0 \text{ for all } a, b \in \mathcal{R}. \quad (31)$$

Substituting bh for b in (31), where $h \in H(\mathcal{R}) \cap Z(\mathcal{R}) \neq \{0\}$, we obtain

$$([b^*, a^*])f(h) + ([a, b])g(h) = 0 \text{ for all } a, b \in \mathcal{R}. \quad (32)$$

Case 1. $f(h) \neq 0$.

If $g(h) = 0$, we have $([b^*, a^*])f(h) = 0$. Because $\mathcal{R}$ is a prime ring and $f(h) \neq 0$, we conclude that $[b, a] = 0$ for all $a, b \in \mathcal{R}$. Thus, $\mathcal{R}$ is commutative.

If $g(h) \neq 0$, replacing b by bs in (32), where $s \in S(\mathcal{R}) \cap Z(\mathcal{R}) \neq \{0\}$, we arrive at $([a, b])g(h) - ([b^*, a^*])f(h) = 0$ for all $a, b \in \mathcal{R}$. Comparing with (32), we obtain $([a, b])g(h) = 0$ for all $a, b \in \mathcal{R}$. Since $\mathcal{R}$ is a prime ring and $g(h) \neq 0$, we conclude that $[a, b] = 0$ for all $a, b \in \mathcal{R}$. Hence, $\mathcal{R}$ is commutative.

Case 2. $f(h) = 0$.

If $g(h) \neq 0$, by Equation (32), we have $([a, b])g(h) = 0$ for all $a, b \in \mathcal{R}$. Using the same proof as above, we conclude that $\mathcal{R}$ is commutative. If $g(h) = 0$, we have $g(z) = 0$ and $f(z) = 0$ by Lemma 1. Replacing b by bs in (31), where $s \in S(\mathcal{R}) \cap Z(\mathcal{R}) \neq \{0\}$, we obtain

$$[\mathcal{F}(a), b] + [a, \mathcal{G}(b)] = 0 \text{ for all } a, b \in \mathcal{R}. \quad (33)$$

By [[9], Theorem 2.5] and [[10], Lemma 3], $\mathcal{R}$ is commutative.

This completes the proof. $\square$

By using the same techniques as previously mentioned, with appropriate adjustments, we are able to obtain the following theorem.

Theorem 6. Let $(\mathcal{R}, *)$ be a two-torsion free prime ring with involution of the second kind, and let $(\mathcal{F}, f)$ and $(\mathcal{G}, g)$ be generalized derivations associated with nonzero derivations f and g of $\mathcal{R}$, respectively; then, the following assertions are equivalent:

- (1) $[\mathcal{F}(a), a^*] - [a, \mathcal{G}(a^*)] = 0$ for all $a \in \mathcal{R}$;
- (2) $\mathcal{R}$ is commutative.

The following example demonstrates that the condition “ $*$ is of second kind” is necessary in Theorems 5 and 6.

Example 3. Let S be the ring of integers and $\mathcal{R} = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}, a, b, c \in S \right\}$. Define maps $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}^* = \begin{bmatrix} c & -b \\ 0 & a \end{bmatrix}$. It is obvious that $\mathcal{R}$ is a two-torsion free prime ring and $*$ is an involution of the first kind. Let us define a derivation f on $\mathcal{R}$ by $f\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$. Then, $\mathcal{F} = f$ is a generalized derivation and $f \neq 0$. It is easy to see that $\mathcal{G} = \mathcal{F}$ and $\mathcal{G} = -\mathcal{F}$ satisfy the condition of Theorems 5 and 6, respectively. But $\mathcal{R}$ is not commutative.

Next, we will generalize the results of the [11] theorem from another perspective.

Theorem 7. Let $(\mathcal{R}, *)$ be a two-torsion free prime ring with involution of the second kind. If $\mathcal{R}$ admits a generalized derivation $\mathcal{F}$ associated with a nonzero derivation d , then the following assertions are equivalent:

- (1) $\mathcal{F}([a, a^*]) + [\mathcal{F}(a), a^*] \in Z(\mathcal{R})$ for all $a \in \mathcal{R}$;
- (2) $\mathcal{R}$ is commutative.

Proof. By the hypothesis, we have

$$\mathcal{F}([a, a^*]) + [\mathcal{F}(a), a^*] \in Z(\mathcal{R}) \text{ for all } a \in \mathcal{R}. \quad (34)$$

The linearization of (34) shows that

$$\mathcal{F}([a, b^*]) + \mathcal{F}([b, a^*]) + [\mathcal{F}(a), b^*] + [\mathcal{F}(b), a^*] \in Z(\mathcal{R}) \text{ for all } a, b \in \mathcal{R}. \quad (35)$$

By replacing b with b^* in (35), we can derive that

$$\mathcal{F}([a, b]) + \mathcal{F}([b^*, a^*]) + [\mathcal{F}(a), b] + [\mathcal{F}(b^*), a^*] \in Z(\mathcal{R}) \text{ for all } a, b \in \mathcal{R}. \quad (36)$$

Substituting bh for b in (36), where $h \in H(\mathcal{R}) \cap Z(\mathcal{R}) \neq \{0\}$, we obtain

$$([a, b] + 2[b^*, a^*])d(h) \in Z(\mathcal{R}) \text{ for all } a, b \in \mathcal{R}. \quad (37)$$

Using the fact that $d(Z(\mathcal{R})) \in Z(\mathcal{R})$ and Lemma 3, we obtain $d(h) = 0$ for all $h \in H(\mathcal{R}) \cap Z(\mathcal{R})$ or $[a, b] + 2[b^*, a^*] \in Z(\mathcal{R})$ for all $a, b \in \mathcal{R}$.

We assume that $[a, b] + 2[b^*, a^*] \in Z(\mathcal{R})$ for all $a, b \in \mathcal{R}$. Substituting bs for b in the above equation, where $s \in S(\mathcal{R}) \cap Z(\mathcal{R}) \neq \{0\}$, we can obtain $[a, b] - 2[b^*, a^*] \in Z(\mathcal{R})$ for all $a, b \in \mathcal{R}$. Combining the above equations, we can obtain $[a, b] \in Z(\mathcal{R})$ and $\mathcal{R}$ is commutative by Lemma 4.

Now, we consider that $d(h) = 0$ for all $h \in H(\mathcal{R}) \cap Z(\mathcal{R})$. By Lemma 1, we obtain $d(z) = 0$ for all $z \in Z(\mathcal{R})$. Replacing b by bs in (36), where $s \in S(\mathcal{R}) \cap Z(\mathcal{R}) \neq \{0\}$, we have

$$\mathcal{F}([a, b]) - \mathcal{F}([b^*, a^*]) + [\mathcal{F}(a), b] - [\mathcal{F}(b^*), a^*] \in Z(\mathcal{R}) \text{ for all } a, b \in \mathcal{R}. \quad (38)$$

Comparing (36) and (38), we have

$$\mathcal{F}([a, b]) + [\mathcal{F}(a), b] \in Z(\mathcal{R}) \text{ for all } a, b \in \mathcal{R}. \quad (39)$$

Substituting ba for b in (39), we obtain

$$\mathcal{F}([a, b])a + [a, b]d(a) + ([\mathcal{F}(a), b])a + b[\mathcal{F}(a), a] \in Z(\mathcal{R}) \text{ for all } a, b \in \mathcal{R}. \quad (40)$$

Replacing b by h in (40), where $h \in H(\mathcal{R}) \cap Z(\mathcal{R}) \neq \{0\}$, we can obtain

$$h[\mathcal{F}(a), a] \in Z(\mathcal{R}) \text{ for all } a \in \mathcal{R}. \quad (41)$$

By Lemma 3, we have $[\mathcal{F}(a), a] \in Z(\mathcal{R})$ for all $a \in \mathcal{R}$, which means that $\mathcal{R}$ is commutative by Lemma 5.

This completes the proof. $\square$

Using the similar techniques, we can prove the following theorem.

Theorem 8. Let $(\mathcal{R}, *)$ be a two-torsion free prime ring with involution of the second kind. If $\mathcal{R}$ admits a generalized derivation $\mathcal{F}$ associated with a nonzero derivation d , then the following assertions are equivalent:

- (1) $\mathcal{F}([a, a^*]) - [\mathcal{F}(a), a^*] \in Z(\mathcal{R})$ for all $a \in \mathcal{R}$;
- (2) $\mathcal{R}$ is commutative.

Corollary 4. Let $(\mathcal{R}, *)$ be a two-torsion free prime ring with involution of the second kind. If $\mathcal{R}$ admits a generalized derivation $\mathcal{F}$ associated with a nonzero derivation d , then the following assertions are equivalent:

- (1) $\mathcal{F}([a, b]) \pm [\mathcal{F}(a), b] \in Z(\mathcal{R})$ for all $a, b \in \mathcal{R}$;
- (2) $\mathcal{R}$ is commutative.

Theorem 9. Let $(\mathcal{R}, *)$ be a noncommutative two-torsion free prime ring with involution of the second kind. If $\mathcal{R}$ admits a generalized derivation $\mathcal{F}$ associated with a derivation d , then the following assertions are equivalent:

- (1) $\mathcal{F}(a \circ a^*) + \mathcal{F}(a) \circ a^* \in Z(\mathcal{R})$ for all $a \in \mathcal{R}$;
- (2) $\mathcal{F} = 0$.

Proof. By the hypothesis, we have

$$\mathcal{F}(a \circ a^*) + \mathcal{F}(a) \circ a^* \in Z(\mathcal{R}) \text{ for all } a \in \mathcal{R}. \quad (42)$$

The linearization of (42) shows that

$$\mathcal{F}(a \circ b^*) + \mathcal{F}(b \circ a^*) + \mathcal{F}(a) \circ b^* + \mathcal{F}(b) \circ a^* \in Z(\mathcal{R}) \text{ for all } a, b \in \mathcal{R}. \quad (43)$$

By replacing b with b^* in (43), we can derive that

$$\mathcal{F}(a \circ b) + \mathcal{F}(b^* \circ a^*) + \mathcal{F}(a) \circ b + \mathcal{F}(b^*) \circ a^* \in Z(\mathcal{R}) \text{ for all } a, b \in \mathcal{R}. \quad (44)$$

Substituting bh for b in (44), where $h \in H(\mathcal{R}) \cap Z(\mathcal{R}) \neq \{0\}$, we obtain

$$(a \circ b + 2b^* \circ a^*)d(h) \in Z(\mathcal{R}) \text{ for all } a, b \in \mathcal{R}. \quad (45)$$

Using the fact that $d(Z(\mathcal{R})) \subseteq Z(\mathcal{R})$ and Lemma 3, we arrive $d(h) = 0$ or $a \circ b + 2b^* \circ a^* \in Z(\mathcal{R})$ for all $a, b \in \mathcal{R}$. In the latter case, as above, we obtain that $a \circ b \in Z(\mathcal{R})$ for all $a, b \in \mathcal{R}$. Replacing b by h , where $h \in H(\mathcal{R}) \cap Z(\mathcal{R}) \neq \{0\}$, we have $ah \in Z(\mathcal{R})$ and $a \in Z(\mathcal{R})$ by Lemma 3; thus, $\mathcal{R}$ is commutative, which is a contradiction.

Now, we consider the case that $d(h) = 0$; replacing b by bs in (44), where $s \in S(\mathcal{R}) \cap Z(\mathcal{R}) \neq \{0\}$, we have

$$\mathcal{F}(a \circ b) - \mathcal{F}(b^* \circ a^*) + \mathcal{F}(a) \circ b - \mathcal{F}(b^*) \circ a^* \in Z(\mathcal{R}) \text{ for all } a, b \in \mathcal{R}. \quad (46)$$

Comparing (46) and (44), we can obtain

$$\mathcal{F}(a \circ b) + \mathcal{F}(a) \circ b \in Z(\mathcal{R}) \text{ for all } a, b \in \mathcal{R}. \quad (47)$$

Replacing b by h in (47), where $h \in H(\mathcal{R}) \cap Z(\mathcal{R}) \neq \{0\}$, we obtain that

$$\mathcal{F}(a)h \in Z(\mathcal{R}) \text{ for all } a \in \mathcal{R}. \quad (48)$$

By Lemma 3, we obtain that $\mathcal{F}(a) \in Z(\mathcal{R})$ for all $a \in \mathcal{R}$. Therefore, we can obtain $[\mathcal{F}(a), b] = 0$, for all $a, b \in \mathcal{R}$. If $d \neq 0$, replacing b by a in $[\mathcal{F}(a), b] = 0$, we can obtain $[\mathcal{F}(a), a] = 0$, for all $a \in \mathcal{R}$ and $\mathcal{R}$ is commutative by Lemma 5, contradiction. Accordingly, we have $d = 0$. Replacing a by ac in the above equation, where $r \in \mathcal{R}$, we have $\mathcal{F}(a)[c, b] = 0$ for all $a, b, c \in \mathcal{R}$. The primeness of $\mathcal{R}$ yields that either $\mathcal{F}(a) = 0$ for all $a \in \mathcal{R}$ or $[c, b] = 0$ for all $b, c \in \mathcal{R}$. If $[b, c] = 0$, then $\mathcal{R}$ is commutative by Lemma 4, which is a contradiction. Therefore, we obtain that $\mathcal{F} = 0$.

This completes the proof. $\square$

Using the similar techniques, we can prove the following theorem.

Theorem 10. Let $(\mathcal{R}, *)$ be a noncommutative two-torsion free prime ring with involution of the second kind. If $\mathcal{R}$ admits a generalized derivation $\mathcal{F}$ associated with a derivation d , then the following assertions are equivalent:

- (1) $\mathcal{F}(a \circ a^*) + \mathcal{F}(a) \circ a^* \in Z(\mathcal{R})$ for all $a \in \mathcal{R}$;
- (2) $\mathcal{F} = 0$.

Corollary 5. Let $(\mathcal{R}, *)$ be a two-torsion free prime ring with involution of the second kind. If $\mathcal{R}$ admits a nonzero generalized derivation $\mathcal{F}$ associated with a derivation d , then the following assertions are equivalent:

- (1) $\mathcal{F}(a \circ a^*) \pm \mathcal{F}(a) \circ a^* \in Z(\mathcal{R})$ for all $a \in \mathcal{R}$;
- (2) $\mathcal{R}$ is commutative.

4. Conclusions

In this article, we generalize the theorem of commutativity of a prime ring with involution of the second kind from a few perspectives in [11], prove a two-torsion free prime ring $\mathcal{R}$ with involution of the second kind admits a generalized derivation $\mathcal{F}$, and $\mathcal{G}$ satisfies one of the following equations:

- (i) $\mathcal{F}([a, a^*]) \pm [\mathcal{G}(a), a^*] = 0$ for all $a \in \mathcal{R}$;
- (ii) $\mathcal{F}(a \circ a^*) \pm \mathcal{G}(a) \circ a^* = 0$ for all $a \in \mathcal{R}$;
- (iii) $[\mathcal{F}(a), a^*] \pm [a, \mathcal{G}(a^*)] = 0$ for all $a \in \mathcal{R}$;
- (iv) $\mathcal{F}([a, a^*]) \pm [\mathcal{F}(a), a^*] \in Z(\mathcal{R})$ for all $a \in \mathcal{R}$;
- (v) $\mathcal{F}(a \circ a^*) \pm \mathcal{F}(a) \circ a^* \in Z(\mathcal{R})$ for all $a \in \mathcal{R}$.

Then, $\mathcal{R}$ is commutative.

At the end of this section, we present the following several problems. These problems not only constitute our upcoming research content but also embody the trajectory of our future research endeavors.

Problem 1. Let $(\mathcal{R}, *)$ be a two-torsion free prime ring with involution of the second kind, and let $(\mathcal{F}, f), (\mathcal{G}, g)$ be generalized derivations associated with derivations f and g of $\mathcal{R}$, respectively, where $g \neq 0$ or $f \neq 0$, such that $\mathcal{F}([a, b]) \pm [\mathcal{G}(a), \mathcal{G}(b)] = 0$ for all $a, b \in \mathcal{R}$. Then, what conclusions can we draw about the structure of $\mathcal{R}$?

Problem 2. Let $(\mathcal{R}, *)$ be a prime ring and $\mathcal{R}$ admit a generalized derivation $\mathcal{F}$ associated with a nonzero derivation d such that $\mathcal{F}([a, b^n]) \pm [\mathcal{F}(a), b^n] \in Z(\mathcal{R})$ for all $a, b \in \mathcal{R}$, where $n \geq 2$ is a fixed integer. Then, what conclusions can we draw about the structure of $\mathcal{R}$?

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References

1. Ferreira, R.N.; Ferreira, B.L. Jordan triple derivation on alternative rings. *Proyecciones* **2018**, *37*, 171–180. [\[CrossRef\]](#)
2. Oukhtite, L.; Salhi, S. On commutativity of σ -prime rings. *Glas. Mat.* **2006**, *41*, 57–64. [\[CrossRef\]](#)
3. Divinsky, N. On commuting automorphisms of rings. *Trans. Roy. Soc. Canada. Sect* **1955**, *3*, 19–22.
4. Posner, E. Derivations in prime rings. *Proc. Am. Math. Soc.* **1957**, *8*, 1093–1100. [\[CrossRef\]](#)
5. Ali, A.; Shah, T. Centralizing and commuting generalized derivations on prime rings. *Mat. Vesn.* **2008**, *60*, 1–2.
6. Boua, A.; Ashraf, M. Differential Identities and Generalized Derivations in Prime Rings with Involution. *Southeast Asian Bull. Math.* **2019**, *43*, 165–181.
7. Nejjar, B.; Kacha, A.; Mamouni, A.; Oukhtite, L. Commutativity theorems in rings with involution. *Commun. Algebra* **2017**, *45*, 698–708. [\[CrossRef\]](#)
8. Nejjar, B.; Kacha, A.; Mamouni, A.; Oukhtite, L. Certain commutativity criteria for rings with involution involving generalized derivations. *Georgian Math. J.* **2020**, *27*, 133–139. [\[CrossRef\]](#)
9. Rehman, N.; Ashraf, M.; Ali, S.; Mozumder, M. On semiprime rings with generalized derivations. *Bol. Soc. Parana. Mat.* **2010**, *28*, 25–32. [\[CrossRef\]](#)
10. Zemzami, O.; Oukhtite, L.; Ali, S.; Muthana, N. On certain classes of generalized derivations. *Math. Slovaca* **2019**, *69*, 1023–1032. [\[CrossRef\]](#)
11. Huang, S. On generalized derivations and commutativity of prime rings with involution. *Commun. Algebra* **2023**, *51*, 3521–3527. [\[CrossRef\]](#)
12. Oukhtite, L.; Mamouni, A. Generalized derivations centralizing on Jordan ideals of rings with involution. *Turk. J. Math.* **2014**, *38*, 225–232. [\[CrossRef\]](#)

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