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The Event-Triggered Resilient Control of Discrete-Time Nonlinear Semi-Markov Jump Systems Based on Incremental Quadratic Constraints

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Abstract: This paper explores resilient control problems for discrete-time nonlinear semi-Markov jump systems characterized by incremental quadratic constraints. Considering the system's uncertainties and external environmental factors, mode-dependent resilient controllers are developed to ensure the system's mean-square stability. A proposed event-triggering mechanism is suggested to alleviate the communication burden within the system. Additionally, the system's nonlinearity is characterized by using incremental quadratic constraints to derive a less conservative feasible solution. Sufficient conditions for the system's mean-square stability are established by employing the Lyapunov stability theory. Finally, a numerical simulation example is given to prove the conclusion's validity.

Keywords: semi-Markov jump system; event-trigger control; resilient control; incremental quadratic constraints

MSC: 93E03



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1. Introduction

As a category of switching systems, Markov jump systems (MJSs) have garnered significant attention from researchers due to their exceptional capability in modeling complex systems subject to abrupt changes. These changes often result from structural alterations and environmental variations [1]. Due to its inherent characteristics, the Markov jump system (MJS) provides an effective framework for representing a wide range of real-world systems, encompassing domains such as network communication [2], economics [3], biology [4], and energy [5]. The transition signals of MJSs are controlled by memoryless transition probabilities (TPs) conforming to a Markov stochastic process. This time-invariant transition probability restricts the application of MJSs in practical systems. In fact, the jumps of most systems do not satisfy this constraint. For example, the fault-tolerant control system with sojourn time proposed in [6] follows the Weibull distribution. To address the limitations of strict time-invariant transition probabilities, researchers have introduced semi-Markov jump systems (SMJSs) with time-varying transition probabilities and have carried out extensive studies on this topic [7,8]. Meanwhile, for discrete-time SMJS, Zhang et al. [9] introduced the definition of semi-Markov kernels (SMKs) and σ mean-square stability (σ -MSS). Subsequently, there have been some interesting studies on SMJS [10–12].

On the other hand, most systems are inherently nonlinear, including SMJSs. In recent years, describing practical systems as simple linear systems has stopped meeting the rapidly evolving needs of the modern industry. Systems based on nonlinear constraints are receiving increasing attention. Nonlinear constraints can be described by a nonlinear function, with common examples including Lipschitz constraints [13] and one-sided Lipschitz constraints [14]. For instance, in [15], the multi-agent problem with Lipschitz constraints was studied. However, in this case, the Lipschitz constraints limited the parameters to positive values, restricting the application scope of Lipschitz constraints. In [16],

the one-sided Lipschitz problem of SMJSs was explored. Unlike Lipschitz constraints, the constants in one-sided Lipschitz constraints can be positive, negative, or zero. Recent work by D'Ito et al. [17] has shown that both Lipschitz constraints and one-sided Lipschitz constraints can be regarded as specific instances of incremental quadratic constraints (IQC). This implies that the application of IQC can more broadly describe nonlinear problems. Therefore, this paper investigates control strategies for discrete-time SMJSs with IQC in order to obtain a feasible solution to the nonlinear control problem that is less conservative.

Meanwhile, the increasing openness and advancement of communication networks have made systems more vulnerable to external malicious attacks [18]. Therefore, in order to reduce the stability of the system in the event of an attack, scholars have proposed the concept of system resilience: system resilience denotes the capacity of a system to uphold a satisfactory degree of regular functionality despite encountering disruptions, encompassing both unforeseen accidents and deliberate malicious attacks [19]. In reality, resilient control can be applied to power systems [20], transport systems [21], and so on. For example, Chen investigated the resilient control of helicopters [22]. As for MJSs, studies on the resilient control problem can be found in [23]. In [24], Bin Zhang et al. illustrate the superior performance of resilient control in MJSs under disturbances through a detailed example of an economic system. Regrettably, to the best of the authors' knowledge, resilient control for SMJS has yet to receive significant attention in the existing literature.

The event-triggering control has been proposed by scholars to save communication resources [25]. Compared to periodic sampling, the sampling time of the event-triggering control is event-driven, meaning sampling occurs when certain threshold conditions are met. Therefore, the event-triggering control demonstrates outstanding performance in reducing communication costs. It has been extensively applied in practical systems, including multi-agent systems [26], networked control systems [27], etc. In [28,29], event-triggered H_∞ control for an MJS with unknown input is proposed. Obviously, employing event-triggering mechanisms for the control of SMJSs is highly necessary.

Motivated by the above research results, this paper aims to solve the resilient control problem of nonlinear discrete-time SMJSs based on IQC. The condition for system stability is expressed by linear matrix inequalities (LMIs) and Lyapunov stability theory [30]. According to Lyapunov theory, if the Lyapunov function V of a system is positive definite and its derivative ΔV is negative definite, then the system is stable. The Lyapunov function provides a rapid means to assess the stability of a system [31,32].

The primary contributions of this paper can be summarized as follows:

- (1) IQC is utilized to characterize the nonlinearity of a discrete-time SMJS. Compared to Lipschitz constraints, the nonlinear systems considered in this paper are more general. In this paper, the mean-square stability (MSS) of a nonlinear discrete-time SMJS based on IQC is proved by the relaxation variable method. Sufficient conditions for stability are given by LMI.
- (2) The mode-dependent resilient controller is designed to eliminate the interference of nonlinear unknowns and system mode transformation. At the same time, the resilience of the system is enhanced. The system can quickly return to normal operation when faced with external disturbances.
- (3) The event-triggering strategy is designed and deployed on the resilient controller to reduce the system communication burden.

Notation 1. In this paper, for a probability space, Ψ denotes the sample space, $\mathcal{F}$ denotes the σ -algebra of sample space subsets, and P_r is the probability measure on $\mathcal{F}$; $\mathbb{R}_n$ denotes the n -dimensional Euclidean space; $\mathbb{E}$ denotes the mathematical expectation; $\|X\| = X^T X$ and $\langle X, Y \rangle = X^T Y$; $\mathbb{Z}_+$ is a set of natural numbers; R is a set of real numbers; $x_{k_m}^h$ is equal to $x(k_m + h)$; $*$ represents the ellipsis of symmetric blocks in a block matrix. $\text{diag}\{q\}$ represents a block-diagonal matrix with q as all its diagonal elements.

2. System Setting and Controller Design

2.1. System Setting

For $k \in \mathbb{Z}_+ \geq 0$, consider the discrete-time SMJS based on a completely known probability space $(\Psi, \mathcal{F}, P_r)$:

$$x_{k+1} = A_{r_k}x_k + B_{r_k}u(k, r_k) + \varrho(x_k) + \omega(k) \quad (1)$$

where $u(k, r_k) \in \mathbb{R}_n^u$ and $x_k \in \mathbb{R}_n$ are the input of the system and the state variables, respectively. The semi-Markov chain $\{r_k\}$ is a stochastic process, which takes values in Y , where $Y = \{1, 2, 3, \dots, M\}$ is a finite set of positive integers that control the switching of the system among M system modes. $A_{r_k} \in \mathbb{R}_n$ and $B_{r_k} \in \mathbb{R}_n^u$ are known matrices representing the SMJS structure. $\omega(k)$ is the unexpected attack or the external disturbance; $\varrho(x_k)$ is an unknown nonlinear function.

Definition 1 ([9]). To elucidate the semi-Markov chain more effectively, the following definitions are presented:

- (1) $\{r_k\}$ is a stochastic process such that $r_k \in Y$ represents the system mode at the k th jump.
- (2) $\{s_k\}$ is a stochastic process such that $s_k \in \mathbb{Z}_+$ denotes the time at the k th jump and $s_0 = 0$.
- (3) $\{t_k\}$ is a stochastic process such that $t_k \in \mathbb{Z}_+$ and $t_k = s_k - s_{k-1}$ denote the sojourn-time between the $(k-1)$ th jump and the k th jump, $t_0 = 0$.

Definition 2 ([9]). The stochastic sequence denoted as $\{r_k, s_k\}$ is referred to as the Markov renewal chain (MRC) in this paper. For $\epsilon, \zeta \in Y$, $n \in \mathbb{Z}_+$ and $\tau \in \mathbb{Z}_+$, the discrete-time SMK is defined as follows:

$$\begin{aligned} &Pr(s_{n+1} - s_n = \tau, r_{n+1} = \epsilon | r_n = \zeta, \dots, r_0 = \zeta_0) \\ &= Pr(s_{n+1} - s_n = \tau, r_{n+1} = \epsilon | r_n = \zeta) = \pi_{\zeta\epsilon}(\tau), \end{aligned} \quad (2)$$

where $\pi_{\zeta\epsilon}(\tau) \in [0, 1]$ and $\sum_{\tau=0}^{\infty} \sum_{\epsilon \in Y} \pi_{\zeta\epsilon}(\tau) = 1$, $\pi_{\zeta\epsilon}(0) = 0$. Additionally, the TPs can be defined as:

$$Pr(r_{n+1} = \epsilon | r_n = \zeta) = \theta_{\zeta\epsilon}, \quad (3)$$

where matrix $\Theta = [\theta_{\zeta\epsilon}]_{\zeta, \epsilon \in Y}$, $\theta_{\zeta\epsilon} \in [0, 1]$ and $\theta_{\zeta\zeta} = 0$. Then, for $\epsilon, \zeta \in Y$, $n \in \mathbb{Z}_+$ and $\tau \in \mathbb{Z}_+$, we denote

$$Pr(s_{n+1} - s_n = \tau | r_{n+1} = \epsilon, r_n = \zeta) = \omega_{\zeta\epsilon}(\tau), \quad (4)$$

where matrix $\Omega = [\omega_{\zeta\epsilon}(\tau)]_{\zeta, \epsilon \in Y, \tau \in \mathbb{Z}_+}$ serves as the probability density function (PDF) that describes the distribution of sojourn-time, and $\omega_{\zeta\zeta} = 0$.

Remark 1. From (2)–(4), we can obtain $\pi_{\zeta\epsilon}(\tau) = \theta_{\zeta\epsilon}\omega_{\zeta\epsilon}(\tau)$. In addition, $\pi_{\zeta\epsilon}(\tau)$ is determined by the current state and the next state because $\omega_{\zeta\epsilon}(\tau)$ is related to both. In other words, the state transition of a semi-Markov stochastic process is determined not only by the transition probability $\theta_{\zeta\epsilon}$, but also by the sojourn-time τ in the current state. The sojourn-time τ is determined by the distribution in state ζ before entering state ϵ , and its probability density is denoted as $\omega_{\zeta\epsilon}$. Therefore, the SMJS overcomes the limitation of the time-invariant state transition probability with no memory in MJS.

Definition 3 ([33]). For the matrix $\mathcal{W} = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$, if $W_{11} > 0$, $W_{12}^T = W_{21}$ and $W_{22} < 0$, the nonlinear function $\varrho(x(k))$ satisfies the incremental quadratic constraint (IQC) if and only if the inequality below holds:

$$\begin{bmatrix} x_k \\ \varrho(x_k) \end{bmatrix}^T \mathcal{W} \begin{bmatrix} x_k \\ \varrho(x_k) \end{bmatrix} \geq 0, \quad (5)$$

Then, we can obtain:

$$x_k^T W_{11} x_k + x_k^T W_{12} \varrho(x_k) + \varrho(x_k) W_{12}^T x_k + \varrho(x_k) W_{22} \varrho(x_k) \geq 0. \quad (6)$$

Remark 2. If the matrix $\mathcal{W}$ takes the following special form, $\mathcal{W} = \kappa \begin{bmatrix} \lambda I & 0 \\ 0 & -I \end{bmatrix}$, where $\kappa \geq 0$, $\lambda > 0$, then (5) is equal to

$$\kappa[\lambda \|x_k\|^2 - \|\varrho(x_k)\|^2] \geq 0,$$

which means that the nonlinear function $\varrho(x_k)$ satisfies the Lipschitz constraint. Or, $\mathcal{W} = \kappa \begin{bmatrix} \lambda I & -\frac{1}{2}I \\ -\frac{1}{2}I & 0 \end{bmatrix}$, where $\kappa, \lambda \in \mathbb{R}$; then, (5) is equal to

$$\kappa[\lambda \|x_k\|^2 - \langle x_k, \varrho(x_k) \rangle] \geq 0.$$

Obviously, this means that the nonlinear function $\varrho(x_k)$ satisfies the one-side Lipschitz constraint.

From the above two cases, it can be seen that the IQC represents a broader range of possibilities than the Lipschitz constraint and the one-sided Lipschitz constraint, as shown in Figure 1.

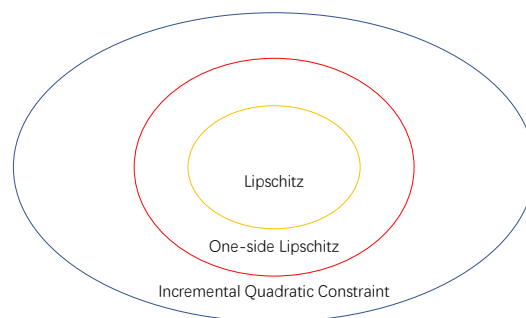


Figure 1. The relationship between Lipschitz constraint, one-side Lipschitz constraint, and incremental quadratic constraint.

Definition 4 ([28]). Under the conditions of $u(k, r_k) = 0$ and $\omega(k) = 0$, with a given initial condition x_0 and reference signal r_0 , the system (1) exhibits mean-square stability (MSS) provided that the subsequent equation is satisfied.

$$\lim_{k \rightarrow \infty} \mathbb{E} \|x_k\|_{x_0, r_0}^2 \rightarrow 0. \quad (7)$$

2.2. Resilient Controller under Time-Trigger

The time-triggered resilient controller is developed to ensure the MSS of the system. The design of the controller is as follows:

$$u(k, r_k) = (\mathcal{K}(r_k) + \mathcal{K}_\delta(r_k))x_k, \mathcal{K}_\delta = \mathcal{M}_\delta(r_k) * \bar{\delta}(r_k) * \mathcal{N}_\delta(r_k). \quad (8)$$

In this equation, $\mathcal{M}_\delta(r_k)$ and $\mathcal{N}_\delta(r_k)$ are matrices of suitable dimensions that need to be determined, while $\bar{\delta}(r_k)$ is a positive constant and $\mathcal{K}(r_k)$ represents the gain matrix that must be designed.

2.3. Resilient Controller under Event-Trigger

For the current triggered signal $x(k_m)$ and the next triggered signal $x(k_{m+1})$, where $h \in \mathbb{Z}_+$, there exist positive matrix $\Phi(r_k)$ and $\gamma(r_k)$; if the following inequality holds, the event is driven.

$$\begin{aligned} e_{k_m}^h{}^T \Phi(r_{k_m}) e_{k_m}^h &> \gamma_{r_{k_m}} x_{k_m}^h{}^T \Phi(r_{k_m}) x_{k_m}^h, \\ e_{k_m}^h{}^T \Phi(r_{k_m}) e_{k_m}^h - \gamma_{r_{k_m}} x_{k_m}^h{}^T \Phi(r_{k_m}) x_{k_m}^h &> 0, e_{k_m}^h = x_{k_m} - x_{k_m}^h, \end{aligned} \quad (9)$$

The resilient controller, implemented with an event-trigger mechanism, is formulated to ensure the MSS of the system. Thus, the controller is devised as outlined below:

$$u(k_m, r_k) = (\mathcal{K}(r_k) + \mathcal{K}_\delta(r_k))x(k_m), \mathcal{K}_\delta(r_k) = \mathcal{M}_\delta(r_k) * \mathcal{D}(r_k) * \mathcal{N}_\delta(r_k). \quad (10)$$

Remark 3. To illustrate the difference between time-triggered strategy and event-triggered strategy, we refer to Figure 2 [34]. When using a time-triggered resilient controller, the time h between k_m and k_{m+1} is fixed at 1. This means the system samples at fixed intervals. In contrast, when using an event-triggered resilient controller, $\tau_1^m + \tau_2^m + \dots + \tau_l^m + \tau_{l+1}^m = h$ is not fixed. The transition from k_m to k_{m+1} is governed by (9). In other words, the system samples after a variable time h . Event triggering reduces the number of triggers and the communication load on the system.

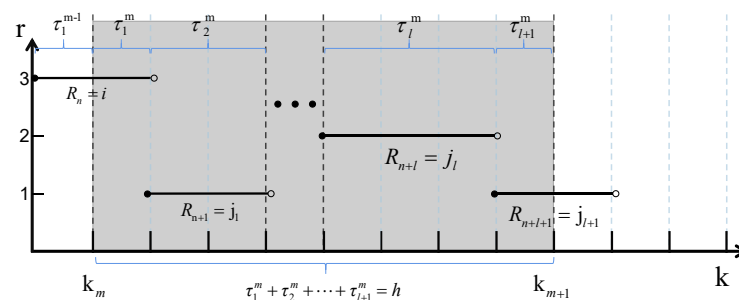


Figure 2. Illustration of h and event-trigger signal k_m, k_{m+1} .

Before proceeding to the proof, we cite two lemmas which are necessary for the following proof.

Lemma 1 ([35]). For given appropriate dimensional matrices z , M , N , and Δ , the following inequality holds:

$$z + \tilde{N}^T \Delta \tilde{M}^T + \tilde{M} \Delta \tilde{N} \leq z + \alpha^{-1} \tilde{N}^T \tilde{N} + \alpha \tilde{M}^T \tilde{M}, \quad (11)$$

where α is a positive constant.

Lemma 2 ([36]). For given real matrices J and G with appropriate dimensions, any positive scalar φ makes the following inequality hold.

$$J^T G + G^T J \leq \varphi J^T J + \varphi^{-1} G^T G. \quad (12)$$

Lemma 3 ([37]). For a symmetric matrix, if $\aleph_{22} < 0$, we have

$$\begin{bmatrix} \aleph_{11} & \aleph_{12} \\ \aleph_{21} & \aleph_{22} \end{bmatrix} < 0 \implies \aleph_{11} - \aleph_{12} \aleph_{22}^{-1} \aleph_{21} < 0,$$

3. Main Results

In this section, the feasibility of the controllers proposed in (8) and (10) is demonstrated. In Theorem 1, sufficient conditions for the system's MSS under resilient control are obtained. In Theorem 2, the MSS of the system under event-triggered resilient control is established.

Theorem 1. For all $\varsigma, \epsilon \in \Upsilon$, the system (1) has MSS; for given the constant $\rho(\varsigma) > 0$, there exist matrices W_{12} , $\mathcal{M}_\delta(\varsigma)$, and $\mathcal{N}_\delta(\varsigma)$, and the definite symmetric matrix $P(\varsigma) > 0$, $P(\epsilon) > 0$, $W_{11} > 0$, $W_{22} < 0$, such that the following inequality holds:

$$\begin{bmatrix} -P(\varsigma) + W_{11} & W_{12} & \hat{\mathcal{A}}_\varsigma^T & 0 & \rho(\varsigma)\mathcal{N}_\delta(\varsigma) \\ * & W_{22} & I & 0 & 0 \\ * & * & -P(\varsigma)^{-1} & B_\varsigma\mathcal{M}_\delta(\varsigma) & 0 \\ * & * & * & -\rho(\varsigma) & 0 \\ * & * & * & * & -\rho(\varsigma) \end{bmatrix} < 0, \quad (13)$$

$$\begin{bmatrix} -P(\varsigma) + W_{11} & W_{12} & \hat{\mathcal{A}}_\varsigma^T & 0 & \rho(\varsigma)\mathcal{N}_\delta(\varsigma) \\ * & W_{22} & I & 0 & 0 \\ * & * & -\sum_{\tau=1}^{T_{max}^\varsigma} Q(\varsigma)^{-1} & B_\varsigma\mathcal{M}_\delta(\varsigma) & 0 \\ * & * & * & -\rho(\varsigma) & 0 \\ * & * & * & * & -\rho(\varsigma) \end{bmatrix} < 0, \quad (14)$$

where $O(\varsigma) = \frac{\sum_{\epsilon \in \Upsilon} \pi_{\varsigma\epsilon}(\tau)P(\epsilon)}{\sum_{\tau=1}^{T_{max}^\varsigma} \sum_{\epsilon \in \Upsilon} \pi_{\varsigma\epsilon}(\tau)}$, $\hat{\mathcal{A}}_\varsigma = A_\varsigma + \mathcal{K}(\varsigma)B_\varsigma$.

Proof. Step 1. Assume in system (1) that $\omega(k) = 0$ and $u(k, r_k) = 0$; then, the Lyapunov function can be constructed:

$$V(x_k, r_k) = x_k^T P(r_k) x_k,$$

where $P(r_k) > 0$, so $V(x_k, r_k) > 0$.

Then:

$$\Delta V = \mathbb{E}[V(x_{k+1}, r_k + 1)] - V(x_k, r_k) \leq 0.$$

When $r_k + 1 = r_k = \varsigma$, we have $P(r_k + 1) = P(r_k) = P(\varsigma)$; thus:

$$\Delta V = x_{k+1}^T P(i) x_{k+1} - x_k^T P(i) x_k \leq 0. \quad (15)$$

Let $\hbar = (15) + (6)$; we have:

$$\begin{aligned} \hbar &= x_k^T [-P(\varsigma) + A_\varsigma^T P(\varsigma) A_\varsigma + W_{11}] x_k + x_k^T [A_\varsigma^T P(\varsigma) + W_{12}] \varrho(x_k) \\ &+ \varrho(x_k)^T [P(\varsigma) A_\varsigma + W_{12}^T] x_k + \varrho(x_k)^T [P(\varsigma) + W_{22}] \varrho(x_k) \leq 0. \end{aligned} \quad (16)$$

Then, (16) can be written in matrix form:

$$\begin{bmatrix} x_k \\ \varrho(x_k) \end{bmatrix}^T \begin{bmatrix} -P(\varsigma) + A_\varsigma^T P(\varsigma) A_\varsigma + W_{11} & A_\varsigma^T P(\varsigma) + W_{12} \\ * & W_{22} \end{bmatrix} \begin{bmatrix} x_k \\ \varrho(x_k) \end{bmatrix} \leq 0$$

$$\begin{bmatrix} -P(\varsigma) + A_\varsigma^T P(\varsigma) A_\varsigma + W_{11} & A_\varsigma^T P(\varsigma) + W_{12} \\ * & W_{22} \end{bmatrix} \leq 0. \quad (17)$$

Applying Lemma 3, (17) is equal to the following inequality:

$$\begin{bmatrix} -P(\varsigma) + W_{11} & W_{12} & A_\varsigma^T \\ * & W_{22} & I \\ * & * & -P(\varsigma)^{-1} \end{bmatrix} \leq 0. \quad (18)$$

When $r_k + 1 = r_{k+1} = \epsilon$, $r_k = \varsigma$, (15) can be written as the following inequality:

$$\Delta V = x_{k+1}^T \sum_{\tau=1}^{T_{max}^\varsigma} O(\varsigma) x_{k+1} - x_k^T P(\varsigma) x_k \leq 0. \quad (19)$$

Based on **Lemma 3**, (19) can be written in the following form:

$$\begin{bmatrix} -P(\varsigma) + W_{11} & W_{12} & A_{\varsigma}^T \\ * & W_{22} & I \\ * & * & -\sum_{\tau=1}^{T_{\max}^{\varsigma}} O(\varsigma)^{-1} \end{bmatrix} \leq 0. \quad (20)$$

Step 2. Assume in system (1) that $\omega(k) = 0$ and $u(k, r_k) = \mathcal{K}(r_k)x_k$; then, we obtain $\hat{\mathcal{A}}_{\varsigma} = A_{\varsigma} + B_{\varsigma}\mathcal{K}(\varsigma)$. Substitute it into (18) and (20):

$$\begin{bmatrix} -P(\varsigma) + W_{11} & W_{12} & \hat{\mathcal{A}}_{\varsigma}^T \\ * & W_{22} & I \\ * & * & -P(\varsigma)^{-1} \end{bmatrix} \leq 0, \quad (21)$$

and

$$\begin{bmatrix} -P(\varsigma) + W_{11} & W_{12} & \hat{\mathcal{A}}_{\varsigma}^T \\ * & W_{22} & I \\ * & * & -\sum_{\tau=1}^{T_{\max}^{\varsigma}} O(\varsigma)^{-1} \end{bmatrix} \leq 0. \quad (22)$$

Step 3. Assume in system (1) that $\omega(k) = 0$ and $u(k, r_k) = \mathcal{K}(r_k) + \mathcal{K}_{\delta}(r_k)$, $\mathcal{K}_{\delta}(r_k) = \mathcal{M}_{\delta}(r_k)\bar{\mathcal{O}}(r_k)\mathcal{N}_{\delta}(r_k)$; substitute it into (17). For k such that $r_k + 1 = r_k = \varsigma$, (21) can be written in the following form:

$$\begin{bmatrix} -P(\varsigma) + W_{11} & W_{12} & \hat{\mathcal{A}}_{\varsigma}^T \\ * & W_{22} & I \\ * & * & -P(\varsigma)^{-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 & \mathcal{N}_{\delta}(\varsigma)^T \bar{\mathcal{O}}(\varsigma)^T \mathcal{M}_{\delta}(\varsigma)^T B_{\varsigma}^T \\ 0 & 0 & 0 \\ * & 0 & 0 \end{bmatrix} \leq 0. \quad (23)$$

Applying **Lemma 3**, (23) is equal to

$$\begin{aligned} & \wp(\varsigma) + \begin{bmatrix} \mathcal{N}_{\delta}(\varsigma)^T \\ 0 \\ 0 \end{bmatrix} \bar{\mathcal{O}}(\varsigma) \begin{bmatrix} 0 & 0 & \mathcal{M}_{\delta}(\varsigma)^T B_{\varsigma}^T \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ B_{\varsigma} \mathcal{M}_{\delta}(\varsigma) \end{bmatrix} \bar{\mathcal{O}}(\varsigma) \begin{bmatrix} \mathcal{N}_{\delta}(\varsigma) & 0 & 0 \end{bmatrix} \\ & = \wp(\varsigma) + \hat{\mathcal{N}}_{\delta}(\varsigma)^T \bar{\mathcal{O}}(\varsigma) \hat{\mathcal{M}}_{\delta}(\varsigma)^T + \hat{\mathcal{M}}_{\delta}(\varsigma) \bar{\mathcal{O}}(\varsigma) \hat{\mathcal{N}}_{\delta}(\varsigma), \end{aligned} \quad (24)$$

where

$$\begin{aligned} \wp(\varsigma) &= \begin{bmatrix} -P(\varsigma) + W_{11} & W_{12} & \hat{\mathcal{A}}_{\varsigma}^T \\ * & W_{22} & I \\ * & * & -P(\varsigma)^{-1} \end{bmatrix}, \\ \hat{\mathcal{M}}_{\delta}(\varsigma) &= \begin{bmatrix} 0 \\ 0 \\ B_{\varsigma} \mathcal{M}_{\delta}(\varsigma) \end{bmatrix}, \hat{\mathcal{N}}_{\delta}(\varsigma) = \begin{bmatrix} \mathcal{N}_{\delta}(\varsigma) & 0 & 0 \end{bmatrix}. \end{aligned}$$

Thus, applying **Lemma 1**, for $\rho(\varsigma) > 0$, we can obtain

$$(24) \leq \wp(\varsigma) + \rho(\varsigma)^{-1} \hat{\mathcal{M}}_{\delta}(\varsigma) \hat{\mathcal{M}}_{\delta}(\varsigma) + \rho(\varsigma) \hat{\mathcal{N}}_{\delta}(\varsigma)^T \hat{\mathcal{N}}_{\delta}(\varsigma). \quad (25)$$

Construct a matrix and apply **Lemma 3**:

$$\begin{aligned}
& \begin{bmatrix} \varphi(\varsigma) & \hat{\mathcal{M}}_\delta(\varsigma)^T & \hat{\mathcal{N}}_\delta(\varsigma)^T \\ * & -\rho(\varsigma) & 0 \\ * & * & -\rho(\varsigma) \end{bmatrix} \\
&= \begin{bmatrix} \varphi(\varsigma) & \hat{\mathcal{M}}_\delta(\varsigma)^T \\ * & -\rho(\varsigma) \end{bmatrix} + \begin{bmatrix} \hat{\mathcal{N}}_\delta(\varsigma)^T \\ 0 \end{bmatrix} \rho(\varsigma)^{-1} [\hat{\mathcal{N}}_\delta(\varsigma)^T \quad 0] \\
&= \begin{bmatrix} \varphi(\varsigma) + \rho(\varsigma)^{-1} \hat{\mathcal{N}}_\delta(\varsigma)^T \hat{\mathcal{N}}_\delta(\varsigma) & \hat{\mathcal{M}}_\delta(\varsigma)^T \\ * & -\rho(\varsigma) \end{bmatrix} \\
&= \varphi(\varsigma) + \rho(\varsigma)^{-1} \hat{\mathcal{M}}_\delta(\varsigma) \hat{\mathcal{M}}_\delta(\varsigma) + \rho(\varsigma) \hat{\mathcal{N}}_\delta(\varsigma)^T \hat{\mathcal{N}}_\delta(\varsigma) \leq 0.
\end{aligned} \tag{26}$$

So, we obtain the sufficient conditions for the system's MSS:

$$\begin{bmatrix} \varphi(\varsigma) & \hat{\mathcal{M}}_\delta(\varsigma)^T & \hat{\mathcal{N}}_\delta(\varsigma)^T \\ * & -\rho(\varsigma) & 0 \\ * & * & -\rho(\varsigma) \end{bmatrix} \leq 0. \tag{27}$$

Obviously, for k such that $r_k + 1 = r_{k+1} = \epsilon$, $r_k = \varsigma$, we also have

$$\begin{bmatrix} \hat{\varphi}(\varsigma) & \hat{\mathcal{M}}_\delta(\varsigma)^T & \hat{\mathcal{N}}_\delta(\varsigma)^T \\ * & -\rho(\varsigma) & 0 \\ * & * & -\rho(\varsigma) \end{bmatrix} \leq 0, \tag{28}$$

where

$$\hat{\varphi}(\varsigma) = \begin{bmatrix} -P(\varsigma) + W_{11} & W_{12} & \hat{\mathcal{A}}_\varsigma^T \\ * & W_{22} & I \\ * & * & -\sum_{\tau=1}^{T_{max}^\varsigma} Q(\varsigma)^{-1} \end{bmatrix}.$$

By employing the LMI in (27) and (28), (13) and (14) can be obtained, which indicates that Theorem 1 is proved. $\square$

Remark 4. In Theorem 1, we divide the system into two cases $r_k + 1 = r_k = \varsigma$ and $r_k + 1 = r_{k+1} = \epsilon$, $r_k = \varsigma$ and then discuss them separately, resulting in inequalities (13) and (14). When (13) and (14) are satisfied, according to the Lyapunov stability theorem, it follows that $V(x_k, r_k) > 0$ and $\Delta V < 0$. Therefore, as $\lim_{k \rightarrow \infty} \mathbb{E} \|x_k\|_{x_0, r_0}^2 \rightarrow 0$, the system satisfies the MSS.

Building upon Theorem 1, we consider event-triggered resilient control to reduce the communication burden of the system. Based on this, we present Theorem 2.

Theorem 2. For all $\varsigma, \epsilon \in \mathcal{Y}$, the system (33) has MSS; for the given positive constants $\rho > 0$, $\epsilon > 0$, $\eta > 0$, if there exist matrices W_{12} , $\mathcal{M}_\delta(\varsigma)$, and $\mathcal{N}_\delta(\varsigma)$, and the definite symmetric matrix $P(\varsigma) > 0$, $P(\epsilon) > 0$, $W_{11} > 0$, $W_{22} < 0$, such that the following inequality holds:

$$\begin{bmatrix} \mathcal{H}(\varsigma) & \chi(\varsigma) & \tilde{\mathcal{M}}_\delta(\varsigma) & \tilde{\mathcal{N}}_\delta(\varsigma) \\ * & \mathcal{P}(\varsigma) & 0 & 0 \\ * & * & \Lambda(\varsigma) & 0 \\ * & * & * & \Lambda(\varsigma) \end{bmatrix} < 0, \tag{29}$$

$$\begin{bmatrix} \mathcal{H}(\varsigma) & \chi(\varsigma) & \tilde{\mathcal{M}}_\delta(\varsigma) & \tilde{\mathcal{N}}_\delta(\varsigma) \\ * & \mathcal{Q}(\varsigma) & 0 & 0 \\ * & * & \Lambda(\varsigma) & 0 \\ * & * & * & \Lambda(\varsigma) \end{bmatrix} < 0, \tag{30}$$

where $\beta = 1 + \varepsilon$, $\zeta = 1 + \eta$, $\mu = 1 + \varepsilon^{-1} + \eta^{-1}$ and

$$\begin{aligned}\mathcal{H}(\varsigma) &= \begin{bmatrix} -P(\varsigma) + W_{11} + \gamma(\varsigma)\Phi(\varsigma) & 0 & W_{12} \\ * & -\Phi(\varsigma) & 0 \\ * & * & W_{22} \end{bmatrix}, \chi(\varsigma) = \begin{bmatrix} \sqrt{\beta}\hat{\mathcal{A}}_{\varsigma}^T & 0 & 0 \\ * & \sqrt{\mu}\mathcal{K}(\varsigma)^T B_{\varsigma}^T & 0 \\ * & * & \sqrt{\zeta}I \end{bmatrix}, \\ \tilde{\mathcal{M}}_{\delta}(\varsigma) &= \begin{bmatrix} 0 & 0 & 0 & \sqrt[4]{\beta}B_{\varsigma}\mathcal{M}_{\delta}(\varsigma) & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt[4]{\mu}B_{\varsigma}\mathcal{M}_{\delta}(\varsigma) & 0 \end{bmatrix}^T, \\ \tilde{\mathcal{N}}_{\delta}(\varsigma) &= \begin{bmatrix} \sqrt[4]{\zeta}\mathcal{N}_{\delta}(\varsigma) & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt[4]{\mu}\mathcal{N}_{\delta}(\varsigma) & 0 & 0 & 0 & 0 \end{bmatrix}, \\ \mathcal{P}(\varsigma) &= \text{diag}\{-P(\varsigma)^{-1}\}, \Lambda(\varsigma) = \text{diag}\{-\rho(\varsigma)\}, \\ \mathcal{Q}(\varsigma) &= \text{diag}\{-\sum_{\tau=1}^{T_{\max}^{\varsigma}} Q(\varsigma)^{-1}\}, Q(\varsigma) = \frac{\sum_{\epsilon \in Y} \pi_{\varsigma\epsilon}(\tau)P(\epsilon)}{\sum_{\tau=1}^{T_{\max}^{\varsigma}} \sum_{\epsilon \in Y} \pi_{\varsigma\epsilon}(\tau)}.\end{aligned}$$

Remark 5. Considering the Event-Trigger Scheme (9), if the system state satisfies condition (9), for the current triggered signal $x(k_m)$ and the current time $k_m + 1$, the system (1) can be expressed in the following form:

$$x_{k_m+1} = A_{r_{k_m}} x_{k_m} + B_{r_{k_m}} u(k_m, r_{k_m}) + \varrho(x_{k_m}) + \omega(k_m). \quad (31)$$

Obviously, the system (31) satisfies Theorem 1.

Proof. Considering the Event-Trigger Scheme (9), for the current triggered signal $x(k_m)$ and the current time $k_m + h + 1$, the system (1) can be expressed in the following form:

$$x_{k_m+h+1} = A_{r_{k_m+h}} x_{k_m+h} + B_{r_{k_m+h}} u(k_m, r_{k_m}) + \varrho(x_{k_m+h}) + \omega(k_m + h). \quad (32)$$

Let $x_{k_m+h+1} = x_{k_m}^{h+1}$, $x_{k_m+h} = x_{k_m}^h$; assume that $u(k_m, r_{k_m}) = (\mathcal{K}(r_{k_m}) + \mathcal{K}_{\delta}(r_{k_m}))x_{k_m}$, $\mathcal{K}_{\delta} = \mathcal{M}_{\delta}(r_k) * \tilde{\mathcal{O}}(r_k) * \mathcal{N}_{\delta}(r_k)$, $\omega(k) = 0$; then, the system (1) can be expressed in the following form:

$$\begin{aligned}x_{k_m}^{h+1} &= A_{r_{k_m}^h} x_{k_m}^h + B_{r_{k_m}^h} \mathbb{K}(i)x_{k_m} + \varrho(x_{k_m}^h) + \omega(k_m^h) \\ &= \mathbb{A}_{r_{k_m}^h} x_{k_m}^h + B_{r_{k_m}^h} \mathbb{K}(i)e_{k_m}^h + \varrho(x_{k_m}^h) + \omega(k_m^h),\end{aligned} \quad (33)$$

where $\mathbb{K}(r_{k_m}^h) = \mathcal{K}(r_{k_m}^h) + \mathcal{M}_{\delta}(r_{k_m}^h)\tilde{\mathcal{O}}(r_{k_m}^h)\mathcal{N}_{\delta}(r_{k_m}^h)$, $\mathbb{A}_{r_{k_m}^h} = A_{r_{k_m}^h} + B_{r_{k_m}^h} \mathbb{K}(r_{k_m}^h)$, and $e_{k_m}^h$ satisfies the following inequality:

$$\gamma_{r_k} x_{k_m}^h{}^T \Phi(r_{k_m}^h) x_{k_m}^h - e_{k_m}^h{}^T \Phi(r_{k_m}^h) e_{k_m}^h > 0, e_{k_m}^h = x_{k_m} - x_{k_m}^h. \quad (34)$$

Construct a Lyapunov function:

$$V(x_{k_m}^h, r_{k_m}^h) = x_{k_m}^h{}^T P(r_{k_m}^h) x_{k_m}^h,$$

where $P(r_{k_m}^h) > 0$, so $V(x_{k_m}^h, r_{k_m}^h) > 0$.

Then:

$$\Delta V = \mathbb{E}[V(x_{k_m}^{h+1}, r_{k_m}^{h+1})] - V(x_{k_m}^h, r_{k_m}^h) \leq 0.$$

For k that $r_{k_m}^{h+1} = r_{k_m}^h = \varsigma$, we have $P(r_{k_m}^{h+1}) = P(r_{k_m}^h) = P(\varsigma)$, thus:

$$\begin{aligned}\Delta V &= [x_{k_m}^h \mathbb{A}_{\varsigma} + B_{\varsigma} \mathbb{K}(\varsigma)e_{k_m}^h + \varrho(x_{k_m}^h)]^T P(\varsigma) [x_{k_m}^h \mathbb{A}_{\varsigma} + B_{\varsigma} \mathbb{K}(\varsigma)e_{k_m}^h + \varrho(x_{k_m}^h)] \\ &\quad - x_{k_m}^h{}^T P(\varsigma) x_{k_m}^h \leq 0.\end{aligned} \quad (35)$$

Let $\tilde{h} = (35) + (6) + (34)$; then, we have

$$\begin{aligned} \tilde{h} = & x_{k_m}^h{}^T [-P(\varsigma) + W_{11} + \mathbb{A}_\varsigma^T P(\varsigma) \mathbb{A}_\varsigma + \gamma(\varsigma) \Phi(\varsigma)] x_{k_m}^h \\ & + e_{k_m}^h{}^T [\mathbb{K}(\varsigma)^T B_\varsigma^T P(\varsigma) B_\varsigma K(\varsigma) - \Phi(\varsigma)] e_{k_m}^h + \varrho(x_{k_m}^h)^T [P(\varsigma) + W_{22}] \varrho(x_{k_m}^h) \\ & + e_{k_m}^h{}^T [\mathbb{K}(\varsigma)^T B_\varsigma^T P(\varsigma) \mathbb{A}_\varsigma] x_{k_m}^h + x_{k_m}^h{}^T [\mathbb{A}_\varsigma^T P(\varsigma) B_\varsigma \mathbb{K}(\varsigma)] e_{k_m}^h \\ & + \varrho(x_{k_m}^h)^T [P(\varsigma) B_\varsigma \mathbb{K}(\varsigma)] e_{k_m}^h + e_{k_m}^h{}^T [\mathbb{K}(\varsigma)^T B_\varsigma^T P(\varsigma)] \varrho(x_{k_m}^h) \\ & + x_{k_m}^h{}^T [\mathbb{A}_\varsigma^T P(\varsigma) + W_{12}] \varrho(x_{k_m}^h) + \varrho(x_{k_m}^h)^T [P(\varsigma) \mathbb{A}_\varsigma + W_{21}] x_{k_m}^h \leq 0. \end{aligned} \quad (36)$$

Applying **Lemma 2**:

$$\begin{aligned} \tilde{h} \leq & x_{k_m}^h{}^T [-P(\varsigma) + W_{11} + \mathbb{A}_\varsigma^T P(\varsigma) \mathbb{A}_\varsigma + \gamma(\varsigma) \Phi(\varsigma)] x_{k_m}^h \\ & + e_{k_m}^h{}^T [\mathbb{K}(\varsigma)^T B_\varsigma^T P(\varsigma) B_\varsigma \mathbb{K}(\varsigma) - \Phi(\varsigma)] e_{k_m}^h + \varrho(x_{k_m}^h)^T [P(\varsigma) + W_{22}] \varrho(x_{k_m}^h) \\ & + \varepsilon^{-1} e_{k_m}^h{}^T \mathbb{K}(\varsigma)^T B_\varsigma^T P(\varsigma) B(\varsigma) \mathbb{K}(\varsigma) e_{k_m}^h + \varepsilon x_{k_m}^h{}^T \mathbb{A}_\varsigma^T P(\varsigma) \mathbb{A}_\varsigma x_{k_m}^h \\ & + \eta^{-1} e_{k_m}^h{}^T \mathbb{K}(\varsigma)^T B_\varsigma^T P(\varsigma) B_\varsigma \mathbb{K}(\varsigma) e_{k_m}^h + \eta \varrho(x_{k_m}^h)^T P(\varsigma) \varrho(x_{k_m}^h) \\ & + x_{k_m}^h{}^T [\mathbb{A}_\varsigma^T P(\varsigma) + W_{12}] \varrho(x_{k_m}^h) + \varrho(x_{k_m}^h)^T [P(\varsigma) \mathbb{A}_\varsigma + W_{21}] x_{k_m}^h. \end{aligned} \quad (37)$$

Then, write it in matrix form:

$$\begin{bmatrix} x_{k_m}^h \\ e_{k_m}^h \\ \varrho(x_{k_m}^h) \end{bmatrix}^T * \begin{bmatrix} \Xi_1 & 0 & \mathbb{A}_\varsigma^T P(\varsigma) + W_{12} \\ 0 & \Xi_2 & 0 \\ 0 & 0 & (1 + \eta)P(\varsigma) + W_{22} \end{bmatrix} * \begin{bmatrix} x_{k_m}^h \\ e_{k_m}^h \\ \varrho(x_{k_m}^h) \end{bmatrix} \leq 0, \quad (38)$$

where

$$\begin{aligned} \Xi_1 &= -P(\varsigma) + (1 + \varepsilon) \mathbb{A}_\varsigma^T P(\varsigma) \mathbb{A}_\varsigma + \gamma(\varsigma) \Phi(\varsigma) \\ \Xi_2 &= (1 + \varepsilon^{-1} + \eta^{-1}) [\mathbb{K}(\varsigma)^T B_\varsigma^T P(\varsigma) B_\varsigma \mathbb{K}(\varsigma)] - \Phi(\varsigma). \end{aligned}$$

For $\beta = 1 + \varepsilon$, $\zeta = 1 + \eta$, $\mu = 1 + \varepsilon^{-1} + \eta^{-1}$, (38) is equal to the following inequalities:

$$\begin{aligned} & \begin{bmatrix} -P(\varsigma) + W_{11} + \gamma(\varsigma) \Phi(\varsigma) & 0 & W_{12} \\ 0 & -\Phi(\varsigma) & 0 \\ 0 & 0 & W_{22} \end{bmatrix} + \\ & \begin{bmatrix} \beta \mathbb{A}_\varsigma^T P(\varsigma) \mathbb{A}_\varsigma & 0 & 0 \\ 0 & \mu \mathbb{K}(\varsigma)^T B_\varsigma^T P(\varsigma) B_\varsigma \mathbb{K}(\varsigma) & 0 \\ 0 & 0 & \zeta P(\varsigma) \end{bmatrix} \leq 0 \quad (39) \\ \Rightarrow & \begin{bmatrix} -P(\varsigma) + W_{11} + \gamma(\varsigma) \Phi(\varsigma) & 0 & W_{12} & \sqrt{\beta} \mathbb{A}_\varsigma^T & 0 & 0 \\ 0 & -\Phi(\varsigma) & 0 & 0 & \sqrt{\mu} \mathbb{K}(\varsigma)^T B_\varsigma^T & 0 \\ * & 0 & W_{22} & 0 & 0 & \sqrt{\zeta} I \\ * & * & * & -P(\varsigma)^{-1} & 0 & 0 \\ * & * & * & * & -P(\varsigma)^{-1} & 0 \\ * & * & * & * & * & -P(\varsigma)^{-1} \end{bmatrix} \leq 0. \quad (40) \end{aligned}$$

Due to $\mathbb{A}_\varsigma = A_\varsigma + B_\varsigma \mathbb{K}(\varsigma)$ and $\mathbb{K}(\varsigma) = \mathcal{K}(\varsigma) + \mathcal{M}_\delta(\varsigma) \bar{\partial}(\varsigma) \mathcal{N}_\delta(\varsigma)$, let $\hat{\mathcal{A}}_\varsigma = A_\varsigma + B_\varsigma \mathcal{K}(\varsigma)$, thus:

$$\begin{aligned}
& \begin{bmatrix} -P(\varsigma) + W_{11} + \gamma(\varsigma)\Phi(\varsigma) & 0 & W_{12} & \sqrt{\beta}\mathcal{A}_{\varsigma}^T & 0 & 0 \\ * & -\Phi(\varsigma) & 0 & 0 & \sqrt{\mu}\mathcal{K}(\varsigma)^T B_{\varsigma}^T & 0 \\ * & * & W_{22} & 0 & 0 & \sqrt{\xi}I \\ * & * & * & -P(\varsigma)^{-1} & 0 & 0 \\ * & * & * & * & -P(\varsigma)^{-1} & 0 \\ * & * & * & * & 0 & -P(\varsigma)^{-1} \end{bmatrix} \\
& + \begin{bmatrix} 0 & 0 & 0 & \sqrt{\beta}[B_{\varsigma}\mathcal{M}_{\delta}(\varsigma)\tilde{\mathcal{O}}(\varsigma)\mathcal{N}_{\delta}(\varsigma)]^T & 0 & 0 \\ * & 0 & 0 & 0 & \sqrt{\mu}[B_{\varsigma}\mathcal{M}_{\delta}(\varsigma)\tilde{\mathcal{O}}(\varsigma)\mathcal{N}_{\delta}(\varsigma)]^T & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \\ * & * & * & * & * & * \end{bmatrix} < 0. \quad (41)
\end{aligned}$$

Let

$$\begin{aligned}
\tilde{\mathcal{M}}_{\delta}(\varsigma) &= \begin{bmatrix} 0 & 0 & 0 & \sqrt[4]{\beta}B_{\varsigma}\mathcal{M}_{\delta}(\varsigma) & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt[4]{\mu}B_{\varsigma}\mathcal{M}_{\delta}(\varsigma) & 0 \end{bmatrix}^T, \\
\tilde{\mathcal{N}}_{\delta}(\varsigma) &= \begin{bmatrix} \sqrt[4]{\xi}\mathcal{N}_{\delta}(\varsigma) & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt[4]{\mu}\mathcal{N}_{\delta}(\varsigma) & 0 & 0 & 0 & 0 \end{bmatrix}, \\
\wp(\varsigma) &= \begin{bmatrix} \mathcal{H}(\varsigma) & \chi(\varsigma) \\ * & \mathcal{P}(\varsigma) \end{bmatrix}, \\
\mathcal{H}(\varsigma) &= \begin{bmatrix} -P(\varsigma) + W_{11} + \gamma(\varsigma)\Phi(\varsigma) & 0 & W_{12} \\ * & -\Phi(\varsigma) & 0 \\ * & * & W_{22} \end{bmatrix}, \chi(\varsigma) = \begin{bmatrix} \sqrt{\beta}\mathcal{A}_{\varsigma}^T & 0 & 0 \\ * & \sqrt{\mu}\mathcal{K}(\varsigma)^T B_{\varsigma}^T & 0 \\ * & * & \sqrt{\xi}I \end{bmatrix}, \\
\mathcal{P}(\varsigma) &= \text{diag}\{-P(\varsigma)^{-1}\}.
\end{aligned}$$

(41) can be written in the following form:

$$\wp(\varsigma) + \tilde{\mathcal{N}}_{\delta}(\varsigma)^T \tilde{\mathcal{O}}(\varsigma) \tilde{\mathcal{M}}_{\delta}(\varsigma)^T + \tilde{\mathcal{M}}_{\delta}(\varsigma) \tilde{\mathcal{O}}(\varsigma) \tilde{\mathcal{N}}_{\delta}(\varsigma). \quad (42)$$

Applying **Lemma 1**, for $\rho(\varsigma) > 0$, we can obtain

$$\begin{aligned}
& \wp(\varsigma) + \tilde{\mathcal{N}}_{\delta}(\varsigma)^T \tilde{\mathcal{O}}(\varsigma) \tilde{\mathcal{M}}_{\delta}(\varsigma)^T + \tilde{\mathcal{M}}_{\delta}(\varsigma) \tilde{\mathcal{O}}(\varsigma) \tilde{\mathcal{N}}_{\delta}(\varsigma) \\
& \leq \wp(\varsigma) + \rho(\varsigma)^{-1} \tilde{\mathcal{N}}_{\delta}(\varsigma)^T \tilde{\mathcal{N}}_{\delta}(\varsigma) + \rho(\varsigma) \tilde{\mathcal{M}}_{\delta}(\varsigma)^T \tilde{\mathcal{M}}_{\delta}(\varsigma).
\end{aligned}$$

Thus, the following matrix inequalities can be obtained:

$$\begin{bmatrix} \wp(\varsigma) & \tilde{\mathcal{M}}_{\delta}(\varsigma) & \tilde{\mathcal{N}}_{\delta}(\varsigma)^T \\ * & -\rho(\varsigma) & 0 \\ * & * & -\rho(\varsigma) \end{bmatrix} \leq 0. \quad (43)$$

Obviously, for k such that $r_k + 1 = r_{k+1} = \epsilon$, $r_k = \varsigma$, we have the matrix inequalities:

$$\begin{bmatrix} \hat{\wp}(\varsigma) & \tilde{\mathcal{M}}_{\delta}(\varsigma) & \tilde{\mathcal{N}}_{\delta}(\varsigma)^T \\ * & -\rho(\varsigma) & 0 \\ * & * & -\rho(\varsigma) \end{bmatrix} \leq 0, \quad (44)$$

where

$$\begin{aligned}\hat{\phi}(\varsigma) &= \begin{bmatrix} \mathcal{H}(\varsigma) & \chi(\varsigma) \\ * & \mathcal{O}(\varsigma) \end{bmatrix}, \\ \mathcal{H}(\varsigma) &= \begin{bmatrix} -P(\varsigma) + W_{11} + \gamma(\varsigma)\Phi(\varsigma) & 0 & W_{12} \\ * & -\Phi(\varsigma) & 0 \\ * & * & W_{22} \end{bmatrix}, \chi(\varsigma) = \begin{bmatrix} \sqrt{\beta}\mathcal{A}_{\varsigma}^T & 0 & 0 \\ * & \sqrt{\mu}\mathcal{K}(\varsigma)^T B_{\varsigma}^T & 0 \\ * & * & \sqrt{\xi}I \end{bmatrix}, \\ \mathcal{Q}(\varsigma) &= \text{diag}\{-\sum_{\tau=1}^{T_{\max}^{\varsigma}} \mathcal{Q}(\varsigma)^{-1}\}, \mathcal{Q}(\varsigma) = \frac{\sum_{\epsilon \in Y} \pi_{\varsigma\epsilon}(\tau)P(\epsilon)}{\sum_{\tau=1}^{T_{\max}^{\varsigma}} \sum_{\epsilon \in Y} \pi_{\varsigma\epsilon}(\tau)}.\end{aligned}$$

By applying the LMI (43) and (44), (29) and (30) can be obtained, which indicates that Theorem 2 is proved. $\square$

4. Numerical Example

In this section, the following numerical example is provided to demonstrate the validity of Theorem 1 and Theorem 2.

Example: Consider an SMJS with three modes.

$$\begin{aligned}A_1 &= \begin{bmatrix} 1 & -0.44 \\ 0.8 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 0 & -0.5 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 0.6 \end{bmatrix}, \\ B_1 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1 \\ -0.5 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 \\ -1.5 \end{bmatrix}.\end{aligned}$$

The transition probabilities and sojourn-time are assumed to be as follows:

$$\Theta = \begin{bmatrix} 0 & 0.25 & 0.75 \\ 0.5 & 0 & 0.5 \\ 0.3 & 0.7 & 0 \end{bmatrix}, \quad \Omega = \begin{bmatrix} 0 & \frac{0.4^{\tau}0.6^{10-\tau}10!}{(10-\tau)!\tau!} & \frac{0.6^{\tau}0.4^{10-\tau}10!}{(10-\tau)!\tau!} \\ 0.9^{(\tau-1)^{1.3}} - 0.9^{\tau^{1.3}} & 0 & 0.1^{(\tau-1)^{1.5}} - 0.1^{\tau^{1.5}} \\ 0.5^{(\tau-1)^{1.3}} - 0.5^{\tau^{1.3}} & 0.5^{(\tau-1)^{1.5}} - 0.5^{\tau^{1.5}} & 0 \end{bmatrix}.$$

By solving the LMIs in Theorem 2, the system control matrix can be obtained as the following:

$$\begin{aligned}\mathcal{K}(1) &= \begin{bmatrix} -0.7555 & -0.1301 \\ -0.1301 & -0.7595 \end{bmatrix}, \quad \mathcal{K}(2) = \begin{bmatrix} 0.6966 & 0.3213 \\ 0.3213 & 2.2903 \end{bmatrix}, \quad \mathcal{K}(3) = \begin{bmatrix} -0.7169 & 0.0549 \\ 0.0549 & -0.3000 \end{bmatrix}, \\ \mathcal{M}_{\delta}(1) &= \begin{bmatrix} -0.1782 & -0.0003 \\ -0.0003 & -0.1748 \end{bmatrix}, \quad \mathcal{M}_{\delta}(2) = \begin{bmatrix} -0.1935 & 0.0133 \\ 0.0133 & -0.5049 \end{bmatrix}, \quad \mathcal{M}_{\delta}(3) = \begin{bmatrix} -0.1557 & 0.0021 \\ 0.0021 & -0.0775 \end{bmatrix}, \\ \mathcal{N}_{\delta}(1) &= \begin{bmatrix} -0.1196 & -0.0014 \\ -0.0014 & -0.1226 \end{bmatrix}, \quad \mathcal{N}_{\delta}(2) = \begin{bmatrix} -0.0825 & 0.0015 \\ 0.0015 & -0.0878 \end{bmatrix}, \quad \mathcal{N}_{\delta}(3) = \begin{bmatrix} 0.1579 & 0.0007 \\ 0.0007 & 0.1534 \end{bmatrix}.\end{aligned}$$

And the IQC is solved as

$$\mathcal{W} = \left[\begin{array}{cc|cc} 0.6943 & -0.2485 & 0 & 0 \\ -0.2485 & 1.4087 & 0 & 0 \\ \hline 0 & 0 & -8.7317 & 0.0497 \\ 0 & 0 & 0.0497 & -8.7096 \end{array} \right].$$

It can be seen that $W_{11} > 0$, $W_{12}^T = W_{21}$, and $W_{22} < 0$. Therefore, the nonlinear function $q(x(k))$ satisfies IQC.

With the event-trigger factor $\gamma(1) = 0.2$, $\gamma(2) = 0.3$, $\gamma(3) = 0.3$, let the initial state $x(0) = [2 \quad -3.5]^T$. Figure 3 illustrates the system's mode transitions. In Figures 4–7, the state of the system response are shown, where $q(x(k)) = \sin(x(k))/10$, and when $k = 33$, $\omega(k) = 5$ corresponds to external disturbances in reality.

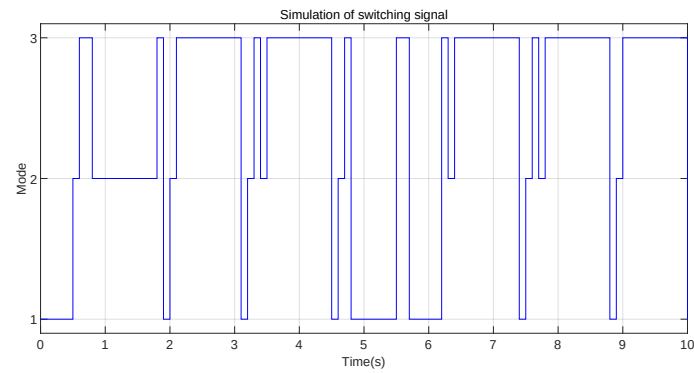


Figure 3. Simulation of switching signal.

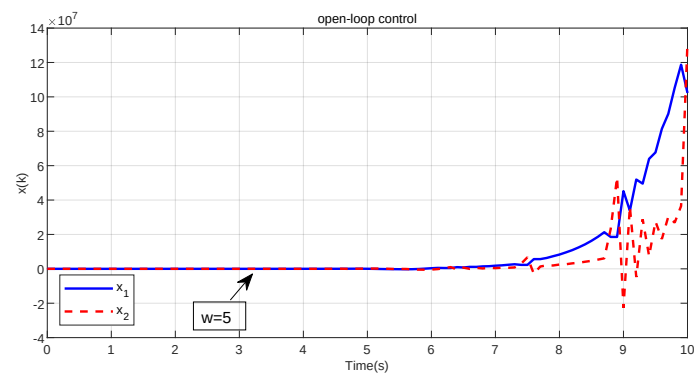


Figure 4. The state response of the open-loop control.

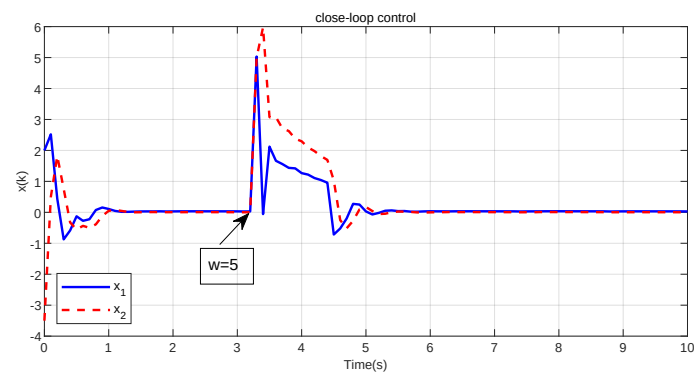


Figure 5. The state response of the close-loop control.

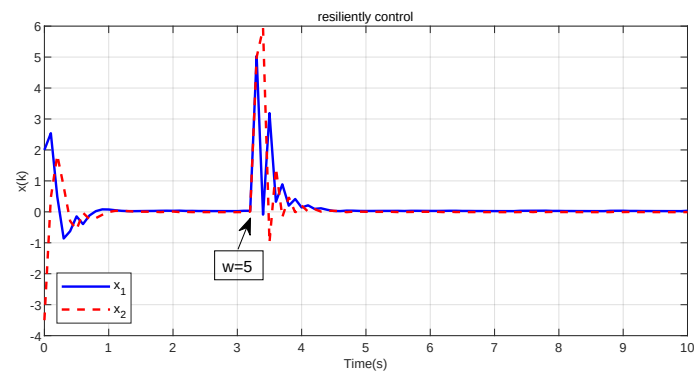


Figure 6. The state response of the resilient control.

From Figure 4, we can see that the system state diverges when using the open-loop control. From Figure 5, we can observe that the system state has MSS when using the

closed-loop control. However, when the system state encounters external disturbances, the system state changes significantly, and the stability of the system recovers slowly.

From Figures 6 and 7, we can observe that the system state has MSS when using the resilient control and the event-triggered resilient control. When subjected to external disturbances, the system's stability recovers rapidly. The event instant is shown in Figure 8. From Figures 7 and 8, we can observe that when employing event-triggered resilient control, the system maintains stability while reducing the number of sampling instances and alleviating communication load. Additionally, the system's resilience is not compromised; it still quickly recovers stability in the presence of external disturbances.

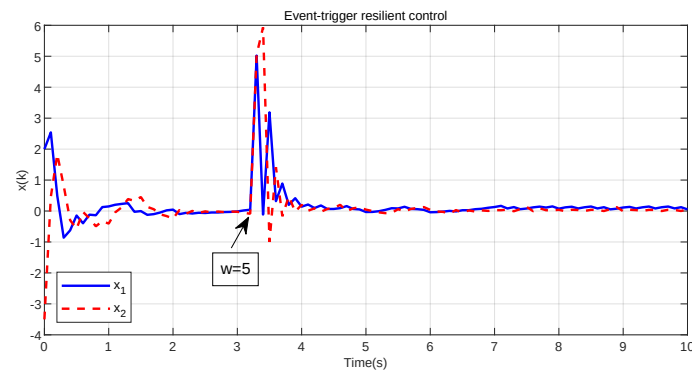


Figure 7. The state response of the resilient control with event-trigger.

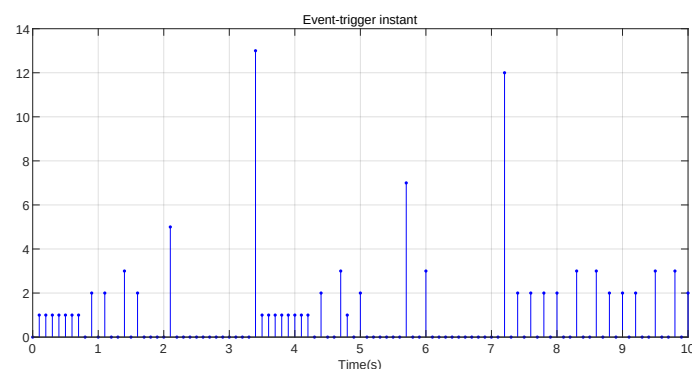


Figure 8. The event instance of a system.

5. Conclusions

This paper presents an event-triggered resilient control strategy to ensure the MSS of nonlinear discrete-time SMJSs. By utilizing (IQC) to represent the system's nonlinear constraints, less conservative feasible solutions are obtained. Furthermore, sufficient conditions for the MSS of the system are derived through Lyapunov stability theory and LMI. The feasibility of the resilient control strategy and the effectiveness of the event-triggered resilient controller are demonstrated through a numerical example.

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