Abstract: Starting from the very first principles we derive explicit parameterizations of the ortho-symplectic matrices in the real four-dimensional Euclidean space. These matrices depend on a set of four real parameters which splits naturally as a union of the real line and the three-dimensional space. It turns out that each of these sets is associated with a separate Lie algebra which after exponentiations generates Lie groups that commute between themselves. Besides, by making use of the Cayley and Fedorov maps, we have arrived at alternative realizations of the ortho-symplectic matrices in four dimensions. Finally, relying on the fundamental structure results in Lie group theory we have derived one more explicit parameterization of these matrices which suggests that the obtained earlier results can be viewed as a universal method for building the representations of the unitary groups in arbitrary dimension.

Keywords: Cayley formula; Cayley map; group factorization; Hamiltonian matrices; Lie algebra; Lie group; orthogonal matrices; rotations; symplectic matrices; unitary matrices; vector parameterization

MSC: 17B81; 22E70; 81R05

1. Introduction

Going back in the history of mathematics one will see that the concept of a group is introduced relatively late, say about 1860 by Cayley [1], who has generalized the older notion of “substitutions”. In the late 1800s, as an outcome of efforts to extend these ideas to differential equations, Lie and Killing developed the idea of “continuous groups” bearing nowadays the name Lie groups [2,3] which are indispensable tools in the integration of the systems with symmetries [4–8]. One of the first cases which attracts serious attention is the group of proper group motions in $\mathbb{R}^3$ which is known presently as the group of the Euclidean motions $E(3)$. This group is not a semi-simple but a semi-direct product of orthogonal transformations and space translations. Both of these groups can be recognized as groups of isometries in the three-dimensional space $\mathbb{R}^3$.

In parallel, developments in classical [9], and quantum mechanics as well [10], have pointed out that the real symplectic groups acting in the phase spaces are also fundamental objects for these disciplines. The same conclusion is true in the fields of optics [11,12] and nuclear physics [13].

So, one comes in front of a general question: which group is more fundamental, the kinematical or dynamical one, or in technical terms, the orthogonal or symplectic one? Actually, the compromise in this dilemma is obvious and can be spelled out by the following

Definition 1. Ortho-Symplectic Matrices (or OSM for short) are those matrices in $\text{SL}(2n, \mathbb{R})$ that are simultaneously orthogonal and symplectic.

In formal mathematical notation, these properties can be expressed via the formulas

$$M^t M = I_{2n}, \quad M^t J_{2n} M = J_{2n}, \quad M \in \text{SL}(2n, \mathbb{R}), \quad n \in \mathbb{Z}$$

(1)
in which $I_{2n}$ denotes the identity matrix in dimension $2n$, and $J_{2n}$ is the so-called symplectic unit of the same size which is of the form

$$I_{2n} := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \tag{2}$$

At Lie algebra level the above conditions can be represented equivalently in the form

$$A^t + A = 0_{2n} \quad \text{and} \quad A^t J_{2n} + J_{2n} A = 0_{2n}, \quad A \in \text{Mat}(2n, \mathbb{R}) \tag{3}$$

where $0_{2n}$ denotes the zeroth matrix in the respective dimension and $\text{Mat}(2n, \mathbb{R})$ is the set of matrices of the specified dimension.

Due to the special form of $J_{2n}$ it is obvious that

$$J_{2n}^2 = -I_{2n} \tag{4}$$

and, therefore, the second condition in (3) can be rewritten in the form

$$A^t = J_{2n} A J_{2n}. \tag{5}$$

Let us mention in passing that matrices of the type (5) are known as Hamiltonian matrices. As it is well known that all orthogonal matrices acting in the Euclidean plane ($n = 1$) are necessarily symplectic we will concentrate our consideration on the next possible case $n \equiv 2$. Comments on the general case will be postponed for the last Section of the paper.

In these settings, the first condition in (3) tells us that we are dealing with the set of real $4 \times 4$ anti-symmetric matrices, i.e.,

$$A = \begin{pmatrix} 0 & -a_6 & -a_5 & -a_1 \\ a_6 & 0 & -a_4 & -a_2 \\ a_5 & a_4 & 0 & -a_3 \\ a_1 & a_2 & a_3 & 0 \end{pmatrix} \tag{6}$$

while the second one (5) ensures the equalities $a_4 \equiv a_1$ along $a_6 \equiv a_3$. Taking into account these conditions, the matrix $A$ can be parameterized, as follows

$$A = \begin{pmatrix} 0 & -z & -\zeta & -x \\ z & 0 & -x & -\chi \\ \zeta & x & 0 & -z \\ x & \chi & z & 0 \end{pmatrix} \quad \text{where} \quad x, \chi, z, \zeta \in \mathbb{R}. \tag{7}$$

2. Exponential Map

One can use the infinitesimal version of the condition (7) which ensures that we are dealing with orthogonal matrices which are at the same time also symplectic, our next task is to find these matrices in explicit form.

It is well known that the most direct way to pass from the Lie Algebras to the Lie groups is via the so-called exponential map, cf [14].

By its very definition, the exponential map for any square matrix is given by the formal series

$$\text{Exp}(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!} \tag{8}$$

which in some cases can be evaluated explicitly.

In most of these cases, this is conducted by taking into account the advantages provided by the Hamilton–Cayley theorem, i.e., that each matrix annihilates its own characteristic polynomial.
The latter is defined by the general formula
\[ p_A(\lambda) := \det(A - \lambda I) = \lambda^n + b_1(A)\lambda^{n-1} + b_2(A)\lambda^{n-2} + \cdots + b_n(A), \quad \dim A = n \] (9)
and the Hamilton–Cayley theorem in this notation states that
\[ p_A(A) \equiv 0. \] (10)
However, in the case under consideration there exists another useful property, namely

**Lemma 1.** The characteristic polynomials of the anti-symmetric matrices of even dimensions contain only even degrees of \(\lambda\), i.e.,
\[ p_A(\lambda) = p_A(-\lambda) \quad \text{if} \quad A^t = -A, \quad \dim A = 2n. \] (11)

**Proof.** We have the following sequence of equalities
\[
p_A(\lambda) = \det(A - \lambda I) = \det((-A - \lambda I)^t) = \det((-A - \lambda I)^t) = \det(-A - \lambda I) = \det((-1)(A + \lambda I)) = (-1)^{2n} \det(A + \lambda I) = \det(A + \lambda I) = p_A(-\lambda)
\]
in which the first one is just the definition, and the second one reflects the fact that the determinant of the transposed matrix coincides with that of the original matrix, the third represents the distributive property of the transposition operation, the fourth accounts for the symmetries of the matrices \(A\) and \(I\), the fifth accounts for the fundamental properties of matrix determinants, the sixth accounts for the even dimensions of the matrices under consideration, and the last one is again the definition read in the opposite direction. \( \square \)

**Lemma 2.** The characteristic polynomial of the matrix \(A\) in (7) is
\[ p_A(\lambda) = \lambda^4 + b_2(A)\lambda^2 + b_4(A) \] (12)
where
\[ b_2(A) = 2x^2 + 2z^2 + \chi^2 + \zeta^2, \quad b_4(A) = (x^2 + z^2 - \chi\zeta)^2. \] (13)

Various formulas regarding the exponential maps of the real \(4 \times 4\) matrices belonging to the Lie algebras \(\mathfrak{so}(4, \mathbb{R})\), \(\mathfrak{so}(3, 1)\), \(\mathfrak{so}(2, 2)\) and \(\mathfrak{sp}(4, \mathbb{R})\) have been derived in [15].

The explicit formulas derived there are based directly on the precise knowledge of the two real coefficients \(b_2(A)\) and \(b_4(A)\). The latter ones have different (explicit!) expressions for each of the above-mentioned algebras and what remains to be conducted is just to put the present parameters inside the respective formulas.

However, one should notice that these formulas include all positive powers (up to the third one) of the generating matrices, and therefore, the final expressions are not so attractive for theoretical considerations.

Let us mention also that in [16], the authors have presented a method for finding such explicit parameterization which is related to the families of periodic solutions of some autonomous Hamiltonian systems.

In the next Section, we will present a constructive procedure by which the higher powers are avoided and this makes the corresponding formulas easy for understanding and application, in general.

### 3. Disentangling of the Exponent Matrix

The idea behind the title of this section is really quite simple and relies on one of the fundamental properties of the exponential map. It is well known (and very easy to check) that, in general,
\[ \exp(A + B) \neq \exp(A)\exp(B), \quad A, B \in L(n, \mathbb{R}). \] (14)
However, if the matrices \( A \) and \( B \) commute, i.e.,

\[
A \cdot B = B \cdot A
\]  

we will have as well

\[
\text{Exp}(A + B) \equiv \text{Exp}(A)\text{Exp}(B) = \text{Exp}(B)\text{Exp}(A).
\]  

So, our first step in deriving our formula for \( OSM \) will be to find the appropriate splitting of \( A \) in (7), namely, such matrices \( P \) and \( Q \) which possess the necessary properties

\[
A = P + Q,
\]

\[
P \cdot Q = Q \cdot P.
\]  

(17)

There is not a general prescription of how to conduct this, but one of the guiding principles to solve the problem is to find the system of equations which has the same number of unknowns.

As the Lie algebra \( \mathfrak{so}(4, \mathbb{R}) \) is six-dimensional, it seems appropriate that each of the summands \( P, Q \) are dependant on three parameters, i.e., a vector \([17, 18]\). Finer results about the structure of the rotational group can be seen in \([19]\).

Going back to our problem, let us try the sought-for decomposition with the following matrices

\[
P := \begin{pmatrix}
0 & -p_3 & p_2 & p_1 \\
-p_3 & 0 & -p_1 & p_2 \\
-p_2 & p_1 & 0 & p_3 \\
-p_1 & -p_2 & -p_3 & 0
\end{pmatrix},
\]

\[
Q := \begin{pmatrix}
0 & -q_3 & q_2 & -q_1 \\
q_3 & 0 & -q_1 & -q_2 \\
-q_2 & q_1 & 0 & -q_3 \\
q_1 & q_2 & q_3 & 0
\end{pmatrix}.
\]  

(18)

One should notice (or check directly) that the second equation in (17) is satisfied automatically while the first one produces the system of six linear equations for the sixth unknowns, \( p_1, p_2, \ldots, q_3 \), i.e.,

\[
\begin{align*}
p_3 + q_3 - z &= 0, & p_2 + q_2 + \zeta &= 0, & p_1 + q_1 - x &= 0 \\
p_1 - q_1 + x &= 0, & p_2 - q_2 + \chi &= 0, & p_3 - q_3 + z &= 0.
\end{align*}
\]  

(19)

Solutions to the above system of equations are

\[
p_1 = 0, \quad p_2 = -\frac{\chi + \zeta}{2}, \quad p_3 = 0, \quad q_1 = x, \quad q_2 = \frac{\chi - \zeta}{2}, \quad q_3 = z.
\]  

(20)

It would be more convenient for the next considerations to exchange some of the notation used to now, say

\[
p_2 = \alpha \quad \text{and} \quad q_2 = y.
\]  

(21)

In this way we will have

\[
P := \begin{pmatrix}
0 & 0 & \alpha & 0 \\
0 & 0 & 0 & \alpha \\
-\alpha & 0 & 0 & 0 \\
0 & -\alpha & 0 & 0
\end{pmatrix},
\]

\[
Q := \begin{pmatrix}
0 & -z & y & -x \\
z & 0 & -x & -y \\
-y & x & 0 & -z \\
x & y & z & 0
\end{pmatrix}.
\]  

(22)

Having these matrices it is easy to see (check) that

\[
p_P(\lambda) = \left(\lambda^2 + \alpha^2\right)^2, \quad p_Q(\lambda) = \left(\lambda^2 + x^2 + y^2 + z^2\right)^2
\]  

(23)

and, therefore, the characteristic polynomials of \( P \) and \( Q \) are reduced to the minimal ones which are of the form
\[ p_A(A) = A^2 + \rho^2 I_4, \quad A = P \text{ or } Q \quad \text{and} \quad \rho = \alpha \text{ or } \sqrt{x^2 + y^2 + z^2} = r. \] (24)

In this situation it is more than appropriate to state

**Lemma 3.** Let \( A \in \text{Mat}(n, \mathbb{R}) \) be a matrix that satisfies the identity \( A^2 = -\rho^2 I_4 \) in which \( \rho \in \mathbb{R} \). Then

(i) If \( \rho \equiv 0 \), \( \exp(A) = I_4 + A \)

(ii) If \( \rho \neq 0 \), \( \exp(A) = \cos \rho I_4 + \frac{\sin \rho}{\rho} A \).

**Proof.** (i) Follows directly from the very definition \( \exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!} \equiv I_4 + A \) of the exponential map as the summation of the series ends at \( k = 1 \).

(ii) Since \( A^2 = -\rho^2 I_4 \), \( A^2k = (-1)^k \rho^{2k} I_4 \) and \( A^{2k+1} = (-1)^{k+1} \rho^{2k+2} I_4 \), we have

\[
\exp(A) = \left( I_4 + \frac{A^2}{2!} + \frac{A^4}{4!} + \cdots \right) + \left( A + \frac{A^3}{3!} + \frac{A^5}{5!} + \cdots \right)
\]
\[ = \left(1 - \frac{\rho^2}{2!} + \frac{\rho^4}{4!} + \cdots \right) I_4 + \left(1 - \frac{\rho^2}{2!} + \frac{\rho^4}{4!} + \cdots \right) A + \frac{1}{\rho} \left( I_4 - \frac{\rho^2}{3!} + \frac{\rho^4}{5!} + \cdots \right) A
\]
\[ = \cos \rho I_4 + \frac{\sin \rho}{\rho} A. \] (25)

In explicit form we have

\[
\exp(P) = R_1 = \cos \alpha I_4 + \frac{\sin \alpha}{\alpha} P = \begin{pmatrix}
\cos \alpha & 0 & \sin \alpha & 0 \\
0 & \cos \alpha & 0 & \sin \alpha \\
-\sin \alpha & 0 & \cos \alpha & 0 \\
0 & -\sin \alpha & 0 & \cos \alpha
\end{pmatrix}
\] (26)

and

\[
\exp(Q) = R_2 = \cos r I_4 + \frac{\sin r}{r} Q = \frac{1}{r} \begin{pmatrix}
r \cos r & -z \sin r & y \sin r & -x \sin r \\
z \sin r & r \cos r & -x \sin r & -y \sin r \\
y \sin r & x \sin r & r \cos r & -z \sin r \\
x \sin r & y \sin r & z \sin r & r \cos r
\end{pmatrix}. \] (27)

Now, it is easy to see (or check directly) that

\[ R_1 \cdot R_1 = R_1 \cdot R_1 = I_4 = R_2 \cdot R_2 = R_2 \cdot R_2 \] (28)

and this tells us that both \( R_1 \) and \( R_2 \) are orthogonal and besides via a slightly longer computation that

\[ R_1^t I_4 R_1 = I_4 = R_2^t I_4 R_2 \] (29)

one proves that they are also symplectic.

Relying on the identities presented in (28) and (29), it is clear that the matrix is

\[ M := R_2 \cdot R_1 = R_1 \cdot R_2 \] (30)
is also OSM as
\[
M^t \cdot M = (R_2 \cdot R_1)^t \cdot R_2 \cdot R_1 = R_1^t \cdot (R_2^t \cdot R_1) \cdot R_2 \cdot R_1 = R_1^t \cdot R_1 = I_4
\]
and
\[
M^t \cdot J_4 \cdot M = (R_2 \cdot R_1)^t \cdot J_4 \cdot R_2 \cdot R_1 = R_1^t \cdot (R_2^t \cdot J_4 \cdot R_2) \cdot R_2 \cdot R_1 = R_1^t \cdot J_4 \cdot R_1 = J_4.
\]

4. Cayley Map

There is an alternative to the exponentiation method for passing from the Lie algebra to the Lie group suggested a long time ago by Cayley [20]. By its very definition, the Cayley map is defined on the set of all non-exceptional matrices \( X \), i.e., those for which \( \det(I - X) \neq 0 \) (33)
is given by the explicit formula
\[
\text{Cay}(X) = \frac{I + X}{I - X} = M(X)
\]
in which \( I \) is the identity matrix of the same dimension as that of \( X \).

Actually, (34) is the short notation for either one of the equivalent forms
\[
M(X) = (I + X)(I - X)^{-1} = (I - X)^{-1}(I + X).
\]

Besides, Cayley has proven (to be precise only in the cases of three and four-dimensional skew-symmetric and orthogonal matrices) that every non-exceptional matrix \( M \) can be expressed in the form (34) in which the matrix \( X \) is also non-exceptional.

And what is even more interesting is that the Cayley map (35), can be inverted explicitly as well. The result of the inversion is given by the formula
\[
X = X(M) = \frac{M - I}{M + I}.
\]

Here, we will follow the same strategy as previously in the case of the exponential map. Let us start by pointing out that further on \( X \) will refer either to \( P \) or \( Q \).

As their characteristic (minimal) polynomials are of second degree, this means that together the series expansion and Hamilton–Cayley theorem ensure that \( M(X) \) can be presented effectively as
\[
M(X) = aX + cI, \quad a, c \in \mathbb{R}.
\]

If we rewrite now the last equation in the form (using the definition (34))
\[
I + X = (I - X)(aX + cI)
\]
and uniformize the right-hand side (using the fact that \( X^2 = -\rho^2 I \)) as the left one, we end up with the expression
\[
(ap^2 + c)I + (a - c)X.
\]

Equating the respective coefficients at both sides produces the system of equations
\[
ap^2 + c = 1, \quad a - c = 1.
\]

Solutions to these equations are given by the formulas
\[
a = \frac{2}{1 + \rho^2}, \quad c = \frac{1 - \rho^2}{1 + \rho^2}
\]
and, therefore,
\[ M(X) = \frac{2}{1 + \rho^2} X + \frac{1 - \rho^2}{1 + \rho^2} I \]  
(42)

specifies the group element associated with \( X \). Respectively, the explicit forms of the group elements associated with \( P \) and \( Q \) are

\[ \text{Cay}(P) = \frac{1}{1 + \alpha^2} \begin{pmatrix} 1 - \alpha^2 & 0 & 2\alpha & 0 \\ 0 & 1 - \alpha^2 & 0 & 2\alpha \\ -2\alpha & 0 & 1 - \alpha^2 & 0 \\ 0 & -2\alpha & 0 & 1 - \alpha^2 \end{pmatrix} \]  
(43)

and

\[ \text{Cay}(Q) = \frac{1}{1 + r^2} \begin{pmatrix} 1 - r^2 & -2z & 2y & -2x \\ 2z & 1 - r^2 & -2x & -2y \\ -2y & 2x & 1 - r^2 & -2z \\ 2x & 2y & 2z & 1 - r^2 \end{pmatrix}. \]  
(44)

Comparing the above results with the formula about the \( 2 \times 2 \) symplectic matrices [21], suggests another formula

\[ M(X) = \frac{2X + (1 - \sqrt{\text{det} X}) I}{1 + \sqrt{\text{det} X}} \]  
(45)

which has been checked to generate unimodular matrices at least in the cases of \( X \) coinciding with \( P \) or \( Q \).

5. Fedorov’s Parameterization

Yet another approach to generating rotational matrices in three- and four-dimensional spaces was developed a long time ago by the Belarusian mathematical physicist Fedorov and presented in his book [22]. Unfortunately, this book has not been translated into English, but interested readers can find some of Fedorov’s and his collaborators’ papers on the subject. Fedorov’s formula associates with the three-dimensional vector a \( 4 \times 4 \) rotational matrix of the form

\[ \text{Fed}(X) := \frac{I + X}{\sqrt{1 + x^2}} \]  
(46)

in which \( X \) is the Lie algebra element of \( \mathfrak{so}(4, \mathbb{R}) \) generated by the vector \( x \) and \( x^2 \) denotes the scalar product \( x \cdot x \).

In our case, we have, respectively,

\[ \text{Fed}(P) = \frac{I + P}{\sqrt{1 + \alpha^2}} = \frac{1}{\sqrt{1 + \alpha^2}} \begin{pmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & 0 & \alpha \\ -\alpha & 0 & 1 & 0 \\ 0 & -\alpha & 0 & 1 \end{pmatrix}, \quad \alpha = (0, \alpha, 0) \]  
(47)

and

\[ \text{Fed}(Q) = \frac{I + Q}{\sqrt{1 + x^2}} = \frac{1}{\sqrt{1 + r^2}} \begin{pmatrix} 1 & -z & y & -x \\ z & 1 & -x & -y \\ -y & x & 1 & -z \\ x & y & z & 1 \end{pmatrix}, \quad x = (x, y, z). \]  
(48)

Despite these matrices being orthogonal by construction, it is not clear that they are really symplectic. An additional check proves that \( \text{Fed}(P) \) and \( \text{Fed}(Q) \) are indeed symplectic, i.e.,

\[ (\text{Fed}(P))^\dagger \cdot J \cdot \text{Fed}(P) = (\text{Fed}(Q))^\dagger \cdot J \cdot \text{Fed}(Q) = J. \]  
(49)
6. The Big Picture

What has been conducted up to now was to derive some explicit realization of the $4 \times 4$ OSM using some specific instruments like exponential and Cayley map or Fedorov’s parameterization in this dimension. Actually, all this is a part of a far away more general context according to which it is clear that OSM, i.e.,

$$\text{SO}(2n, \mathbb{R}) \cap \text{Sp}(2n, \mathbb{R}) \equiv U(n), \quad n \in \mathbb{N}$$  \hspace{1cm} (50)

where $U(n)$ denotes the unitary group in dimension $n$ defined by the condition

$$U^+(n) \cdot U(n) = I_n. \hspace{1cm} (51)$$

The latter implies that its adjoint (Hermitian conjugate) is equal to its inverse, i.e.,

$$U^+(n) = U^{-1}(n). \hspace{1cm} (52)$$

From this viewpoint, our results could be spelled out by saying that we have built various four-dimensional representations of the group

$$U(2) \cong U(1) \times SU(2) \cong S^1 \times S^3$$  \hspace{1cm} (53)

where $SU(2)$ denotes the special unitary group in two dimensions while $S^1$ and $S^3$ notation refers to their characterization as topological spaces. Actually, we have found the following four-dimensional representations of $U(2)$

1. Exponential

$$\text{Exp}(P) \cdot \text{Exp}(Q) = \text{Exp}(Q) \cdot \text{Exp}(P)$$

$$\begin{pmatrix}
    rca - ysr & (xa - zca) sr & ycasr + rsacr & -(xa + zsa) sr \\
    (zca + xa) sr & rca + ysr & (zas - xca) sr & rsacr - yca sr \\
    -ycasr - rsacr & (xca + zsa) sr & rca - ysr & (xa - zca) sr \\
    (xca - zsa) sr & ycasr - rsacr & (zca + xa) sr & rca + ysa sr
\end{pmatrix}$$

in which we have introduced the following shorthand notation

$$sa = \sin a, \quad ca = \cos a, \quad sr = \sin r \quad \text{and} \quad cr = \cos r.$$  \hspace{1cm} (55)

2. Cayley

$$\text{Cay}(P) \cdot \text{Cay}(Q) = \text{Cay}(Q) \cdot \text{Cay}(P) = \frac{1}{(1 + a^2)(1 + r^2)} \begin{pmatrix} S & K \\ -K & S \end{pmatrix}$$  \hspace{1cm} (56)

where

$$S = \begin{pmatrix}
    (a^2 - 1)(r^2 - 1) - 4ay & 4ax + 2(a^2 - 1)z \\
    4ax + 2(1 - a^2)z & (a^2 - 1)(r^2 - 1) + 4ay
\end{pmatrix}$$  \hspace{1cm} (57)

and

$$K = \begin{pmatrix}
    2(a(1 - r^2) + (1 - a^2)y) & 2(a^2 - 1)x - 4az \\
    2(a^2 - 1)x + 4az & 2(a(1 - r^2) + (a^2 - 1)y)
\end{pmatrix}.$$  \hspace{1cm} (58)

3. Fedorov
Fed(P) ⋅ Fed(Q) = Fed(Q) ⋅ Fed(P) = \frac{1}{\sqrt{1 + \alpha^2}} \cdot \frac{1}{\sqrt{1 + r^2}} \begin{pmatrix} 1 - ay & ax - z & \alpha - y \\ ax + z & 1 + ay & az - x \\ -\alpha - y & x + az & 1 - ay \end{pmatrix}.

Relying on the fundamental fact (50) we can interpret the situation in two ways, i.e.,

(1) As a possibility to build from the first principles, the representations of the OSM which are rational, in general, (cf. (56) or (59)) and then to use them via the formula

$$\begin{pmatrix} S & K \\ -K & S \end{pmatrix} \rightarrow U = S + iK, \quad S, K \in Mat(n, \mathbb{R})$$

(60)

to construct the representations of the unitary group $U(n)$.

(2) To use this fact in the reverse direction and build the OSM from the available representations of $U(n)$ which will be demonstrated below.

7. Orthogonal Factorization of $U(n)$

This possibility is based on another fundamental property of the unitary matrices, i.e., that they can be factored into the form

$$U(n) = \mathcal{O}(n)D(n)\mathcal{O}(n)$$

(61)

where $\mathcal{O}(n)$ denotes the orthogonal matrices in $\mathbb{R}^n$ and $D(n)$ is the set of the diagonal unitary matrices of the respective dimension. For more details see [23].

In the case under consideration, we have

$$\mathcal{O}(2) = \left\{ \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix} ; \varphi \in \mathbb{R} \right\} \quad \text{and} \quad \mathcal{D}(2) = \left\{ \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{pmatrix} ; \theta_1, \theta_2 \in \mathbb{R} \right\}$$

(62)

which implies immediately that

$$U(2) \cong \begin{pmatrix} \cos \varphi_1 & \sin \varphi_1 \\ -\sin \varphi_1 & \cos \varphi_1 \end{pmatrix} \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{pmatrix} \begin{pmatrix} \cos \varphi_2 & \sin \varphi_2 \\ -\sin \varphi_2 & \cos \varphi_2 \end{pmatrix}, \quad \varphi_1, \varphi_2, \theta_1, \theta_2 \in \mathbb{R}.$$  

(63)

Expanding the diagonal unitary matrix, i.e,

$$\begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{pmatrix} = \begin{pmatrix} \cos \theta_1 + i \sin \theta_1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & \cos \theta_2 + i \sin \theta_2 \end{pmatrix} = \begin{pmatrix} \cos \theta_1 & 0 \\ 0 & \cos \theta_2 \end{pmatrix} + i \begin{pmatrix} \sin \theta_1 & 0 \\ 0 & \sin \theta_2 \end{pmatrix}$$

it is easy to find directly the submatrices $S_2$ and $K_2$ which appear in (56) in the form

$$S_2 = \begin{pmatrix} c\varphi_1c\varphi_2c\varphi_2 - s\varphi_1c\varphi_2c\varphi_2 & s\varphi_1c\varphi_2c\varphi_2 + c\varphi_1c\varphi_2c\varphi_2 \\ -s\varphi_1c\varphi_2c\varphi_2 - c\varphi_1c\varphi_2c\varphi_2 & c\varphi_1c\varphi_2c\varphi_2 - s\varphi_1c\varphi_2c\varphi_2 \end{pmatrix}$$

(64)

$$K_2 = \begin{pmatrix} c\varphi_1s\varphi_1c\varphi_2c\varphi_2 - s\varphi_1s\varphi_1c\varphi_2c\varphi_2 & s\varphi_1s\varphi_1c\varphi_2c\varphi_2 + c\varphi_1s\varphi_1c\varphi_2c\varphi_2 \\ -c\varphi_1s\varphi_1c\varphi_2c\varphi_2 - c\varphi_1s\varphi_1c\varphi_2c\varphi_2 & c\varphi_1s\varphi_1c\varphi_2c\varphi_2 - s\varphi_1s\varphi_1c\varphi_2c\varphi_2 \end{pmatrix}.$$  

(65)

Having them, it is possible to check immediately that the matrix

$$M_4 = \begin{pmatrix} S_2 & K_2 \\ -K_2 & S_2 \end{pmatrix}$$  

(66)
is both orthogonal and symplectic, i.e.,
\[ M_4^t M_4 = M_4 M_4^t \equiv I_4, \quad M_4^t J_4 M_4 \equiv J_4 \] (67)
and, therefore, it presents an example of OSM in hyperspherical parameterization along
the genuine representation of the unitary matrix of order two given by the formula
\[ U(2) = S_2 + iK_2. \] (68)

8. Conclusions

The principal purpose of the present paper was to construct explicit parameterizations
of the four-dimensional ortho-symplectic matrices in a rigorous and constructive ab initio
manner.

Obviously, what remains to be conducted is to solve the tempting problem of finding
out the relationships between these independently constructed representations and to
explore them further.

Besides, the effect of these parameterizations can be traced directly in the case of
the planar Landau problem modeled via two-dimensional harmonic oscillator [24] or in
studying the representations of the orthosymplectic Lie superalgebra \(osp(1|4)\) in the spirit
of [25].

Finally, one of the most important implications of these representations is the incorpo-
ration of the representations of all unitary groups in a general and easily tractable scheme.

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