

Generalized Fuzzy Rough Approximations on Hypergroups

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Abstract: In this paper, we define the fuzzy set-valued homomorphisms of the canonical hypergroups as a generalization of fuzzy congruences and investigate some of their features. This structure is an extension of the definition of set-valued homomorphism defined for groups to hypergroups. With this extension, it has become possible to study generalized fuzzy rough approximations in hyperalgebraic structures such as semihypergroups, polygroups, hyperrings, hypermodules, etc. This paper presents the generalized fuzzy rough approximations based on two-universe (I, T) -fuzzy model on canonical hypergroups.

Keywords: canonical hypergroup; fuzzy set-valued homomorphism; fuzzy rough approximation

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1. Introduction

Rough set theory is asserted by Pawlak [1] as a tool to search structural relationships within uncertain and imprecise data. Pawlak refers to a pair consisting of a universal set and an indiscernibility relation (equivalence relation) on this set as a rough approximation space. The lower and upper approximations are definable subsets of the universal set. They can characterize any subset of the universal set, reveal hidden patterns within information systems and express these patterns as decision rules. The lower approximation contains objects that definitely belong to the subset, while the upper approximation contains objects that possibly belong to the subset. The difference between the upper and lower approximations is called the boundary region. If the boundary region is empty, the subset is crisp; otherwise, it is rough. As fundamental concepts, the approximations in Pawlak's approximation space are constructed via an equivalence relation.

Because of the various restrictions of equivalence relations, many extensions have replaced equivalence relations with notions such as neighborhood systems, general binary relations, covering-based systems, Boolean algebras, etc. [2–4]. Yao [5,6] started to redefine the approximation operators and their related notions from neighborhoods. Since the neighborhoods are very efficient at mitigating uncertainty and address a variety of practical applications, researchers have been interested in the novel sorts of granular computing derived from rough neighborhoods [7–11]. Examining a particular problem choice of hypothesis is crucial in mathematical research. To start under some conditions enhances the theory's applicability to real-world problems. It leads to more precise computations and improves decision making and data analysis efficiency. However, generalizations are to develop stronger theorems and address a broad spectrum of problems.

Fuzzy set theory is a theory on uncertainties through gradation. Since its introduction by Zadeh in 1965 [12], it has complemented numerous theories examining datasets with uncertainties and has been the subject of extensive research. Considering fuzzy relations

instead of crisp binary relations, Dubois and Prade [13] introduced fuzzy rough sets. As one of the various fuzzy generalizations of rough approximations, Wu et al. [14], following Radzikowska and Kerre [15], redefined (I, T) -fuzzy rough sets, where I is an implication and T is a t-norm. Recent research includes studies on approximation operators defined by fuzzy neighborhoods using fuzzy coverings. In 2022, Yang extended fuzzy coverings to two universal sets [4].

Algebraic structures are the fundamental concepts of mathematics that have a significant role in disciplines such as information science, theoretical physics, computer science, engineering, coding theory, etc., showcasing their broad applicability. This situation motivates researchers to analyze algebraic aspects of the various extensions of theories modeling uncertainties. Following Zadeh's [12] introduction to fuzzy sets, Rosenfeld [16] pioneered the study of fuzzy subgroups within group theory. This initial groundwork has encouraged significant interest in extending abstract algebraic concepts to the fuzzy setting [17,18]. As group theory is one of the algebraic structures widely used and discussed in mathematics and its applications, we focus on an extension of group theory in this paper.

A hyperbinary operation, which generalizes binary operations, is a function defined on the Cartesian product of a non-empty set with itself, taking values in the non-empty subsets of that set. A non-empty set equipped with a hyperoperation is called a hyperstructure or a hypergroupoid. By satisfying some properties, hyperbinary operations can define other algebraic hyperstructures such as hypersemigroups, hypergroups and hyperrings. After Marty [19] initiated the concept of the hypergroup, both the theories and applications of the hyperstructures have made achievements in pure and applied mathematics and computer science [20–30]. A canonical hypergroup is a natural generalization of the concept of an abelian group [31,32]. The class of hypergroups compasses a broader range of structures from groups, making the investigation of fuzzy hypergroups particularly compelling [25,31]. Numerous extensions of this topic to hyperalgebraic structures are available in the literature [23,25].

Algebraic concepts of rough set theory include algebraic structures. Rough sets have various compatible extensions on algebraic structures. Biswas and Nanda [33] proposed the rough subgroups as the first rough application of algebra. Kuroki [34] introduced rough ideals in semigroups using congruences instead of equivalence relations. Mordeson applied rough sets to rings [35]. Rough sets were extended to two universal sets by Yao in 1996 [5]. Davvaz applied rough sets, extended to two universal sets, to groups using his proposed set-valued homomorphisms [36]. Yamak et al. applied these concepts to rings and modules [37,38]. Rough sets have extensions on algebraic structures with fuzzy equivalence relations in fuzzy sets. Li et al. (2007) studied TL-fuzzy rough ideals in rings [39]. Li and Yin (2012) defined the ν -lower and T -upper fuzzy rough approximation operators on semigroups [40]. Wu et al. [14] extended fuzzy rough sets to a two-universe (I, T) -fuzzy model, then Ekiz et al. applied (I, T) -L-fuzzy rough sets to semigroups, groups and rings as an extension of (I, T) -fuzzy rough sets to lattices [41–43]. Some extensions of rough set theory were studied on hyperstructures [44,45].

This paper discusses applying (I, T) -fuzzy rough sets as a basic rough set model to hypergroups, extending beyond their application to groups. The importance of this paper is that it proposes a method for extending (I, T) -fuzzy rough sets to hyperalgebraic structures. In this paper, we introduce a concept of fuzzy set-valued homomorphisms as an extension of the fuzzy congruences of hypergroups [23] and examine some of their features. While fuzzy congruence is a kind of fuzzy relation on a single hypergroup, fuzzy set-valued homomorphism is a kind of fuzzy set-valued mapping from a hypergroup to the fuzzy sets of another hypergroup. Herein, we generate fuzzy set-valued homomorphisms from fuzzy congruences. This paper extends Ameri's study [23] to two different hypergroups as universal sets. In this way, it becomes possible to apply the results of [43] for hypergroups and investigate them in this broader context. In this paper, we also constitute the generalized fuzzy rough approximations by a fuzzy set-valued homomorphism on canonical hypergroups and examine some related properties. Some other similar research

topics may be to define the fuzzy set-valued homomorphisms in various hyperalgebraic systems such as semihypergroups and hyperrings and analyze their properties. In this way, extensions of other hyperalgebraic structures may be possible in which fuzzy set-valued homomorphisms can apply to the generalized fuzzy rough structures. For instance, [24,27,28] can be extended in a fuzzy sense and [40–42] in a hyper sense.

2. Preliminaries

Here, we provide a concise overview of the essential notions and preliminary evidence necessary for the subsequent results.

2.1. Fuzzy Logical Operators

Let $x, y \in U$. Then $x \wedge y$ denotes $\inf\{x, y\}$ and $x \vee y$ denotes $\sup\{x, y\}$. On the unit-interval $[0, 1]$, an associative and commutative binary relation T is a triangular norm (or t -norm) if it is an increasing function and satisfies the boundary condition $T(\alpha, 1) = \alpha$ for all $\alpha \in [0, 1]$. The minimum t -norm T_M and the drastic product t -norm T_D , respectively, are:

$$T_M(\alpha, \beta) = \alpha \wedge \beta \text{ and } T_D(\alpha, \beta) = \begin{cases} \beta, & \text{if } \alpha = 1; \\ \alpha, & \text{if } \beta = 1; \\ 0, & \text{otherwise} \end{cases}$$

Let T be a t -norm. Then, for all $(x, y) \in [0, 1]$, $T_M(x, y) \leq T(x, y) \leq T_D(x, y)$. D_T denotes the set $\{x \in [0, 1] | T(x, x) = x\}$.

On the unit-interval $[0, 1]$, an associative and commutative binary relation S is a triangular conorm (or t -conorm) if it is an increasing function and satisfies the boundary condition $S(\alpha, 0) = \alpha$ for all $\alpha \in [0, 1]$. The maximum operator $S_M(\beta, \alpha) = \beta \vee \alpha$ is the smallest t -conorm on $[0, 1]$.

A decreasing mapping $N : [0, 1] \rightarrow [0, 1]$ is a negator if it is providing the conditions $N(1) = 0, N(0) = 1$. The negator $N_s(x) = 1 - x$ for all $x \in [0, 1]$ is the standard negator. An implication is a binary relation on $[0, 1]$ which is decreasing with respect to the first component, increasing concerning the second component and providing the conditions $J(1, 1) = J(0, 0) = 1, J(1, 0) = 0$. An S -implication based on a t -conorm S and a negator N is the implication $J(x, y) = S(N(x), y)$ for all $x, y \in [0, 1]$, and R -implication (residual implication) based on a t -norm T is the implication $J(x, y) = \bigvee_{T(x, \lambda) \leq y} \lambda$ for all $x, y \in [0, 1]$ (see [40,46,47]).

2.2. Fuzzy Sets and Relations

Let U, W, U', W' and Z be non-empty sets as universes of discourses and $\alpha, \beta \in [0, 1]$.

A fuzzy subset of U is any function from U into $[0, 1]$ (see [12]). $P(U)$ and $F(U)$, respectively, denote the class of all subsets and fuzzy subsets of U . $P^*(U)$ denotes all of the non-empty subsets of U . Let $\alpha \in [0, 1]$. Then the set $\mu_\alpha = \{x \in U | \mu(x) \geq \alpha\}$ is α -cut (or level) subsets of μ . Let $\emptyset \neq B \subseteq U$. Then $(\alpha, \beta)_B$ is a fuzzy subset of U with the value β if $x \in B$ and α elsewhere. Then $(0, 1)_B$ is called the characteristic function of B .

Let $\mu \in F(U)$, $\nu \in F(W)$ and $f : U \rightarrow W$ be a function. Then $f(\mu) \in F(W)$ and $f^{-1}(\nu) \in F(U)$ are $f(\mu)(y) = \bigvee_{f(x)=y} \mu(x)$, $(\forall y \in W)$ and $f^{-1}(\nu) = \nu \circ f$. For any fuzzy subset $\mu \in F(U)$, μ^{-1} is a fuzzy subset of U such that $\mu^{-1}(x) = \mu(x^{-1})$ for all $x \in U$.

Let $\mu, \nu \in F(U)$ and T, S, I be a t -norm, t -conorm and an implication, respectively. Then $(\mu T \nu)(x) = \mu(x) T \nu(x)$, $(\mu S \nu)(x) = \mu(x) S \nu(x)$ and $(\mu I \nu)(x) = \mu(x) I \nu(x)$ for all $x \in U$.

A fuzzy subset $R \in F(U \times W)$ is a fuzzy relation from U to W . $R(x, y)$ is the degree of relation between x and y , where $(x, y) \in U \times W$. If $U = W$, then R is a fuzzy relation on U . Let $R : U \rightarrow F(W)$ be a function and $x \in U$. Then $R(x)$ denotes a fuzzy set of W .

Let $R : U \rightarrow F(W), P : U' \rightarrow F(W'), g : W \rightarrow W'$ and $f : U \rightarrow U'$ be functions. Then the fuzzy set-valued mappings $R^{-1} : W \rightarrow F(U), P_{(f,g)}^{-1} : U \rightarrow F(W)$ and $P_f : U \rightarrow F(W')$,

respectively, are $R^{-1}(y)(x) = R(x)(y)$, $P_{(f,g)}^{-1}(x)(y) = P(f(x))(g(y))$ and $P_f(x)(y') = P(f(x))(y')$ for all $x \in U, x' \in U', y \in W, y' \in W'$. The set-valued function $R(f)_{(\alpha,\beta)} : U \rightarrow F(U')$ is $R(f)_{(\alpha,\beta)}(x)(y) = \begin{cases} \beta, & \text{if } f(x) = y; \\ \alpha, & \text{if } f(x) \neq y, \end{cases}$ for all $y \in U'$.

The T- and J-compositions of the fuzzy set-valued functions $R : U \rightarrow F(W)$ and $P : W \rightarrow F(Z)$ are the fuzzy set-valued functions $P \circ_T R : U \rightarrow F(Z)$ and $P \circ_J R : U \rightarrow F(Z)$;

$$(P \circ_T R)(x)(z) = \bigvee_{y \in Y} (R(x)(y)TP(y)(z)),$$

$$(P \circ_J R)(x)(z) = \bigwedge_{y \in Y} (R(x)(y)JP(y)(z))$$

for all $x \in U, z \in Z$ (see [17]).

Let $R \in F(U \times W)$ and $K : U \rightarrow F(W)$ be a set-valued mapping. Then, in this paper, R and K are referred to as connected, if $K(x)(y) = R(x, y)$ for all $x \in U, y \in W$.

2.3. Generalized Fuzzy Rough Approximation Spaces

Let $R : U \rightarrow F(W)$ be a set-valued mapping, T be a t-norm and J be an implication on the unit-interval. For any fuzzy subset μ of W , the T-upper and J-lower fuzzy rough approximations of μ are fuzzy sets of U , $\bar{R}^T(\mu)$ and $\underline{R}_J(\mu)$, respectively. Their membership functions are

$$\bar{R}^T(\mu)(x) = \bigvee_{y \in Y} (R(x)(y)T\mu(y)),$$

$$\underline{R}_J(\mu)(x) = \bigwedge_{y \in Y} (R(x)(y)J\mu(y)), (\forall x \in U).$$

The triple (U, W, R) is a generalized fuzzy approximation space. The operators $\bar{R}$ and $\underline{R}$ from $F(W)$ to $F(U)$ are T-upper and J-lower fuzzy rough approximation operators of (U, W, R) , respectively. The pair $(\underline{R}(\mu), \bar{R}(\mu))$ is a (J, T) -fuzzy rough set of μ with respect to (U, W, R) (see [14,40–43]). If R is a fuzzy relation on U , then (U, R) is a fuzzy approximation space.

2.4. Hypergroups

An algebraic hyperstructure (or hypergroupoid) is a non-empty set H together with hyperoperation, i.e., a map $\circ : H \times H \rightarrow P^*(H)$.

A hyperstructure $(H, \circ)$ is a semihypergroup if, for all $a, b, c \in H$, $(a \circ b) \circ c = a \circ (b \circ c)$. $(a \circ b) \circ c$ means $\bigcup_{x \in a \circ b} x \circ c$ and $a \circ (b \circ c)$ means $\bigcup_{y \in b \circ c} a \circ y$. If $a \in H$ and $\emptyset \neq A, B \subseteq H$, then $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$, $A \circ a = \bigcup_{x \in A} x \circ a$ and $a \circ B = \bigcup_{y \in B} a \circ y$.

$\emptyset \neq B \subseteq H$ is a subsemihypergroup of H if $B \circ B \subseteq B$ and in this case H is a supersemihypergroup of B .

Let $(H, \circ)$ be a semihypergroup. Then H is a hypergroup if it satisfies the reproduction axiom: $a \circ H = H \circ a = H$ ($\forall a \in H$). An element e in a semihypergroup H is identity if, for all $a \in H$, $a \circ e = e \circ a = a$. An element 0 in a semihypergroup H is zero element if $a \circ 0 = 0 \circ a = \emptyset$.

A semihypergroup $(H, \circ)$ is a canonical hypergroup if

- (i) $x \circ y = y \circ x$ for all $x, y \in H$,
- (ii) there exists $e \in H$ such that $e \circ x = \{x\}$ for all $x \in H$ (e is scalar identity),
- (iii) there exists a unique element $x' \in H$ for any $x \in H$ such that $e \in x \circ x'$ (x' is the opposite of x and x^{-1} denotes it),
- (iv) $z \in x \circ y$ implies $y \in x^{-1} \circ z$ and $x \in z \circ y^{-1}$ for every $x, y, z \in H$.

A canonical subhypergroup of a canonical hypergroup $(H, \circ)$ is a non-empty subset K of H containing the scalar identity of $(H, \circ)$ if it forms a canonical hypergroup under the hyperoperation $\circ$ on H (see [19,20,25,31,32,48]).

Example 1. Let H be a non-empty set.

- (i) Let $(H, \cdot)$ be an abelian group and a mapping $\circ : H \times H \rightarrow P^*(H)$ be defined by $x \circ y = \{x \cdot y\}$ for all $x, y \in H$. Then $(H, \circ)$ is a canonical hypergroup.
- (ii) If $(H, +)$ is an abelian group and ρ is an equivalence relation in H which has classes $\bar{x} = \{x, -x\}$, then for all $\bar{x}, \bar{y} \in H/\rho$, we define $\bar{x} \circ \bar{y} = \{\bar{x} + \bar{y}, \bar{x} - \bar{y}\}$. Then $(H/\rho, \circ)$ is a canonical hypergroup [32].

Example 2. Let $H = \{x, y, z, t, e\}$ with the following multiplication table (Figure 1):

$\circ$	x	y	z	t	e
x	H	$\{x, y, z, t\}$	$\{x, y, z, t\}$	$\{x, y, z, t\}$	$\{x\}$
y	$\{x, y, z, t\}$	H	$\{x, y, z, t\}$	$\{x, y, z, t\}$	$\{y\}$
z	$\{x, y, z, t\}$	$\{x, y, z, t\}$	H	$\{x, y, z, t\}$	$\{z\}$
t	$\{x, y, z, t\}$	$\{x, y, z, t\}$	$\{x, y, z, t\}$	H	$\{t\}$
e	$\{x\}$	$\{y\}$	$\{z\}$	$\{t\}$	$\{e\}$

Figure 1. $(H, \circ)$ multiplication table.

Then H is a canonical hypergroup with the scalar element e .

Definition 1 ([22,25,29]). Let G, H be hypergroups. Then a function $f : G \rightarrow H$ is

- (i) an inclusion homomorphism if $f(x)f(y) \subseteq f(xy)$ for all $x, y \in G$;
- (ii) a strong homomorphism (or good homomorphism) if $f(x)f(y) = f(xy)$ for all $x, y \in G$.

2.5. T-Fuzzy Subhypergroups

Definition 2. Let G be a canonical hypergroup and T be a t -norm. Then a fuzzy subset μ of G is a T -fuzzy subhypergroup of G if the following conditions hold:

- (i) $\mu(a)T\mu(b) \leq \wedge_{x \in ab} \mu(x)$,
- (ii) $\mu(a) \leq \mu(a^{-1})$ for all $a, b \in G$ (see [23,49]).

Definition 3 ([23]). Let μ, ν be two fuzzy subsets of semihypergroup H . Then $\mu \circ \nu$ is the fuzzy subset of H ,

$$(\mu \circ \nu)(x) = \begin{cases} \bigvee_{x \in pt} (\mu(p) \wedge \nu(t)), & \text{if } \exists p, t \in H : x \in p \cdot t; \\ 0, & \text{otherwise.} \end{cases}$$

Definition 4 ([23]). Let H be a canonical hypergroup. Then a fuzzy binary relation R on H is

- (i) fuzzy compatible if

$$\left(\bigwedge_{u \in a \circ c} \bigvee_{v \in b \circ d} R(u, v) \right) \wedge \left(\bigwedge_{v \in b \circ d} \bigvee_{u \in a \circ c} R(u, v) \right) \geq R(a, b) \wedge R(c, d)$$

for all $a, b, c, d \in H$.

- (ii) Fuzzy strong compatible if

$$\bigwedge_{x \in a \circ c, y \in b \circ d} R(x, y) \geq R(a, b) \wedge R(c, d)$$

for all $a, b, c, d \in H$.

2.6. Set-Valued Homomorphisms of Canonical Hypergroups

Definition 5 ([28]). Let G and H be canonical hypergroups. Then a mapping from $\theta : G \rightarrow P^*(H)$ which preserves the operations $\theta(a)\theta(b) \subseteq \theta(ab)$ and $(\theta(a))^{-1} \subseteq \theta(a^{-1})$ for all $a, b \in G$ is called a set-valued homomorphism of canonical hypergroups. A set-valued homomorphism of canonical hypergroups θ is called strong if $\theta(a)\theta(b) = \theta(ab)$ for all $a, b \in G$.

3. T-Fuzzy Set-Valued Homomorphisms of Canonical Hypergroups

In this section, we introduce the concept of fuzzy set-valued homomorphism of hypergroups and demonstrate some of their features.

Definition 6. Let μ, ν be two fuzzy subsets of the semihypergroup H and T be a t -norm. Then we define the fuzzy subset $\mu \vee_T \nu$ of H by

$$(\mu \vee_T \nu)(x) = \begin{cases} \bigvee_{x \in pt} (\mu(p)Tv(t)), & \text{if } \exists p, t \in H : x \in pt; \\ 0, & \text{otherwise.} \end{cases}$$

for all $x \in H$. Clearly " $\vee_T$ " is a binary relation on $F(H)$ and $\mu \vee_T \nu$ is called the $\vee_T$ -product of μ and ν . If $(H, \cdot)$ is a canonical hypergroup, then we have $(\mu \vee_T \nu)(x) = \bigvee_{x \in pt} (\mu(p)Tv(t))$ since $x \in ex$ for all $x \in H$.

Definition 7. Let G, H be canonical hypergroups and $R : G \rightarrow F(H)$ be a mapping. Then

- (i) R is a T -fuzzy set-valued homomorphism of semihypergroups if, for all $a, b \in G$, $R(a) \vee_T R(b) \leq \bigvee_{u \in ab} R(u)$.
- (ii) R is a strong T -fuzzy set-valued homomorphism of semihypergroups if, for all $a, b \in G$, $(R(a) \vee_T R(b)) \leq \bigwedge_{u \in ab} R(u)$.
- (iii) R is a completely strong T -fuzzy set-valued homomorphism of semihypergroups if, for all $a, b \in G$, $R(a) \vee_T R(b) = \bigwedge_{u \in ab} R(u)$.

$H_T(G, F(H))$ denotes the set of all the T -fuzzy set-valued homomorphisms, $sH_T(G, F(H))$ denotes the set of all the strong T -fuzzy set-valued homomorphisms, and $chH_T(G, F(H))$ denotes the set of all the completely strong T -fuzzy set-valued homomorphisms from G to H . If $T = T_M$, then a T -fuzzy set-valued homomorphism is called a fuzzy set-valued homomorphism and the set of all the fuzzy set-valued homomorphisms from G to H is denoted by $H(G, F(H))$. If $T = T_M$, then the statement also applies analogously to $sH_T(G, F(H))$ and $chH_T(G, F(H))$.

Definition 8. Let $(G, \circ)$ and $(H, \cdot)$ be canonical hypergroups and $R : G \rightarrow F(H)$ be a mapping satisfying $(R(a))^{-1} \leq R(a^{-1})$ for all $a \in G$. Then

- (i) $R \in H_T(G, F(H))$ is a T -fuzzy set-valued homomorphism of canonical hypergroups.
- (ii) $R \in sH_T(G, F(H))$ is a strong T -fuzzy set-valued homomorphism of canonical hypergroups.
- (iii) $R \in chH_T(G, F(H))$ is a completely strong T -fuzzy set-valued homomorphism of canonical hypergroups.

Example 3. Let $G = \{a, b, c, e\}$ be a Klein-4 group and $(G, \circ)$ the canonical hypergroup in Example 1 (i) and let $(H, \cdot)$ be the canonical hypergroup in Example 2. Then $R : G \rightarrow F(H)$ defined by Figure 2 is a completely strong T_D -fuzzy set-valued homomorphism of canonical hypergroups but it is not a T_M -fuzzy set-valued homomorphism since

h	x	y	z	t	e
$R(a)(h)$	0,4	0,5	0,2	0	1
$R(b)(h)$	0,4	0,5	0,2	0	1
$R(c)(h)$	0,4	0,5	0,2	0	1
$R(e)(h)$	0,4	0,5	0,2	0	1

Figure 2. Definition table of R .

$$(R(a) \vee_{T_M} R(b))(x) = 0,5 \not\leq 0,4 = R(c)(x) = \bigwedge_{u \in \bar{a} \circ \bar{b}} R(u)(x)$$

Example 4. Let $(G, \circ)$ and $(H, \cdot)$ be canonical hypergroups and $\alpha \leq \beta, \alpha, \beta \in [0, 1]$. $P, R : G \rightarrow F(H)$ are defined by the following:

$$P(a)(x) = \begin{cases} \alpha, & \text{if } a \neq e; \\ \beta, & \text{if } a = e, \end{cases} \quad R(a)(x) = \begin{cases} \alpha, & \text{if } x \neq e; \\ \beta, & \text{if } x = e. \end{cases}$$

Then $P, R \in H_T(G, F(H))$.

Proposition 1. Let $(G, \circ)$, $(H, \cdot)$ be canonical hypergroups and $R \in H_T(G, F(H))$. Then $(R(a))^{-1} = R(a^{-1})$ for all $a \in G$.

Proof. Straightforward. $\square$

Theorem 1. Let $f : G \rightarrow H$ be an inclusion homomorphism of semihypergroups. Then $R(f)_{(\alpha, \beta)} \in H_T(G, F(H))$ if $\alpha \leq \beta$.

Proof. Let $a, x \in G$ and let $(R(f)_{(\alpha, \beta)}(x) \vee_T R(f)_{(\alpha, \beta)}(a))(v) = \beta$ for any $v \in H$. Then there exist $y, b \in H$ such that $v \in yb$ and $R(f)_{(\alpha, \beta)}(x)(y) = \beta$, and $R(f)_{(\alpha, \beta)}(a)(b) = \beta$. Thus $f(x) = y$ and $f(a) = b$. So $v \in yb \subseteq f(x)f(a) \subseteq f(x \circ a)$ since $f : G \rightarrow H$ is an inclusion homomorphism. There exist $u \in x \circ a$ such that $f(u) = v$. Then $\bigvee_{u \in x \circ a} R(f)_{(\alpha, \beta)}(u)(v) = \beta$. Thus $R(f)_{(\alpha, \beta)} \in H_T(G, F(H))$. $\square$

Example 5. Let G be the canonical hypergroup in Example 2 and $f : G \rightarrow G$ be defined as $f(a) = a$ for all $a \in G$. Then f is an inclusion homomorphism of semihypergroups and the mapping $R(f)_{(0, 1)} : G \rightarrow F(G)$ is a T-fuzzy set-valued homomorphism but is not strong since

$$(R(f)_{(0, 1)}(x) \vee_T R(f)_{(0, 1)}(x))(e) = 1 \not\leq 0 = \left(\bigwedge_{u \in x \cdot x} R(f)_{(0, 1)}(u) \right)(e)$$

Theorem 2. Let $\theta : G \rightarrow P^*(H)$ be a set-valued homomorphism, $\alpha, \beta \in [0, 1]$, $\beta \in D_T$ and $\alpha \leq \beta$. Then $R : G \rightarrow F(H)$ defined by $R(x) = (\alpha, \beta)_{\theta(x)}$ for all $x \in G$ is a T-fuzzy set-valued homomorphism.

Proof. Let $a, b \in G$ and let $(R(a) \vee_T R(b))(v) = \beta$ for any $v \in H$. Then there exists $k, t \in H$ such that $v \in kt$ and $(\alpha, \beta)_{\theta(a)}(k) = \beta$, and $(\alpha, \beta)_{\theta(b)}(t) = \beta$. Thus $k \in \theta(a)$ and $t \in \theta(b)$. Hence $v \in \theta(a \circ b)$ since θ is a set-valued homomorphism. Thus, there exists an $u \in a \circ b$ such that $v \in \theta(u)$. Then

$$\bigvee_{u \in a \circ b} (\alpha, \beta)_{\theta(u)}(v) = \beta$$

Hence $R : G \rightarrow F(H)$ is a T-fuzzy set-valued homomorphism. $\square$

The following example demonstrates that the fuzzy set-valued homomorphism obtained by the method in Theorem 2 may not be strong.

Example 6. Let G be the canonical hypergroup in Example 2 and $\theta : G \rightarrow P^*(H)$ be defined as $\theta(a) = \{a\}$ for all $a \in G$. Then θ is a strong set-valued homomorphism and the mapping $R : G \rightarrow F(H)$ defined by $R(a) = (0, 1)_{\theta(a)}$ for all $a \in G$ is not a strong T -fuzzy set-valued homomorphism since

$$(R(x) \vee_T R(x))(x) = 1 \not\leq 0 = \left(\bigwedge_{u \in x.x} R(u) \right)(x)$$

The following theorem shows that the fuzzy congruences are a special type of the set-valued homomorphisms.

Theorem 3. Let G be a semihypergroup, R be a fuzzy (strong) compatible on G and $P : G \rightarrow F(G)$ be the connected T -fuzzy set-valued mapping. Then P is a (strong) T -fuzzy set-valued homomorphism.

Proof. Let $a, b \in G$. Since R is fuzzy compatible, then, for any $p \in G$, we have

$$\begin{aligned} (P(a) \vee_T P(b))(p) &= \bigvee_{p \in kt} (P(a)(k)TP(b)(t)) \leq \bigvee_{p \in kt} (R(a, k) \wedge R(b, t)) \\ &\leq \bigvee_{p \in kt} \left(\left(\bigwedge_{u \in ab} \bigvee_{v \in kt} R(u, v) \right) \wedge \left(\bigwedge_{v \in kt} \bigvee_{u \in ab} R(u, v) \right) \right) \\ &\leq \bigvee_{p \in kt} \left(\bigwedge_{v \in kt} \bigvee_{u \in ab} R(u, v) \right) \leq \bigvee_{p \in kt} \left(\bigvee_{u \in ab} R(u, p) \right) = \bigvee_{u \in ab} R(u, p) \\ &= \bigvee_{u \in ab} P(u)(p) = \left(\bigvee_{u \in ab} P(u) \right)(p). \end{aligned}$$

Hence P is a T -fuzzy set-valued homomorphism since $P(a) \vee_T P(b) \leq \bigvee_{u \in ab} P(u)$ for all $a, b \in G$. Let R be a fuzzy strong compatible on G . Then, for any $p \in G$, we have

$$\begin{aligned} (P(a) \vee_T P(b))(p) &= \bigvee_{p \in kt} (P(a)(k)TP(b)(t)) \leq \bigvee_{p \in kt} (R(a, k) \wedge R(b, t)) \\ &\leq \bigvee_{p \in kt} \left(\bigwedge_{x \in ab, y \in kt} R(x, y) \right) \leq \bigvee_{p \in kt} \left(\bigwedge_{x \in ab} R(x, p) \right) = \bigwedge_{x \in ab} R(x, p) \\ &= \left(\bigwedge_{x \in ab} R(x) \right)(p). \end{aligned}$$

Hence P is a strong T -fuzzy set-valued homomorphism since

$$P(a) \vee_T P(b) \leq \bigwedge_{u \in ab} P(u)$$

for all $a, b \in G$. $\square$

Example 7.

- (i) $R(f)_{(0,1)}$ in Example 5 is a T -fuzzy set-valued homomorphism. Moreover, its connected fuzzy relation is fuzzy compatible. However, it is not strong fuzzy compatible since it is not a strong T -fuzzy set-valued homomorphism.
- (ii) Even though the connected fuzzy relation of R in Example 6 is fuzzy compatible, R is not a strong T -fuzzy set-valued homomorphism.

Theorem 4. Let G and H be canonical hypergroups and let $R : G \rightarrow F(H)$ be a fuzzy set-valued mapping and $\theta : G \rightarrow \mathcal{P}(H)$ be the set-valued mapping defined by $\theta(a) = R(a)_\alpha$ for all $a \in G$.

- (i) If $R \in H_T(G, F(H))$ and $\alpha \in D_T$, then θ is a set-valued homomorphism.
- (ii) If $R \in H_T(G, F(H))$ is completely strong, then $\bigcap_{u \in a \circ b} \theta(u) \subseteq \theta(a)\theta(b)$ for all $a, b \in G$.
- (iii) If θ is a set-valued homomorphism for all $\alpha \in [0, 1]$, then $R \in H_T(G, F(H))$.

Proof. Let $a, b \in G$.

- (i) Let $v \in \theta(a).\theta(b)$. Then there exist $p \in \theta(a)$ and $t \in \theta(b)$ such that $v \in p.t$. Hence $p \in R(a)_\alpha$ and $t \in R(b)_\alpha$. Thus $R(a)(p) \geq \alpha$ and $R(b)(t) \geq \alpha$. So $R(a)(p)TR(b)(t) \geq \alpha T \alpha = \alpha$ since $\alpha \in D_T$. Therefore, $(R(a) \vee_T R(b))(v) \geq \alpha$. So $\bigvee_{u \in a \circ b} R(u)(v) \geq \alpha$ since $R \in H_T(G, F(H))$. Hence there exists an element $u \in a \circ b$ such that $R(u)(v) \geq \alpha$. Thus $v \in R(u)_\alpha = \theta(u)$. So $v \in \theta(a \circ b)$ since $\theta(u) \subseteq \theta(a \circ b)$. Thus $\theta(a).\theta(b) \subseteq \theta(a \circ b)$ is obtained. Let $x \in (\theta(a))^{-1}$. Thus $x^{-1} \in \theta(a)$ and so $x^{-1} \in R(a)_\alpha$. Hence $R(a)(x^{-1}) \geq \alpha$. So $(R(a))^{-1}(x) \geq \alpha$. Since $R \in H_T(G, F(H))$. Hence $R(a^{-1})(x) \geq \alpha$. Thus $x \in R(a^{-1})_\alpha$. So $x \in \theta(a^{-1})$. Therefore $(\theta(a))^{-1} \subseteq \theta(a^{-1})$ is obtained. Finally, $\theta : G \rightarrow \mathcal{P}(H)$ is a set-valued homomorphism.
- (ii) Let $v \in \bigcap_{u \in a \circ b} \theta(u)$. Then $v \in \theta(u)$ for all $u \in a \circ b$. Hence $v \in R(u)_\alpha$. So $R(u)(v) \geq \alpha$. Therefore $\bigwedge_{u \in a \circ b} R(u)(v) \geq \alpha$. Thus $(R(a) \vee_T R(b))(v) \geq \alpha$ since $R \in H_T(G, F(H))$ is completely strong. Then $\bigvee_{p \in \theta(a)} R(a)(p)TR(b)(t) \geq \alpha$. There exist $p, t \in H$ such that $v \in p.t$ and $R(a)(p) \wedge R(b)(t) \geq \alpha$. Thus $R(a)(p), R(b)(t) \geq \alpha$. Hence $p \in R(a)_\alpha$ and $t \in R(b)_\alpha$. Then $p \in \theta(a)$ and $t \in \theta(b)$. So $v \in \theta(a)\theta(b)$ since $v \in p.t \subseteq \theta(a)\theta(b)$. Finally, $\bigcap_{u \in a \circ b} \theta(u) \subseteq \theta(a)\theta(b)$ for all $a, b \in G$.
- (iii) Let $v \in H$ and let $\alpha := R(a)(p)TR(b)(t)$ for any $p, t \in H$ such that $v \in p.t$. Thus $R(a)(p) \geq \alpha$ and $R(b)(t) \geq \alpha$. Hence $p \in \theta(a)$ and $t \in \theta(b)$. So $p.t \in \theta(a).\theta(b)$. Since θ is a set-valued homomorphism, then $p.t \in \theta(a \circ b)$. Hence $v \in \theta(a \circ b)$. Thus there exists an $u \in a \circ b$ such that $v \in \theta(u)$. Then $R(u)(v) \geq \alpha$. Hence, $\bigvee_{u \in a \circ b} R(u)(v) \geq R(a)(p)TR(b)(t)$ is obtained. Thus, we have $\bigvee_{u \in a \circ b} R(u)(v) \geq \bigvee_{p \in \theta(a)} R(a)(p)TR(b)(t)$. Therefore $R(a) \vee_T R(b) \leq \bigvee_{u \in a \circ b} R(u)$ is verified. Let $v \in H$ and $(R(a))^{-1}(v) \geq \alpha$. Then $(R(a))^{-1}(v) \geq \alpha$. So $R(a)(v^{-1}) \geq \alpha$. Hence $v^{-1} \in R(a)_\alpha$. Thus $v^{-1} \in \theta(a)$. Hence $v \in (\theta(a))^{-1}$. Since θ is a set-valued homomorphism, then $v \in \theta(a^{-1})$. Hence $v \in R(a^{-1})_\alpha$, i.e., $R(a^{-1})(v) \geq \alpha$. Thus $R(a^{-1})(v) \geq (R(a))^{-1}(v)$. Therefore $R(a^{-1}) \geq (R(a))^{-1}$. Finally, $R \in H_T(G, F(H))$ is obtained. $\square$

Theorem 5. Let G, H be canonical hypergroups, $R_1 : G \rightarrow F(H)$ and $R_2 : H \rightarrow F(K)$. Then $R_2 \circ_T R_1 \in H_T(G, F(K))$ if $R_1 \in H_T(G, F(H))$ and $R_2 \in H_T(H, F(K))$.

Proof. Let $a, b \in G$ and $k \in H$. We have

$$\begin{aligned}
((R_2 \circ_T R_1)(a) \vee_T (R_2 \circ_T R_1)(b))(k) &= \bigvee_{k \in k_1 k_2} ((R_2 \circ_T R_1)(a)(k_1) T (R_2 \circ_T R_1)(b)(k_2)) \\
&= \bigvee_{k \in k_1 k_2} \left(\left(\bigvee_{h_1 \in H} R_1(a)(h_1) T R_2(h_1)(k_1) \right) T \left(\bigvee_{h_2 \in H} R_1(b)(h_2) T R_2(h_2)(k_2) \right) \right) \\
&= \bigvee_{k \in k_1 k_2} \bigvee_{h_1, h_2 \in H} (R_1(a)(h_1) T R_2(h_1)(k_1) T R_1(b)(h_2) T R_2(h_2)(k_2)) \\
&= \bigvee_{h_1, h_2 \in H} (R_1(a)(h_1) T R_1(b)(h_2) T \bigvee_{k \in k_1 k_2} (R_2(h_1)(k_1) T R_2(h_2)(k_2)) \\
&= \bigvee_{h_1, h_2 \in H} (R_1(a)(h_1) T R_1(b)(h_2) T (R_2(h_1) \vee_T R_2(h_2))(k)) \\
&\leq \bigvee_{h_1, h_2 \in H} \left(R_1(a)(h_1) T R_1(b)(h_2) T \left(\bigvee_{v \in h_1 h_2} R_2(v) \right) (k) \right) \\
&\leq \bigvee_{h_1, h_2 \in H} \left(\bigvee_{v \in h_1 h_2} (R_1(a)(h_1) T R_1(b)(h_2) T \left(\bigvee_{v \in h_1 h_2} R_2(v) \right) (k)) \right) \\
&= \bigvee_{h_1, h_2 \in H} \left((R_1(a) \vee_T R_1(b))(v) T \left(\bigvee_{v \in h_1 h_2} R_2(v) \right) (k) \right) \\
&\leq \bigvee_{h_1, h_2 \in H} \left(\left(\bigvee_{u \in a \circ b} R_1(u) \right) (v) T \left(\bigvee_{v \in h_1 h_2} R_2(v) \right) (k) \right) \\
&= \bigvee_{u \in a \circ b} \left(\bigvee_{v \in H} R_1(u)(v) T R_2(v)(k) \right) \\
&= \bigvee_{u \in a \circ b} (R_2 \circ_T R_1)(u)(k).
\end{aligned}$$

Also

$$\begin{aligned}
((R_2 \circ_T R_1)(a))^{-1}(k) &= ((R_2 \circ_T R_1)(a))(k^{-1}) = \bigvee_{h \in H} (R_1(a)(h) T R_2(h)(k^{-1})) \\
&= \bigvee_{h \in H} ((R_1(a))^{-1}(h^{-1}) T (R_2(h))^{-1}(k)) \\
&\leq \bigvee_{h \in H} (R_1(a^{-1})(h^{-1}) T R_2(h^{-1})(k)) = ((R_2 \circ_T R_1)(a^{-1}))(k).
\end{aligned}$$

□

Theorem 6. Let G, H be canonical hypergroups and $f : G \rightarrow G'$ and $g : H \rightarrow H'$ be inclusion homomorphisms.

- (i) If $P \in H_T(G', F(H'))$, then $P^{-1}_{(f,g)} \in H_T(G, H)$.
- (ii) If $S \in H_T(G, F(H'))$, then $S_f \in H_T(G, H')$.

Proof. Let $x, a \in G$ and $h \in H$. Thus

(i)

$$\begin{aligned}
(P^{-1}_{(f,g)}(x) \vee_T P^{-1}_{(f,g)}(a))(h) &= \bigvee_{h \in yb} (P^{-1}_{(f,g)}(x)(y) T P^{-1}_{(f,g)}(a)(b)) \\
&= \bigvee_{h \in yb} (P(f(x))(g(y)) T P(f(a))(g(b))) \\
&\leq \bigvee_{g(h) \in g(y)g(b)} (P(f(x))(g(y)) T P(f(a))(g(b))) \\
&= (P(f(x)) \vee_T P(f(a)))(g(h)) \leq \bigvee_{t \in f(xa)} P(t)(g(h)) \\
&= \bigvee_{u \in x \circ a} P(f(u))(g(h)) = \left(\bigvee_{u \in x \circ a} P^{-1}_{(f,g)}(u) \right) (h).
\end{aligned}$$

Hence, we have

$$P^{-1}_{(f,g)}(x) \vee_T P^{-1}_{(f,g)}(a) \leq \bigvee_{u \in x \circ a} P^{-1}_{(f,g)}(u).$$

(ii)

$$\begin{aligned} (S_f(x) \vee_T S_f(a))(h) &= \bigvee_{h \in yb} (S_f(x)(y)TS_f(a)(b)) = \bigvee_{h \in yb} (S(f(x))(y)TS(f(a))(b)) \\ &= (S(f(x)) \vee_T S(f(a)))(h) \leq \bigvee_{u \in x \circ a} S(f(u))(h) \\ &= \left(\bigvee_{u \in x \circ a} S_f(u) \right)(h). \end{aligned}$$

Hence, we have

$$S_f(x) \vee_T S_f(a) \leq \bigvee_{u \in x \circ a} S_f(u).$$

□

Definition 9. Let G, H be canonical hypergroups and $R \in H_T(G, F(H))$. Then

- (i) the kernel of R is a fuzzy subset of G denoted by $\text{Ker}R$ such that $\text{Ker}R(x) = R(x)(e)$ for all $x \in G$.
- (ii) the image of R is a fuzzy subset of H denoted by $\text{Im}R$ such that $\text{Im}R(y) = \bigvee_{x \in G} R(x)(y)$ for all $y \in H$ (see [43]).

Theorem 7. Let $R \in H_T(G, F(H))$. Then $\text{Ker}R$ is a T -fuzzy subhypergroup of G if R is strong.

Proof. Let $x, a \in G$. Then $\text{Ker}R$ is a T -fuzzy subhypergroup of G since

$$\begin{aligned} \bigwedge_{u \in x \circ a} \text{Ker}R(u) &= \bigwedge_{u \in x \circ a} R(u)(e) \geq (R(x) \vee_T R(a))(e) \geq R(x)(e)TR(a)(e) \\ &= \text{Ker}R(x)TKerR(a), \\ \text{Ker}R(x^{-1}) &= R(x^{-1})(e) \geq (R(x))^{-1}(e) = R(x)(e^{-1}) = R(x)(e) = \text{Ker}R(x). \end{aligned}$$

□

Theorem 8. Let $R \in \text{Hom}T(G, F(H))$. Then $\text{Im}R$ is a T -fuzzy subhypergroup of H .

Proof. Let $y, b \in H$. Then

$$\begin{aligned} \text{Im}R(y)T\text{Im}R(b) &= \left(\bigvee_{x \in G} R(x)(y) \right) T \left(\bigvee_{a \in G} R(a)(b) \right) = \bigvee_{x, a \in G} (R(x)(y)TR(a)(b)) \\ &\leq \bigvee_{x, a \in G} \left(\bigvee_{u \in x \circ a} R(u)(t) \right) \leq \bigwedge_{t \in yb} \left(\bigvee_{g \in G} R(g)(t) \right) = \bigwedge_{t \in yb} (\text{Im}R(t)) \end{aligned}$$

Thus $\text{Im}R(y)T\text{Im}R(b) \leq \bigwedge_{t \in yb} (\text{Im}R(t))$ is obtained and clearly $\text{Im}R(y) \leq \text{Im}R(y^{-1})$. Then $\text{Im}R$ is a T -fuzzy subhypergroup of H . □

Theorem 9. Let G, H be canonical hypergroups and $R \in H_T(G, F(H))$. Then $R(e)$ is a T -fuzzy subhypergroup of H .

Proof. Let $a, b \in H$. Then

$$\begin{aligned} R(e)(a)TR(e)(b) &\leq \bigvee_{p \in ab} (R(e)(a)TR(e)(b)) = (R(e) \vee_T R(e))(p) \\ &\leq \left(\bigvee_{u \in ee} R(u) \right)(p) = R(e)(p) \end{aligned}$$

Hence

$$R(e)(a)TR(e)(b) \leq \bigwedge_{p \in ab} R(e)(p)$$

Also, $R(e)(a) \leq R(e)(a^{-1})$ since $R(e)(a) = (R(e))^{-1}(a^{-1}) \leq R(e^{-1})(a^{-1}) = R(e)(a^{-1})$. Thus $R(e)$ is a T-fuzzy subhypergroup of H . $\square$

Definition 10. Let G, H be semihypergroups $f : G \rightarrow H$ be a function and $R : G \rightarrow F(G)$ and $P : H \rightarrow F(H)$ be set-valued mappings. Then the set-valued function $f(R) : H \rightarrow F(H)$ is

$$f(R)(h_1)(h_2) = \begin{cases} \bigvee R(g_1)(g_2), & \exists g_1, g_2 \in G : f(g_1) = h_1, f(g_2) = h_2 \\ f(g_1) = h_1 \\ f(g_2) = h_2 \\ 0, & \text{otherwise} \end{cases}$$

and the set-valued function $f^{-1}(P) : G \rightarrow F(G)$ is $f^{-1}(P)(g_1)(g_2) = P(f(g_1))(f(g_2))$ for all $g_1, g_2 \in G$ and $h_1, h_2 \in H$.

Theorem 10. Let $f : G \rightarrow H$ be a good homomorphism and $R : G \rightarrow F(G)$, $P : H \rightarrow F(H)$ be set-valued homomorphisms. Then:

- (i) $f(R) : H \rightarrow F(H)$ is a set-valued homomorphism if $f : G \rightarrow H$ is a bijection.
- (ii) $f^{-1}(P) : G \rightarrow F(G)$ is a set-valued homomorphism.

Proof.

- (i) Let $h, h_1, h_2 \in H$. Then $\exists g_1, g_2 \in G : f(g_1) = h_1, f(g_2) = h_2$ and $f(g) = h$. Hence

$$\begin{aligned} (f(R)(h_1) \vee_T f(R)(h_2))(h) &= \bigvee_{h \in x_1 x_2} (f(R)(h_1)(x_1)Tf(R)(h_2)(x_2)) \\ &= \bigvee_{h \in x_1 x_2} R(g_1)(a_1)TR(g_2)(a_2) \leq \bigvee_{g \in a_1 a_2} R(g_1)(a_1)TR(g_2)(a_2) \\ &= (R(g_1) \vee_T R(g_2))(g) \leq \left(\bigvee_{s \in g_1 g_2} R(s) \right)(g) = \bigvee_{s \in g_1 g_2} R(s)(g) \\ &= \bigvee_{f(s) \in h_1 h_2} f(R)(f(s))(h) = \bigvee_{t \in h_1 h_2} f(R)(t)(h), \end{aligned}$$

where $f(a_1) = x_1, f(a_2) = x_2$. Thus $f(R) : H \rightarrow F(H)$ is a set-valued homomorphism.

- (ii) Let $a, b, c \in G$. Then

$$\begin{aligned}
& (f^{-1}(P)(a)Tf^{-1}(P)(b))(c) \\
&= \bigvee_{c \in xy} f^{-1}(P)(a)(x)Tf^{-1}(P)(b)(y) \\
&= \bigvee_{c \in xy} P(f(a))(f(x))TP(f(b))(f(y)) \\
&\leq \bigvee_{f(c) \in f(x)(y)} P(f(a))(f(x))TP(f(b))(f(y)) \\
&\leq P(f(a)) \vee_T P(f(b))(f(c)) \leq \left(\bigvee_{s \in f(a)f(b)} P(s) \right) (f(c)) \\
&= \bigvee_{s \in f(ab)} P(s)(f(c)) = \bigvee_{t \in ab} P(f(t))(f(c)) \\
&= \bigvee_{t \in ab} f^{-1}(P)(t)(c),
\end{aligned}$$

where $\exists t \in ab$ such that $f(t) = s$ since $s \in f(ab)$. Hence $f^{-1}(P) : G \rightarrow F(G)$ is a set-valued homomorphism. $\square$

Theorem 11. Let $P, R : G \rightarrow F(H)$ define $(PTR)(g) := P(g)TR(g)$ for all $g \in G$. Then $PTR \in H_T(G, F(H))$ if $P, R \in H_T(G, F(H))$.

Proof. Let $a, b \in G$ and $h \in H$. Then

$$\begin{aligned}
& ((PTR)(a) \vee_T (PTR)(b))(h) \\
&= \bigvee_{h \in xy} (PTR)(a)(x)T(PTR)(b)(y) \\
&= \bigvee_{h \in xy} (P(a)(x)TR(a)(x))T(P(b)(y)TR(b)(y)) \\
&= \bigvee_{h \in xy} (P(a)(x)TP(b)(y))T(R(a)(x)TR(b)(y)) \\
&= \bigvee_{h \in xy} (P(a)(x)TP(b)(y))T \bigvee_{h \in xy} (R(a)(x)TR(b)(y)) \\
&= (P(a) \vee_T P(b))(h)T(R(a) \vee_T R(b))(h) \\
&\leq \left(\bigvee_{u \in ab} P(u) \right) (h)T \left(\bigvee_{u \in ab} R(u) \right) (h) \\
&= \left(\bigvee_{u \in ab} (PTR)(u) \right) (h)
\end{aligned}$$

We have $(PTR)(a) \vee_T (PTR)(b) \leq \bigvee_{u \in ab} (PTR)(u)$. We also have $((PTR)(a))^{-1} \leq (PTR)(a^{-1})$ since

$$\begin{aligned}
((PTR)(a))^{-1}(h) &= (PTR)(a)(h^{-1}) = P(a)(h^{-1})TR(a)(h^{-1}) \\
&= (P(a))^{-1}(h)T(R(a))^{-1}(h) \leq P(a^{-1})(h)TR(a^{-1})(h) \\
&= (PTR)(a^{-1})(h).
\end{aligned}$$

Therefore $PTR \in H_T(G, F(H))$.

The following example shows that the above theorem may not be true for some implications. $\square$

Example 8. Let H be the canonical hypergroup in Example 2 and $P, R \in H_T(H, F(H))$ be in Example 4. Let J be an R -implicator based on T . Then $RJP \notin H_T(H, F(H))$ since $((RJP)(e) \vee_T (RJP)(x))(e) = 1 \not\leq \alpha = (\bigvee_{u \in ex} (RJP)(u))(e)$.

Theorem 12. Let $\{R_i | i \in I\} \subseteq H_T(G, F(H))$. Then we have

- (i) $\bigwedge_{i \in I} R_i \in H_T(G, F(H))$,
- (ii) $\bigvee_{i \in I} R_i \in H_T(G, F(H))$.

Proof.

(i) Let $a, b \in G$ and $h \in H$. Then we have

$$\left(\bigwedge_{i \in I} R_i\right)(a) \vee_T \left(\bigwedge_{i \in I} R_i\right)(b) \leq \bigvee_{u \in ab} \left(\bigwedge_{i \in I} R_i\right)(u)$$

since

$$\begin{aligned} \left(\left(\bigwedge_{i \in I} R_i\right)(a) \vee_T \left(\bigwedge_{i \in I} R_i\right)(b)\right)(h) &= \bigvee_{h \in xy} \left(\bigwedge_{i \in I} R_i\right)(a)(x) T \left(\bigwedge_{i \in I} R_i\right)(b)(y) \\ &= \bigvee_{h \in xy} \left(\bigwedge_{i \in I} R_i(a)(x)\right) T \left(\bigwedge_{i \in I} R_i(b)(y)\right) \\ &= \bigvee_{h \in xy} \left(\bigwedge_{i \in I} R_i(a)(x) T R_i(b)(y)\right) \\ &= \bigwedge_{i \in I} \left(\bigvee_{h \in xy} R_i(a)(x) T R_i(b)(y)\right) = \bigwedge_{i \in I} (R_i(a) \vee_T R_i(b)(h)) \\ &\leq \bigwedge_{i \in I} \left(\bigvee_{u \in ab} R_i(u)\right)(h) = \left(\bigvee_{u \in ab} \left(\bigwedge_{i \in I} R_i(u)\right)\right)(h) \\ &= \left(\bigvee_{u \in ab} \left(\bigwedge_{i \in I} R_i\right)(u)\right)(h). \end{aligned}$$

Also

$$\left(\left(\bigwedge_{i \in I} R_i\right)(a)\right)^{-1} \leq \left(\bigwedge_{i \in I} R_i\right)(a^{-1})$$

since

$$\begin{aligned} \left(\left(\bigwedge_{i \in I} R_i\right)(a)\right)^{-1}(h) &= \left(\bigwedge_{i \in I} R_i\right)(a)(h^{-1}) = \bigwedge_{i \in I} R_i(a)(h^{-1}) = \bigwedge_{i \in I} (R_i(a))^{-1}(h) \\ &\leq \bigwedge_{i \in I} R_i(a^{-1})(h) = \left(\bigwedge_{i \in I} R_i\right)(a^{-1})(h). \end{aligned}$$

(ii) It is proved similarly to part (i). $\square$

4. Generalized Fuzzy Rough Approximations on Canonical Hypergroups

In this section, some properties provided for groups [43] are tested for hypergroups. Thus, we investigate some features of the generalized fuzzy rough approximation spaces that occurred by a T-fuzzy set-valued homomorphism of canonical hypergroups.

Theorem 13. Let $\mu, \nu \in F(H)$. Then

$$\bar{R}^T(\mu)(a) T \bar{R}^T(\nu)(b) \leq \bigwedge_{u \in a \circ b} \bar{R}^T(\mu \vee_T \nu)(u)$$

for all $a, b \in G$ if $R \in sH_T(G, F(H))$.

Proof. Let $a, b \in G$. Thus

$$\begin{aligned}
 \bar{R}^T(\mu)(a) \bar{R}^T(\nu)(b) &= \left(\bigvee_{k \in H} R(a)(k) T\mu(k) \right) T \left(\bigvee_{t \in H} R(b)(t) T\nu(t) \right) \\
 &= \bigvee_{k, t \in H} ((R(a)(k) TR(b)(t)) T(\mu(k) T\nu(t))) \\
 &\leq \bigvee_{k, t \in H} \left(\left(\bigvee_{p \in k \circ t} R(a)(k) TR(b)(t) \right) T \left(\bigvee_{p \in k \circ t} (\mu \vee_T \nu)(p) \right) \right) \\
 &= \bigvee_{p \in H} ((R(a) \vee_T R(b))(p) T(\mu \vee_T \nu)(p)) \\
 &\leq \bigvee_{p \in H} \left(\left(\bigwedge_{u \in a \circ b} R(u) \right) (p) T(\mu \vee_T \nu)(p) \right) \\
 &\leq \bigwedge_{u \in a \circ b} \left(\bigvee_{p \in H} (R(u)(p) T(\mu \vee_T \nu)(p)) \right) \leq \bigwedge_{u \in a \circ b} \bar{R}^T(\mu \vee_T \nu)(u).
 \end{aligned}$$

Example 9 shows that Theorem 13 may not be true for the J-lower fuzzy rough approximations. $\square$

Example 9. Let G and H be considered as in Example 3, $R \in sH_{T_M}(G, F(H))$ be considered as in Example 4, with the values $\alpha = 0, 2$ and $\beta = 0, 6$, and let fuzzy subhypergroups μ, η and ν of H be taken as follows (Figure 3):

h	x	y	z	t	e
$\mu(h)$	0,1	0,1	0,1	0,1	1
$\nu(h)$	0,4	0,4	0,4	0,4	0,5
$\eta(h)$	0,4	0,4	0,4	0,4	1

Figure 3. Definition table of μ, ν and η .

Then we have

$$\underline{R}_J(\eta)(a) \underline{R}_J(\nu)(e) = 0,5 \not\leq 0,4 = \underline{R}_J(\eta \vee_T \nu)(a) = \bigwedge_{\bar{u} \in \bar{a} \circ \bar{e}} \underline{R}_J(\eta \vee_T \nu)(u)$$

where J is the R -implication of T_M . Moreover, we have

$$\begin{aligned}
 \underline{R}_J(\mu)(e) \underline{R}_J(\nu)(e) &= 0,8 \not\leq 0,5 = \bigvee_{\bar{u} \in \bar{e} \circ \bar{e}} \underline{R}_J(\mu \vee_T \nu)(u) = \underline{R}_J(\mu \vee_T \nu)(e) \\
 &= \bigwedge_{\bar{u} \in \bar{e} \circ \bar{e}} \underline{R}_J(\mu \vee_T \nu)(e)
 \end{aligned}$$

where J is the S_M -implication based on the standard negator.

Corollary 1. $R \in sH_T(G, F(H))$. If μ is a T -fuzzy subsemihypergroup of H , then

$$\bar{R}^T(\mu)(a) \bar{R}^T(\mu)(b) \leq \bigwedge_{t \in a \circ b} \bar{R}^T(\mu)(t)$$

for all $a, b \in G$.

Proof. Let $a, b \in G$. Then, we have

$$\bar{R}^T(\mu)(a) \bar{R}^T(\mu)(b) \leq \bigwedge_{t \in a \circ b} \bar{R}^T(\mu \vee_T \mu)(t)$$

for all $a, b \in G$ by Theorem 13. Therefore, we have

$$\bar{R}^T(\mu)(a)T\bar{R}^T(\mu)(b) \leq \bigwedge_{t \in a \circ b} \bar{R}^T(\mu)(t)$$

since $\mu \vee_T \mu \leq \mu$. $\square$

Theorem 14. Let $R \in sH_T(G, F(H))$. If μ is a T -fuzzy subhypergroup of H , then $\bar{R}^T(\mu)$ is a T -fuzzy subhypergroup of G .

Proof. We have

$$\bar{R}^T(\mu)(a)T\bar{R}^T(\mu)(b) \leq \bigwedge_{t \in a \circ b} \bar{R}^T(\mu)(t)$$

for all $a, b \in G$ by Corollary 34. Let $a \in G$. Then

$$\begin{aligned} \bar{R}^T(\mu)(a) &= \bigvee_{b \in Y} R(a)(b)T\mu(b) = \bigvee_{b \in Y} (R(a))^{-1}(b^{-1})T\mu(b) \\ &\leq \bigvee_{b^{-1} \in Y} R(a^{-1})(b^{-1})T\mu(b^{-1}) = \bar{R}^T(\mu)(a^{-1}). \end{aligned}$$

Hence $\bar{R}^T(\mu)$ is a T -fuzzy subhypergroup of G . $\square$

Theorem 15. Let $\mu, \nu \in F(H)$. Then $\bar{R}^T(\mu) \vee_T \bar{R}^T(\nu) \leq \bar{R}^T(\mu \vee_T \nu)$ if $R \in H_T(G, F(H))$.

Proof. Let $x \in G$. Then $\bar{R}^T(\mu) \vee_T \bar{R}^T(\nu) \leq \bar{R}^T(\mu \vee_T \nu)$ since

$$\begin{aligned} (\bar{R}^T(\mu) \vee_T \bar{R}^T(\nu))(x) &= \bigvee_{x \in ab} (\bar{R}^T(\mu)(a)T\bar{R}^T(\nu)(b)) \\ &= \bigvee_{x \in ab} \left(\left(\bigvee_{k \in H} R(a)(k)T\mu(k) \right) T \left(\bigvee_{t \in H} R(b)(t)T\nu(t) \right) \right) \\ &= \bigvee_{x \in ab} \left(\bigvee_{k, t \in H} ((R(a)(k)TR(b)(t))T(\mu(k)T\nu(t))) \right) \\ &= \bigvee_{x \in ab} \left(\bigvee_{s \in H} ((R(a) \vee_T R(b))(s)T(\mu \vee_T \nu)(s)) \right) \\ &\leq \bigvee_{x \in ab} \left(\bigvee_{s \in H} \left(\left(\bigvee_{u \in ab} R(u) \right) (s)T(\mu \vee_T \nu)(s) \right) \right) \\ &= \bigvee_{x \in ab} \left(\bigvee_{s \in H} (R(x)(s)T(\mu \vee_T \nu)(s)) \right) \\ &= \bigvee_{x \in ab} \bar{R}^T(\mu \vee_T \nu)(x) \bar{R}^T(\mu \vee_T \nu)(x). \end{aligned}$$

Example 10 shows that Theorem 14 and Theorem 15 may be false for the J -lower fuzzy rough approximations. $\square$

Example 10. Let H be considered as in Example 2, $T = T_M$, J be the R -implication of T_M and $R : H \rightarrow F(H)$ be defined as

$$R(a)(b) = \begin{cases} 1, & \text{if } a = e \\ 0, & \text{if } a = x \\ \frac{1}{2}, & \text{if } a = y \\ 0, & \text{if } a = z \\ 0, & \text{if } a = t \end{cases}$$

for all $a, b \in H$. Then $R \in H_{T_M}(H, F(H))$. Let $\mu, \nu \in F(H)$ be considered as in Example 9. Hence $\underline{R}_J(\mu) \vee_T \underline{R}_J(\nu) \not\leq \underline{R}_J(\mu \vee_T \nu)$ since $(\underline{R}_J(\mu) \vee_T \underline{R}_J(\nu))(e) = 1 \not\leq 0.4 = \underline{R}_J(\mu \vee_T \nu)(e)$. However, even though μ is a T_M -fuzzy subhypergroup of H , $\underline{R}_J(\mu)$ is not a T_M -fuzzy subhypergroup of H since $\underline{R}_J(\mu)(x)T\underline{R}_J(\mu)(x) = 1 \not\leq 0, 1 = \bigwedge_{u \in xx} \underline{R}_J(\mu)(x)$.

Theorem 16. Let G, H, K be canonical hypergroups, J be an implication and let $R \in H_T(G, F(H))$ and $S \in H_T(H, F(K))$ for a t -norm T . Then

- (i) $\overline{R}^T(R(x))(a) \leq \bigvee_{u \in ax^{-1}} \text{Ker}(u)$ for all $x, a \in G$,
- (ii) $\overline{R}^T(\text{Ker}S) = \text{Ker}(S \circ_T R)$,
- (iii) $\underline{R}_J(\text{Ker}S) = \text{Ker}(S \circ_J R)$,
- (iv) $\overline{S^{-1}}^T(\text{Ker}R^{-1}) = \text{Ker}(S \circ_T R)^{-1}$,
- (v) $\underline{S^{-1}}_J(\text{Ker}R^{-1}) = \text{Ker}(R^{-1} \circ_J S^{-1})$,
- (vi) $\overline{R}^T(\text{Im}S^{-1}) = \text{Im}(S \circ_T R)^{-1}$,
- (vii) $\overline{S^{-1}}^T(\text{Im}R) = \text{Im}(S \circ_T R)$,
- (viii) $\text{Im}(S \circ_J R)^{-1} \leq \underline{R}_J(\text{Im}S^{-1})$,
- (ix) $\text{Im}(R^{-1} \circ_J S^{-1})^{-1} \leq \underline{S^{-1}}_J(\text{Im}R)$.

Proof. Straightforward. $\square$

Theorem 17. Let a bijection $f : G \rightarrow H$ be a good homomorphism and $R : G \rightarrow F(G)$ be a set-valued homomorphism. Then

$$\overline{f(R)}^T(f(\mu)) = f(\overline{R}^T(\mu)) \text{ and } \underline{f(R)}_J(f(\mu)) = f(\underline{R}_J(\mu))$$

Proof. Let $h \in H$. Then $\exists g \in G : f(g) = h$. Hence

$$\begin{aligned} \overline{f(R)}^T(f(\mu))(h) &= \bigvee_{y \in H} f(R)(h)(y) T f(\mu)(y) = \bigvee_{x \in H} R(g)(x) T \mu(x) = \overline{R}^T(\mu)(g) \\ &= f(\overline{R}^T(\mu))(h) \end{aligned}$$

and

$$\begin{aligned} \underline{f(R)}_J(f(\mu))(h) &= \bigwedge_{y \in H} f(R)(h)(y) I f(\mu)(y) = \bigwedge_{x \in H} R(g)(x) I \mu(x) = \underline{R}_J(\mu)(g) \\ &= f(\underline{R}_J(\mu))(h), \end{aligned}$$

where $f(x) = y$. $\square$

Theorem 18. Let $f : G \rightarrow H$ be a mapping, $\mu \in F(G)$ and $P : H \rightarrow F(H)$ be a set-valued homomorphism. Then:

- (i) $f^{-1}(\underline{P}_J(\mu)) \leq \underline{f^{-1}(P)}_J(f^{-1}(\mu))$.
- (ii) $\overline{f^{-1}(P)}^T(f^{-1}(\mu)) \leq f^{-1}(\overline{P}^T(\mu))$.
- (iii) $f^{-1}(\underline{P}_J(\mu)) = \underline{f^{-1}(P)}_J(f^{-1}(\mu))$ and $\overline{f^{-1}(P)}^T(f^{-1}(\mu)) = f^{-1}(\overline{P}^T(\mu))$ if $f : G \rightarrow H$ is surjective.

Proof.

- (i) Let $a \in G$. Then

$$\begin{aligned} f^{-1}(\underline{P}_J(\mu))(a) &= \underline{P}_J(\mu)(f(a)) = \bigwedge_{x \in H} P(f(a))(x) T \mu(x) \leq \bigwedge_{b \in G} P(f(a))(f(b)) T \mu(f(b)) \\ &= \bigwedge_{b \in G} f^{-1}(P)(a)(b) T f^{-1}(\mu)(b) = \underline{f^{-1}(P)}_J(f^{-1}(\mu))(a) \end{aligned}$$

- (ii) Let $a \in G$. Then

$$\begin{aligned}\overline{f^{-1}(P)}^T(f^{-1}(\mu))(a) &= \bigvee_{b \in G} f^{-1}(P)(a)(b)Tf^{-1}(\mu)(b) = \bigvee_{b \in G} P(f(a))(f(b))T\mu(f(b)) \\ &\leq \bigvee_{x \in H} P(f(a))(x)T\mu(x) = \overline{P}^T(\mu)(f(a)) = f^{-1}(\overline{P}^T(\mu))(a)\end{aligned}$$

(iii) It follows immediately from (i) and (ii). $\square$

Theorem 19. Let $P, R : G \rightarrow F(H)$. Then $\overline{PTR}^T(\mu) \leq \overline{P}^T(\mu)T\overline{R}^T(\mu)$ and $\underline{P}_J(\mu)T\underline{R}_J(\mu) \leq \underline{PTR}_J(\mu)$ if $\text{Im}\mu \subseteq D_T$.

Proof. Straightforward. $\square$

Theorem 20. Let $\{R_i | i \in I\} \subseteq H_T(G, F(H))$. Then $\overline{\bigwedge_{i \in I} R_i}^T(\mu) \leq \bigwedge_{i \in I} \overline{R_i}^T(\mu)$ and $\underline{\bigvee_{i \in I} R_i}_J(\mu) \leq \bigvee_{i \in I} \underline{R_i}_J(\mu)$.

Proof. Straightforward. $\square$

Theorem 21. Let $R : G \rightarrow F(H)$ and $P : H \rightarrow F(Z)$. Then

- (i) $\overline{P \circ_T R}^T(\mu) = \overline{R}^T(\overline{P}^T(\mu))$,
- (ii) $\underline{P \circ_T R}_J(\mu) = \underline{R}_J(\underline{P}_J(\mu))$ if J is an R -implication based on the t -norm T .

Proof. Let $a \in G$. Then we have

(i)

$$\begin{aligned}\overline{P \circ_T R}^T(\mu)(a) &= \bigvee_{z \in Z} (P \circ_T R)(a)(z)T\mu(z) = \bigvee_{z \in Z} \left(\bigvee_{y \in H} R(a)(y)TP(y)(z) \right) T\mu(z) \\ &= \bigvee_{y \in H} R(a)(y)T \left(\bigvee_{z \in Z} P(y)(z)T\mu(z) \right) = \bigvee_{y \in H} R(a)(y)T(\overline{P}^T(\mu)(y)) \\ &= \overline{R}^T(\overline{P}^T(\mu))(a)\end{aligned}$$

(ii)

$$\begin{aligned}\underline{P \circ_T R}_J(\mu)(a) &= \bigwedge_{z \in Z} (P \circ_T R)(a)(z)J\mu(z) = \bigwedge_{z \in Z} \left(\bigvee_{y \in H} R(a)(y)TP(y)(z) \right) J\mu(z) \\ &= \bigwedge_{z \in Z} \bigwedge_{y \in H} (R(a)(y)TP(y)(z))J\mu(z) = \bigwedge_{z \in Z} \bigwedge_{y \in H} R(a)(y)J(P(y)(z)J\mu(z)) \\ &= \bigwedge_{y \in H} R(a)(y)T \left(\bigwedge_{z \in Z} P(y)(z)J\mu(z) \right) = \bigwedge_{y \in H} R(a)(y)J(\underline{P}_J(\mu)(y)) \\ &= \underline{R}_J(\underline{P}_J(\mu))(a).\end{aligned}$$

$\square$

5. Conclusions

In this paper, we introduce fuzzy set-valued homomorphisms as an extension of the fuzzy congruences on hypergroups. This extension enables the broadening of Ameri's work [23] to two different universal sets. In this way, we have an opportunity to investigate for hypergroups some results of Ekiz et al. [43]. This expansion of the fuzzy set-valued homomorphisms allows for a broader examination of algebraic structures and offers new research opportunities.

Using the findings from this paper, the Figure 4 can be constructed:

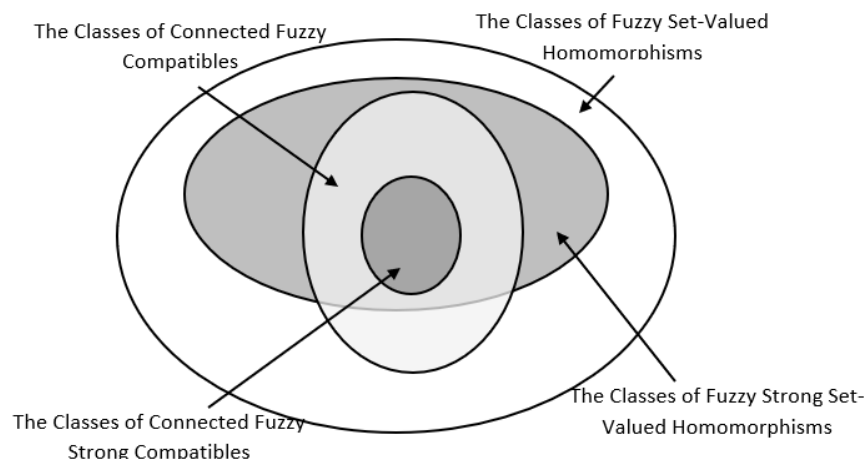


Figure 4. Comparison between the class of connected fuzzy and fuzzy strong compatibles and the class of fuzzy and fuzzy strong set-valued homomorphisms.

The main contributions and future work in this paper are summarized as follows:

1. A T-fuzzy set-valued homomorphism of hypergroups is introduced. T-fuzzy set-valued homomorphisms on groups are extended to hypergroups [43]. It has been shown that a set-valued function connected to a fuzzy compatible is a fuzzy set-valued homomorphism. With this definition, some results of Ameri's study are extended to two different hypergroups [23].
2. The relations between T-fuzzy set-valued homomorphisms and certain concepts such as the inclusion homomorphism or the set-valued homomorphisms were investigated. In this way, some important T-fuzzy set-valued homomorphisms were obtained.
3. We investigated the properties in the two-universe (I,T)-fuzzy rough set model by taking hypergroups as universal sets and T-fuzzy set-valued homomorphisms instead of a fuzzy relation. The definition of a T-fuzzy set-valued homomorphism is very important for the analysis of a hyperalgebraic structure in the two-universe (I,T)-fuzzy rough set model. In this paper, we give a quite harmonious definition with the fuzzy congruences and the crisp version of the T-fuzzy set-valued homomorphism.
4. We study some properties of lower and upper approximations of a T-fuzzy subsemi-hypergroup.
5. Following this study, it becomes possible to define a T-fuzzy set-valued homomorphism on various algebraic structures such as hyperrings (hypernearings), hypermodules, polygroups, etc. and explore their properties. Moreover, these studies can expand to a lattice L by defining a TL-fuzzy set-valued homomorphism. With the T-fuzzy set-valued homomorphisms, we try to examine the two-universe (I,T)-fuzzy rough set model in semihypergroups.

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