

Article

On Convoluted Forms of Multivariate Legendre-Hermite Polynomials with Algebraic Matrix Based Approach

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Abstract: The main purpose of this article is to construct a new class of multivariate Legendre-Hermite-Apostol type Frobenius-Euler polynomials. A number of significant analytical characterizations of these polynomials using various generating function techniques are provided in a methodical manner. These enactments involve explicit relations comprising Hurwitz-Lerch zeta functions and λ -Stirling numbers of the second kind, recurrence relations, and summation formulae. The symmetry identities for these polynomials are established by connecting generalized integer power sums, double power sums and Hurwitz-Lerch zeta functions. In the end, these polynomials are also characterized via an algebraic matrix based approach.

Keywords: Legendre polynomials; Hermite polynomials; Legendre-Hermite polynomials; Apostol type Legendre-Hermite Frobenius-Euler polynomials

MSC: 33E20; 33C45; 33B10; 33E30; 11T23



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1. Introduction and Preliminaries

The Hermite and Legendre polynomials are pivotal in classical boundary value problems, particularly within parabolic regions and coordinate systems. These problems arise in diverse fields ranging from heat conduction to fluid dynamics, where understanding the behavior of functions within specific geometrical boundaries is essential and also offer elegant solutions to such problems, aiding in formulating and analyzing mathematical models in various physical and engineering disciplines [1–7].

A lot of research has been done on Frobenius-Euler polynomials, also known as Eulerian polynomials, and their generalizations, called as the Apostol type Frobenius-Euler polynomials, see [8–17]. These inquiries have covered a broad range of subjects, including identities, recurrence relations, differential equations, integral representations, explicit formulas, extensions in single and double variables, and fractional operational approach, among others. A modern and useful area of research is the study of the fusion of various polynomial types to produce creative multivariate generalized polynomials. The noteworthy characteristics of these polynomials which include recurrence and explicit relationships, functional and differential equations, summation formulas, symmetric and convolution properties, and determinant representations make this field of study especially pertinent. The significance of these characteristics extends across various academic domains, making multivariate convoluted special polynomials a compelling subject of exploration.

The generalized Apostol type Frobenius-Euler polynomials $\mathcal{F}_n^{(\alpha)}(\phi; u; \lambda)$ are generated by the function [13] is given by

$$\left(\frac{1-u}{\lambda e^t - u}\right)^\alpha e^{\phi t} = \sum_{n=0}^{\infty} \mathcal{F}_n^{(\alpha)}(\phi; u; \lambda) \frac{t^n}{n!} \quad (u, \alpha \in \mathbb{C}, u \neq 1). \quad (1)$$

When $\phi = 0$,

$$\mathcal{F}_n^{(\alpha)}(u; \lambda) := \mathcal{F}_n^{(\alpha)}(0; u; \lambda) \quad (2)$$

are called the generalized Apostol type Frobenius-Euler numbers.

The explicit representation for the generalized Apostol type Frobenius-Euler polynomials $\mathcal{F}_n^{(\alpha)}(\phi; u; \lambda)$ [13] is given by

$$\mathcal{F}_n^{(\alpha)}(\phi; u; \lambda) = \sum_{k=0}^n \binom{n}{k} \mathcal{F}_k^{(\alpha)}(u; \lambda) \phi^{n-k}, \quad (3)$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the binomial coefficient.

The generalized Apostol-Euler polynomials $\mathfrak{E}_n^{(\alpha)}(\phi; \lambda)$ [18] are given by

$$\mathfrak{E}_n^{(\alpha)}(\phi; \lambda) := \mathcal{F}_n^{(\alpha)}(\phi; -1; \lambda). \quad (4)$$

In the special case when $\lambda = 1$, the generalized Apostol type Frobenius-Euler polynomials reduced to the generalized Frobenius-Euler polynomials [19,20] given by

$$F_n^{(\alpha)}(\phi; u) := \mathcal{F}_n^{(\alpha)}(\phi; u; 1) \quad (5)$$

and generalized Apostol-Euler polynomials reduce to generalized Euler polynomials given by

$$E_n^{(\alpha)}(\phi) := \mathfrak{E}_n^{(\alpha)}(\phi; 1). \quad (6)$$

For $\alpha = 1$, the polynomials $F_n^{(\alpha)}(\phi; u)$ and $E_n^{(\alpha)}(\phi)$ reduce to the Eulerian polynomials $F_n(\phi; u)$ [19,20] and classical Euler polynomials $E_n(x)$ [21].

In addition to introducing new families of special polynomials, several helpful identities can be derived by utilizing the majority of special functions of mathematical physics and their generalizations as proposed by physical problems. Several other fields, including quantum mechanics and statistics, have also made use of these special polynomials for the description and analysis of complex systems. Polynomial sequences are essential to many areas of mathematics, including combinatorics, entropy, and algebraic combinatorics. The Legendre, Hermite, Chebyshev, Laguerre, and Jacobi polynomials are a few instances of polynomial sequences that are solutions to particular ordinary differential equations. The Legendre polynomials defined on the interval $[0, +\infty)$ with regard to the weight function e^{-u} are the significant class of orthogonal polynomials with numerous applications in mathematics and physics [22]. One key feature of the Legendre polynomials is their ability to satisfy a recurrence relation. This property makes it possible to calculate higher-degree polynomials from lower-degree ones, which facilitates effective polynomial generation and numerical computations. In addition, the generating function of Legendre polynomials aids in solving differential equations and closed-form formulae [23]. Among the applications of the Legendre polynomials are the solutions of the Schrodinger equation for the hydrogen atom and other quantum systems with spherical symmetry in mathematics, physics, and engineering. These polynomials also arise from problems related to diffusion equations, wave propagation, and heat conduction, see also [6,7].

The 2-variable Legendre polynomials (2VLeP) $S_n(\phi, \xi)$ are generated by the function (see [24])

$$e^{\xi t} J_0(2t\sqrt{-\phi}) = \sum_{n=0}^{\infty} S_n(\phi, \xi) \frac{t^n}{n!}, \quad (7)$$

where $J_\eta(\phi t)$ is the Bessel function of first kind of order η defined by (see [22])

$$J_\eta(\phi) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(\eta + k + 1)} \left(\frac{\phi}{2}\right)^{\eta+2k} \quad (\phi \in \mathbb{C} \setminus (-\infty, 0]; \eta \in \mathbb{C}). \quad (8)$$

The 2VLeP $S_n(\phi, \xi)$ are represented by the following series expansion:

$$S_n(\phi, \xi) = n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\phi^k \xi^{n-2k}}{k! \Gamma(k+1) (n-2k)!}, \quad (9)$$

where $[n]$ is the greatest integer function.

Additionally, we observe that

$$\exp(-tD_\phi^{-1}) = J_0(2\sqrt{t\phi}); \quad D_\phi^{-n}\{1\} := \frac{\phi^n}{n!} \quad (10)$$

is the inverse derivative operator.

In view of relation $P_n(\phi) = S_n\left(-\frac{1-\phi^2}{4}, \phi\right)$, we get the classical Legendre polynomials $P_n(\phi)$ (see [24]).

With roots in classical mathematics, the Hermite polynomials are regarded as one of the most important and ancient special functions in mathematics. Because of their many uses and versatility, these polynomials have proven to be highly useful in a wide range of fields. Hermite polynomials are used extensively in quantum mechanics, primarily as solutions to differential equations controlling harmonic oscillators. These equations, like the Schrödinger equation, are essential to comprehending how particles behave in quantum systems. Hermite polynomials are essential tools in theoretical physics because they solve these kinds of equations, which makes it easier to analyze and forecast quantum mechanical phenomena, see [1,2,4,5,25].

Apart from their importance in quantum mechanics, Hermite polynomials are essential in classical boundary-value problems, especially in coordinate systems and parabolic regions. These issues come up in a variety of disciplines, such as fluid dynamics and heat conduction, where it's crucial to comprehend how functions behave within particular geometrical bounds. Hermite polynomials provide sophisticated solutions to these kinds of issues, supporting the development and examination of mathematical models in a range of scientific and technical fields. Furthermore, Hermite polynomials are used in wavelet transform analysis as basis functions in signal processing. Wavelets are mathematical functions that are applied to various scales of signal analysis and information extraction, see [3,26–28]. Researchers can effectively analyze and process complex data by representing signals in terms of Hermite wavelets, which makes Hermite polynomials indispensable in domains like data compression, communication systems, and image processing. Hermite polynomials also appear in contexts like the Edgeworth series and Brownian motion, which have broad implications in probabilistic studies. Understanding random phenomena and modeling uncertainty in a variety of scientific and financial applications require an understanding of these stochastic processes. Researchers can create intricate probabilistic models and precisely analyze complex systems by utilizing Hermite polynomials. Hermite polynomials are also very important in combinatorics, where they are used to examine discrete structures and solve counting problems. Because of their combinatorial qualities, researchers can study combinatorial structures and algorithms more easily by counting arrangements, permutations, and combinations in various contexts, see also [28–30].

The 2-variable Hermite Kampé de Fériet polynomials (2VHKdFP) $H_n(\xi, \omega)$ are generated by the function [31]

$$e^{\xi t + \omega t^2} = \sum_{n=0}^{\infty} H_n(\xi, \omega) \frac{t^n}{n!}. \quad (11)$$

The 2VHKdFP $H_n(\xi, \omega)$ are also defined by the following series expansion:

$$H_n(\xi, \omega) = n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\omega^k \xi^{n-2k}}{k! (n-2k)!}. \quad (12)$$

In view of relation $H_n(\xi) = H_n(\xi, -1/2)$, we get the classical Hermite polynomials $H_n(\xi)$ (see [22]).

Also, in view of the Equations (7), (10) and (11), we find that

$$S_n(\phi, \xi) = H_n(\xi, D_\phi^{-1}), \quad (13)$$

where $H_n(\xi, D_\phi^{-1})$ are the 2-variable Hermite-Kampé de Fériet polynomials.

The ability of generalized convoluted polynomials to establish explicit relationships and recurrence is one of their key characteristics. This indicates that specific behaviors or patterns repeat, giving researchers a valuable tool for understanding and predicting mathematical phenomena. Furthermore, these polynomials' capacity to be used to create differential and functional equations increases their usefulness in the solution of challenging mathematical issues. This feature is particularly helpful in applications that require the modeling and analysis of dynamic relationships or rates of change. Another important feature of summation formulas is their ability to concisely represent series or sequences, which makes complicated mathematical expressions easier to understand. The symmetric and convolutional characteristics of these polynomials increase their adaptability and allow researchers to investigate a wide range of mathematical operations and manipulations.

Motivated by the above facts, the article is organized as follows. In Section 2, the convolution of multivariate Legendre-Hermite polynomials is taken with generalized Apostol type Frobenius-Euler polynomials to construct a new class, namely the multivariate generalized Apostol type Legendre-Hermite-Frobenius-Euler polynomials (MVGATLeHFEP) and its operational view is presented. Sections 3 and 4 are devoted for providing several recurrence relations, connection and summation formulae and symmetric identities for these polynomials. In Section 5, some algebraic notions are imposed to provide a nice standard to these polynomials.

2. Convoluted Multivariate Legendre-Hermite Polynomials

The Hermite polynomials have far-reaching implications in probabilistic studies, appearing in contexts such as the Edgeworth series and Brownian motion. These stochastic processes are essential in understanding random phenomena and modeling uncertainty in various scientific and financial applications. By leveraging Hermite polynomials, researchers can develop sophisticated probabilistic models and analyze complex systems with precision. Additionally, Hermite polynomials play a crucial role in combinatorics, where they are employed to solve counting problems and analyze discrete structures. Their combinatorial properties enable researchers to enumerate arrangements, permutations, and combinations in diverse settings, facilitating the study of combinatorial structures and algorithms.

To construct the convoluted form, we expand the exponential function and replace the powers ξ^n ($n = 0, 1, 2, \dots$) in equation (11) by the polynomials $S_n(\phi, \xi)$ ($n = 0, 1, 2, \dots$) and after using Equation (7), we find the multivariate Legendre-Hermite polynomials (MVLeHP) ${}_S H_n(\phi, \xi, \omega)$.

The generating function for the MVLeHP ${}_sH_n(\phi, \xi, \omega)$ is given by

$$e^{\xi t + \omega t^2} J_0(2t\sqrt{-\phi}) = \sum_{n=0}^{\infty} {}_sH_n(\phi, \xi, \omega) \frac{t^n}{n!}. \quad (14)$$

Using Equations (11) and (8) with Equation (7) and expansion of $e^{\omega t^2}$ in Equation (14), we find that the explicit series expansions for the MVLeHP ${}_sH_n(\phi, \xi, \omega)$ are given by

$${}_sH_n(\phi, \xi, \omega) = n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{H_{n-2k}(\xi, \omega) \phi^k}{k! \Gamma(k+1) (n-2k)!}, \quad (15)$$

$${}_sH_n(\phi, \xi, \omega) = n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{S_{n-2k}(\phi, \xi) \omega^k}{k! (n-2k)!}, \quad (16)$$

which in view of Equations (9) and (12) becomes:

$${}_sH_n(\phi, \xi, \omega) = n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{l=0}^{\lfloor \frac{n-2k}{2} \rfloor} \frac{\omega^l \xi^{n-2k-2l} \phi^k}{l! (n-2k-2l)! k! \Gamma(k+1)}. \quad (17)$$

Now, fix the parameter ω and choose a concrete value $\omega = 6$ and present the shapes by drawing the surface plots of the MVLeHP ${}_sH_n(\phi, \xi, 6)$ for even and odd index n of polynomials (Figures 1 and 2).

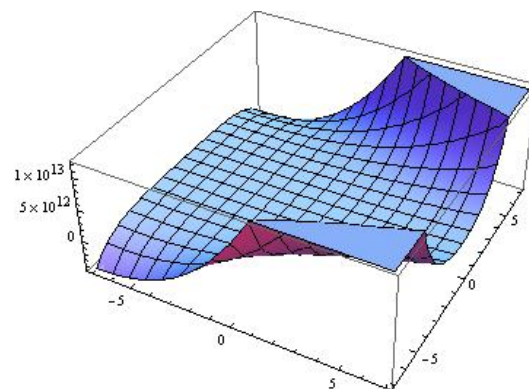


Figure 1. Plot of ${}_sH_{12}(\phi, \xi, 6)$.

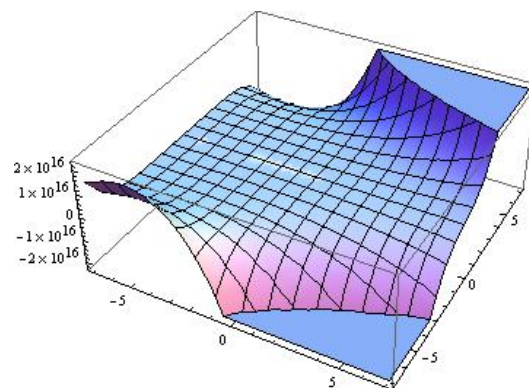


Figure 2. Plot of ${}_sH_{15}(\phi, \xi, 6)$.

Remark 1. Considering (10) and (14) with relation (13), we find that

$${}_sH_n(\phi, \xi, \omega) = {}_H H_n(\xi, D_\phi^{-1}, \omega), \quad (18)$$

where ${}_H H_n(\xi, D_\phi^{-1}, \omega)$ denotes the multivariate double Hermite polynomials. The generating function and the series expansion for the multivariate double Hermite polynomials are given by

$$e^{\xi t} e^{(D_\phi^{-1} + \omega)t^2} = \sum_{n=0}^{\infty} {}_H H_n(\xi, D_\phi^{-1}, \omega) \frac{t^n}{n!}, \quad (19)$$

$${}_H H_n(\xi, D_\phi^{-1}, \omega) = n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{l=0}^{\lfloor \frac{n-2k}{2} \rfloor} \frac{\omega^l D_\phi^{-(n-2k-2l)} \xi^k}{l! (n-2k-2l)! k! \Gamma(k+1)}, \quad (20)$$

$$\equiv n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{l=0}^{\lfloor \frac{n-2k}{2} \rfloor} \frac{\omega^l \phi^{(n-2k-2l)} \xi^k}{l! ((n-2k-2l)!)^2 k! \Gamma(k+1)}. \quad (21)$$

For some concrete value $\omega = 6$, the shapes of the multivariate double Hermite polynomials ${}_H H_n(\xi, D_\phi^{-1}, 6)$ for even and odd index n of polynomials are as follows (Figures 3 and 4):

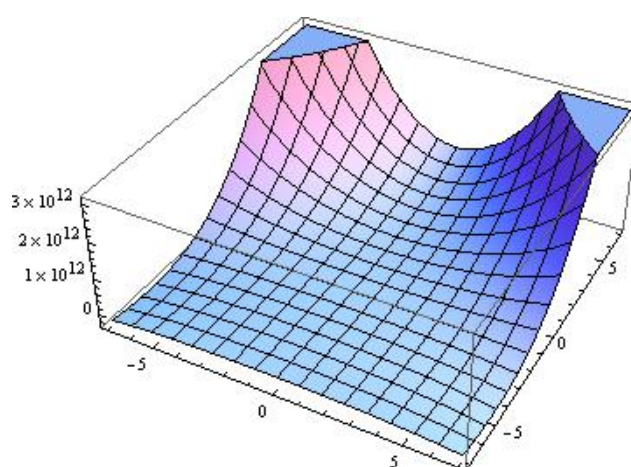


Figure 3. Plot of ${}_H H_{12}(\xi, D_\phi^{-1}, 6)$.

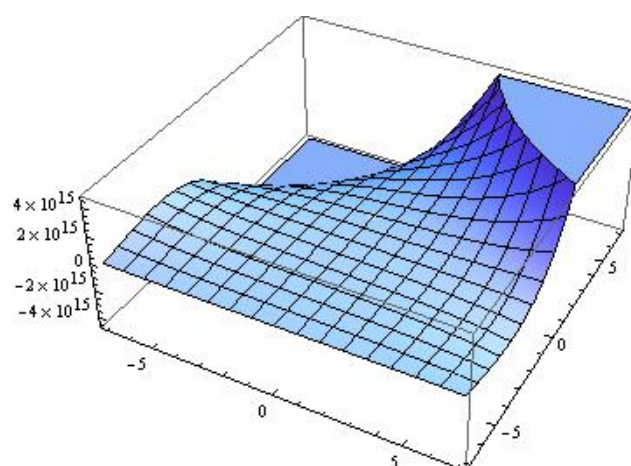


Figure 4. Plot of ${}_H H_{15}(\xi, D_\phi^{-1}, 6)$.

Further, by convoluting the MVLeHP ${}_S H_n(\phi, \xi, \omega)$ with the ATFEP $\mathcal{F}_n^{(\alpha)}(\phi; u; \lambda)$ using replacement method, we generate the multivariate generalized Apostol type Legendre-Hermite Frobenius-Euler polynomials (MVGATLeHFEP) denoted by ${}_S H \mathcal{F}_n^{(\alpha)}(\phi, \xi, \omega; u; \lambda)$.

The MVGATLeHFEP ${}_S H \mathcal{F}_n^{(\alpha)}(\phi, \xi, \omega; u; \lambda)$ are generated by the function

$$\left(\frac{1-u}{\lambda e^t - u} \right)^\alpha e^{\xi t + \omega t^2} J_0(2t\sqrt{-\phi}) = \sum_{n=0}^{\infty} {}_S H \mathcal{F}_n^{(\alpha)}(\phi, \xi, \omega; u; \lambda) \frac{t^n}{n!}, \quad (u, \lambda, \alpha \in \mathbb{C}, u \neq 1). \quad (22)$$

Through utilizing Equations (1) in conjunction with (2) and (17) within Equation (22), we find that the series expansion for the MVGATLeHFEP is given by

$${}_sH\mathcal{F}_n^{(\alpha)}(\phi, \zeta, \omega; u; \lambda) = n! \sum_{m=0}^n \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{l=0}^{\lfloor \frac{m-2k}{2} \rfloor} \frac{\mathcal{F}_{n-m}^{(\alpha)}(u; \lambda) \omega^l \zeta^{m-2k-2l} \phi^k}{(n-m)! l! (m-2k-2l)! k! \Gamma(k+1)}. \quad (23)$$

The multivariate generalized Legendre-Hermite-Apostol-Euler polynomials (MVGLLeHAEP) are given by

$${}_sH\mathfrak{E}_n^{(\alpha)}(\phi, \zeta, \omega; \lambda) := {}_sH\mathcal{F}_n^{(\alpha)}(\phi, \zeta, \omega; -1; \lambda). \quad (24)$$

In the special case when $\lambda = 1$, the MVGLLeHATFEP reduced to the multivariate generalized Legendre-Hermite-Frobenius-Euler polynomials (3VGLLeHFEP) given by

$${}_sHF_n^{(\alpha)}(\phi, \zeta, \omega; u) := {}_sH\mathcal{F}_n^{(\alpha)}(\phi, \zeta, \omega; u; 1) \quad (25)$$

and MVGLLeHAEP reduce to the multivariate generalized Legendre-Hermite-Euler polynomials (MVGLLeHEP) given by

$${}_sHE_n^{(\alpha)}(\phi, \zeta, \omega) := {}_sH\mathfrak{E}_n^{(\alpha)}(\phi, \zeta, \omega; 1). \quad (26)$$

For $\alpha = 1$, the MVGLLeHFEP reduce to the multivariate Legendre-Hermite-Frobenius-Euler polynomials ${}_sHF_n(\phi, \zeta, \omega; u)$ and MVGLLeHEP multivariate Legendre-Hermite-Euler polynomials ${}_sHE_n(\phi, \zeta, \omega)$. We reflect the shapes of the multivariate Legendre-Hermite-Euler polynomials ${}_sHE_n(\phi, \zeta, \omega)$ by utilizing its series expansion given as follows:

$${}_sHE_n(\phi, \zeta, \omega) = n! \sum_{m=0}^n \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{l=0}^{\lfloor \frac{m-2k}{2} \rfloor} \frac{E_{n-m} \omega^l \zeta^{m-2k-2l} \phi^k}{(n-m)! l! (m-2k-2l)! k! \Gamma(k+1)}, \quad (27)$$

where E_n are the Euler numbers. For a concrete value $\omega = 6$, the shapes of the multivariate Legendre-Hermite-Euler polynomials ${}_sHE_n(\phi, \zeta, 6)$ for even and odd index n of polynomials are as follows (Figures 5 and 6):

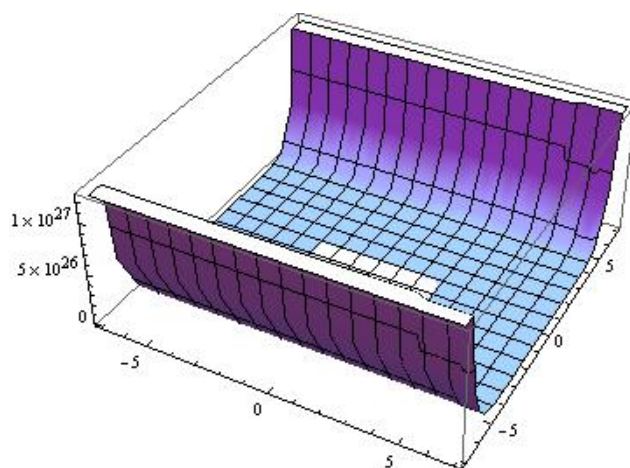


Figure 5. Plot of ${}_sHE_{12}(\phi, \zeta, 6)$.

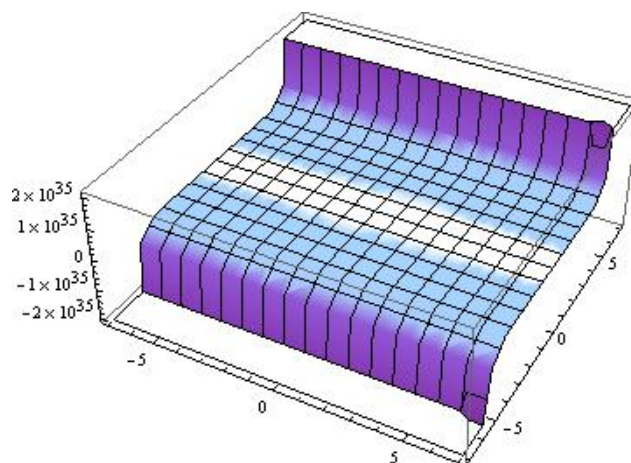


Figure 6. Plot of ${}_sH E_{15}(\phi, \zeta, 6)$.

Remark 2. In view of Equations (10) and (22) with relation (13), we find that

$${}_sH \mathcal{F}_n^{(\alpha)}(\phi, \zeta, \omega; u; \lambda) = {}_H H \mathcal{F}_n^{(\alpha)}(\zeta, D_\phi^{-1}, \omega; u; \lambda), \quad (28)$$

where ${}_H H \mathcal{F}_n^{(\alpha)}(\zeta, D_\phi^{-1}, \omega; u; \lambda)$ denotes the multivariate generalized Apostol type double Hermite Frobenius-Euler polynomials (MVGATDHFEP). The generating function and the series expansion for these polynomials are given by

$$\left(\frac{1-u}{\lambda e^t - u} \right)^\alpha e^{\zeta t} e^{(D_\phi^{-1} + \omega)t^2} = \sum_{n=0}^{\infty} {}_H H \mathcal{F}_n^{(\alpha)}(\zeta, D_\phi^{-1}, \omega; u; \lambda) \frac{t^n}{n!}, \quad (29)$$

$${}_H H \mathcal{F}_n^{(\alpha)}(\zeta, D_\phi^{-1}, \omega; u; \lambda) = n! \sum_{m=0}^n \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{l=0}^{\lfloor \frac{m-2k}{2} \rfloor} \frac{\mathcal{F}_{n-m}^{(\alpha)}(u; \lambda) \omega^l D_\phi^{-(m-2k-2l)} \zeta^k}{(n-m)! l! (m-2k-2l)! k! \Gamma(k+1)}. \quad (30)$$

The multivariate generalized double Hermite-Apostol-Euler polynomials (MVG DHAEP) are given by

$${}_H H \mathfrak{E}_n^{(\alpha)}(\zeta, D_\phi^{-1}, \omega; \lambda) := {}_H H \mathcal{F}_n^{(\alpha)}(\zeta, D_\phi^{-1}, \omega; -1; \lambda). \quad (31)$$

In the special case when $\lambda = 1$, the MVGATDHFEP reduced to the multivariate generalized double Hermite-Frobenius-Euler polynomials (MVG MHFEP) given by

$${}_H H F_n^{(\alpha)}(\zeta, D_\phi^{-1}, \omega; u) := {}_H H \mathcal{F}_n^{(\alpha)}(\zeta, D_\phi^{-1}, \omega; u; 1) \quad (32)$$

and MVGDHAEP reduce to the multivariate generalized double Hermite-Euler polynomials (MVG DHEP) given by

$${}_H H E_n^{(\alpha)}(\zeta, D_\phi^{-1}, \omega) := {}_H H \mathfrak{E}_n^{(\alpha)}(\zeta, D_\phi^{-1}, \omega; 1). \quad (33)$$

For $\alpha = 1$, the MVGDHFEP reduce to the multivariate double Hermite-Frobenius-Euler polynomials ${}_H H F_n(\zeta, D_\phi^{-1}, \omega; u)$ and MVGDHEP multivariate double Hermite-Euler polynomials ${}_H H E_n(\zeta, D_\phi^{-1}, \omega)$. We reflect the shapes of the multivariate double Hermite-Euler polynomials ${}_H H E_n(\zeta, D_\phi^{-1}, \omega)$ by utilizing its series expansion given as follows:

$${}_H H E_n(\zeta, D_\phi^{-1}, \omega) = n! \sum_{m=0}^n \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{l=0}^{\lfloor \frac{m-2k}{2} \rfloor} \frac{E_{n-m} \omega^l \phi^{(m-2k-2l)} \zeta^k}{(n-m)! l! ((m-2k-2l)!)^2 k! \Gamma(k+1)}. \quad (34)$$

For a concrete value $\omega = 6$, the shapes of the multivariate double Hermite-Euler polynomials ${}_{HH}E_n(\xi, D_\phi^{-1}, 6)$ for even and odd index n of polynomials are as follows (Figures 7 and 8):

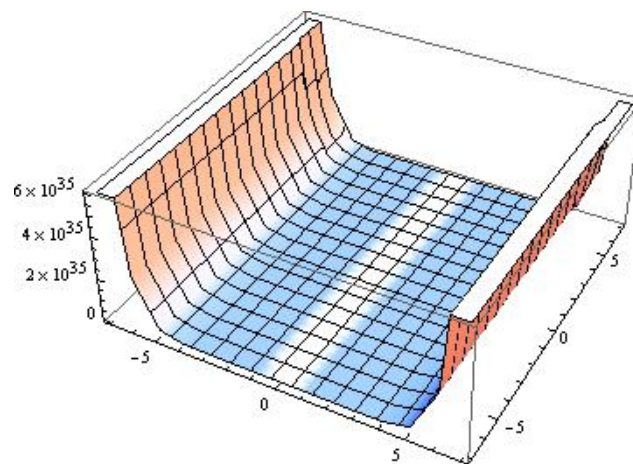


Figure 7. Plot of ${}_{HH}E_{12}(\xi, D_\phi^{-1}, 6)$.

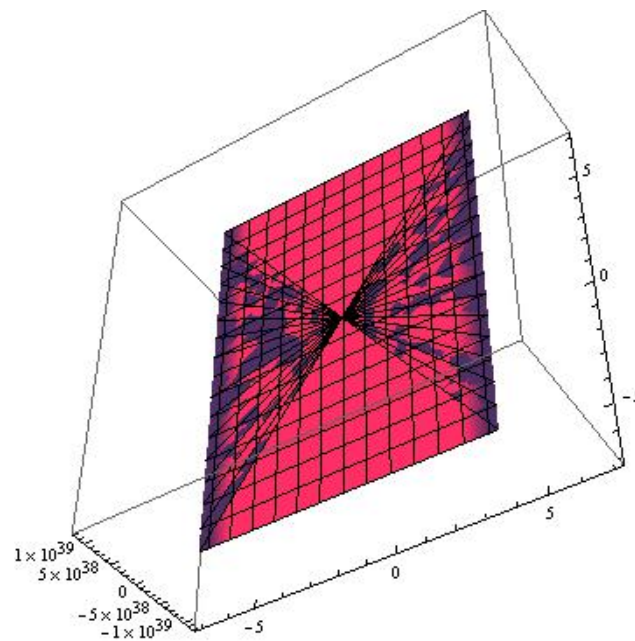


Figure 8. Plot of ${}_{HH}E_{13}(\xi, D_\phi^{-1}, 6)$.

Remark 3. By taking $\phi = 0$ in generating Equation (22) such that

$${}_sH\mathcal{F}_n^{(\alpha)}(0, \xi, \omega; u; \lambda) := {}_H\mathcal{F}_n^{(\alpha)}(\xi, \omega; u; \lambda) \quad (35)$$

yields the 2-variable generalized Apostol type Hermite-Frobenius-Euler polynomials (2VGATHFEP) ${}_H\mathcal{F}_n^{(\alpha)}(\xi, \omega; u; \lambda)$ [10].

Remark 4. By taking $\omega = 0$ in generating Equation (22) such that

$${}_sH\mathcal{F}_n^{(\alpha)}(\phi, \xi, 0; u; \lambda) := {}_s\mathcal{F}_n^{(\alpha)}(\phi, \xi; u; \lambda) \quad (36)$$

yields the 2-variable generalized Apostol type Legendre-Frobenius-Euler polynomials (2VGATLe-FEP) ${}_s\mathcal{F}_n^{(\alpha)}(\phi, \xi; u; \lambda)$.

Note. We note that for $u = -1$, the results for the 2VGATHFEP and 2VGATLeFEP reduce to the results for the 2-variable generalized Hermite-Apostol-Euler polynomials (2VGHAEP) ${}_H\mathfrak{E}_n^{(\alpha)}(\xi, \omega; \lambda)$ and 2-variable generalized Legendre-Apostol-Euler polynomials (2VGLeAEP) ${}_S\mathfrak{E}_n^{(\alpha)}(\phi, \xi; \lambda)$.

The properties of both ordinary and generalized special functions can be derived using the operational techniques associated with the study of generalized special functions, including multivariate cases, which can offer a unified study of these polynomials. The majority of special functions and their generalizations have their roots in physical issues.

We recall that the 2VLeP $S_n(\phi, \xi)$ are demonstrated to be natural solutions of a specific set of partial differential equations, and that they are of considerable interest because of their numerous applications in physics and mathematics:

$$\frac{\partial^2}{\partial \xi^2} S_n(\phi, \xi) = \left(\frac{\partial}{\partial \phi} \phi \frac{\partial}{\partial \phi} \right) S_n(\phi, \xi); \quad S_n(\phi, 0) = \begin{cases} n! \frac{\phi^{[n/2]}}{([n/2]!)^2}, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases} \quad (37)$$

and are defined by the following operational representation [24]:

$$S_n(\phi, \xi) = \left[\exp \left(D_\phi^{-1} \frac{\partial^2}{\partial \xi^2} \right) \right] (\xi^n) \quad (38)$$

with $\frac{\partial}{\partial D_\phi^{-1}} = \frac{\partial}{\partial \phi} \phi \frac{\partial}{\partial \phi}$.

In many areas of pure and applied mathematics and physics, the 2VHKdFP $H_n(\xi, \omega)$ are widely employed. They have been used to solve issues with quantum mechanics and optical beam transport. These polynomials are the solutions of the heat equation:

$$\frac{\partial}{\partial \omega} H_n(\xi, \omega) = \frac{\partial^2}{\partial \xi^2} H_n(\xi, \omega), \quad H_n(\xi, 0) = \xi^n \quad (39)$$

and are defined by means of the operational representation:

$$H_n(\xi, \omega) = \left[\exp \left(\omega \frac{\partial^2}{\partial \xi^2} \right) \right] (\xi^n). \quad (40)$$

We establish the operational representation for the MVLeHP ${}_S H_n(\phi, \xi, \omega)$. Use of Equations (40) with (9) in (15) yields the following operational representation for the MVLeHP:

$${}_S H_n(\phi, \xi, \omega) = \left[\exp \left(\omega \frac{\partial^2}{\partial \xi^2} \right) \right] (S_n(\phi, \xi)). \quad (41)$$

Again, with use of Equations (38) with (12) in (16) yields the following:

$${}_S H_n(\phi, \xi, \omega) = \left[\exp \left(D_\phi^{-1} \frac{\partial^2}{\partial \xi^2} \right) \right] (H_n(\xi, \omega)). \quad (42)$$

Also, using Equation (38) (or (40)) in (41) (or (42)) yields the following operational representation for the MVLeHP:

$${}_S H_n(\phi, \xi, \omega) = \left[\exp \left(D_\phi^{-1} \frac{\partial^2}{\partial \xi^2} + \omega \frac{\partial^2}{\partial \xi^2} \right) \right] (\xi^n). \quad (43)$$

Next, we find the operational representation for the MVGATLeHFEP ${}_sH\mathcal{F}_n^{(\alpha)}(\phi, \xi, \omega; u; \lambda)$. We note that in view of generating Equation (22), the MVGATLeHFEP are the solutions of the following equations:

$$\frac{\partial^2}{\partial \xi^2} {}_sH\mathcal{F}_n^{(\alpha)}(\phi, \xi, \omega; u; \lambda) = \frac{\partial}{\partial \omega} {}_sH\mathcal{F}_n^{(\alpha)}(\phi, \xi, \omega; u; \lambda) \quad (44)$$

under the following initial condition:

$${}_sH\mathcal{F}_n^{(\alpha)}(\phi, \xi, 0; u; \lambda) = {}_s\mathcal{F}_n^{(\alpha)}(\phi, \xi; u; \lambda). \quad (45)$$

Consequently, we determine the operational representation between the MVGATLeHFEP and the MVLeATFEP as follows in light of the aforementioned equations:

$${}_sH\mathcal{F}_n^{(\alpha)}(\phi, \xi, \omega; u; \lambda) = \left[\exp\left(\omega \frac{\partial^2}{\partial \xi^2}\right) \right] ({}_s\mathcal{F}_n^{(\alpha)}(\phi, \xi; u; \lambda)). \quad (46)$$

Again, from generating Equation (22), we find that the MVGATLeHFEP are the solutions of the following equations:

$$\frac{\partial^2}{\partial \xi^2} {}_sH\mathcal{F}_n^{(\alpha)}(\phi, \xi, \omega; u; \lambda) = \left(\frac{\partial}{\partial \phi} \phi \frac{\partial}{\partial \phi} \right) {}_sH\mathcal{F}_n^{(\alpha)}(\phi, \xi, \omega; u; \lambda) \quad (47)$$

under the following initial condition:

$${}_sH\mathcal{F}_n^{(\alpha)}(0, \xi, \omega; u; \lambda) = {}_H\mathcal{F}_n^{(\alpha)}(\xi, \omega; u; \lambda). \quad (48)$$

Consequently, we determine the operational representation between the MVGATLeHFEP and the MVHATFEP as follows in light of the aforementioned equations:

$${}_sH\mathcal{F}_n^{(\alpha)}(\phi, \xi, \omega; u; \lambda) = \left[\exp\left(D_\phi^{-1} \frac{\partial^2}{\partial \xi^2}\right) \right] ({}_H\mathcal{F}_n^{(\alpha)}(\xi, \omega; u; \lambda)). \quad (49)$$

Note. As we can see, for $u = -1$, the operational representations (46) and (49) for the MVGATLeHFEP gives the following operational representation between the MVGATLeHFEP and 2VGLAEP and MVGATLeHFEP and 2VGHAEP such that

$${}_sH\mathfrak{E}_n^{(\alpha)}(\phi, \xi, \omega; \lambda) = \left[\exp\left(\omega \frac{\partial^2}{\partial \xi^2}\right) \right] ({}_s\mathfrak{E}_n^{(\alpha)}(\phi, \xi; \lambda)), \quad (50)$$

$${}_sH\mathfrak{E}_n^{(\alpha)}(\phi, \xi, \omega; \lambda) = \left[\exp\left(D_\phi^{-1} \frac{\partial^2}{\partial \xi^2}\right) \right] ({}_H\mathfrak{E}_n^{(\alpha)}(\xi, \omega; \lambda)). \quad (51)$$

We derive some formulas and recurrence relations for the MVGATLeHFEP and their relatives in the following section.

3. Recurrence Relations and Connection Formulae

We establish the following findings in order to obtain the recurrence relations for the MVGATLeHFEP ${}_sH\mathcal{F}_n^{(\alpha)}(\phi, \xi, \omega; u; \lambda)$.

Theorem 1. For $\alpha, \beta \in \mathbb{Z}$, the following recurrence relation for the multivariate generalized Apostol type Legendre-Hermite Frobenius-Euler polynomials holds true:

$${}_sH\mathcal{F}_n^{(\alpha \pm \beta)}(\phi, \xi, \omega; u; \lambda) = \sum_{k=0}^n \binom{n}{k} \mathcal{F}_k^{(\pm \beta)}(u; \lambda) {}_sH\mathcal{F}_{n-k}^{(\alpha)}(\phi, \xi, \omega; u; \lambda). \quad (52)$$

Proof. In view of Equations (1) (with $\phi = 0$) and (22), we have

$$\sum_{n=0}^{\infty} {}_sH\mathcal{F}_n^{(\alpha \pm \beta)}(\phi, \xi, \omega; u; \lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} {}_sH\mathcal{F}_n^{(\alpha)}(\phi, \xi, \omega; u; \lambda) \frac{t^n}{n!} \sum_{k=0}^{\infty} \mathcal{F}_k^{(\pm \beta)}(u; \lambda) \frac{t^k}{k!}, \quad (53)$$

which, when simplified, produces the statement (52). $\square$

Theorem 2. For the multivariate generalized Apostol type Legendre-Hermite Frobenius-Euler polynomials, the following recurrence relation is valid:

$$\begin{aligned} {}_sH\mathcal{F}_{n+1}(\phi, \xi, \omega; u; \lambda) = & y {}_sH\mathcal{F}_n(\phi, \xi, \omega; u; \lambda) + 2n(\omega + D_{\phi}^{-1}) {}_sH\mathcal{F}_{n-1}(\phi, \xi, \omega; u; \lambda) \\ & - \frac{\lambda}{1-u} \sum_{k=0}^n \binom{n}{k} {}_sH\mathcal{F}_{n-k}(\phi, \xi, \omega; u; \lambda) \mathcal{F}_k(1; u; \lambda). \end{aligned} \quad (54)$$

Proof. Taking $\alpha = 1$ and then differentiating generating function (22) with respect to t and after simplifying the equation and then using Equations (22) and (1) in the resulting equation, it follows that

$$\begin{aligned} \sum_{n=0}^{\infty} {}_sH\mathcal{F}_{n+1}(\phi, \xi, \omega; u; \lambda) \frac{t^n}{n!} = & \xi \sum_{n=0}^{\infty} {}_sH\mathcal{F}_n(\phi, \xi, \omega; u; \lambda) \frac{t^n}{n!} + 2 \sum_{n=0}^{\infty} (\omega + D_{\phi}^{-1}) {}_sH\mathcal{F}_n(\phi, \xi, \omega; u; \lambda) \frac{t^{n+1}}{n!} \\ & - \frac{\lambda}{1-u} \sum_{n=0}^{\infty} {}_sH\mathcal{F}_n(\phi, \xi, \omega; u; \lambda) \frac{t^n}{n!} \sum_{k=0}^{\infty} \mathcal{F}_k(1; u; \lambda) \frac{t^k}{k!}, \end{aligned} \quad (55)$$

which, when simplified, produces the statement (54). $\square$

Theorem 3. For multivariate generalized Apostol type Legendre-Hermite Frobenius-Euler polynomials, the following recurrence relation holds for $\gamma > 0$:

$$(1-u)^{\gamma} {}_sH\mathcal{F}_n^{(\alpha-\gamma)}(\phi, \xi, \omega; u; \lambda) = \sum_{k=0}^n \binom{n}{k} {}_sH\mathcal{F}_{n-k}^{(\alpha)}(\phi, \xi, \omega; u; \lambda) \sum_{p=0}^{\gamma} \binom{\gamma}{p} \lambda^p p^k (-u)^{\gamma-p}. \quad (56)$$

Proof. The generating Equation (22) with use of formula $(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$ (see [32]) can be simplified to yield

$$\sum_{n=0}^{\infty} {}_sH\mathcal{F}_n^{(\alpha-\gamma)}(\phi, \xi, \omega; u; \lambda) \frac{t^n}{n!} = (1-u)^{-\gamma} \sum_{n=0}^{\infty} {}_sH\mathcal{F}_n^{(\alpha)}(\phi, \xi, \omega; u; \lambda) \frac{t^n}{n!} \sum_{k=0}^{\gamma} \sum_{p=0}^{\gamma} \binom{\gamma}{p} \lambda^p p^k (-u)^{\gamma-p} \frac{t^k}{k!}, \quad (57)$$

which, when simplified, produces the statement (56). $\square$

Corollary 1. The following recurrence relation for the multivariate generalized Legendre-Hermite-Apostol-Euler polynomials holds true:

$$2^{\gamma} {}_sH\mathfrak{E}_n^{(\alpha-\gamma)}(\phi, \xi, \omega; \lambda) = \sum_{k=0}^n \binom{n}{k} {}_sH\mathfrak{E}_{n-k}^{(\alpha)}(\phi, \xi, \omega; \lambda) \sum_{p=0}^{\gamma} \binom{\gamma}{p} \lambda^p p^k. \quad (58)$$

To determine the relationship formulas between the MVGATLeHFEP ${}_sH\mathcal{F}_n^{(\alpha)}(\phi, \xi, \omega; u; \lambda)$ and the generalized Hurwitz-Lerch Zeta function $\Psi_{\phi}(z, p, a)$ (refer to [33]), we provide the following definition:

Definition 1. The generalized Hurwitz-Lerch Zeta function $\Psi_{\phi}(z, p, a)$ is defined as follows:

$$\Psi_{\phi}(z, p, a) = \sum_{n=0}^{\infty} \frac{(\phi)_n}{n!} \frac{z^n}{(n+a)^p}, \quad (59)$$

where $(\varphi)_n = \varphi(\varphi - 1)(\varphi - 2) \cdots (\varphi - n + 1)$ denotes the pochhammer function, which for $\varphi = 1$ $((1)_n = n!)$ becomes the Hurwitz-Lerch Zeta function $\Phi(z, p, a)$ (see [34]).

Theorem 4. For $u, \alpha \in \mathbb{C}, u \neq 1$, the following connection formula for the multivariate generalized Apostol type Legendre-Hermite Frobenius-Euler polynomials holds true:

$${}_sH\mathcal{F}_n^{(\alpha)}(\phi, \xi, \omega; u; \lambda) = \left(\frac{u-1}{u}\right)^\alpha \sum_{l=0}^n \binom{n}{l} \Psi_\alpha\left(\frac{\lambda}{u}, l-n, y\right) {}_sH_l(\phi, 0, \omega). \quad (60)$$

Proof. With use of the following formula (see [32])

$$(1-z)^{-\lambda} = \sum_{k=0}^{\infty} (\lambda)_k \frac{z^k}{k!}$$

in generating function (22), we find

$$\sum_{n=0}^{\infty} {}_sH\mathcal{F}_n^{(\alpha)}(\phi, \xi, \omega; u; \lambda) \frac{t^n}{n!} = (1-u)^\alpha (-u)^{-\alpha} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} \left(\frac{\lambda}{u}\right)^k \frac{(k+\xi)^n t^n}{n!} e^{\omega t^2} J_0(2\sqrt{-\phi t^2}), \quad (61)$$

which on using Equations (14) and (59) and after simplification produces the statement (60). $\square$

Corollary 2. The following connection formula for the multivariate generalized Legendre-Hermite-Apostol-Euler polynomials holds true:

$${}_sH\mathfrak{E}_n^{(\alpha)}(\phi, \xi, \omega; \lambda) = 2^\alpha \sum_{l=0}^n \binom{n}{l} \Psi_\alpha(-\lambda, l-n, \xi) {}_sH_l(\phi, 0, \omega). \quad (62)$$

Corollary 3. The following connection formula for the multivariate Legendre-Hermite-Apostol-Euler polynomials holds true:

$${}_sH\mathfrak{E}_n(\phi, \xi, \omega; \lambda) = 2 \sum_{l=0}^n \binom{n}{l} \Psi(-\lambda, l-n, \xi) {}_sH_l(\phi, 0, \omega). \quad (63)$$

Corollary 4. The following connection formula for the multivariate Legendre-Hermite-Euler polynomials holds true:

$${}_sHE_n(\phi, \xi, \omega) = 2 \sum_{l=0}^n \binom{n}{l} L(l-n, \xi) {}_sH_l(\phi, 0, \omega), \quad (64)$$

where L -function: $L(p, \phi) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+\phi)^p}$ ($\Re(p) > 1; \phi \in \mathbb{C} \setminus \mathbb{Z}_0^-$) (see [34]).

To establish the connection formulas between the MVGATLeHFEP ${}_sH\mathcal{F}_n^{(\alpha)}(\phi, \xi, \omega; u; \lambda)$ and the λ -Stirling numbers of second kind $S(n, k, \lambda)$ (see [35]), we give the following definition:

Definition 2. The λ -Stirling numbers of second kind $S(n, \vartheta, \lambda)$ is given by

$$\frac{(\lambda e^t - 1)^\vartheta}{\vartheta!} = \sum_{n=1}^{\infty} S(n, \vartheta, \lambda) \frac{t^n}{n!}, \quad (65)$$

which for $\lambda = 1$ gives the Stirling numbers of second kind $S(n, \vartheta)$ (see [35]).

Theorem 5. The following connection formulae for the multivariate generalized Apostol type Legendre-Hermite Frobenius-Euler polynomials holds true for $\alpha, \gamma \in \mathbb{N}_0$:

$$\alpha! \sum_{l=0}^n \binom{n}{l} {}_sH\mathcal{F}_{n-l}^{(\alpha)}(\phi, \xi, \omega; u; \lambda) S\left(l, \alpha, \frac{\lambda}{u}\right) = \left(\frac{1-u}{u}\right)^\alpha {}_sH_n(\phi, \xi, \omega), \quad (66)$$

$${}_sH\mathcal{F}_n^{(\alpha-\gamma)}(\phi, \xi, \omega; u; \lambda) = \gamma! \left(\frac{u}{1-u}\right)^\gamma \sum_{l=0}^n \binom{n}{l} {}_sH\mathcal{F}_{n-l}^{(\alpha)}(\phi, \xi, \omega; u; \lambda) S\left(l, \gamma, \frac{\lambda}{u}\right). \quad (67)$$

Proof. The generating function (22) can be simplified to give

$$\sum_{n=0}^{\infty} {}_sH\mathcal{F}_n^{(\alpha)}(\phi, \xi, \omega; u; \lambda) \frac{t^n}{n!} = \frac{1}{\alpha!} \left(\frac{1-u}{u}\right)^\alpha \frac{\alpha!}{\left(\frac{\lambda}{u} e^t - 1\right)^\alpha} e^{\xi t + \omega t^2} J_0(2\sqrt{-\phi t^2}), \quad (68)$$

which on using Equations (14) and (65) and after simplification produces the statement (66).

Again, we consider generating function (22) in the following form:

$$\sum_{n=0}^{\infty} {}_sH\mathcal{F}_n^{(\alpha-\gamma)}(\phi, \xi, \omega; u; \lambda) \frac{t^n}{n!} = \left(\frac{1-u}{\lambda e^t - u}\right)^\alpha e^{\xi t + \omega t^2} J_0(2\sqrt{-\phi t^2}) \left(\frac{u}{1-u}\right)^\gamma \gamma! \frac{\left(\frac{\lambda}{u} e^t - 1\right)^\gamma}{\gamma!}, \quad (69)$$

which on using Equations (22) and (65) and after simplification produces the statement (68). $\square$

Corollary 5. The following connection formulae for the multivariate generalized Legendre-Hermite-Apostol-Euler polynomials holds true for $\alpha, \gamma \in \mathbb{N}_0$:

$$\alpha! \sum_{l=0}^n \binom{n}{l} {}_sH\mathfrak{E}_{n-l}^{(\alpha)}(\phi, \xi, \omega; \lambda) S(l, \alpha, -\lambda) = (-2)^\alpha {}_sH_n(\phi, \xi, \omega), \quad (70)$$

$${}_sH\mathfrak{E}_n^{(\alpha-\gamma)}(\phi, \xi, \omega; \lambda) = \gamma! 2^\gamma \sum_{l=0}^n \binom{n}{l} {}_sH\mathfrak{E}_{n-l}^{(\alpha)}(\phi, \xi, \omega; \lambda) S(l, \gamma, -\lambda). \quad (71)$$

Corollary 6. The following connection formulae for the multivariate generalized Legendre-Hermite-Euler polynomials holds true $\alpha, \gamma \in \mathbb{N}_0$:

$$\alpha! \sum_{l=0}^n \binom{n}{l} {}_sHE_{n-l}^{(\alpha)}(\phi, \xi, \omega) (-1)^\alpha S(l, \alpha) = (-2)^\alpha {}_sH_n(\phi, \xi, \omega), \quad (72)$$

$${}_sHE_n^{(\alpha-\gamma)}(\phi, \xi, \omega) = \gamma! 2^\gamma \sum_{l=0}^n \binom{n}{l} {}_sHE_{n-l}^{(\alpha)}(\phi, \xi, \omega) (-1)^\gamma S(l, \gamma). \quad (73)$$

In the next section, we derive some summation formulae and symmetric identities for the MVGATLeHFEP ${}_sH\mathcal{F}_n^{(\alpha)}(x, y, z; u; \lambda)$.

4. Summation Formulae and Symmetric Identities

To derive summation formulae for the MVGATLeHFEP ${}_sH\mathcal{F}_n^{(\alpha)}(\phi, \xi, \omega; u; \lambda)$, we prove the following results:

Theorem 6. The following explicit summation formula for the multivariate generalized Apostol type Legendre-Hermite Frobenius-Euler polynomials holds true:

$${}_sH\mathcal{F}_n^{(\alpha)}(\phi, \xi + w, \omega + r; u; \lambda) = \sum_{k=0}^n \binom{n}{k} {}_sH\mathcal{F}_{n-k}^{(\alpha)}(\phi, \xi, \omega; u; \lambda) H_k(w, r). \quad (74)$$

Proof. Replacing $\xi \rightarrow \xi + w$ and $\omega \rightarrow \omega + r$ then using Equations (22) and (11) in the l.h.s. of generating function (22), so that we have

$$\sum_{n=0}^{\infty} {}_sH\mathcal{F}_n^{(\alpha)}(\phi, \xi + w, \omega + r; u; \lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} {}_sH\mathcal{F}_n^{(\alpha)}(\phi, \xi, \omega; \lambda; u) H_k(w, r) \frac{t^{n+k}}{n! k!}, \quad (75)$$

which, when simplified produces statement (74). $\square$

Corollary 7. For $r = 0$ in Equation (74), we have

$${}_sH\mathcal{F}_n^{(\alpha)}(\phi, \xi + w, \omega; u; \lambda) = \sum_{k=0}^n \binom{n}{k} {}_sH\mathcal{F}_k^{(\alpha)}(\phi, \xi, \omega; u; \lambda) w^{n-k}. \quad (76)$$

Theorem 7. The following implicit summation formula for the multivariate generalized Apostol type Legendre-Hermite Frobenius-Euler polynomials holds true:

$${}_sH\mathcal{F}_{n+m}^{(\alpha)}(\phi, v, \omega; u; \lambda) = \sum_{s=0}^m \sum_{k=0}^n \binom{n}{k} \binom{m}{s} (v - \xi)^{k+s} {}_sH\mathcal{F}_{n+m-k-s}^{(\alpha)}(\phi, \xi, \omega; u; \lambda). \quad (77)$$

Proof. Replacing $t \rightarrow t + v$ and then using the following rule:

$$\sum_{N=0}^{\infty} f(N) \frac{(x+y)^N}{N!} = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} f(l+m) \frac{x^l y^m}{l! m!} \quad (78)$$

in generating function (22), so that we have

$$e^{-\xi(t+v)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} {}_sH\mathcal{F}_{n+m}^{(\alpha)}(\phi, \xi, \omega; \lambda; u) \frac{t^n v^m}{n! m!} = \left(\frac{1-u}{\lambda e^{t+v} - u} \right)^{\alpha} e^{\omega(t+v)^2} J_0(2\sqrt{-\phi(t+v)^2}). \quad (79)$$

By changing ξ to v in the previous equation and subsequently equating the resulting formula to the initial equation, we can determine

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} {}_sH\mathcal{F}_{n+m}^{(\alpha)}(\phi, v, \omega; \lambda; u) \frac{t^n v^m}{n! m!} = e^{(v-\xi)(t+v)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} {}_sH\mathcal{F}_{n+m}^{(\alpha)}(\phi, \xi, \omega; \lambda; u) \frac{t^n v^m}{n! m!}, \quad (80)$$

which, when the r.h.s. exponential is expanded, yields

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} {}_sH\mathcal{F}_{n+m}^{(\alpha)}(\phi, v, \omega; \lambda; u) \frac{t^n v^m}{n! m!} = \sum_{k=0}^{\infty} \sum_{s=0}^{\infty} (v - \xi)^{k+s} \frac{t^k v^s}{k! s!} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} {}_sH\mathcal{F}_{n+m}^{(\alpha)}(\phi, \xi, \omega; \lambda; u) \frac{t^n v^m}{n! m!}, \quad (81)$$

We arrive at assertion (77) by substituting $n \rightarrow n - k$ and $m \rightarrow m - s$ in the r.h.s. of the above equation and then equating the coefficients of the same powers of t and v in both sides of the resulting equation. $\square$

Next, we derive some symmetric identities for the MVGATLeHFEP ${}_sH\mathcal{F}_n^{(\alpha)}(\phi, \xi, \omega; u; \lambda)$. For this, we recall the following definitions:

Definition 3. Given any real or complex parameter λ , the generalized sum of integer powers $S_k(q; \lambda)$ can be found using the following formula:

$$\frac{\lambda e^{(q+1)t} - 1}{\lambda e^t - 1} = \sum_{k=0}^{\infty} S_k(q; \lambda) \frac{t^k}{k!}. \quad (82)$$

Definition 4. The double power sums $\mathcal{S}_k^{(l)}(\chi; \lambda)$ for any real or complex parameter λ are defined by the generating function that follows:

$$\left(\frac{1 - \lambda^\chi e^{\chi t}}{1 - \lambda e^t} \right)^l = \frac{1}{\lambda^l} \sum_{q=0}^{\infty} \left\{ \sum_{p=0}^q \binom{q}{p} (-l)^{q-p} \mathcal{S}_k^{(l)}(\chi; \lambda) \right\} \frac{t^q}{q!}. \quad (83)$$

Theorem 8. For all integers $c, d > 0$ and $n \geq 0, \alpha \geq 1, \lambda, u \in \mathbb{C}$, the following symmetry identity for the multivariate generalized Apostol type Legendre-Hermite Frobenius-Euler polynomials holds true:

$$\begin{aligned} & \sum_{k=0}^n \binom{n}{k} c^{n-k} {}_sH\mathcal{F}_{n-k}^{(\alpha)}(d^2\phi, d\zeta, d^3\omega; \lambda; u) \sum_{l=0}^k \binom{k}{l} d^k \mathcal{S}_l\left(c-1; \frac{\lambda}{u}\right) {}_sH\mathcal{F}_{k-l}^{(\alpha-1)}(c^2\hat{\phi}, c\hat{\zeta}, c^3\hat{\omega}; \lambda; u) \\ &= \sum_{k=0}^n \binom{n}{k} d^{n-k} {}_sH\mathcal{F}_{n-k}^{(\alpha)}(c^2\phi, c\zeta, c^3\omega; \lambda; u) \sum_{l=0}^k \binom{k}{l} c^k \mathcal{S}_l\left(d-1; \frac{\lambda}{u}\right) {}_sH\mathcal{F}_{k-l}^{(\alpha-1)}(d^2\hat{\phi}, d\hat{\zeta}, d^3\hat{\omega}; \lambda; u). \end{aligned} \quad (84)$$

Proof. Let

$$G(t) := \frac{(1-u)^{2\alpha-1} e^{\zeta(cdt)+\omega(cdt)^2} J_0(2\sqrt{-\phi(cdt)^2}) (\lambda e^{cdt} - u) e^{\hat{\zeta}(cdt)+\hat{\omega}(cdt)^2} J_0(2\sqrt{-\hat{\phi}(cdt)^2})}{(\lambda e^{ct} - u)^\alpha (\lambda e^{dt} - u)^\alpha}, \quad (85)$$

which on rearranging the powers and then utilizing Equations (22) and (82) and further using the Cauchy product rule, it can be deduced to

$$\begin{aligned} G(t) = & \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \binom{n}{k} c^{n-k} d^k {}_sH\mathcal{F}_{n-k}^{(\alpha)}(d^2\phi, d\zeta, d^3\omega; \lambda; u) \sum_{l=0}^k \binom{k}{l} \mathcal{S}_l\left(c-1; \frac{\lambda}{u}\right) \right. \\ & \left. \times {}_sH\mathcal{F}_{k-l}^{(\alpha-1)}(c^2\hat{\phi}, c\hat{\zeta}, c^3\hat{\omega}; \lambda; u) \right) \frac{t^n}{n!}. \end{aligned} \quad (86)$$

Similar to that, we also get

$$\begin{aligned} G(t) = & \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \binom{n}{k} d^{n-k} c^k {}_sH\mathcal{F}_{n-k}^{(\alpha)}(c^2\phi, c\zeta, c^3\omega; \lambda; u) \sum_{l=0}^k \binom{k}{l} \mathcal{S}_l\left(d-1; \frac{\lambda}{u}\right) \right. \\ & \left. \times {}_sH\mathcal{F}_{k-l}^{(\alpha-1)}(d^2\hat{\phi}, d\hat{\zeta}, d^3\hat{\omega}; \lambda; u) \right) \frac{t^n}{n!}. \end{aligned} \quad (87)$$

We arrive at assertion (84) by equating the coefficients of same powers of t in the r.h.s. of expansions (86) and (87). $\square$

Theorem 9. For each pair of positive integers c, d and for all integers $n \geq 0, \alpha \geq 1, \lambda, u \in \mathbb{C}$, the following symmetry identity for the multivariate generalized Apostol type Legendre-Hermite Frobenius-Euler polynomials holds true:

$$\begin{aligned} & \sum_{k=0}^n \binom{n}{k} \sum_{i=0}^{c-1} \sum_{j=0}^{d-1} u^{c+d-2} \left(\frac{\lambda}{u} \right)^{i+j} c^{n-k} d^k {}_sH\mathcal{F}_k^{(\alpha)}\left(c^2\hat{\phi}, c\hat{\zeta} + \frac{c}{d}j, c^2\hat{\omega}; \lambda; u\right) {}_sH\mathcal{F}_{n-k}^{(\alpha)}\left(d^2\phi, d\zeta + \frac{d}{c}i, d^2\omega; \lambda; u\right) \\ &= \sum_{k=0}^n \binom{n}{k} \sum_{i=0}^{d-1} \sum_{j=0}^{c-1} u^{c+d-2} \left(\frac{\lambda}{u} \right)^{i+j} d^{n-k} c^k {}_sH\mathcal{F}_k^{(\alpha)}\left(d^2\hat{\phi}, d\hat{\zeta} + \frac{d}{c}j, d^2\hat{\omega}; \lambda; u\right) {}_sH\mathcal{F}_{n-k}^{(\alpha)}\left(c^2\phi, c\zeta + \frac{c}{d}i, c^2\omega; \lambda; u\right). \end{aligned} \quad (88)$$

Proof. Let

$$H(t) := \frac{(1-u)^{2\alpha} e^{\zeta(cdt)+\omega(cdt)^2} J_0(2\sqrt{-\phi(cdt)^2}) (\lambda^c e^{cdt} - u^c) (\lambda^d e^{cdt} - u^d) e^{\hat{\zeta}(cdt)+\hat{\omega}(cdt)^2} J_0(2\sqrt{-\hat{\phi}(cdt)^2})}{(\lambda e^{ct} - u)^{\alpha+1} (\lambda e^{dt} - u)^{\alpha+1}}, \quad (89)$$

which on rearranging and then using the expansions for $\left(\frac{\lambda^c e^{cdt} - u^c}{\lambda e^{dt} - u}\right)$ and $\left(\frac{\lambda^d e^{cdt} - u^d}{\lambda e^{ct} u}\right)$ yields

$$H(t) = \left(\frac{1-u}{\lambda e^{ct} - u}\right)^\alpha e^{d\zeta(ct) + d^2\omega(ct)^2} J_0(2\sqrt{-\phi(cdt)^2}) u^{c-1} \sum_{i=0}^{c-1} \left(\frac{\lambda}{u}\right)^i e^{dti} \quad (90)$$

$$\times \left(\frac{1-u}{\lambda e^{dt} - u}\right)^\alpha e^{c\widehat{\zeta}(dt) + c^2\widehat{\omega}(dt)^2} J_0(2\sqrt{-\widehat{\phi}(cdt)^2}) u^{d-1} \sum_{j=0}^{d-1} \left(\frac{\lambda}{u}\right)^j e^{ctj},$$

Using Equation (22) in above equation and then applying the Cauchy product rule in the r.h.s. of resulting equation, we find

$$H(t) = \sum_{k=0}^n \binom{n}{k} \sum_{i=0}^{c-1} \sum_{j=0}^{d-1} u^{c+d-2} \left(\frac{\lambda}{u}\right)^{i+j} c^{n-k} d^k {}_sH\mathcal{F}_k^{(\alpha)}\left((c^2\widehat{\phi}, c\widehat{\zeta} + \frac{c}{d}j, c^2\widehat{\omega}; \lambda; u)\right) \quad (91)$$

$$\times {}_sH\mathcal{F}_{n-k}^{(\alpha)}\left(d^2\phi, d\zeta + \frac{d}{c}i, d^2\omega; \lambda; u\right).$$

In a comparable scheme, we get

$$H(t) = \sum_{k=0}^n \binom{n}{k} \sum_{i=0}^{d-1} \sum_{j=0}^{c-1} u^{d+c-2} \left(\frac{\lambda}{u}\right)^{i+j} d^{n-k} c^k {}_sH\mathcal{F}_k^{(\alpha)}\left((d^2\widehat{\phi}, d\widehat{\zeta} + \frac{d}{c}j, d^2\widehat{\omega}; \lambda; u)\right) \quad (92)$$

$$\times {}_sH\mathcal{F}_{n-k}^{(\alpha)}\left(c^2\phi, c\zeta + \frac{c}{d}i, c^2\omega; \lambda; u\right).$$

We arrive at assertion (88) by equating the coefficients of same powers of t in the r.h.s. of expansions (91) and (92). $\square$

Theorem 10. For each pair of positive integers c, d and for all integers $n \geq 0, \alpha \geq 1, \lambda, u \in \mathbb{C}$, the following symmetry identity for the multivariate generalized Apostol type Legendre-Hermite Frobenius-Euler polynomials holds true:

$$\sum_{l=0}^n \binom{n}{l} {}_sH\mathcal{F}_{n-l}(d^2\phi, d\zeta, d^3\omega; \lambda; u) u^{d\alpha} \lambda^{-\alpha} \sum_{m=0}^l \binom{l}{m} \sum_{r=0}^m \binom{m}{r} (-\alpha)^{m-r} \mathcal{S}_k^{(\alpha)}\left(d; \frac{\lambda}{u}\right) \quad (93)$$

$$\times {}_sH\mathcal{F}_{l-m}^{(\alpha+1)}(c^2\widehat{\phi}, c\widehat{\zeta}, c^3\widehat{\omega}; \lambda; u) c^{n-l+m} d^{l-m}$$

$$= \sum_{l=0}^n \binom{n}{l} {}_sH\mathcal{F}_{n-l}(c^2\phi, c\zeta, c^3\omega; \lambda; u) u^{c\alpha} \lambda^{-\alpha} \sum_{m=0}^l \binom{l}{m} \sum_{r=0}^m \binom{m}{r} (-\alpha)^{m-r} \mathcal{S}_k^{(\alpha)}\left(c; \frac{\lambda}{u}\right)$$

$$\times {}_sH\mathcal{F}_{l-m}^{(\alpha+1)}(d^2\widehat{\phi}, d\widehat{\zeta}, d^3\widehat{\omega}; \lambda; u) d^{n-l+m} c^{l-m}.$$

Proof. Let

$$F(t) := \frac{(1-u)^{\alpha+2} e^{dy(ct) + d^2z(ct)^2} J_0(2\sqrt{-c^2x(dt)^2}) (\lambda^d e^{dct} - u^d)^\alpha e^{cY(dt) + c^2Z(dt)^2} J_0(2\sqrt{-d^2X(ct)^2})}{(\lambda e^{dt} - u)^{\alpha+1} (\lambda e^{ct} - u)^{\alpha+1}}, \quad (94)$$

which on rearranging the powers becomes, using Equations (22) and (83) and after simplification yields

$$F(t) := \sum_{n=0}^{\infty} \sum_{l=0}^n \binom{n}{l} {}_sH\mathcal{F}_{n-l}(d^2\phi, d\zeta, d^3\omega; \lambda; u) c^{n-l} u^{d\alpha} \lambda^{-\alpha} \sum_{m=0}^l \binom{l}{m} \sum_{r=0}^m \binom{m}{r} (-\alpha)^{m-r} \mathcal{S}_k^{(\alpha)}\left(d; \frac{\lambda}{u}\right) \quad (95)$$

$$\times {}_sH\mathcal{F}_{l-m}^{(\alpha+1)}(c^2\widehat{\phi}, c\widehat{\zeta}, c^3\widehat{\omega}; \lambda; u) c^m d^{l-m} \frac{t^n}{n!}.$$

Similar to that, we have

$$F(t) := \sum_{n=0}^{\infty} \sum_{l=0}^n \binom{n}{l} {}_sH\mathcal{F}_{n-l}(c^2\phi, c\zeta, c^3\omega; \lambda; u) d^{n-l} u^{c\alpha} \lambda^{-\alpha} \sum_{m=0}^l \binom{l}{m} \sum_{r=0}^m \binom{m}{r} (-\alpha)^{m-r} \mathcal{S}_k^{(\alpha)}\left(c; \frac{\lambda}{u}\right) \quad (96)$$

$$\times {}_sH\mathcal{F}_{l-m}^{(\alpha+1)}(d^2\widehat{\phi}, d\widehat{\zeta}, d^3\widehat{\omega}; \lambda; u) d^m c^{l-m} \frac{t^n}{n!}.$$

We arrive at assertion (93) by equating the coefficients of same powers of t in the r.h.s. of expansions (95) and (96). $\square$

5. Algebraic Matrix Approach

Determinants have wide range of applications in linear algebra, geometry, calculus, and other fields of mathematics. They provide valuable tools for solving problems, characterizing matrix properties, and understanding the behavior of mathematical systems. Numerical computations and the solution of linear interpolation problems to ascertain the coefficients of polynomial interpolation benefit from the use of the determinant representations of the special polynomials. Determinants offer a methodical and effective approach to calculating the coefficients of the Lagrange polynomial, thereby transforming interpolation into a feasible and dependable method for approximating functions with restricted data. In order to establish the orthogonality connections between the functions which are essential for numerous applications in physics and mathematics determinants are also used. They guarantee that the squared magnitudes of the functions integrate into unity and aid in normalizing the functions. They are employed in the development of generating functions for particular functions. Recursion relations, difference equations, and other characteristics of special functions can be obtained effectively with their help. For computational purposes, the algebraic method involving determinants for the complex special polynomials can be useful. Several important research to explore algebraic properties of various polynomials can be seen in [36–40].

The 2-variable Legendre-Hermite polynomials (2VLeHP) ${}_sH_\theta(\phi, \xi)$ are first represented by their determinant. From the generating equation of multivariate Legendre-Hermite polynomials ${}_sH_n(\phi, \xi, \omega)$, we deduce the following generating function for the 2-variable Legendre-Hermite polynomials provided $(\xi \rightarrow \xi, \omega \rightarrow -1/2)$, which is given by

$$e^{-t^2/2} e^{\xi t} J_0(2\sqrt{-\phi t^2}) = \sum_{\theta=0}^{\infty} {}_sH_\theta(\phi, \xi) \frac{t^\theta}{\theta!}. \quad (97)$$

Now, to establish the determinants for the 2-variable polynomials, we are in search of classical polynomials determinants forms.

Lets recall the determinant form for the classical Hermite polynomials $H_\theta(\phi)$ [41,42]:

$$H_0(\phi) = 1, \quad H_\theta(\phi) = (-1)^\theta \begin{vmatrix} 1 & \phi & \phi^2 & \dots & \phi^{\theta-1} & \phi^\theta \\ 1 & 0 & 1 & \dots & \frac{(\theta-1)! \left(\frac{1}{2}\right)^{\theta-1/2}}{\left(\frac{\theta-1}{2}\right)!} & \frac{\theta! \left(\frac{1}{2}\right)^{\theta/2}}{\left(\frac{\theta}{2}\right)!} \\ 0 & 1 & 0 & \dots & \frac{(\theta-1)! \left(\frac{1}{2}\right)^{\theta-2/2}}{\left(\frac{\theta-2}{2}\right)!} & \frac{(n)! \left(\frac{1}{2}\right)^{\theta-1/2}}{\left(\frac{\theta-1}{2}\right)!} \\ 0 & 0 & 1 & \dots & \frac{(\theta-1)! \left(\frac{1}{2}\right)^{\theta-3/2}}{\left(\frac{\theta-3}{2}\right)!} & \frac{(\theta)! \left(\frac{1}{2}\right)^{\theta-2/2}}{\left(\frac{\theta-2}{2}\right)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{vmatrix}. \quad (98)$$

In the r.h.s. of Equation (98), replace the powers ϕ^θ ($\theta = 0, 1, 2, \dots$) by the polynomials $S_\theta(\phi, \xi)$ ($\theta = 0, 1, 2, \dots$), and ϕ by the polynomial $S_1(\phi, \xi)$ in the l.h.s. Using the following relation:

$$H_\theta(S_1(\phi, \xi)) = {}_sH_\theta(\phi, \xi)$$

in the l.h.s. of resultant equation, we find the following determinant form of the 2VLeHP ${}_sH_\theta(\phi, \xi)$:

$${}_sH_0(\phi, \xi) = 1,$$

$${}_sH_\theta(\phi, \xi) = (-1)^\theta \begin{vmatrix} 1 & S_1(\phi, \xi) & S_2(\phi, \xi) & \cdots & S_{\theta-1}(\phi, \xi) & S_\theta(\phi, \xi) \\ 1 & 0 & 1 & \cdots & \frac{(\theta-1)! \left(\frac{1}{2}\right)^{\theta-1/2}}{\left(\frac{\theta-1}{2}\right)!} & \frac{\theta! \left(\frac{1}{2}\right)^{\theta/2}}{\left(\frac{\theta}{2}\right)!} \\ 0 & 1 & 0 & \cdots & \binom{\theta-1}{1} \frac{(\theta-2)! \left(\frac{1}{2}\right)^{\theta-2/2}}{\left(\frac{\theta-2}{2}\right)!} & \binom{n}{1} \frac{(\theta-1)! \left(\frac{1}{2}\right)^{\theta-1/2}}{\left(\frac{\theta-1}{2}\right)!} \\ 0 & 0 & 1 & \cdots & \binom{\theta-1}{2} \frac{(\theta-3)! \left(\frac{1}{2}\right)^{\theta-3/2}}{\left(\frac{\theta-3}{2}\right)!} & \binom{\theta}{2} \frac{(\theta-2)! \left(\frac{1}{2}\right)^{\theta-2/2}}{\left(\frac{\theta-2}{2}\right)!} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{vmatrix}. \quad (99)$$

In view of relation $S_\theta(\phi, \xi) = H_\theta(\xi, D_\phi^{-1})$, we find that

$${}_sH_\theta(\phi, \xi) = {}_HH_\theta(\xi, D_\phi^{-1}), \quad (100)$$

where ${}_HH_\theta(\xi, D_\phi^{-1})$ denotes the 2-variable double Hermite polynomials (2VMHP). The generating function for the 2VMHP is given by

$$e^{\xi t} e^{(D_\phi^{-1} - \frac{1}{2})t^2} = \sum_{\theta=0}^{\infty} {}_HH_\theta(\xi, D_\phi^{-1}) \frac{t^\theta}{\theta!}. \quad (101)$$

Now, using Equations (100) and (13) in determinant form (99) yields the following determinant form for the 2-variable double Hermite polynomials:

$${}_HH_0(\xi, D_\phi^{-1}) = 1,$$

$${}_HH_\theta(\xi, D_\phi^{-1}) = (-1)^\theta \begin{vmatrix} 1 & H_1(\xi, D_\phi^{-1}) & H_2(\xi, D_\phi^{-1}) & \cdots & H_{\theta-1}(\xi, D_\phi^{-1}) & H_\theta(\xi, D_\phi^{-1}) \\ 1 & 0 & 1 & \cdots & \frac{(\theta-1)! \left(\frac{1}{2}\right)^{\theta-1/2}}{\left(\frac{\theta-1}{2}\right)!} & \frac{\theta! \left(\frac{1}{2}\right)^{\theta/2}}{\left(\frac{\theta}{2}\right)!} \\ 0 & 1 & 0 & \cdots & \binom{\theta-1}{1} \frac{(\theta-2)! \left(\frac{1}{2}\right)^{\theta-2/2}}{\left(\frac{\theta-2}{2}\right)!} & \binom{n}{1} \frac{(\theta-1)! \left(\frac{1}{2}\right)^{\theta-1/2}}{\left(\frac{\theta-1}{2}\right)!} \\ 0 & 0 & 1 & \cdots & \binom{\theta-1}{2} \frac{(\theta-3)! \left(\frac{1}{2}\right)^{\theta-3/2}}{\left(\frac{\theta-3}{2}\right)!} & \binom{\theta}{2} \frac{(\theta-2)! \left(\frac{1}{2}\right)^{\theta-2/2}}{\left(\frac{\theta-2}{2}\right)!} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{vmatrix}. \quad (102)$$

In order to determine the determinant form of the multivariate generalized Legendre-Hermite-Euler polynomials (MVGLHeP) ${}_sHE_\theta^{(\alpha)}(\phi, \xi, \omega)$, we substitute the polynomials ${}_sH_\theta(\phi, \xi, \omega)$ ($\theta = 0, 1, 2, \dots$) in the r.h.s. and ϕ by the polynomial ${}_sH_1(\phi, \xi, \omega)$ in the l.h.s. of the determinant form of generalized Euler polynomials (see [42]) and then using relation

$${}_sHE_\theta^{(\alpha)}(\phi, \xi, \omega) := E_\theta^{(\alpha)}({}_sH_1(\phi, \xi, \omega))$$

in the l.h.s. of the resulting equation, we find the following determinant form of the MVGLHeP:

$${}_sH E_0^{(\alpha)}(\phi, \xi, \omega) = 1$$

$${}_sH E_\theta^{(\alpha)}(\phi, \xi, \omega) = (-1)^\theta$$

$$\begin{vmatrix}
 1 & {}_sH_1(\phi, \xi, \omega) & {}_sH_2(\phi, \xi, \omega) & \cdots & {}_sH_\theta(\phi, \xi, \omega) \\
 1 & \frac{\alpha}{2} & \frac{\alpha^2}{4} + \frac{\alpha}{4} & \cdots & \sum_{\rho=0}^{\theta} (\alpha)_\rho 2^{-\rho} S(\theta, \rho) \\
 0 & 1 & \alpha & \cdots & \binom{\theta}{1} \sum_{\rho=0}^{\theta-1} (\alpha)_\rho 2^{-\rho} S(\theta-1, \rho) \\
 0 & 0 & 1 & \cdots & \binom{\theta}{2} \sum_{\rho=0}^{\theta-2} (\alpha)_\rho 2^{-\rho} S(\theta-2, \rho) \\
 \cdot & \cdot & \cdot & \cdots & \cdot \\
 \cdot & \cdot & \cdot & \cdots & \cdot \\
 \cdot & \cdot & \cdot & \cdots & \cdot \\
 0 & 0 & 0 & \cdots & \binom{\theta}{\theta-1} \frac{\alpha}{2}
 \end{vmatrix}, \quad (103)$$

where $S(\theta, \rho)$ are the Stirling numbers of the second kind defined by

$$S(\theta, \rho) = \frac{1}{\rho!} \sum_{l=0}^{\rho} (-1)^{\rho-l} \binom{\rho}{l} l^\theta. \quad (104)$$

By finding the values of $S(\theta, \rho)$ from Equation (104) and substituting it in Equation (103) together with $\alpha = 1$, we find the following determinant form for the multivariate Legendre-Hermite-Euler polynomials (MVLeHEP) ${}_sH E_\theta(\phi, \xi, \omega)$:

$${}_sH E_0(\phi, \xi, \omega) = 1,$$

$${}_sH E_\theta(\phi, \xi, \omega) = (-1)^\theta$$

$$\begin{vmatrix}
 1 & {}_sH_1(\phi, \xi, \omega) & {}_sH_2(\phi, \xi, \omega) & \cdots & {}_sH_{\theta-1}(\phi, \xi, \omega) & {}_sH_\theta(\phi, \xi, \omega) \\
 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\
 0 & 1 & \frac{1}{2} \binom{2}{1} & \cdots & \frac{1}{2} \binom{\theta-1}{1} & \frac{1}{2} \binom{\theta}{1} \\
 0 & 0 & 1 & \cdots & \frac{1}{2} \binom{\theta-1}{2} & \frac{1}{2} \binom{\theta}{2} \\
 \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
 \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
 \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
 0 & 0 & 0 & \cdots & 1 & \frac{1}{2} \binom{\theta}{\theta-1}
 \end{vmatrix}, \quad (105)$$

where $\theta = 1, 2, \dots$.

In view of the following relation:

$${}_sH E_\theta^{(\alpha)}(\phi, \xi, \omega) = {}_H H E_\theta^{(\alpha)}(\xi, D_\phi^{-1}, \omega),$$

we find the following determinant form for the multivariate generalized double Hermite-Euler polynomials (MVGdHEP) ${}_H H E_\theta^{(\alpha)}(\xi, D_\phi^{-1}, \omega)$:

$${}_{HH}E_0^{(\alpha)}(\xi, D_\phi^{-1}, \omega) = 1$$

$${}_{HH}E_\theta^{(\alpha)}(\xi, D_\phi^{-1}, \omega) = (-1)^\theta$$

$$\begin{vmatrix}
 1 & {}_HH_1(\xi, D_\phi^{-1}, \omega) & {}_HH_2(\xi, D_\phi^{-1}, \omega) & \cdots & {}_HH_\theta(\xi, D_\phi^{-1}, \omega) \\
 1 & \frac{\alpha}{2} & \frac{\alpha^2}{4} + \frac{\alpha}{4} & \cdots & \sum_{\rho=0}^{\theta} (\alpha)_\rho 2^{-\rho} S(\theta, \rho) \\
 0 & 1 & \alpha & \cdots & \binom{\theta}{1} \sum_{\rho=0}^{\theta-1} (\alpha)_\rho 2^{-\rho} S(\theta-1, \rho) \\
 0 & 0 & 1 & \cdots & \binom{\theta}{2} \sum_{\rho=0}^{\theta-2} (\alpha)_\rho 2^{-\rho} S(\theta-2, \rho) \\
 \vdots & \vdots & \vdots & \cdots & \vdots \\
 \vdots & \vdots & \vdots & \cdots & \vdots \\
 0 & 0 & 0 & \cdots & \binom{\theta}{\theta-1} \frac{\alpha}{2}
 \end{vmatrix}, \quad (106)$$

By finding the values of $S(\theta, \rho)$ from Equation (104) and substituting it in Equation (106) together with $\alpha = 1$, we find the following determinant form for the multivariate double Hermite-Euler polynomials (MVDHEP) ${}_{HH}E_\theta(\xi, D_\phi^{-1}, \omega)$:

$${}_{HH}E_0(\xi, D_\phi^{-1}, \omega) = 1,$$

$${}_{HH}E_\theta(\xi, D_\phi^{-1}, \omega) = (-1)^\theta$$

$$\begin{vmatrix}
 1 & {}_HH_1(\xi, D_\phi^{-1}, \omega) & {}_HH_2(\xi, D_\phi^{-1}, \omega) & \cdots & {}_HH_{\theta-1}(\xi, D_\phi^{-1}, \omega) & {}_HH_\theta(\xi, D_\phi^{-1}, \omega) \\
 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\
 0 & 1 & \frac{1}{2} \binom{2}{1} & \cdots & \frac{1}{2} \binom{\theta-1}{1} & \frac{1}{2} \binom{\theta}{1} \\
 0 & 0 & 1 & \cdots & \frac{1}{2} \binom{\theta-1}{2} & \frac{1}{2} \binom{\theta}{2} \\
 \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
 \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
 0 & 0 & 0 & \cdots & 1 & \frac{1}{2} \binom{\theta}{\theta-1}
 \end{vmatrix}, \quad (107)$$

where $\theta = 1, 2, \dots$.

Note. We also that the analytical approach developed in previous Section 2 can also be utilized to provide matrix representations for the two variable and three variable polynomials. If we expand the determinant (98) of Hermite polynomials along first row and then applying operational Equations (38) and (42) in resultant equation and after summing up the terms, we can obtain the matrix representation (99) of the 2VLeHP. In the same way, by appropriately applying the operational rules of the polynomials, we can construct the matrix representations of the polynomials, which we have derived by replacement technique in Section 5. Hence, both the analytical and matrix approaches are connected with each other in the sense that one can derive the matrix representations of the convoluted polynomials by utilizing operational rules of polynomials on respective matrix representations. Such techniques of obtaining matrix representations are carried in various researches, see [43,44].

6. Conclusions

This work presents a novel family of convoluted Legendre-Hermite polynomials known as the Legendre-Hermite Apostol type Frobenius-Euler polynomials. It begins with the construction of multivariate Legendre-Hermite polynomials. By using various generating function techniques, it is possible to obtain several significant analytical characterizations for these polynomials, including operational representations, explicit expressions,

recurrence relations, and complex identities and formulae. The recurrence relations, symmetric identities, and summation formulae that have been painstakingly derived enhance our understanding of these polynomial families. The brief algebraic description adds to a more sophisticated mathematical structure. The unique advantage of this study is that it makes a significant contribution to the field of special polynomials, which is always expanding and provides mathematicians and researchers with an invaluable resource. In summary, the key points and discoveries highlight the deep complexity of these polynomial families and their broad ramifications. This research opens up new avenues for investigation and application while also deepening our understanding and stimulating mathematical discourse. Further investigation could focus on the various applications of these polynomials in different mathematical fields, which would increase their relevance and adaptability. An appropriate combination of the operational approach with the integral transforms will lead to several interesting results to the theory of fractional operational calculus and the following study related to the generalized class of MVGATLeHFEP and their relatives will be considered as the next investigation.

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