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# Rigidity and Triviality of Gradient *r*-Almost Newton-Ricci-Yamabe Solitons

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**Abstract:** In this paper, we develop the concept of gradient $r$-Almost Newton-Ricci-Yamabe solitons (in brief, gradient $r$-ANRY solitons) immersed in a Riemannian manifold. We deduce the minimal and totally geodesic criteria for the hypersurface of a Riemannian manifold in terms of the gradient $r$-ANRY soliton. We also exhibit a Schur-type inequality and discuss the triviality of the gradient $r$-ANRY soliton in the case of a compact manifold. Finally, we demonstrate the completeness and noncompactness of the $r$-Newton-Ricci-Yamabe soliton on the hypersurface of the Riemannian manifold.

**Keywords:** $r$-Almost Newton-Ricci-Yamabe soliton; Riemannian manifold; triviality; Schur-type inequality

**MSC:** 53C20; 53C21; 53C25; 53C42

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## 1. Introduction

Every steady or increasing compact soliton is rigid for compact manifolds [1–4]. Furthermore, all diminishing compact solitons in dimensions 2 [5] and 3 [1] are rigid. According to Eminenti et al. [6], compact shrinking solitons are rigid in any dimension precisely when their scalar curvature is constant.

It should be noted that when a soliton is rigid, the scalar curvature remains constant and the "radial" curvatures vanish, meaning that $\mathcal{R}ic(\quad,\gamma)\gamma = 0$ [7]. On the other hand, we have observed that rigidity on compact solitons is implied by continuous scalar curvature or radial Ricci flatness, where $\mathcal{R}ic(\nabla\gamma, \nabla\gamma) = 0$ for each. It is easy to demonstrate that constant scalar curvature also implies stiffness in the noncompact steady situation [2,3].

For the first time, Hamilton concurrently proposed the theories of Ricci flow [5] and Yamabe flow [8] in 1988. The Ricci soliton and Yamabe soliton are the limiting solutions of the Ricci flow and Yamabe flow. In fact, the Yamabe soliton [9] coincides with the Ricci soliton for dimension $n = 2$, but when $n > 2$, the Yamabe and Ricci solitons are not the same, and the Yamabe soliton retains the conformal class [10].

For many geometers over the past twenty years, the theory of geometric flows, including Ricci flow, Yamabe flow, Einstein flow, and Ricci-Bourguignon has served as a source of inspiration [10–14]. A certain group of solutions in which the metric evolves through dilation and diffeomorphisms play a crucial role in investigating the singularities of the flows because they appear as acceptable singularity analogs [15].

The construction of Ricci-Yamabe solitons from a geometric flow that is a scalar composition of Ricci and Yamabe flow [16] was recently discussed by Siddiqi et al. in [17]. The Ricci-Yamabe flow of the form $(\delta, \varepsilon)$ is another name for this. The Riemannian multiple metric that gives rise to the Ricci-Yamabe flow is represented by

$$\partial_t g(t) = -2\delta\mathcal{R}ic(t) + \varepsilon\mathcal{R}(t)g(t), \quad g_0 = g(0), \quad t \in (a,b), \tag{1}$$

where the terms $\mathcal{R}ic$ and $\mathcal{R}$ refer to the Ricci tensor and scalar curvature, respectively. Additionally, the authors in [16] treated the Ricci-Yamabe flow of type $(\delta, \varepsilon)$, which is precisely

- Ricci flow [5] if $\delta = 1, \varepsilon = 0$ (Ricci solitons [5]).
- Yamabe flow [5] if $\delta = 0, \varepsilon = 1$ (Yamabe solitons [5]).
- Einstein flow [18] if $\delta = 1, \varepsilon = -1$ (Einstein solitons [18]).

As the limit of the Ricci-Yamabe flow solution, Ricci-Yamabe solitons naturally occur. This serves as a significant source of motivation for learning Ricci-Yamabe solitons. In the Ricci-Yamabe flow, a Ricci-Yamabe soliton is one that evolves exclusively by diffeomorphism and scales by a single parameter group. A Ricci-Yamabe soliton is a data $(g, F, \Lambda, \delta, \varepsilon)$ obeying the Riemannian manifold $(M, g)$.

$$\frac{1}{2}\mathfrak{L}_F g + \delta\mathcal{R}ic = \left(\Lambda + \frac{\varepsilon}{2}\mathcal{R}\right)g, \tag{2}$$

where $\mathfrak{L}_F$ shows the Lie derivative along the vector field $F$, and $\Lambda, \delta$, and $\varepsilon$ are real numbers. A Ricci-Yamabe soliton is called *shrinking*, *expanding*, or *steady*, depending on whether $\Lambda > 0$, $\Lambda < 0$, or $\Lambda = 0$, respectively.

Also, if (2) holds for $\Lambda$, $\delta$, $\varepsilon$ smooth functions, then, the soliton is called almost Ricci-Yamabe soliton [19,20].

If there exists a smooth function $\gamma : M \to \mathbb{R}$ such that $F = \nabla\gamma$, then the $(\delta, \varepsilon)$-type Ricci-Yamabe soliton is called a *gradient Ricci-Yamabe soliton* of type $(\delta, \varepsilon)$, denoted by $(M, g, \gamma, \Lambda, \delta, \varepsilon)$, and, in this case, (2) takes the form

$$\hbar ess(\gamma) + \delta\mathcal{R}ic = \left(\Lambda + \frac{\varepsilon}{2}\mathcal{R}\right)g, \tag{3}$$

where $\hbar ess$ is the Hessian of function $\gamma$, and $\gamma$ is called potential of the gradient Ricci-Yamabe soliton of type $(\delta, \varepsilon)$.

**Example 1.** *Let us take the example of the Einstein soliton, which produces solutions to the Einstein flow that are self-similar in such a manner that [18]*

$$\partial_t g(t) = -2\left(\mathcal{R}ic - \frac{\mathcal{R}}{2}g\right).$$

*Therefore, an Einstein soliton occurs as the limit of the Einstein flow solution, such that*

$$\frac{1}{2}\mathfrak{L}_F g + \mathcal{R}ic = \left(\Lambda - \frac{\mathcal{R}}{2}\right)g. \tag{4}$$

*Comparing Equations (2) and (4), we find a $(1, 1)$-type Ricci-Yamabe soliton.*

**Example 2.** *Let us take the example of the Riemann soliton, which produces solutions to the Riemann flow that are self-similar in such a manner that [21]*

$$\frac{\partial}{\partial t}G(t) = -2Rie(g(t)), \tag{5}$$

*where Rie is a $(0, 4)$-type Riemann curvature tensor generated by metric $g$ and $G = \frac{1}{2}g \odot g$ with Kulkarni–Nomizu product $\odot$, defined by*

$$\begin{aligned}(E \odot F)(U, V, W, Z) = {} & E(U, Z)F(V, W) + E(V, W)F(U, Z) \\ & -E(U, W)F(V, Z) - E(V, Z)F(U, W),\end{aligned} \tag{6}$$

*for any vector field $U, V, W, Z \in M$.*

**Definition 1.** *A Riemann soliton on a manifold M is specifically a particular solution of the Riemann flow equation and it is given by*

$$Rie + \frac{1}{2} g \odot \mathfrak{L}_F g + \frac{1}{2} \lambda g \odot g = 0. \tag{7}$$

*For a Riemann soliton, Equations (6) and (7) together entail the following*

$$\frac{2}{(n-2)} S_{ric}(U,V) + (\mathfrak{L}_F g)(U,V) + \frac{2[divU + (n-1)\lambda]}{(n-2)} g(U,V) = 0, \tag{8}$$

*where* $divU = -\frac{\mathcal{R}+n(n-1)\lambda}{2(n-1)}$*, where* $\mathcal{R}$ *indicates the scalar curvature of n-dimensional manifolds. Consequently, in light of Equation (8), the Riemann soliton is expressed by the following shape*

$$\frac{2}{(n-2)} S_{ric}(U,V) + (\mathfrak{L}_F g)(U,V) = \left\{ \frac{\mathcal{R}}{(n-1)(n-2)} - \lambda \right\} g(U,V) = 0, \tag{9}$$

*Now, after comparing Equations (2) and (8), we find a* $\left( \frac{2}{(n-2)}, \frac{1}{(n-1)(n-2)} \right)$*-type Ricci-Yamabe soliton.*

**Remark 1.** *In view of the above example, we can state that the Riemann soliton is a Ricci-Yamabe soliton.*

Based on the ideas of Cunha et al. [22,23], consider a connected and oriented hypersurface $\mathcal{M}^n$ that is immersed into a $(n+1)$-dimensional Riemannian manifold $\mathcal{N}^{n+1}$. For some $0 \leq r \leq n$, we declare that $\mathcal{M}^n$ is a gradient $r$-Almost Newton-Ricci-Yamabe soliton (gradient $r$-ANRY soliton) if the smooth function $\gamma : \mathcal{M}^n \to \mathbb{R}$ exists and the following equation holds:

$$\mathcal{P}_r \circ \hbar ess(\gamma) + \delta \mathcal{R}ic = \left( \Lambda + \frac{\varepsilon}{2} \mathcal{R} \right) g, \tag{10}$$

where $g$ denotes the Riemannian metric , $\Lambda$ indicates a smooth function on $\mathcal{M}$, and $\mathcal{R}$ symbolizes the scalar curvature of $\mathcal{M}$ with respect to g. In addition, $\mathcal{P}_r \circ \hbar ess(\gamma)$ illustrates the tensor generated by

$$\mathcal{P}_r \circ \hbar ess(\gamma)(U,V) = \langle \mathcal{P}_{\mathrm{r}} \nabla_U \nabla \gamma, V \rangle, \tag{11}$$

for tangent vector fields $U, V \in \mathfrak{X}(\mathcal{M})$. In [24], Shaikh et al. discussed the triviality in terms of Ricci solitons, which are closed in this paper. Moreover, Siddiqi et al. [25–27] also studied the notions of $r$-ANR solitons and $r$-ANY solitons.

The study of Equation (10) is fascinating since a gradient $r$-ANRY soliton is reduced to a gradient RY soliton when $r = 0$. Trivial refers to the gradient $r$-ANRY soliton whenever the potential $\gamma$ is constant. It is considered nontrivial if not. Additionally, we refer to the gradient $r$-ANRY soliton as a *gradient r-NRY soliton* when $\Lambda$ is a constant.

The structure of his manuscript is as follows. We review several fundamental details and notations that will appear throughout the work in Section 1. We approach the compact situation in Section 3 and demonstrate some trivial results. We also provide a Schur-type inequality. In Section 4, we investigate the entire case and, for some conditions on the potential function, find constant scalar curvature. Finally, in Section 5, we present some minimal $r$-Almost Newton-Ricci-Yamabe soliton nonexistence results. Additionally, we discuss that the gradient $r$-almost Newton-Yamabe soliton must be steady, totally geodesic, and flat, and, in particular circumstances, we discover that an immersed $r$-almost Newton-Yamabe soliton is isometric to the Euclidean sphere.

## 2. Preliminaries

Let $\mathcal{M}^n$ be a connected and oriented hypersurface immersed into a Riemannian manifold $\mathcal{N}^{(n+1)}$. The Gauss formula for immersion is well known to be given by

$$Rie(U,V)W = (\overline{Rie}(U,V)W)^\top + \langle \mathcal{A}U, W\rangle \mathcal{A}V - \langle \mathcal{A}V, W\rangle \mathcal{A}U$$

for tangent vector fields $U, V, W \in \mathfrak{X}(\mathcal{M})$, where $(\ )^\top$ stands for a vector field's tangential component in $\mathfrak{X}(\mathcal{M})$ along $\mathcal{M}^n$. In this case, $\mathcal{A} : \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ denotes the second fundamental form (or shape operator) of $\mathcal{M}^n$ in $\mathcal{N}^{n+1}$ with respect to a given orientation, and $\overline{Rie}$ and $Rie$ stand for the curvature tensors of $\mathcal{N}^{n+1}$ and $\mathcal{M}^n$, respectively. Specifically, the scalar curvature $\mathcal{R}$ of hypersurface $\mathcal{M}^n$ fulfills the requirements.

$$\mathcal{R} = \sum_{1\le i,j\le n} \langle \overline{Rie}(v_i, v_j)v_j, v_i\rangle + n^2\mathcal{H}^2 - |\mathcal{A}|^2, \tag{12}$$

where $\{v_1, \ldots, v_n\}$ is an orthonormal frame on $T\mathcal{M}$ and $|\cdot|$ indicates the Hilbert–Schmidt norm. In case of a space form $\mathcal{N}^{n+1}(c)$ of constant sectional curvature $c$, we have the value

$$\mathcal{R} = n(n-1)c + n^2\mathcal{H}^2 - |\mathcal{A}|^2. \tag{13}$$

Associated to the second fundamental form $\mathcal{A}$ of hypersurface $M^n$, there are $n-$ algebraic invariants, which are the elementary symmetric functions $\mathcal{R}_r$ of its principal curvatures $k_1, \ldots, k_n$, given by

$$\mathcal{R}_0 = 1 \ \text{ and } \ \mathcal{R}_r = \sum_{i_1<\ldots<i_r} k_{i_1}\cdots k_{i_r}.$$

The following equation describes the $r$-th mean curvature of the immersion

$$\binom{n}{r}\mathcal{H}_r = \mathcal{R}_r.$$

If $r = 1$, we have $\mathcal{H}_1 = \frac{1}{n}tr(\mathcal{A}) = \mathcal{H}$, the mean curvature of $\mathcal{M}^n$.

The $r$-th *Newton transformation* is defined as $\mathcal{P}_r : \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ for each $0 \le r \le n$. On the hypersurface $\mathcal{M}^n$ by using the identity operator ($\mathcal{P}_0 = I$) and the recurrence relation for $1 \le r \le n$

$$\mathcal{P}_r = \sum_{j=0}^{r} (-1)^{r-j}\binom{n}{j}\mathcal{H}_j\mathcal{A}^{(r)}, \tag{14}$$

where $j$ times ($\mathcal{A}^{(0)} = I$) represent the composition of $\mathcal{A}$ with $r$. Observe that the second order linear differential operator $\mathcal{L}_r$ is connected to each Newton transformation $\mathcal{P}_r$, defined by $\mathcal{L}_r : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$

$$\mathcal{L}_r(\gamma) = \mathrm{tr}(\mathcal{P}_r \circ \hbar\mathrm{ess}\ \gamma).$$

We observe that $\mathcal{L}_0$ is just the Laplacian operator for $r = 0$. Additionally, it is apparent that

$$\begin{aligned} \mathrm{div}_{\mathcal{M}}(\mathcal{P}_r\nabla\gamma) &= \sum_{i=1}^{n}\langle(\nabla_{v_i}\mathcal{P}_r)(\nabla\gamma), v_i\rangle + \sum_{i=1}^{n}\langle\mathcal{P}_r(\nabla_{v_i}\nabla\gamma), v_i\rangle \\ &= \langle \mathrm{div}_{\mathcal{M}}\mathcal{P}_r, \nabla\gamma\rangle + \mathcal{L}_r(\gamma), \end{aligned} \tag{15}$$

where the equation for the divergence of $\mathcal{P}_r$ on $\mathcal{M}^n$ is

$$\mathrm{div}_{\mathcal{M}}\mathcal{P}_r = \mathrm{tr}(\nabla\mathcal{P}_r) = \sum_{i=1}^{n}(\nabla_{v_i}\mathcal{P}_r)(v_i).$$

Because $\mathrm{div}_{\mathcal{M}}\mathcal{P}_r = 0$, Equation (15) is reduced to $\mathcal{L}_r(\gamma) = \mathrm{div}_{\mathcal{M}}(\mathcal{P}_r\nabla\gamma)$, in particular when the ambient space has constant sectional curvature (see [28] for more information). The following lemma gives useful conclusions.

**Lemma 1** ([28])**.** *If $\mathcal{M}$ is compact without boundary or if $\mathcal{M}$ is noncompact and $\gamma$ has compact support, then*

(i) $\int_{\mathcal{M}} \mathcal{L}_r(\gamma) = 0,$

(ii) $\int_{\mathcal{M}} \gamma\mathcal{L}_r(\gamma) = -\int_{\mathcal{M}} \langle \mathcal{P}_r \nabla\gamma, \nabla\gamma \rangle.$

The so-called traceless second fundamental form of the hypersurface, denoted by $\Phi = \mathcal{A} - \mathcal{H}I$, will likewise work for our purposes. Take into account that tr $(\Phi) = 0$ and $|\Phi|^2 = \mathrm{tr}(\Phi^2) = |\mathcal{A}|^2 - n\mathcal{H}^2 \geq 0$ are equivalent if and only if $\mathcal{M}^n$ is totally umbilical [29]. Let us study Yau's lemma, which is Theorem 3 of [30], to conclude this topic.

**Lemma 2.** *Let $\gamma$ be a non-negative smooth subharmonic function on a complete Riemannian manifold $\mathcal{M}^n$. If $\gamma \in \mathcal{L}^p(\mathcal{M})$, for some $p > 1$, then $\gamma$ is constant.*

Here, we adopt the symbol $\mathcal{L}^p(\mathcal{M}) = \{\gamma : \mathcal{M}^n \to \mathbb{R} | \int_{\mathcal{M}} |\gamma|^p <\}$, for each $p \geq 1$. Additionally, if the scalar curvature of $\mathcal{M}^n$ is constant, Equation (10) becomes valid.

$$\delta\mathcal{R}ic + \mathcal{P}_r \circ \hbar ess(\gamma) = \mu g, \tag{16}$$

where $\mu = \Lambda - \frac{\varepsilon}{2}\mathcal{R}$. So, we can recall Example 2 of [22] as another example of a gradient $r$-Almost-Newton Ricci-Yamabe soliton.

## 3. Results of Triviality

With the gradient $r$-Newton-Ricci-Yamabe soliton (gradient $r$-NRYS) closed and $\Lambda$ constant, we spend this part presenting our key findings. The Riemannian manifold with constant sectional curvature $c$ is denoted by the symbol $\mathcal{N}_c^{n+1}$ throughout the text. More specifically:

**Theorem 1.** *Let $(\mathcal{M}^n, g, \gamma, \Lambda, \delta, \varepsilon)$ be a closed gradient r-NRYS immersed into a Riemannian manifold $\mathcal{N}_c^{n+1}$ of constant sectional curvature c, such that $\mathcal{P}_r$ is bounded from above or from below (in the sense of quadratic forms). If any one of the following holds,*

*(i)* $\delta > \frac{-n\varepsilon}{2}$ *and* $\mathcal{R} \geq 0$ *and* $\Lambda \geq 0$, *or* $\mathcal{R} \leq 0$ *and* $\Lambda \leq 0$,

*(ii)* $\delta < \frac{-n\varepsilon}{2}$ *and* $\mathcal{R} \geq 0$ *and* $\Lambda \leq 0$, *or* $\mathcal{R} \leq 0$ *and* $\Lambda \geq 0$,

*(iii)* $\delta \neq \frac{-n\varepsilon}{2}$ *and either* $\mathcal{R} \geq \frac{2n\Lambda}{2\delta + n\varepsilon}$ *or* $\mathcal{R} \leq \frac{2n\Lambda}{2\delta + n\varepsilon}$,

*the scalar curvature of $\mathcal{M}^n$ is constant and $\mathcal{M}^n$ is trivial.*

**Proof.** In light of Lemma 1 and the structural equation, we obtain

$$0 = \int_{\mathcal{M}} \mathcal{L}_r(\gamma) = \int_M (n\Lambda - (2\delta + n\varepsilon)\mathcal{R}).$$

Therefore, if (i) and (ii) are true, we derive $\mathcal{R} = \Lambda = 0$ and $\mathcal{L}_r(\gamma) = 0$ from the structural equation. There is a positive constant $C > 0$ such that because the quadratic form of $\mathcal{P}_r$ is bounded above or below,

$$0 = \mathcal{L}_r(\gamma) \leq C\Delta\gamma \ \text{ or } \ 0 = \mathcal{L}_r(\gamma) \geq -C\Delta\gamma,$$

respectively. $\gamma$ is a subharmonic function as a result. Hopf's theorem tells us that $\gamma$ is a constant function since $\mathcal{M}$ is compact. Therefore, the soliton is trivial. Finally, (iii) follows identically to (i) and (ii). □

**Remark 2.** *Items (i) and (ii) in the above theorem entail that M is steady and* $\mathcal{R} = 0$. *Since* $\mathcal{M}^n$ *is trivial, we obtain* $\mathcal{R}ic \equiv 0$. *Finally, (iii) implies* $\mathcal{R} = \frac{-\Lambda n}{\rho n - 1}$. *Since* $\mathcal{M}$ *is trivial, we obtain*

$$\delta \mathcal{R}ic = \left(\Lambda - \frac{n\varepsilon\Lambda}{(2\delta + \varepsilon)}\right) g = \frac{2\delta\Lambda}{(2\delta + n\varepsilon)} g = \delta \frac{\mathcal{R}}{n} g,$$

*i.e.,* $\mathcal{M}^n$ *is Einstein.*

**Theorem 2.** *If* $(\mathcal{M}^n, \gamma, \Lambda, \delta, \varepsilon)$ *is a closed gradient r-NRYS immersed into a* $\mathcal{N}_c^{n+1}$, *such that* $\mathcal{P}_r$ *is bounded above or bounded below (in the sense of quadratic form) and* $\delta \neq \frac{-n\varepsilon}{2}$, *then, the scalar curvature of* $\mathcal{M}^n$ *is constant, and* $\mathcal{M}^n$ *is Einstein and trivial.*

**Proof.** In view of structural equation Lemma 1, we have

$$\int_{\mathcal{M}} |n\Lambda - (2\delta + n\varepsilon)\mathcal{R}|^2 = \int_{\mathcal{M}} (n\Lambda - (2\delta + n\varepsilon)\mathcal{R})\mathcal{L}_r(\gamma) = (n\Lambda - (2\delta + n\varepsilon)\mathcal{R}) \int_{\mathcal{M}} \mathcal{L}_r(\gamma) = 0.$$

Hence, we obtain $\mathcal{R} = \frac{2n\Lambda}{2\delta + n\varepsilon}$ and $\mathcal{L}_r(\gamma) = 0$. Adopting that $\mathcal{P}_r$ is bounded above or bounded below (in the sense of quadratic form) to demonstrate that $\mathcal{M}$ is trivial, we can adopt the same steps as in the proof of Theorem 1. Last but not least, since $M^n$ is trivial, we can move on to Remark 2 to conclude that $\mathcal{M}^n$ is Einstein. □

We established a Schur-type inequality in the following theorem.

**Theorem 3.** *Let* $(\mathcal{M}^n, g, \gamma, \Lambda, \delta, \varepsilon)$ *be a closed gradient r-NRYS immersed into a Riemannian manifold* $\mathcal{N}_c^{n+1}$ *of constant sectional curvature c, such that* $\mathcal{P}_\mathrm{r}$ *is bounded from below (in the sense of quadratic forms) and* $\delta > \frac{-n\varepsilon}{2}$. *Then,*

$$\int_{\mathcal{M}} |\mathcal{R} - \overline{\mathcal{R}}|^2 \leq \frac{nC}{(n-2)\left(\delta + \frac{n\varepsilon}{2}\right)} \, \|\mathring{\mathrm{Ric}}\|_{\mathcal{L}^2} \left\| \nabla^2\gamma - \frac{\Delta\gamma}{n} g \right\|_{L^2}. \tag{17}$$

**Proof.** The contracted second Bianchi identity states

$$\mathrm{div}(\mathcal{R}ic) - \frac{1}{2}\nabla\mathcal{R} = 0,$$

and hence

$$\mathrm{div}(\mathring{\mathcal{R}ic}) = \frac{n-2}{2n}\nabla\mathcal{R},$$

where $\mathring{\mathcal{R}ic}$ is the traceless Ricci tensor. Since $\mathcal{M}$ is compact, we obtain, using our assumption on $\mathcal{P}_r$, that $\langle \mathring{\mathcal{R}ic}, g \rangle = 0$. Provided that $\mathcal{M}$ is compact, we obtain

$$n\Lambda = (2\delta + n\varepsilon)\overline{\mathcal{R}},$$

where $\overline{\mathcal{R}}$ indicates the average of $\mathcal{R}$. Therefore,

$$(2\delta + n\varepsilon)^2 \int_M |\mathcal{R} - \overline{\mathcal{R}}|^2 = \int_M |n\Lambda - (2\delta + n\varepsilon)\mathcal{R}|^2,$$

i.e.,

$$(2\delta + n\varepsilon)^2 \int_{\mathcal{M}} |\mathcal{R} - \overline{\mathcal{R}}|^2 \leq \frac{2nC(2\delta + n\varepsilon)}{n-2} \| \mathring{\mathcal{R}ic} \|_{\mathcal{L}^2} \left\| \nabla^2\gamma - \frac{\Delta\gamma}{n} g \right\|_{\mathcal{L}^2},$$

i.e.,

$$\int_{\mathcal{M}} |\mathcal{R} - \overline{\mathcal{R}}|^2 \leq \frac{2nC}{(n-2)(2\delta + n\varepsilon)} \|(\mathring{\mathcal{R}ic})\|_{\mathcal{L}^2} \left\| \nabla^2\gamma - \frac{\Delta\gamma}{n} g \right\|_{\mathcal{L}^2}. \tag{18}$$

This completes the proof. □

**Remark 3.** *Due to the fact that both sides of the expression (17) diminish in the foregoing theorem if $M^{n\prime\prime}$ is Einstein, the equality is maintained. To demonstrate the rigidity would be a fascinating problem.*

## 4. Complete Noncompact *r*-Newton-Ricci-Yamabe Solitons

This section begins with the following finding.

**Theorem 4.** *Let $(\mathcal{M}^n, g, \gamma, \lambda, \delta, \varepsilon)$ be a complete r-NRYS immersed into a Riemannian manifold $\mathcal{N}_c^{n+1}$ of constant sectional curvature c, such that the potential function is non-negative and $\gamma \in \mathcal{L}^p(\mathcal{M})$, for some $p > 1$. If any one of the following holds,*

*(i) $\delta > \frac{-n\varepsilon}{2}$, $\mathcal{P}_r$ is bounded above (in the sense of quadratic forms), and $\mathcal{R} \geq \frac{2n\Lambda}{(2\delta+n\varepsilon)}$,*
*(ii) $\delta > \frac{-n\varepsilon}{2}$, $\mathcal{P}_r$ is bounded below (in the sense of quadratic forms), and $\mathcal{R} \leq \frac{2n\Lambda}{(2\delta+n\varepsilon)}$,*
*(iii) $\delta < \frac{-n\varepsilon}{2}$, $\mathcal{P}_r$ is bounded below (in the sense of quadratic forms), and $\mathcal{R} \geq \frac{2n\Lambda}{(2\delta+n\varepsilon)}$,*
*(iv) $\delta < \frac{-n\varepsilon}{2}$, $\mathcal{P}_r$ is bounded above (in the sense of quadratic forms), and $\mathcal{R} \leq \frac{2n\Lambda}{(2\delta+n\varepsilon)}$,*

*then, $\mathcal{R} = \frac{2n\Lambda}{(2\delta+n\varepsilon)}$, and $\mathcal{M}^n$ is Einstein trivial.*

**Proof.** Let $\mathcal{P}_r$ be bounded above (in the sense of quadratic form); there exists a positive constant $\sigma > 0$ such that

$$0 \leq n\Lambda - (2\delta + n\varepsilon)\mathcal{R} = \mathcal{L}_r(\gamma) \leq \sigma\Delta\gamma.$$

Thus, $\gamma$ is a subharmonic function, so in light of Lemma 2, we obtain that $\gamma$ is a constant. Therefore, $\mathcal{R} = \frac{2n\Lambda}{(2\delta+n\varepsilon)}$ and $\mathcal{M}$ is trivial. Since $\mathcal{M}$ is trivial, we have from the structural equation that

$$\delta\mathcal{R}ic = \left(\Lambda - \frac{n\varepsilon\Lambda}{(2\delta+\varepsilon)}\right)g = \frac{2\delta\Lambda}{(2\delta+n\varepsilon)}g = \delta\frac{\mathcal{R}}{n}g,$$

i.e., $\mathcal{M}^n$ is Einstein.

Eventually, if (ii) holds, there exists a positive constant $\sigma > 0$ such that

$$\mathcal{L}_r(\gamma) \geq -\sigma\Delta\gamma.$$

Hence, the same steps as for (i) are followed. Cases (iii) and (iv) are analogous. □

**Theorem 5.** *Let $(\mathcal{M}^n, g, \gamma, \Lambda, \delta, \varepsilon)$ be a complete gradient r-ARYS of dimension n immersed into a Riemannian manifold $\mathcal{N}_c^{n+1}$ of constant sectional curvature c, such that $\mathcal{M}$ has non-negative scalar curvature. Assume that the potential function f satisfies the condition*

$$\int_{\mathcal{M}_{\mathcal{B}(b,r)}} \frac{\gamma}{d(x,q)^2} < \infty, \tag{19}$$

*where $\mathcal{B}(b,r)$ is a ball with radius $r > 0$ and center at q, and $d(x,q)$ is the distance function from $q \in \mathcal{M}$. If any one of following holds,*

*(i) $\mathcal{M}$ is non-expanding, $\mathcal{P}_r$ is bounded above (in the sense of quadratic forms), and $\delta > \frac{-n\varepsilon}{2}$,*
*(ii) $\mathcal{M}$ is expanding, $\mathcal{P}_r$ is bounded below (in the sense of quadratic forms), and $\delta < \frac{-n\varepsilon}{2}$,*

*then, $\mathcal{R} = 0$.*

**Proof.** Let us look at item (i); item (ii) is analogous. Taking the trace in the structural equation, we obtain

$$n\Lambda - (2\delta + n\varepsilon)\mathcal{R} = \mathcal{L}_r(\gamma). \tag{20}$$

Consider a cut-off function that was proposed in [2], $\Psi_r \in C_0^\infty(\mathcal{B}(b,2r))$ for $r > 0$ such that

$$\begin{cases} 0 \le \Psi_r \le 1 & \text{in } \mathcal{B}(b,2r) \\ \Psi_r = 1 & \text{in } \mathcal{B}(b,r) \\ |\nabla\Psi_r|^2 \le \frac{C}{r^2} & \text{in } \mathcal{B}(b,2r) \\ \Delta\Psi_r \le \frac{C}{r^2} & \text{in } \mathcal{B}(b,2r), \end{cases} \tag{21}$$

wherein $C > 0$ is a constant. Now adopting (20), integration by parts, and that $\mathcal{P}_r$ is bounded from above (in the sense of quadratic forms), we obtain

$$\begin{aligned} 0 \le \int_{\mathcal{B}(b,2r)} \Psi_r \mathcal{R} &= \int_{\mathcal{B}(b,2r)} \Psi_r \left( \frac{1}{-(2\delta + n\varepsilon)} \mathcal{L}_r(\gamma) + \frac{n\Lambda}{(2\delta + n\varepsilon)} \right) \\ &\le \frac{1}{-(2\delta + n\varepsilon)} \int_{\mathcal{B}(b,2r)} \Psi_r \mathcal{L}_r(\gamma) \le \frac{C_1}{-(2\delta + n\varepsilon)} \int_{\mathcal{B}(b,2r)} \Psi_r \Delta\gamma \\ &\le \frac{C_1}{-(2\delta + n\varepsilon)} \int_{\mathcal{B}(b,2r) - \mathcal{B}(b,r)} \gamma \Delta\Psi_r \\ &\le \frac{C_1}{-(2\delta + n\varepsilon)} \int_{B(q,2r) - B(q,r)} \frac{C_2}{r^2}\gamma \to 0, \end{aligned}$$

as $r \to \infty$. Since $\psi_r = 1$ in $B(q,r)$, with inequality from above, we have $\mathcal{R} = 0$. This concludes the proof. □

**Remark 4.** *The theorem above still guarantees that a gradient r-ANRYS is in fact a gradient r-ANRS in [22]. Therefore, any gradient r-ANRYS satisfying the conditions of Theorem 5 is a gradient r-ANRS with scalar curvature $\mathcal{R} = 0$.*

**Theorem 6.** *Let $(\mathcal{M}^n, g, \gamma, -\frac{n\varepsilon}{2})$ be a non-expanding gradient traceless r-NRYS, such that $\mathcal{P}_r$ is bounded from above (in the sense of quadratic form) with a non-negative potential function $\gamma$. If $\gamma \in \mathcal{L}^p(\mathcal{M})$, for some $p > 1$, then $\mathcal{M}$ is steady, Einstein, and trivial.*

**Proof.** Given that for non-expanding solitons, $\Lambda \ge 0$, it follows from the structural equation that

$$\mathcal{L}_r(\gamma) = n\Lambda \ge 0.$$

From the hypothesis on $P_r$, there exists a positive constant $C > 0$ such that

$$0 \le n\Lambda = L_r(\gamma) \le C\Delta\gamma,$$

i.e., $\gamma$ is a subharmonic non-negative function. Hence, from Lemma 2, we have $\gamma$ constant, and $0 = \Delta\gamma \ge n\Lambda \ge 0$. Therefore $\Lambda = 0$ and $\mathcal{M}$ is trivial. Finally, since $\mathcal{M}$ is trivial, we obtain $\mathcal{R}ic = \frac{\mathcal{R}}{n} g$, so $M$ is Einstein. □

## 5. Nonexistence Results

The following lemma from Caminha et al. [31] will be used in this section:

**Lemma 3.** *Let $E$ be a smooth vector field on the n-dimensional, non-compact, complete, oriented Riemannian manifold $\mathcal{M}^n$, such that $div_{\mathcal{M}} E$ does not change the sign on $\mathcal{M}$. If $|E| \in \mathcal{L}^1(\mathcal{M})$, then $div_{\mathcal{M}} E = 0$.*

By following the idea of Cunha et al.'s theory [22],

**Theorem 7.** *If $(\mathcal{M}^n, g, \gamma, \Lambda, \delta, \varepsilon)$ is a complete r-ANRYS immersed into a Riemannian manifold $\mathcal{N}_c^{n+1}$ of constant sectional curvature $c$, with bounded second fundamental form and potential function $\gamma : \mathcal{M}^n \to \mathbb{R}$ such that $|\nabla\gamma| \in \mathcal{L}^1(\mathcal{M})$, then, we have*

(i) *if* $c \leq 0$, $\Lambda > 0$, *and* $\delta < \frac{-n\varepsilon}{2}$, *then* $\mathcal{M}^n$ *cannot be minimal,*
(ii) *if* $c < 0$, $\Lambda \geq 0$, *and* $\delta < \frac{-n\varepsilon}{2}$, *then* $\mathcal{M}^n$ *cannot be minimal,*
(iii) *if* $c = 0$, $\Lambda \geq 0$, $\delta < \frac{-n\varepsilon}{2}$, *and* $\mathcal{M}^n$ *is minimal, then* $\mathcal{M}^n$ *is steady and isometric to the Euclidean space.*

**Proof.** By using Equation (15), we can determine that the operator $\mathcal{L}_r$ is a divergent-type operator if the ambient space has a constant sectional curvature. On the other hand, since the Newton transformation $\mathcal{P}_r$ has a bounded norm, it follows from (14) that $\mathcal{M}^n$ has a bounded second fundamental form. More specifically,

$$|\mathcal{P}_r \nabla \gamma| \leq |\mathcal{P}_r||\nabla \gamma| \in \mathcal{L}^1(\mathcal{M}). \tag{22}$$

Since $\mathcal{M}^n$ is minimal, and using (i) and (ii), the scalar curvature of $\mathcal{M}^n$ thus fulfills $\mathcal{R} \leq 0 (\mathcal{R} < 0)$ according to equation (13) and the assumption that $c \leq 0$ $(c < 0)$. Thus, contracting (10), we find that

$$L_r(\gamma) = n\Lambda - (2\delta + n\varepsilon)\mathcal{R} > 0.$$

In both situations, the fact that follows contradicts Lemma 3. The first two claims are now validated by this. Given that the ambient space has constant sectional curvature $c = 0$ and that $\mathcal{M}^n$ is minimal [32] for claim (iii), Equation (13) becomes applicable.

$$\mathcal{R} = -|\mathcal{A}|^2 \leq 0. \tag{23}$$

So, since $\Lambda \geq 0$ and $\rho < \frac{1}{n}$, we have

$$L_r(\gamma) = n\Lambda - (2\delta + n\varepsilon)\mathcal{R} \geq 0.$$

Now, using the fact that $\mathcal{L}_r$ is a divergent-type operator and $|\mathcal{P}_r \nabla \gamma| \in \mathcal{L}^1(\mathcal{M})$, again from Lemma 3, we have $\mathcal{L}_r \gamma = 0$ on $\mathcal{M}^n$. So, we obtain the conclusion that $\mathcal{R} \geq 0$, i.e., $\frac{2n\Lambda}{(2\delta + n\varepsilon)} \geq 0$, and, $\mathcal{R} = \Lambda = 0$. This means that $|\mathcal{A}|^2 = 0$. Hence, the gradient $r$-ANRYS is steady, totally geodesic, and flat. □

Additionally, we may prove the following conclusion, which is valid when the ambient space is an arbitrary Riemannian manifold.

**Theorem 8.** *Let* $(\mathcal{M}^n, g, \gamma, \Lambda, \delta, \varepsilon)$ *be a complete r-ANRYS immersed into a Riemannian manifold* $\mathcal{N}^{n+1}$ *of sectional curvature* $\kappa$*, such that* $\mathcal{P}_r$ *is bounded from above (in the sense of quadratic form) and its potential function* $\gamma : \mathcal{M}^n \to \mathbb{R}$ *is non-negative, and* $\gamma \in \mathcal{L}^p(\mathcal{M})$*, for some* $p > 1$*. Then, we have*

(i) *if* $\kappa \leq 0$, $\Lambda > 0$, *and* $\delta < \frac{-n\varepsilon}{2}$, *then* $\mathcal{M}^n$ *cannot be minimal,*
(ii) *if* $\kappa < 0$, $\Lambda \geq 0$, *and* $\delta < \frac{-n\varepsilon}{2}$, *then* $\mathcal{M}^n$ *cannot be minimal,*
(iii) *if* $\kappa \leq 0$, $\Lambda \geq 0$, $\delta < \frac{-n\varepsilon}{2}$, *and* $\mathcal{M}^n$ *is minimal, then* $\mathcal{M}^n$ *is steady, flat, and totally geodesic.*

**Proof.** For proving (i), the sectional curvature of the ambient space and the Equation (12) suggest that $\mathcal{R} \leq 0$ because we assume that $\mathcal{M}^n$ is minimal. Consequently, by reducing Equation (10), we have

$$\mathcal{L}_r(\gamma) = n\Lambda - \left(\delta + \frac{n\varepsilon}{2}\right)\mathcal{R} > 0. \tag{24}$$

There exists a positive constant $C > 0$ such that, given that we are assuming that $\mathcal{P}_r$ is bounded above,

$$C\Delta\gamma \geq \mathcal{L}_r(\gamma) > 0. \tag{25}$$

Lemma 2, in particular, leads to the conclusion that $\gamma$ must be a constant, which is contradictory. Therefore, using the same logic used to prove Theorem 7, we can easily derive (ii) and (iii). □

**Example 3.** *Let us consider the standard immersion of $\mathbb{S}^n$ into $\mathbb{S}^{n+1}(c)$, which we know is totally geodesic, for $c = 0$ and $c = 1$. In particular, $P_r \equiv 0$, for all $1 \leq r \leq n$, and choosing $\Lambda = \frac{2\delta-(n-1)\varepsilon}{2(n-1)}$, we obtain that Equation (10) is fulfilled by the immersion. Moreover, for $c = -1$, it is well known that the hyperbolic space $\mathbb{H}^n$ is the only totally geodesic hypersurface immersed into the $\mathbb{S}^{n+1}$.*

For the situation when $U = \nabla\gamma$, we re-establish Theorem 1.5 of [33], providing the prerequisites for a $r$-ANRYS to be totally umbilical because it has a bounded second fundamental form. As a result, we establish the following theorem.

**Theorem 9.** *Let $(\mathcal{M}^n, g, \gamma, \Lambda, \delta, \varepsilon)$ be a complete r-ANRYS immersed into a Riemannian manifold $\mathcal{N}^{n+1}_{(c)}$ of constant sectional curvature c, with bounded second fundamental form and potential function $\gamma : M^n \to \mathbb{R}$, such that $|\nabla\gamma| \in \mathcal{L}^1(\mathcal{M})$. Then, we have*

*(i) if $\delta < \frac{-n\varepsilon}{2}$ and $\Lambda \geq n(\delta + \frac{n\varepsilon}{2})H^2 - (n-1)(\delta + \frac{n\varepsilon}{2})c$, then, $\mathcal{M}^n$ is totally geodesic, such as $\Lambda = (\delta + \frac{n\varepsilon}{2})(n-1)c$, and scalar curvature $\mathcal{R} = n(n-1)c$,*

*(ii) if $\mathcal{M}^n$ is compact, $\delta < \frac{-n\varepsilon}{2}$, and $\Lambda \geq n(\delta + \frac{n\varepsilon}{2})H^2 - (n-1)(\delta + \frac{n\varepsilon}{2})c$, then, $\mathcal{M}^n$ is isometric to a Euclidean sphere $\mathbb{S}^n$,*

*(iii) if $\delta < \frac{-n\varepsilon}{2}$ and $\Lambda \geq (n+1)(\delta + \frac{n\varepsilon}{2})H^2 - (n-1)(\delta + \frac{n\varepsilon}{2})c$, then, $\mathcal{M}^n$ is totally umbilical and the scalar curvature $\mathcal{R} = n(n-1)c - n(n+1)\kappa_{\mathcal{M}}$ is constant, where $\kappa_{\mathcal{M}} = \left[\frac{\Lambda}{(n+1)(\delta+\frac{n\varepsilon}{2})} + \frac{n-1}{n+1}c\right]$ is the sectional curvature of $\mathcal{M}^n$.*

**Proof.** For proving (i), we can use Equation (13) and the structural equation to obtain

$$\mathcal{L}_r(\gamma) = n\Big[\Lambda + (n-1)\Big(\delta + \frac{n\varepsilon}{2}\Big)c - n\Big(\delta + \frac{n\varepsilon}{2}\Big)\mathcal{H}^2\Big] - \Big(\delta + \frac{n\varepsilon}{2}\Big)|\mathcal{A}|^2. \tag{26}$$

After that, we conclude from our analysis of $\Lambda$ that $\mathcal{L}_r(\gamma)$ is a non-negative function on $\mathcal{M}^n$. Lemma 3 enables us to determine that $\mathcal{L}_r(\gamma) = 0$. Therefore, we conclude from Equation (26) that $Mn$ is totally geodesic and $\Lambda = (\delta + \frac{n\varepsilon}{2})(n-1)c$. Additionally, it is evident from the structural equation that

$$\mathcal{R} = \frac{n\Lambda}{\delta + \frac{n\varepsilon}{2}} = n(n-1)c,$$

that conclusively proves (i).

Given that $\mathcal{M}^n$ is totally geodesic and therefore compact, $\mathcal{M}^n$ is isometric to the Euclidean sphere ($\mathbb{S}^n$), demonstrating that the ambient space must necessarily be a sphere (ii).

For assertion (iii), we start with Equation (26), which can be expressed in terms of the traceless second basic form $\Phi$

$$\begin{aligned}\mathcal{L}_r(\gamma) = n[\Lambda + (n-1)\Big(\delta + \frac{n\varepsilon}{2}\Big)c \\ -(n+1)\Big(\delta + \frac{n\varepsilon}{2}\Big)\mathcal{H}^2] - \Big(\delta + \frac{n\varepsilon}{2}\Big)|\Phi|^2.\end{aligned} \tag{27}$$

Consequently, $\mathcal{L}_r(\gamma) \geq 0$ is the result of our assumption that $\Lambda$ and $\delta + \frac{n}{\varepsilon^2}$ are equal. Then, by once more using Lemma 3, we obtain $\mathcal{L}_r(\gamma) = 0$. This suggests that $\mathcal{M}^n$ is a totally umbilical hypersurface since $|\Phi|^2 = 0$. In particular, the principal curvature $\rho$ of $M^n$ is constant and $\mathcal{M}^n$ has a constant sectional curvature provided by $\kappa_{\mathcal{M}} = c + \rho^2$. This, along with (27), implies

$$\Lambda = (n+1)\Big(\delta + \frac{n\varepsilon}{2}\Big)\mathcal{H}^2 - (n-1)\Big(\delta + \frac{n\varepsilon}{2}\Big)c \tag{28}$$

$$= (n+1)\left(\delta + \frac{n\varepsilon}{2}\right)(c+\rho^2) - (n-1)\left(\delta + \frac{n\varepsilon}{2}\right)c$$
$$= (n+1)\left(\delta + \frac{n\varepsilon}{2}\right)\kappa_{\mathcal{M}} - (n-1)\left(\delta + \frac{n\varepsilon}{2}\right)c.$$

Since $\mathcal{L}_r(\gamma) = 0$, we obtain

$$\mathcal{R} = \frac{n(n-1)\Lambda}{\delta + \frac{n\varepsilon}{2}} = n(n-1)c - n(n+1)K_M,$$

as desired. □

**Example 4.** *Let us consider the standard immersion of the n-sphere $\mathbb{S}^n$ into Euclidean space $\mathbb{R}^{n+1}$ endowed with induced metric g. According to [22], by choosing the functions*

$$\Lambda_\alpha(x) = -g(x,\alpha) + n - 1$$

*and*

$$\gamma(\alpha)(x) = -\Lambda_\alpha + c,$$

*where $\alpha \in \mathbb{R}^{n+1}$, $\alpha \neq 0$, $c \in \mathbb{R}$, and $x = (x_1, ..., x_{n+1}) \in \mathbb{S}^n$ is the position vector, we obtain that $(\mathbb{S}^n, g, \nabla\gamma_\alpha, \Lambda_\alpha)$ satisfies (10).*

*On the other hand, it is well known that $\mathbb{S}^{n+1}$ is totally umbilical with r-th mean curvature $\mathcal{H}_r = 1$ and second fundamental form $\mathcal{A} = I$. In particular, for every $0 \leq r \leq (n-1)$, the Newton tensors are given by $\mathcal{P}_r = aI$, where $a = \sum_{j=0}^{r}(-1)^{r-j}\binom{n}{j}$. Hence, taking the smooth function $\gamma = a^{-1}\gamma_\alpha$, we obtain that the immersion satisfies Equation (10).*

We can now assert the following result of Theorem 9.

**Corollary 1.** *Let $(\mathcal{M}^n, g, \gamma, \Lambda, \delta, \varepsilon)$ be a compact r-ANRYS immersed into $\mathbb{R}^{n+1}$, such that $\delta < \frac{-n\varepsilon}{2}$. If $\Lambda \geq (n+1)(\delta + \frac{n\varepsilon}{2})\mathcal{H}^2$, then $\mathcal{M}^n$ is isometric to $\mathbb{S}^n$.*

In Theorem 1.6 [33], it was proved that a nontrivial ARS $M^n$ minimally immersed into $\mathbb{S}^{n+1}$ with $\mathcal{R} \geq n(n-2)$ and the norm of second fundamental form $\mathcal{A}$ must be isometric to $\mathbb{S}^n$ in order for it to gain its maximum value [34]. We now establish a generalization of this result by using Theorem 9.

**Corollary 2.** *Let $(\mathcal{M}^n, g, \gamma, \Lambda, \delta, \varepsilon)$ be a complete r-ANRYS minimally immersed into $\mathbb{S}^{n+1}$, such that $\delta < \frac{-n\varepsilon}{2}$. Assume that $\mathcal{R} \geq n(n-2)$, the norm of the second fundamental form attains its maximum, and $\Lambda \geq (n-1)(\delta + \frac{n\varepsilon}{2})$. Then, $\mathcal{M}^n$ is isometric to $\mathbb{S}^n$.*

**Proof.** Using the minimality of the immersion and that $\mathcal{R} \geq n(n-2)$, from (13) we obtain that

$$|\mathcal{A}|^2 = n(n-1) - \mathcal{R} \leq n.$$

By Simons's formula [35], we obtain

$$\Delta|\mathcal{A}|^2 = |\nabla\mathcal{A}|^2 + (n - |\mathcal{A}|^2)|\mathcal{A}|^2 \geq 0. \tag{29}$$

Hopf's strong maximum principle can be used to prove that $\nabla\mathcal{A} = 0$ on $\mathcal{M}^n$. As a result, Proposition 1 of [36] provides that $\mathcal{M}^n$ must be compact, and we infer the conclusions from Theorem 9. □

We can also arrive at the following theorem by applying Lemma 2.

**Theorem 10.** *Let $(\mathcal{M}^n, g, \gamma, \Lambda, \delta, \varepsilon)$ be a complete r-ANRYS immersed into a Riemannian manifold $\mathcal{N}^{n+1}_{(c)}$ of constant sectional curvature c , such that $\mathcal{P}_r$ is bounded from above (in the sense of quadratic form) with non-negative potential function $\gamma \in \mathcal{L}^p(M)$, for some $p > 1$. Then, we have the following:*

*(i)* *if $\delta < \frac{-n\varepsilon}{2}$ and $\Lambda \geq n(\delta + \frac{n\varepsilon}{2})H^2 - (n-1)(\delta + \frac{n\varepsilon}{2})c$, then, $\mathcal{M}^n$ is totally geodesic, with $\Lambda = (\delta + \frac{n\varepsilon}{2})(n-1)c$, and scalar curvature $\mathcal{R} = n(n-1)c$,*

*(ii)* *if $\delta < \frac{-n\varepsilon}{2}$ and $\Lambda \geq (n+1)(\delta + \frac{n\varepsilon}{2})H^2 - (n-1)(\delta + \frac{n\varepsilon}{2})c$, then, $\mathcal{M}^n$ is totally umbilical. In particular, the scalar curvature $\mathcal{R} = n(n-1)c - n(n+1)\kappa_{\mathcal{M}}$ is constant, where $\kappa_{\mathcal{M}} = \frac{\Lambda}{(n+1)(\delta + \frac{n\varepsilon}{2})} + \frac{n-1}{n+1}c$ is the sectional curvature of $\mathcal{M}^n$.*

**Proof.** Notice that from Equation (26) and the assumption on $\Lambda$, we obtain

$$\mathcal{L}_r(\gamma) = n\left[\Lambda + (n-1)\left(\delta + \frac{n\varepsilon}{2}\right)c - n\left(\delta + \frac{n\varepsilon}{2}\right)\mathcal{H}^2\right] - \left(\delta + \frac{n\varepsilon}{2}\right)|\mathcal{A}|^2 \geq 0. \tag{30}$$

Since we are assuming that $\mathcal{P}_r$ is bounded from above, there exists a positive constant $C > 0$ such that

$$C\Delta\gamma \geq \mathcal{L}_r(\gamma) \geq 0. \tag{31}$$

It follows from Lemma 2 that $\gamma$ must be a constant. In light of the fact that $\mathcal{L}_r(\gamma) = 0$, Equation (30) leads us to the conclusion that $\mathcal{M}^n$ is totally geodesic with

$$\Lambda = -(n-1)\left(\delta + \frac{n\varepsilon}{2}\right)c \text{ and } \mathcal{R} = n(n-1)c,$$

which provides the proof for (i). In conclusion, assertion (ii) is easily proven by using the same logic as in Theorem 9. □

## 6. Conclusions

We introduced the concept of gradient type $r$-almost-Newton-Ricci-Yamabe solitons immersed into a Riemannian manifold, which extends the notion of Ricci solitons and Yamabe solitons to immersions to constant sectional curvature space. These new objects were approached through nonexistence result and characterizations. We also proved some triviality results for the compact case and, under some conditions, we obtained constant scalar curvature.

$r$-almost Newton Ricci-Yamabe solitons submerged into a Riemannian manifold were the framework of this research. We gained the triviality criteria for compact gradient $r$-Almost Newton-Ricci-Yamabe solitons. Our computation concentrated on the hypersurface of a Riemannian manifold that has a bounded second fundamental form, and the conditions for a $r$-Almost Newton-Ricci-Yamabe soliton on the hypersurface to be totally umbilical were met. It was also shown that the steady $r$-Almost Newton-Ricci-Yamabe soliton admits a complete $r$-almost Newton-Ricci-Yamabe soliton on the hypersurface of Riemannian manifolds. Furthermore, we deduced Hopf's strong maximum principle and a Schur-type inequality in terms of the immersed $r$-Almost Newton-Ricci-Yamabe soliton in Riemannian manifolds which is compact and totally geodesic. Additionally, our findings contribute to understanding that the Euclidean sphere $\mathcal{S}^{4m}$ is isometric to the immersed $r$-Almost Newton-Ricci-Yamabe soliton in Riemannian manifolds.

**Future Work:** We can anticipate studying submanifolds in ambient space forms, such as Lorentzain manifolds, almost normal contact manifolds, and paraconatct manifolds, in the characterization of our primary conclusion. Furthermore, we can investigate the setting of Riemann solitons [21] in different ambient spaces that have some kind of induced connection, such as non-metric and semi-symmetric connections. Furthermore, $r$-almost-Newton-Ricci-Yamabe solitons can be established. Golden Riemannian manifolds with constant sectional curvature space will be a new and interesting problem.

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