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Almost Periodic Solutions of Differential Equations with Generalized Piecewise Constant Delay

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Abstract: In this paper, we investigate differential equations with generalized piecewise constant delay, DEGPCD in short, and establish the existence and stability of a unique almost periodic solution that is exponentially stable. Our results are derived by utilizing the properties of the (μ_1, μ_2)-exponential dichotomy, Cauchy and Green matrices, a Gronwall-type inequality for DEGPCD, and the Banach fixed point theorem. We apply these findings to derive new criteria for the existence, uniqueness, and convergence dynamics of almost periodic solutions in both the linear inhomogeneous and quasilinear DEGPCD systems through the (μ_1, μ_2)-exponential dichotomy for difference equations. These results are novel and serve to recover, extend, and improve upon recent research.

Keywords: almost periodic solutions; almost periodic sequences; almost periodic functions; piecewise constant argument of generalized type; exponential dichotomy; stability of solutions; Gronwall integral inequality

MSC: 34C27; 34D20; 34A38; 34D09; 26D10

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1. Introduction

Almost periodic differential equations have diverse applications across various fields, modeling complex phenomena in systems such as electrical networks, ecological models, mechanical vibrations, celestial mechanics, and engineering technologies. Their broad applicability in both scientific and engineering contexts has driven significant advancements in these equations over the past three decades. Although numerous studies have focused on the qualitative properties of periodic solutions [1–5], research on almost periodic solutions offers even wider-ranging, adaptable applications by incorporating the variable, time-dependent coefficients typical in non-periodic intervals (see [6–10]).

In mathematical analysis, almost periodic solutions describe functions that exhibit a regularity akin to periodic functions but lack an exact repeating period. This concept applies to solutions of differential equations or dynamical systems where behavior approximates periodicity, showing slight temporal variations. Essentially, these solutions remain "close" to being periodic, maintaining boundedness and recurrence without a fixed period, making them particularly useful in modeling complex or quasi-repetitive phenomena in physics, biology, and engineering (see [11–16]).

Traditional tools for resolving qualitative problems in periodic models are generally unsuitable for almost periodic cases due to operator compactness constraints. Although progress has been made over recent decades, many challenges remain, with several issues scarcely explored in the literature. Thus, advancing the study of almost periodic differential equations is both timely and valuable.

The investigation of differential equations with piecewise constant arguments (DEPCA) was first introduced by A. D. Myshkis in 1977 [17]. The formal exploration of piecewise constant systems began in the early 1980s. Since then, DEPCAs have garnered significant attention from researchers across various fields, including mathematics, biology, and engineering, as they offer valuable models for numerous real-world phenomena. These phenomena are often best represented by systems governed by differential equations with piecewise constant arguments, which are typically classified as discontinuous systems. Their behavior tends to be more complex and richer compared to conventional continuous systems governed by smooth differential equations. The interest in DEPCA systems stems from their nature as hybrids between discrete and continuous dynamical systems, blending characteristics of both differential and differential-difference equations. DEPCAs are closely related to impulse and difference equations, and their study has become increasingly prominent over the past two decades.

The first significant contribution to the study of differential equations with piecewise constant arguments (DEPCAs) was made by S. M. Shah and J. Wiener in 1983 [18]. In the same year, J. Wiener extended this work by investigating DEPCAs with delayed arguments (DEPCDs) in [19]. In 1984, K. L. Cooke and J. Wiener explored the existence and uniqueness of solutions for the DEPCD, as well as its backward continuation on $(-\infty, 0]$, and they examined the asymptotic stability of the zero solution [20]. Research on the oscillation properties and stability of DEPCAs can be found in [4,21–23] and the references therein. The existence of periodic solutions has been addressed in works such as [2,3,24,25] and the references provided in these studies. In 1991, K. L. Cooke and J. Wiener presented a survey summarizing the results in the areas of DEPCD and DEPCA, encompassing stability, oscillation properties, and the existence of periodic solutions [26]. Further exploration of DEPCAs was conducted by J. Wiener in 1993 and L. Dai in 2008, who compiled the advancements and applications of DEPCAs in their respective books [27,28]. Book [27] introduces techniques for assessing and investigating nonlinear systems with piecewise constant characteristics, widely applicable in domains including mathematics, physics, and engineering, grounded in both practical and theoretical perspectives. It reviews recent strategies for examining and numerically resolving nonlinear dynamics of such systems. A newly proposed greatest-integer argument utilizing a piecewise constant function serves to analyze nonlinear dynamics and introduces an innovative criterion to differentiate chaotic and irregular behaviors from structured solutions in nonlinear systems.

Since 2009, Dr. Manuel Pinto, a pioneer in the study of DEPCAs and their generalized types, along with his collaborators has laid the groundwork for what has now developed into a significant body of research. In 2010, Dr. Manuel Pinto and his team extended Gronwall's lemma for DEPCAs with a generalized type in [2], comparing two existing Gronwall-type inequalities and demonstrating improved estimates essential for analyzing the existence and uniqueness of solutions for DEPCAs of generalized type (DEPCAGs). This finding was later broadened to include impulsive effects in [25], marking a critical advancement in applying Gronwall's inequality to impulsive DEPCAGs. In their work [2], the authors expanded the parameter variation formula originally developed in [26] for generalized piecewise constant argument and extended the well-known Gronwall inequality for application to DEPCAGs, notably achieving this without requiring typical positivity conditions. Later, in 2013, the authors achieved the first generalization of the work in [29] to investigate oscillatory and periodic solutions within DEPCAG systems [30]. In 2014, through collaboration with J.-C. Jeng, they utilized MATLAB to simulate and approximate DEPCAG solutions, thereby facilitating the study of the global convergence of periodic solutions in DEPCAG neural network models [31]. In 2015, Dr. Manuel Pinto and his collaborators [32,33] conducted pioneering research on the existence, computational viability, and stability of solutions for diffusion equations incorporating a generalized piecewise constant argument. Additionally, they examined essential and adequate conditions for the controllability and observability of linear time-varying control systems featuring piecewise constant state variables. In 2018, the team employed a Green-type matrix to examine asymptotic equivalence in DEPCAG systems, as discussed in [34]. Subsequently, in [35], they established sufficient conditions for the existence of a uniformly Hölder continuous homeomorphism between the solutions of a linear differential system with piecewise constant argument and those of a perturbed system. This analysis relied primarily on a newly introduced concept of exponential dichotomy. Then, in 2022, applying the technique developed in [31] and the Green's function, the impulsive effect was incorporated to analyze neural network models with DEPCAGs, addressing both delay-type [3] and advanced-retarded alternating arguments [36], without relying on Razumikhin-type techniques or constructing a Lyapunov function from previous literature. Also, an optimal condition was established to ensure the existence of periodic solutions. Additionally, in [37], a sufficient condition was introduced to ensure global exponential stability of equilibrium points in BAM neural network models with DEGPCD systems using linearization methods and fixed-point theorem approaches. Numerous subsequent studies have continued this research path, as evidenced by Dr. Manuel Pinto and his collaborators' contributions in works such as [7,30–33,35,38–43]. This progress has significantly boosted academic publications on DEPCAs, as reflected in these references.

In 1997, R. Yuan and colleagues [44] formulated adequate criteria to ensure the existence and uniqueness of almost periodic solutions for the subsequent systems of differential equations with piecewise constant delay (DEPCD):

$$x'(t) = A(t)x(t) + B(t)x([t]) + f(t), \quad t \in \mathbb{R}$$
$$x'(t) = A(t)x(t) + B(t)x([t]) + g(t, x(t), x([t])), \quad t \in \mathbb{R},$$

where A and B are $q \times q$ matrix-valued functions, f and g are continuous functions, and $[\cdot]$ denotes the floor function.

In a noteworthy study, Xia et al. [45] (2007) applied the concepts of exponential dichotomy and the Banach fixed-point theorem to establish sufficient conditions ensuring the existence and uniqueness of almost periodic solutions for the DEPCD of the form

$$y'(t) = A(t)y(t) + B(t)y([t]) + h(t) + \kappa g(t, y(t), y([t]), \kappa), \quad t \in \mathbb{R},$$

as well as for the more general nonlinear DEPCA,

$$y'(t) = f(t, y(t), y([t])) + \kappa g(t, y(t), y([t]), \kappa), \quad t \in \mathbb{R},$$

where A and B are $q \times q$ matrix-valued functions, and h, f, and g are continuous functions. The parameter κ represents a small real value within an interval $I \subset \mathbb{R}$, with $0 \in I$.

In 2012, R.-K. Zhuang [46] established sufficient conditions for the existence and uniqueness of almost periodic solutions to Nth-order neutral DEPCD systems with almost periodic time dependence, formulated as

$$(x(t) + px(t-1))^N = qx([t]) + f(t), \quad t \in \mathbb{R},$$

where p and q are nonzero constants, N is a positive integer, and $f(t)$ is Bohr almost periodic. In [47], Li and He analyzed second-order equations within DEPCA systems involving the argument $2([(t+1)/2])$, focusing on the existence of almost periodic solutions. In [48], G. Seifert thoroughly studied the special case where $N = 2$ and $|p| < 1$. Additionally, in [49], H.-X. Li focused on the case where $N = 2$ and $p = 1$.

Recently, numerous researchers have examined the existence of almost periodic solutions in differential equations with piecewise constant arguments (see [10,44–47,49–53]). However, to the best of our knowledge, there are relatively few studies addressing stability theory and the existence of almost periodic solutions for differential equations with a generalized piecewise constant argument.

We let $\mathbb{N}$ represent the set of natural numbers, $\mathbb{Z}$ denote the set of all integers, $\mathbb{R}$ signify the set of real numbers. Additionally, we let $|\cdot|$ denote the Euclidean norm. We assume that

$$\Theta = \left\{ \{t_k\} : t_k \in \mathbb{R},\ t_k < t_{k+1},\ k \in \mathbb{Z},\ \lim_{k \to \pm\infty} t_k = \pm\infty \right\}$$

is the set of all sequences that are unbounded and strictly increasing. We fix a real-valued sequence $\{t_k\} \in \Theta$, $k \in \mathbb{Z}$, such that $t_k < t_{k+1}$. We let $\beta : \mathbb{R} \to \mathbb{R}$ be a specified general step function defined by $\beta|_{I_k} = t_k$, where $I_k = [t_k, t_{k+1})$, and $\mathbb{R} = \cup_{k\in\mathbb{Z}} I_k$, with $t_k \to \pm\infty$ as $k \to \pm\infty$. These step functions β possess this general property, hence differential equations with a general β are referred to as DEPCDs of generalized type. It is worth noting that the research community focusing on DEPCDs of generalized type remains relatively small. For additional references, see [3,6,37,38,54].

Motivated by the above discussion, in this paper, we contribute to the study of existence, uniqueness, and stability analysis of almost periodic solutions for differential equations involving generalized piecewise constant delay (DEGPCD). We begin by considering the following forms:

1. The linear DEGPCD:
$$x'(t) = \mathcal{A}(t)x(t) + \mathcal{B}(t)x(\beta(t)). \tag{1}$$
2. The linear inhomogeneous DEGPCD:
$$y'(t) = \mathcal{A}(t)y(t) + \mathcal{B}(t)y(\beta(t)) + f(t). \tag{2}$$
3. The quasilinear DEGPCD:
$$z'(t) = \mathcal{A}(t)z(t) + \mathcal{B}(t)z(\beta(t)) + g\big(t, z(t), z(\beta(t))\big), \tag{3}$$

where $t \in \mathbb{R}$, $\mathcal{A}, \mathcal{B} : \mathbb{R} \to \mathbb{R}^{q\times q}$ and $f : \mathbb{R} \to \mathbb{R}^q$ are continuous functions, and β is a general step function. We define a function $y : \mathbb{R} \to \mathbb{R}^q$ as a solution to the linear inhomogeneous DEGPCD (2) if it meets the following criteria:

(i) y is continuous on $\mathbb{R}$;
(ii) The derivative y' is defined on $\mathbb{R}$, except potentially at points $t = t_n$ for $n \in \mathbb{Z}$, only one-sided derivatives exist;
(iii) The function y fulfills (2) within each interval (t_n, t_{n+1}), $n \in \mathbb{Z}$.

We let I_q denote the $q \times q$ identity matrix and we let $\Phi(t, s)$, with $\Phi(s, s) = I_q$ for $s \in \mathbb{R}$, denote the fundamental matrix solution of the following system:
$$x'(t) = \mathcal{A}(t)x(t),\ t \in \mathbb{R}. \tag{4}$$

For $t \in I_k$, where $k \in \mathbb{Z}$ and $s \leq t$, we define
$$C(t, t_k) = \Phi(t, t_k)J(t, t_k),$$
where
$$J(t, t_k) = I_p + \int_{t_k}^{t} \Phi(t_k, s)B(s)ds$$
and assume the condition
$$J(t, t_k) \text{ is invertible for all } k \in \mathbb{Z} \text{ and } t \in [t_k, t_{k+1}). \tag{5}$$

If $y(t)$ is a solution of the linear inhomogeneous DEGPCD (2), then $\{\widetilde{y}(k)\}_{k\in\mathbb{Z}} = \{y(t_k)\}_{k\in\mathbb{Z}}$ satisfies the difference equation:
$$\widetilde{y}(k+1) = C(k)\widetilde{y}(k) + \widetilde{f}(k),\ k \in \mathbb{Z}, \tag{6}$$
where
$$C(k) = C_k(t_{k+1}, t_k) = \Phi(t_{k+1}, t_k)J(t_{k+1}, t_k) := \Phi_k \cdot J_k,$$
and
$$\widetilde{f}(k) = \int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, u)f(u)du.$$

If $z(t)$ is a solution of the quasilinear DEGPCD (3), then the sequence $\{\widetilde{z}(k)\}_{k\in\mathbb{Z}} = \{z(t_k)\}_{k\in\mathbb{Z}}$ satisfies the difference equation:

$$\widetilde{z}(k+1) = C(k)\widetilde{z}(k) + p(k, \widetilde{z}(k), \widetilde{z}(k+1)),\ k \in \mathbb{Z}, \tag{7}$$

where

$$p(k, \widetilde{z}(k), \widetilde{z}(k+1)) = \int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, u) g(u, z(u), z(t_k)) du.$$

By Condition (5), $C(k)$ is an invertible $q \times q$ matrix for each $k \in \mathbb{Z}$. We define the linear difference equation,

$$\widetilde{y}(k+1) = C(k)\widetilde{y}(k), \quad \widetilde{y}(k) = y(t_k) \tag{8}$$

which admits a (μ_1, μ_2)-exponential dichotomy on $\mathbb{Z}$, if there are positive constants K, μ_1, μ_2, with $0 < \mu_1 < 1$, $\mu_2 > 1$, and a projection matrix $\mathcal{P}$ satisfying $\mathcal{P}^2 = \mathcal{P}$ such that

$$\begin{cases} \left|\widetilde{Y}(k)\mathcal{P}\widetilde{Y}^{-1}(h)\right| \leq K\mu_1^{(k-h)}, & k \geq h, \\ \left|\widetilde{Y}(k)(I-\mathcal{P})\widetilde{Y}^{-1}(h)\right| \leq K\mu_2^{(k-h)}, & k < h, \end{cases} \tag{9}$$

where $\widetilde{Y}(k)$ is the fundamental matrix solution of the linear difference Equation (8), such that $\widetilde{Y}(0) = I$. We note that if $\mu_1 = e^{-\sigma_1}$, $\mu_2 = e^{\sigma_2}$ where $\sigma_1, \sigma_1 > 0$, this leads to the concept of exponential dichotomy.

In particular, the linear difference Equation (8) is said to be exponentially stable as $k \to +\infty$ if it exhibits an exponential dichotomy with $P = I$, i.e,

$$\left|\widetilde{Y}(k)\widetilde{Y}^{-1}(h)\right| \leq K\mu_1^{(k-h)}, \qquad k \geq h. \tag{10}$$

We determine that the linear DEGPCD (1) has a (μ_1, μ_2)-exponential dichotomy if linear difference Equation (8) possesses a (μ_1, μ_2)-exponential dichotomy. In 1994, G. Papaschinopoulos made significant contributions to the DEPCD theory [55–57], defining exponential dichotomy with $\mu_1 = e^{-\sigma}$ and $\mu_2 = e^{\sigma}$, where $\sigma > 0$, for the linear DEPCD (1) when the corresponding discrete System (8) exhibits such behavior.

The novelty of this work lies in presenting new and simplified adequate conditions that assure the existence, uniqueness, and exponential stability of almost periodic solutions for the DEGPCD system. The approach relies on properties of (μ_1, μ_2)-exponential dichotomy, Cauchy and Green-type matrices, a Gronwall-type inequality specific to DEGPCD, and the Banach contraction principle. We extend the classical results of [44] to DEPCD of generalized type.

The structure of this paper is organized as follows: in Section 2, we provide key definitions and preliminary results essential for establishing the existence of a unique solution to the DEGPCD system. Cauchy and Green-type matrices are introduced, which play a pivotal role in deriving variation of parameters formulas. The solutions to DEGPCD Systems (1)–(3) are discussed, with a focus on the utility of a Gronwall-type inequality for DEGPCD. In Section 3, we present criteria concerning almost periodic sequences and functions within the DEGPCD system. Section 4 focuses on the existence of almost periodic solutions for both the linear inhomogeneous DEGPCD (2) and the quasilinear DEGPCD (3). Additionally, we establish sufficient conditions for the exponential stability of these almost periodic solutions. The paper concludes with Section 5, where we summarize the findings and draw final conclusions.

2. Preliminaries

2.1. Definitions and Assumptions

In this subsection, we present some well-established classical definitions of Bohr almost periodicity (see, for instance, [5,8]) along with the necessary assumptions.

Definition 1 ([8] p. 7). *(i) A function $f_1 : \mathbb{R} \to \mathbb{R}^q$ is called an almost periodic function if the ε-translation set of f_1*

$$T(f_1, \varepsilon) = \Big\{ \tau \in \mathbb{R} \big| |f_1(t+\tau) - f_1(t)| < \varepsilon \text{ for all } t \in \mathbb{R} \Big\}$$

is a relatively dense set in $\mathbb{R}$ for all $\varepsilon > 0$, τ is called the ε-period for f_1.

(ii) A function $f_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^q$ is called a double almost periodic function if the ε-translation set of f_2

$$T(f_2, \varepsilon) = \Big\{ \tau \in \mathbb{R} \big| |f_2(t+\tau, s+\tau) - f_2(t,s)| < \varepsilon \text{ for all } t \in \mathbb{R} \Big\}$$

is a relatively dense set in $\mathbb{R}$ for all $\varepsilon > 0$, τ is called the ε-period for f_2.

Definition 2 ([5] p. 183). *(i) We let $\{x_i\}$ be a sequence in $\mathbb{R}^q$, defined for $i \in \mathbb{Z}$. An integer p is called an ε-almost period of a sequence $\{x_i\}$ if, for any $i \in \mathbb{Z}$,*

$$|x_{i+p} - x_i| < \varepsilon.$$

(ii) A sequence $\{x_i\}$ is called almost periodic if for any $\varepsilon > 0$ there exists a relatively dense set of its ε-periods.

Definition 3 ([5] p. 195). *We let, for any integers i and p, $t_{i,p} = t_{i+p} - t_i$, where $\{t_i\} \in \Theta$. We consider the sequences $\{t_{i,p}\}$, $i \in \mathbb{Z}$, $p \in \mathbb{Z}$. The family of the sequences $\{t_{i,p}\}$ is called equipotentially almost periodic if for an arbitrary $\varepsilon > 0$ there exists a relatively dense set of ε-almost periods that are common for all $\{t_{i,p}\}$.*

Remark 1. *Trofimchuk ([5], p. 377) demonstrated that any sequence $\{t_i\}$ belonging to a family of equipotentially almost periodic sequences can be expressed as*

$$t_i = i\tau + a_i, \tag{11}$$

where $\tau \in \mathbb{R}$ and $\{a_i\}$ is an almost periodic sequence.

Remark 2. *For $\epsilon > 0$, we let $\mathcal{T}(\{t_{i,p}\}, \epsilon)$ represent the set of $\tau \in \mathbb{R}$ for which there exists $p \in \mathbb{Z}$ such that*

$$\sup_{i \in \mathbb{Z}} |t_{i,p} - \tau| \le \epsilon.$$

By [5], Lemma 25, we can see that $\{t_{i,p}\}$, $i \in \mathbb{Z}$, $p \in \mathbb{Z}$ are equipotentially almost periodic if and only if the set of $\mathcal{T}(\{t_{i,p}\}, \varepsilon)$ is relatively dense for any $\varepsilon > 0$.

According to ([5], Lemma 25), it is established that the sequence $\{t_{i,p}\}$, where $i \in \mathbb{Z}$ and $p \in \mathbb{Z}$, is equipotentially almost periodic if and only if the set $\mathcal{T}(\{t_{i,p}\}, \varepsilon)$ is relatively dense for any $\varepsilon > 0$.

The following assumptions are needed throughout the paper.

($\mathcal{AP}$) Almost periodic conditions:
(1) $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are Bohr almost periodic matrix functions;
(2) $f(t)$ is a Bohr almost periodic vector function;
(3) The set of sequence $\{t_{i,p}\}$, $t_{i,p} = t_{i+p} - t_i$, where $i \in \mathbb{Z}$, $p \in \mathbb{Z}$, and $\{t_i\} \in \Theta$, is equipotentially almost periodic;
(4) The function $g(t, z_1, z_2)$ is Bohr almost periodic in t uniformly with respect to $z_1, z_2 \in \Im$, where $\Im \subseteq \mathbb{R}^q \times \mathbb{R}^q$ is a compact subset.

($\mathcal{L}_g$) Lipschitz condition:
There exist constants $\mathcal{L}_1^g, \mathcal{L}_2^g > 0$ such that

$$|g(t, u_1, v_1) - g(t, u_2, v_2)| \le \mathcal{L}_1^g |u_1 - u_2| + \mathcal{L}_2^g |v_1 - v_2|, \tag{12}$$

for $t \in \mathbb{R}$ and $(u_1, v_1), (u_2, v_2) \in \Im$.

($\mathcal{D}$) Dichotomy condition:
Linear difference Equation (8) has a (μ_1, μ_2)-exponential dichotomy.

2.2. Cauchy and Green-Type Matrices for the DEGPCD System

In this subsection, we introduce the Cauchy and Green-type matrices for the DEGPCD system (2). Following a similar approach as in [42], we first solve the linear DEGPCD system (1) on on $I_i = [t_i, t_{i+1})$ for $i = i(\varsigma)$, i.e., $x'(t) = A(t)x(t) + B(t)x_i$, where $x_i = x(t_i)$, This solution is then extended to the entire interval $[\varsigma, \infty)$ to derive a representation formula. We define the backward and forward products as follows:

$$\overleftarrow{\prod_{k=i}^{j}} C_k = C_j C_{j-1} \cdots C_i \qquad \text{and} \qquad \overrightarrow{\prod_{k=i}^{j}} C_k = C_i \cdots C_{j-1} C_j,$$

where $i, j \in \mathbb{Z}$. In case where $i = j$, the product is simply the value of the single factor C_j. If $i > j$, the product is an empty product with a value of 1.

We let I_p denote the $p \times p$ identity matrix. We denote $\Phi(t, s)$ as the fundamental matrix for system $x'(t) = \mathcal{A}(t)x(t)$, satisfying $\Phi(s, s) = I_p$ for $s \in \mathbb{R}$.

We now define the matrices $J(t, t_k)$ and $C(t, t_k)$ as follows:

$$J(t, t_k) = I_q + \int_{t_k}^{t} \Phi(t_k, s)B(s)ds \quad \text{and} \quad C(t, t_k) = \Phi(t, t_k)J(t, t_k), \quad t \in [t_k, t_{k+1}),$$

so that the function $v(t) = C(t, s)\xi$ is the unique solution to the initial value problem $x'(t) = \mathcal{A}(t)x + \mathcal{B}(t)\xi$, $x(s) = \xi$, in $\mathbb{R}$.

For the matrix J, we impose the following **invertibility condition**, which is assumed throughout the paper:

($\mathcal{I}$) For every fixed $k \in \mathbb{Z}$, $t \in [t_k, t_{k+1})$, the matrix $J(t, t_k)$ is nonsingular.

For any $k \in \mathbb{Z}$, the solution of the linear DEGPCD system (1) on the interval $I_k = [t_k, t_{k+1})$ with $x_k = x(t_k)$ is given by the function

$$x(t) = C(t, t_k)x_k, \tag{13}$$

where, by choosing $i = i(\varsigma)$ and setting $t = \varsigma$, we determine that $x(\varsigma)$ is uniquely related to $x_{i(\varsigma)} = x(\beta(\varsigma))$ through the relation

$$x(\varsigma) = C(\varsigma, \beta(\varsigma))x(\beta(\varsigma)) \iff x(\beta(\varsigma)) = C(\varsigma, \beta(\varsigma))^{-1}x(\varsigma) \tag{14}$$

due to condition ($\mathcal{I}$), with $t_{i(\varsigma)} = \beta(\varsigma)$. Thus, from the equations above, we derive that the unique solution $x(t)$ of the linear DEGPCD system (1) on the entire interval I_i, for $i = i(\varsigma)$ is given by

$$x(t) = C(t, \beta(\varsigma))C(\varsigma, \beta(\varsigma))^{-1}x(\varsigma). \tag{15}$$

It should be noted that this relation holds for every $t, \varsigma \in I_i$ and $i \in \mathbb{Z}$, thus solving (1) over the entire interval I_i. Specifically, we just solved the linear DEGPCD system:

$$x'(t) = \mathcal{A}(t)x(t) + \mathcal{B}(t)x(\beta(\varsigma)), \;\; t \in [\beta(\varsigma), t_{i(\varsigma)+1}) = [t_{i(\varsigma)}, t_{i(\varsigma)+1}).$$

Therefore, the solution $x(t)$ can be continued both forwards and backwards. However, we focus on continuing the solution for $t \geq t_{i(\varsigma)}$.

If $\varsigma \in I_i = [t_{i+1}, t_i)$ then, by (15),

$$x(t_{i+1}) = C(t_{i+1}, t_i)C(\varsigma, t_i)^{-1}x(\varsigma)$$

For $t \in I_{i+1}$ and $\varsigma \in I_i$, we have

$$x(t) = C(t, t_{i+1})x(t_{i+1}) = C(t, t_{i+1})C(t_{i+1}, t_i)C(\varsigma, t_i)^{-1}x(\varsigma).$$

More generally, by induction, for any $n > i$ and $t \in I_n$, $\varsigma \in I_i$, we obtain

$$x(t) = C(t, \beta(t)) \cdot \overleftarrow{\prod}_{k=i}^{n-1} \Big[C(t_{k+1}, \beta(t_k)) \Big] \cdot C(\varsigma, \beta(\varsigma))^{-1}x(\varsigma).$$

Thus, by defining $i = i(t) \in \mathbb{Z}$ as the unique integer for which $t \in I_i = [t_i, t_{i+1})$, the following is obtained:

$$\Psi(t,s) = C(t, \beta(t_{i(t)})) \cdot \overleftarrow{\prod}_{k=i(s)}^{i(t)-1} \Big[C(t_{k+1}, \beta(t_k)) \Big] \cdot C(s, \beta(t_{i(s)}))^{-1}, \tag{16}$$

which represents the solution $x = x(t, \varsigma, x(\varsigma))$ as

$$x(t) = \Psi(t, \varsigma)x(\varsigma), \tag{17}$$

for $t \in I_{i(t)} and \varsigma \in I_{i(\varsigma)}$.

The matrix $\Psi(t,s)$ is nonsingular, making it a fundamental solution matrix of the linear DEGPCD system (1), satisfying $\Psi(\varsigma, \varsigma) = I$ and the properties

$$\Psi(t,s)\Psi(s,\varsigma) = \Psi(t,\varsigma), \quad \text{and} \quad \Psi(t,s) = \Psi(s,t)^{-1}, \quad \text{for} \quad t \geq s \geq \varsigma.$$

In other words, $\Psi(t, \varsigma)$ is a *Cauchy type matrix* for the DEGPCD system.

Similarly, we can solve the linear DEGPCD system (1) for $t \leq s \leq t_k$, as follows:

$$x(t) = \hat{\Psi}(t, t_k)x_k, \tag{18}$$

where

$$\hat{\Psi}(t,s) = C(t, \beta(t_{i(t)+1})) \cdot \overrightarrow{\prod}_{k=i(t)}^{i(s)-1} [C(t_{k+1}, \beta(t_k))^{-1}] \cdot C(s, \beta(t_{i(s)}))^{-1}. \tag{19}$$

Moreover, if $\varsigma \in \Theta$ is given and fixed, we can derive the following expression:

$$x(t) = \begin{cases} C(t, \beta(t_{i(t)})) \cdot \overleftarrow{\prod}_{k=i(\varsigma)}^{i(t)-1} \Big[C(t_{k+1}, \beta(t_k)) \Big] x(\varsigma), & \text{for} \quad t \geq \varsigma, \\ C(t, \beta(t_{i(t)+1})) \cdot \overrightarrow{\prod}_{k=i(t)}^{i(\varsigma)-1} \Big[C(t_{k+1}, \beta(t_k))^{-1} \Big] x(\varsigma), & \text{for} \quad t < \varsigma. \end{cases}$$

Following the ideas presented in [42] for delay-type situations, we define a Green-type matrix $G = G(t,s)$ for every $\varsigma \in \mathbb{R}$ and $s, t \in \mathbb{R}$.

Definition 4 (Green matrix type for $s \in I_k$)**.** *For $t \in \mathbb{R}$ and $s \in I_k$, we define the Green matrix type as follows:*

$$G_k(t,s) = \begin{cases} \Psi(t, t_{i(\varsigma)+1})\Phi(t_{i(\varsigma)+1}, s) - \Psi(t, \varsigma)\Phi(\varsigma, s), & \textit{for} \quad s < \varsigma \neq t_k, \quad i(s) < i(t), \\ \Psi(t, t_{k+1})\Phi(t_{k+1}, s), & \textit{for} \quad \varsigma = t_k \leq s, \quad i(s) < i(t), \\ \Phi(t,s), & \textit{for} \quad i(t) = i(s), \\ \hat{\Psi}(t, t_{i(\varsigma)})\Phi(t_{i(\varsigma)}, s) - \hat{\Psi}(t, \varsigma)\Phi(\varsigma, s), & \textit{for} \quad s < \varsigma \neq t_k, \quad i(t) < i(s), \\ \hat{\Psi}(t, t_k)\Phi(t_k, s), & \textit{for} \quad s \leq \varsigma = t_{k+1}, \; i(t) < i(s). \end{cases}$$

Definition 5 (Green matrix type for the DEGPCD system)**.** *Let us define the Green matrix type for the DEGPCD system as follows:*

$$\begin{aligned} G(t,s) &= G_{i(\varsigma)}(t,s) + \sum_{k=i(\varsigma)+1}^{i(t)-1} G_k(t,s) + G_{i(t)}(t,s), \quad i(s) < i(t), \\ G(t,s) &= G_{i(t)}(t,s) + \sum_{k=i(t)+1}^{i(\varsigma)-1} G_k(t,s) + G_{i(\varsigma)}(t,s), \quad i(s) > i(t), \end{aligned} \tag{20}$$

and

$$G(t,s) = G_{i(s)}(t,s), \quad \textit{for} \quad t,s \in I_{i(s)}.$$

2.3. A Representation Formula

The purpose of this subsection is to present a representation formula for the solutions of Equation (2) in systems of differential equations with generalized piecewise constant delay. This formula is a variation of parameters formula for the theory of DEGPCD systems. We recall that Condition $(\mathcal{I})$ holds. Following the approach in [34], we arrive at one of the main results.

Theorem 1. *We assume that Condition $(\mathcal{I})$ is satisfied. Then, for any $(\varsigma, y_0) \in \mathbb{R} \times \mathbb{R}^q$, the solution $y(t) = y(t,\varsigma,y_0)$ of the DEGPCD system (2) that satisfies $y(\varsigma) = y_0$ is defined on $\mathbb{R}$ and is expressed as follows:*

$$y(t) = Z(t,\varsigma)y_0 + \int_{\beta(\varsigma)}^{t} G(t,s)f(s)ds, \tag{21}$$

where

$$Z(t,\varsigma) = \begin{cases} \Psi(t,\varsigma), & \textit{if} \quad t \geq \varsigma, \\ \hat{\Psi}(t,\varsigma), & \textit{if} \quad t < \varsigma, \end{cases} \tag{22}$$

and $G(t,s)$ is the Green matrix type for the DEGPCD system.

Proof. Case 1. If $t \in [\varsigma, \infty)$.

We consider $t \in [t_i, t_{i+1})$ and $\beta(t) = t_i$, so that the solution to the linear inhomogeneous DEGPCD $y'(t) = \mathcal{A}(t)y(t) + \mathcal{B}(t)y(t_i) + f(t)$ is uniquely given by

$$\begin{aligned} y(t) &= \Phi(t,t_i)y(t_i) + \int_{t_i}^{t} \Phi(t,s)B(s)y(t_i)ds + \int_{t_i}^{t} \Phi(t,s)f(s)ds \\ &= C(t,t_i)y(t_i) + \int_{t_i}^{t} \Phi(t,s)f(s)ds, \quad t \in I_i = [t_i, t_{i+1}). \end{aligned} \tag{23}$$

Taking the limit $t \to t_{i+1}$, we obtain

$$y(t_{i+1}) = C(t_{i+1},t_i)y(t_i) + \int_{t_i}^{t_{i+1}} \Phi(t_{i+1},s)f(s)ds.$$

For $j > i(\varsigma) + 1$, we have

$$y(t_j) = C(t_j,t_{j-1})y(t_{j-1}) + \int_{t_{j-1}}^{t_j} \Phi(t_j,s)f(s)ds. \tag{24}$$

Similarly, for (23) and $t \in I_{i(\varsigma)}$, we have

$$y(t) = C(t,t_{i(\varsigma)})C(\varsigma,t_{i(\varsigma)})^{-1}\left(y(\varsigma) - \int_{t_{i(\varsigma)}}^{\varsigma} \Phi(\varsigma,s)f(s)ds\right) + \int_{t_{i(\varsigma)}}^{t} \Phi(t,s)f(s)ds$$

and

$$y(t_{i(\varsigma)+1}) = C(t_{i(\varsigma)+1}, t_{i(\varsigma)})C(\varsigma, t_{i(\varsigma)})^{-1}\left(y(\varsigma) - \int_{t_{i(\varsigma)}}^{\varsigma} \Phi(\varsigma, s)f(s)ds\right) + \int_{t_{i(\varsigma)}}^{t_{i(\varsigma)+1}} \Phi(t_{i(\varsigma)+1}, s)f(s)ds. \tag{25}$$

From (23)–(25), for $i \geq i(\varsigma) + 1$, we deduce

$$y(t_i) = \overleftarrow{\prod}_{j=i(\varsigma)}^{i-1} C(t_{j+1}, t_j)C(\varsigma, t_{i(\varsigma)})^{-1}\left(y(\varsigma) - \int_{t_{i(\varsigma)}}^{\varsigma} \Phi(\varsigma, s)f(s)ds\right) + \sum_{j=i(\varsigma)}^{i-1}\left(\overleftarrow{\prod}_{k=j+1}^{i-1} C(t_{k+1}, t_k)\right)\int_{t_j}^{t_{j+1}} \Phi(t_{j+1}, s)f(s)ds. \tag{26}$$

Thus, from (23) and (26), for $t \in I_i$, we obtain the solution

$$y(t) = \Psi(t, \varsigma)y(\varsigma) - \Psi(t, \varsigma)\int_{t_{i(\varsigma)}}^{\varsigma} \Phi(\varsigma, s)f(s)ds + \sum_{k=i(\varsigma)}^{i-1} \Psi(t, t_{k+1})\int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, s)f(s)ds + \int_{t_i}^{t} \Phi(t, s)f(s)ds. \tag{27}$$

Considering the integral part of the first term in the sum for $k = i(\varsigma)$, we write

$$\Psi(t, t_{i(\varsigma)+1})\int_{t_{i(\varsigma)}}^{t_{i(\varsigma)+1}} \Phi(t_{i(\varsigma)+1}, s)f(s)ds = \Psi(t, t_{i(\varsigma)+1})\int_{t_{i(\tau)}}^{\varsigma} \Phi(t_{i(\varsigma)+1}, s)f(s)ds + \Psi(t, t_{i(\varsigma)+1})\int_{\varsigma}^{t_{i(\varsigma)+1}} \Phi(t_{i(\varsigma)+1}, s)f(s)ds,$$

The conclusion follows from Definition 5.

Case 2. If $t \in (-\infty, \varsigma]$. The proof follows a similar approach as in Case 1. □

Analogous to Theorem 1, we deduce the following variation of the parameters formula for the quasilinear DEGPCD system by setting $f(s) = g(s, z(s), z(\beta(s)))$, where $s \in \mathbb{R}$.

Corollary 1. *We assume that Condition ($\mathcal{I}$) holds, and we let $f = g(t, \cdot, \cdot) \in C(\mathbb{R} \times \mathbb{R}^q \times \mathbb{R}^q, \mathbb{R}^q)$. Then, for each $(\varsigma, z_0) \in \mathbb{R} \times \mathbb{R}^q$, any solution $z(t) = z(t, \varsigma, z_0)$ of the quasilinear DEGPCD system*

$$z'(t) = \mathcal{A}(t)z(t) + \mathcal{B}(t)z(\beta(t)) + g\big(t, z(t), z(\beta(t))\big)$$

satisfies the integral Equation (27), with $f(s) = g(s, z(s), z(\beta(s)))$ or equivalently

$$z(t) = Z(t, \varsigma)z_0 + \int_{\beta(\varsigma)}^{t} G(t, s)g(s, z(s), z(\beta(s)))ds, \tag{28}$$

where $Z(t, \varsigma) = \Psi(t, \varsigma)$ if $t \geq \varsigma$, and $Z(t, \varsigma) = \hat{\Psi}(t, \varsigma)$, if $t < \varsigma$.

Conversely, any solution z of the integral Equation (28) is also a solution of the quasilinear DEGPCD system (3).

2.4. Existence and Uniqueness of Solutions to the Quasilinear DEGPCD System

In this subsection, we utilize integral inequalities of Gronwall's type specifically adapted for the DEGPCD system. Integral inequalities have historically played a pivotal role in studying the qualitative behavior of solutions to differential equations. For further reading, see, for instance, [58]. Now, we recall a Gronwall-type inequality tailored for

DEGPCD systems, as demonstrated in ([54], Lemma 2.1). This lemma is instrumental in proving the existence and uniqueness of solutions.

Lemma 1 (DEGPCD's Gronwall Inequality)**.** *We let $u : \mathbb{R} \to \mathbb{R}^+$ be a continuous function satisfying*

$$u(t) \le \alpha + \left| \int_{\ell}^{t} [\eta_1(s)u(s) + \eta_2(s)u(\beta(s))]ds \right|, \tag{29}$$

where $\alpha \ge 0$ and $\eta_i : \mathbb{R} \to \mathbb{R}^+$, $i \in \{1, 2\}$ is a piecewise continuous function.

Then,

- *If $t \ge \ell$,*

$$u(t) \le \alpha \exp\left(\int_{\ell}^{t} [\eta_1(s) + \eta_2(s)]ds \right). \tag{30}$$

- *If $t < \ell$,*

$$u(t) \le \alpha \exp\left(\int_{t}^{\ell} \left[\eta_1(s) + \frac{\eta_2(s)}{1-\sigma} \right] ds \right), \tag{31}$$

where

$$\sigma := \max_{j \le i(\ell)} \int_{t_j}^{t_{j+1}} [\eta_1(s) + \eta_2(s)]ds \le \kappa < 1.$$

We let $\eta_1(t) = \eta_j^1(t)$ and $\eta_2(t) = \eta_j^2(t)$ for $t \in [t_j, t_{j+1})$ and $j \in \mathbb{Z}$. By applying the Gronwall inequality for DEGPCD systems, we can deduce the following result.

Lemma 2. *We let $u : \mathbb{R} \to \mathbb{R}^+$ be a continuous function satisfying*

$$\begin{aligned} u(t) \le \alpha + \Bigg| \int_{\ell}^{t_{i(l)+1}} \left[\eta_{i(l)}^1(s)u(s) + \eta_{i(l)}^2(s)u(\beta(s)) \right] ds \\ + \sum_{k=i(l)+1}^{i(t)-1} \int_{t_k}^{t_{k+1}} \left[\eta_k^1(s)u(s) + \eta_k^2(s)u(\beta(s)) \right] ds \\ + \int_{t_{i(t)}}^{t} \left[\eta_{i(t)}^1(s)u(s) + \eta_{i(t)}^2(s)u(\beta(s)) \right] ds \Bigg|, \end{aligned} \tag{32}$$

where $\alpha \ge 0$ and η_j^1, η_j^2 are continuous functions in $[t_j, t_{j+1})$, $j \in \mathbb{Z}$.

Then,

- *If $t \ge \ell$,*

$$\begin{aligned} u(t) \le \alpha \exp\Bigg(\int_{\ell}^{t_{i(l)+1}} (\eta_{i(l)}^1(s) + \eta_{i(l)}^2(s))ds + \sum_{k=i(l)+1}^{i(t)-1} \int_{t_k}^{t_{k+1}} (\eta_k^1(s) + \eta_k^2(s))ds \\ + \int_{t_{i(t)}}^{t} (\eta_{i(t)}^1(s) + \eta_{i(t)}^2(s))ds \Bigg). \end{aligned} \tag{33}$$

- *If $t < \ell$,*

$$\begin{aligned} u(t) \le \alpha \exp\Bigg(\int_{t}^{t_{i(t)+1}} \left(\eta_{i(t)}^1(s) + \frac{\eta_{i(t)}^2(s)}{1-\sigma} \right) ds \\ + \sum_{k=i(t)+1}^{i(l)-1} \int_{t_k}^{t_{k+1}} \left(\eta_k^1(s) + \frac{\eta_k^2(s)}{1-\sigma} \right) ds \\ + \int_{t_{i(l)}}^{l} \left(\eta_{i(l)}^1(s) + \frac{\eta_{i(l)}^2(s)}{1-\sigma} \right) ds \Bigg), \end{aligned} \tag{34}$$

where

$$\sigma := \max_{j \le i(\ell)} \int_{t_j}^{t_{j+1}} \left[\eta_j^1(s) + \eta_j^2(s)\right] ds \le \kappa < 1.$$

Similarly to [42], Lemma 4.3 in the context of purely delayed systems, we state the following result, omitting the proof.

Lemma 3. *We asume that condition $(\mathcal{I})$ is fulfilled. Accordingly, for any t and s, both the matrix $\Psi(t,s)$ and its inverse $\Psi(t,s)^{-1}$ are well defined. Additionally, there exist positive constant numbers α and ρ such that*

$$|\Phi(t,s)| \le \rho, \quad |\Psi(t,s)| \le \alpha, \quad |G_i(t,s)| \le \alpha\rho, \quad \text{for} \quad s,t \in [t_i, t_{i+1}), \quad i \in \mathbb{Z}. \tag{35}$$

Now, in order to study the quasilinear DEGPCD system (3), we introduce the following assumption:

(E) Given α and ρ from (35), we assume that $\alpha\rho(\mathcal{L}_1^g + \mathcal{L}_2^g)(t_{k+1} - t_k) < 1, \quad \text{for} \quad k \le i(\varsigma).$

Using the techniques developed in [42,54], we obtain the following result.

Theorem 2. *Under conditions $(\mathcal{L}_g)$, $(\mathcal{I})$ and (E), for each $(\varsigma, z_0) \in \mathbb{R} \times \mathbb{R}^q$, there exists a unique solution $z(t) = z(t, \varsigma, z_0)$ to the quasilinear DEGPCD system (3) with $z(\varsigma) = z_0$ for $t \in \mathbb{R}$.*

2.5. Difference Equations with Exponentially Dichotomy

We present two results related to difference equations exhibiting exponential dichotomy, which can be applied to study the existence of almost periodic solutions for the DEGPCD system.

Proposition 1 ([59], Proposition 2.6)**.** *We assume that linear difference Equation (8) possesses an exponential dichotomy on $\mathbb{Z}$. Under this condition, linear difference Equation (8) admits no nontrivial solutions that are bounded on $\mathbb{Z}$.*

Using a similar approach to the one presented in [59], Lemma 2.7, we can prove the following result.

Proposition 2. *We consider linear difference Equation (8) to have a (μ_1, μ_2)-exponential dichotomy on $\mathbb{Z}$, characterized by constants K, μ_1, and μ_2 where $0 < \mu_1 < 1$ and $1 < \mu_2$. We let $\{\widetilde{f}(i)\}_{i \in \mathbb{Z}}$ represent a bounded sequence. Then, the associated inhomogeneous difference Equation (6) possesses a unique solution, which remains bounded on $\mathbb{Z}$. Moreover, for all k,*

$$|\widetilde{y}(k)| \le K\left(\frac{1}{1-\mu_1} + \frac{1}{\mu_2 - 1}\right) \sup_{k \in \mathbb{Z}} \left|\widetilde{f}(k)\right|. \tag{36}$$

Proof. We let $\widetilde{Y}(k,h)$ denote the transition matrix for linear difference Equation (8). We then define the solution as

$$\widetilde{y}(k) = \sum_{h=-\infty}^{\infty} G(k, h+1)\widetilde{f}(h) \tag{37}$$

where $G(k, h+1)$ is the Green's function, defined by

$$G(k,h) = \begin{cases} \widetilde{Y}(k)P\widetilde{Y}^{-1}(h), & \text{if} \quad k \ge h, \\ -\widetilde{Y}(k)(I-P)\widetilde{Y}^{-1}(h), & \text{if} \quad k < h. \end{cases} \tag{38}$$

The sequence $\widetilde{y}(k)$ given by (37) is bounded and satisfies both (8) and (36).

Indeed, using exponential dichotomy Condition (9), we obtain

$$
\begin{aligned}
|\widetilde{y}(k)| &= \left| \sum_{h=-\infty}^{\infty} G(k,h+1)\widetilde{f}(h) \right| \\
&\leq \left| \sum_{h=-\infty}^{k} G(k,h+1)\widetilde{f}(h) \right| + \left| \sum_{h=k+1}^{\infty} G(k,h+1)\widetilde{f}(h) \right| \\
&\leq K \left| \sum_{h=-\infty}^{k} \mu_1^{k-h} \right| \sup_{h\in\mathbb{Z}} \left|\widetilde{f}(h)\right| + K \left| \sum_{h=k+1}^{\infty} \mu_2^{k-h} \right| \sup_{h\in\mathbb{Z}} \left|\widetilde{f}(h)\right| \\
&\leq K \left(\frac{1}{1-\mu_1} + \frac{1}{\mu_2 - 1} \right) \sup_{h\in\mathbb{Z}} \left|\widetilde{f}(h)\right|.
\end{aligned}
$$

This completes the proof. □

Remark 3. *We consider the linear difference Equation (8) exhibiting an exponential dichotomy on* $\mathbb{Z}$*, and define* $\mu 1 = e^{-\sigma}$ *and* $\mu 2 = e^{\sigma}$ *for a constant* $\sigma > 0$*. By applying Proposition 2, the outcomes from [59], Lemma 2.7 follow, indicating that associated inhomogeneous difference Equation (6) has a unique solution that remains bounded over* $\mathbb{Z}$*. Moreover, for all k, the following inequality holds:*

$$
|\widetilde{y}(k)| \leq K \left(\frac{e^{\sigma}+1}{e^{\sigma}-1} \right) \sup_{k\in\mathbb{Z}} \left|\widetilde{f}(k)\right|. \tag{39}
$$

If linear difference Equation (8) is exponentially stable, then inhomogeneous difference Equation (6) also has a unique bounded solution on $\mathbb{Z}$*, satisfying, for all k,*

$$
|\widetilde{y}(k)| \leq K \left(\frac{1}{1-e^{-\sigma}} \right) \sup_{k\in\mathbb{Z}} \left|\widetilde{f}(k)\right|. \tag{40}
$$

3. Almost Periodic Sequences and Functions

To study the existence and uniqueness of almost periodic solutions for linear inhomogeneous DEGPCD system (2) and quasilinear DEGPCD system (3), we must first explore the basic properties of almost periodic sequences and functions.

3.1. Almost Periodic Sequences

In this subsection, we provide the criteria for almost periodic sequences and functions within the DEGPCD system. Fundamental results regarding almost periodic functions can be found in the book by A. M. Samoilenko and N. A. Perestyuk [5].

Lemma 4 ([5], Teorem 67)**.** *An almost periodic function is bounded.*

Lemma 5 ([5], Lemma 29)**.** *We let the* $\{t_{i,p}\}$ *be equipotentially almost periodic and a function* $\varphi(t)$ *be Bohr almost periodic. Then, for any* $\varepsilon > 0$ *there exists such* $l = l(\varepsilon) > 0$ *that for any interval* $\Im$ *of length l there are* $\tau \in \Im$ *and an integer p such that*

$$
\left|t_{i,p} - \tau\right| < \varepsilon, \quad |\varphi(t+\tau) - \varphi(t)| < \varepsilon,
$$

for all integers $i \in \mathbb{Z}$ *and all* $t \in \mathbb{R}$*.*

Using the standard techniques of Lemma 5, we can derive the following result.

Corollary 2. *If the conditions* $(\mathcal{A}P_1)$*,* $(\mathcal{A}P_2)$*,* $(\mathcal{A}P_3)$ *hold, the set* $\mathcal{T}(A,\varepsilon) \cap \mathcal{T}(B,\varepsilon) \cap \mathcal{T}(f,\varepsilon) \cap \mathcal{T}(\{t_{i,p}\},\varepsilon)$ *is relatively dense.*

Lemma 6 ([5], Lemma 30)**.** *We let $\varphi(t)$ be Bohr almost periodic function. If a sequence $\{t_i\}$ is such that sequences $\{t_{i,p}\}$, $p \in \mathbb{Z}$, are equipotentially almost periodic, then the sequence of numbers $\{\varphi(t_i)\}$ is almost periodic.*

3.2. Almost Periodicity in DEGPCD

In the following subsection, we consider that $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are Bohr almost periodic functions. Consequently, there exist constants $|\mathcal{A}|_\infty > 0$ and $|\mathcal{B}|_\infty > 0$, where $|\mathcal{A}|_\infty = \sup_{t\in\mathbb{R}} |\mathcal{A}(t)|$ and $|\mathcal{B}|_\infty = \sup_{t\in\mathbb{R}} |\mathcal{B}(t)|$.

For simplicity, we define

(i) $\mathcal{T}(\{t_{i,p}\},\varepsilon) =: \Big\{\tau \in \mathbb{R} \big| |t_{i,p} - \tau| < \varepsilon,\ p \in \mathbb{Z} \text{ fixed},\ i \in \mathbb{Z}\Big\}.$

(ii) $\Psi_k(t) := \Psi(t,t_k) = C(t,t_i)\overleftarrow{\prod}_{j=k}^{i-1} C_j,\ \ t \geq t_k,\ \ t \in I_i.$

Lemma 7. *We assume conditions ($\mathcal{A}P_1$) and ($\mathcal{A}P_3$) are satisfied. Then, for any $\varepsilon > 0$, there exist a real number τ, an integer p, and positive constants ρ, K_i, $|\mathcal{A}|_\infty$, and $|\mathcal{B}|_\infty$ such that the following properties hold:*

(a) *There exist K_i, $\rho > 0$, and $i \in \mathbb{Z}$, such that*

$$|\Phi(t,s)| \leq K_i \leq \rho,\ t,s \in I_i,$$

where $K_i = \max_{t,s\in I_i} |\Phi(t,s)|$ *and* $\rho = \sup_{t,s\in I_i,\ \forall i\in\mathbb{Z}} |\Phi(t,s)|.$

(b) *The following inequality holds for any $t,s \in I_i$:*

$$|\Phi(t+\tau,s+\tau) - \Phi(t,s)| \leq \varepsilon K_i (t_{i+1} - t_i) e^{|\mathcal{A}|_\infty (t_{i+1}-t_i)}.$$

(c) $\big|\Phi(t+\tau,t_i+\tau) - \Phi(t+\tau,t_{i+p})\big| \leq \varepsilon|\mathcal{A}|_\infty \rho.$

(d) $\big|\Phi(t+\tau,t_{i+p}) - \Phi(t,t_i)\big| \leq \varepsilon\rho\Big(|\mathcal{A}|_\infty + (t_{i+1}-t_i)e^{|\mathcal{A}|_\infty(t_{i+1}-t_i)}\Big),\ t,s \in I_i.$

(e) $|\Phi(t+\tau,s+\tau)\mathcal{B}(s+\tau) - \Phi(t,s)\mathcal{B}(s)| \leq \varepsilon\rho\Big(1 + |\mathcal{B}|_\infty (t_{i+1}-t_i)e^{|\mathcal{A}|_\infty(t_{i+1}-t_i)}\Big),$ $t,s \in I_i.$

(f) $\big|\Phi(t_{i+p},s+\tau) - \Phi(t_i,s)\big| \leq \varepsilon\rho(|\mathcal{A}|_\infty + (t_{i+1}-t_i))e^{|\mathcal{A}|_\infty(t_{i+1}-t_i)}.$

(g) $\big|\Phi(t_{i+1},t_{i+p}-\tau) - \Phi(t_{i+1},t_i)\big| \leq \varepsilon|\mathcal{A}|_\infty\rho.$

(h) $\big|\Phi(t_{i+p+1},t_{i+p}) - \Phi(t_{i+1},t_i)\big| \leq \varepsilon\rho(2|\mathcal{A}|_\infty + (t_{i+1}-t_i))e^{|\mathcal{A}|_\infty(t_{i+1}-t_i)}.$

(i)

$$\begin{aligned}|\Phi(t_{i+p+1},s+\tau)\mathcal{B}(s+\tau) - \Phi(t_{i+1},s)\mathcal{B}(s)| \leq &\varepsilon\rho\Big(1 + \big[|\mathcal{A}|_\infty + (t_{i+1}-t_i)\big] \\ &\times |\mathcal{B}|_\infty e^{|\mathcal{A}|_\infty(t_{i+1}-t_i)}\Big).\end{aligned}$$

Proof. From Corollary 2, it follows that $\tau \in T(\mathcal{A},\varepsilon) \cap T(\mathcal{B},\varepsilon) \cap T(\{t_{i,p}\},\varepsilon)$ is relatively dense. For $\tau \in T(\mathcal{A},\varepsilon) \cap T(\mathcal{B},\varepsilon) \cap T(\{t_{i,p}\},\varepsilon)$, we prove the following relations:
(a) Since $\Phi(t)$ is the fundamental matrix of the liner ODE system (4), $\Phi^{-1}(t)$ is a fundamental matrix of the adjoint equation

$$x'(t) = -\mathcal{A}(t)x(t).$$

Consequently, we have

$$\Phi^{-1}(t) - \Phi^{-1}(s) = -\int_s^t \Phi^{-1}(u)\mathcal{A}(u)du, \quad s < t.$$

Thus, we have

$$\Phi(r,t) - \Phi(r,s) = -\int_s^t \Phi(r,u)\mathcal{A}(u)du, \quad s < t,$$

$$\Phi(t,s) = I + \int_s^t \Phi(t,u)\mathcal{A}(u)du, \quad s < t,$$

$$\Phi(t+\tau, s+\tau) = I + \int_s^t \Phi(t+\tau, u+\tau)\mathcal{A}(u+\tau)du, \quad s < t.$$

By applying Gronwall's inequality, we obtain

$$|\Phi(t,s)| \leq |I|e^{|\mathcal{A}|_\infty(t_{i+1}-t_i)} := K_i \leq \rho, \ t,s \in I_i,$$

where $\rho = \sup\limits_{t,s\in I_i,\ \forall i\in\mathbb{Z}} |\Phi(t,s)|$.

(b) From the formulas above, we derive the following:

$$|\Phi(t+\tau, s+\tau) - \Phi(t,s)| = \left|\int_s^t \Phi(t+\tau, u+\tau)\mathcal{A}(u+\tau) - \Phi(t,u)\mathcal{A}(u)du\right|$$

$$\leq \int_s^t |\Phi(t+\tau, u+\tau) - \Phi(t,u)||\mathcal{A}(u+\tau)|du + \int_s^t |\Phi(t,u)||\mathcal{A}(u+\tau) - \mathcal{A}(u)|du$$

$$\leq K_i\varepsilon(t_{i+1} - t_i) + |\mathcal{A}|_\infty \int_s^t |\Phi(t+\tau, u+\tau) - \Phi(t,u)|du.$$

Using Gronwall's inequality, we obtain

$$|\Phi(t+\tau, s+\tau) - \Phi(t,s)| \leq K_i\varepsilon(t_{i+1} - t_i)e^{|\mathcal{A}|_\infty(t_{i+1}-t_i)}.$$

(c) Similarly,

$$\begin{aligned}
\left|\Phi(t+\tau, t_i+\tau) - \Phi(t+\tau, t_{i+p})\right| &\leq \left|\int_{t_{i+p}}^{t_i+\tau} \Phi(t+\tau, u)\mathcal{A}(u)du\right| \\
&\leq \left|\int_{t_{i+p}-\tau}^{t_i} \Phi(t+\tau, u+\tau)\mathcal{A}(u+\tau)du\right| \\
&\leq |\mathcal{A}|_\infty K_{i+p}\left|t_i - (t_{i+p} - \tau)\right| \\
&\leq |\mathcal{A}|_\infty \rho\varepsilon.
\end{aligned}$$

(d) It follows, by combining results (b) and (c), that

$$\begin{aligned}
&\left|\Phi(t+\tau, t_{i+p}) - \Phi(t,t_i)\right| \\
&\leq \left|\Phi(t+\tau, t_{i+p}) - \Phi(t+\tau, t_i+\tau)\right| + |\Phi(t+\tau, t_i+\tau) - \Phi(t,t_i)|.
\end{aligned}$$

(e) Similarly, it follows, by (a), (b) and condition ($\mathcal{A}P_1$), that

$$\begin{aligned}
&|\Phi(t+\tau, s+\tau)\mathcal{B}(s+\tau) - \Phi(t,s)\mathcal{B}(s)| \\
&\leq |\Phi(t+\tau, s+\tau)||\mathcal{B}(s+\tau) - \mathcal{B}(s)| + |\Phi(t+\tau, s+\tau) - \Phi(t,s)||\mathcal{B}(s)|.
\end{aligned}$$

(f) From $(\mathcal{A}P_3)$, we have $\left|t_{i+p+1} - (t_{i+1} + \tau)\right| < \varepsilon$. For $s \in I_i$, we note

$$\Phi(t_{i+p+1}, s+\tau) - \Phi(t_{i+1}, s) = \int_{s+\tau}^{t_{i+p+1}} \Phi(t_{i+p+1}, u)\mathcal{A}(u)du - \int_{s}^{t_{i+1}} \Phi(t_{i+1}, u)\mathcal{A}(u)du$$
$$= \int_{t_{i+1}}^{t_{i+p+1}-\tau} \Phi(t_{i+p+1}, u+\tau)\mathcal{A}(u+\tau)du + \int_{s}^{t_{i+1}} \Phi(t_{i+p+1}, u+\tau)[\mathcal{A}(u+\tau) - \mathcal{A}(u)]du$$
$$+ \int_{s}^{t_{i+1}} \left[\Phi(t_{i+p+1}, u+\tau) - \Phi(t_{i+1}, u)\right]\mathcal{A}(u)du.$$

Thus,

$$\begin{aligned}|\Phi(t_{i+p+1}, s+\tau) - \Phi(t_{i+1}, s)| \le& \left|\int_{t_{i+1}}^{t_{i+p+1}-\tau} \Phi(t_{i+p+1}, u+\tau)\mathcal{A}(u+\tau)du\right| \\ &+ \int_{t_i}^{t_{i+1}} |\Phi(t_{i+p+1}, u+\tau)||\mathcal{A}(u+\tau) - \mathcal{A}(u)|du \\ &+ \int_{s}^{t_{i+1}} \left|\Phi(t_{i+p+1}, u+\tau) - \Phi(t_{i+1}, u)\right||\mathcal{A}(u)|du \\ \le& \rho\varepsilon(|\mathcal{A}|_\infty + (t_{i+1} - t_i)) \\ &+ \int_{s}^{t_{i+1}} \left[\Phi(t_{i+p+1}, u+\tau) - \Phi(t_{i+1}, u)\right]\mathcal{A}(u)du.\end{aligned}$$

By applying Gronwall's inequality, we obtain

$$\begin{aligned}\left|\Phi(t_{i+p+1}, s+\tau) - \Phi(t_{i+1}, s)\right| &\le \rho\varepsilon(|\mathcal{A}|_\infty + (t_{i+1} - t_i)) \exp\left(\int_{s}^{t_{i+1}} \mathcal{A}(u)du\right) \\ &\le \rho\varepsilon(|\mathcal{A}|_\infty + (t_{i+1} - t_i)) \exp(|\mathcal{A}|_\infty (t_{i+1} - t_i)).\end{aligned}$$

(g) It follows by $(\mathcal{A}P_3)$ that

$$\begin{aligned}\left|\Phi(t_{i+1}, t_{i+p} - \tau) - \Phi(t_{i+1}, t_i)\right| &= \left|\int_{t_{i+p}-\tau}^{t_{i+1}} \Phi(t_{i+1}, u)\mathcal{A}(u)du - \int_{t_i}^{t_{i+1}} \Phi(t_{i+1}, u)\mathcal{A}(u)du\right| \\ &\le \left|\int_{t_{i+p}-\tau}^{t_i} \Phi(t_{i+1}, u)\mathcal{A}(u)du\right| \le \varepsilon\rho|\mathcal{A}|_\infty.\end{aligned}$$

(h) It follows by (f) and (g) that

$$\begin{aligned}|\Phi(t_{i+p+1}, t_{i+p}) - \Phi(t_{i+1}, t_i)| \le& |\Phi(t_{i+p+1}, t_{i+p}) - \Phi(t_{i+1}, t_{i+p} - \tau)| \\ &+ |\Phi(t_{i+1}, t_{i+p} - \tau) - \Phi(t_{i+1}, t_i)|.\end{aligned}$$

(i) Finally, it follows by $(\mathcal{A}P_1)$ and (f) from

$$\begin{aligned}&\left|\Phi(t_{i+p+1}, s+\tau)\mathcal{B}(s+\tau) - \Phi(t_{i+1}, s)\mathcal{B}(s)\right| \\ &\le \left|\Phi(t_{i+p+1}, s+\tau)\right||\mathcal{B}(s+\tau) - \mathcal{B}(s)| + \left|\Phi(t_{i+p+1}, s+\tau) - \Phi(t_{i+1}, s)\right||\mathcal{B}(s)|.\end{aligned}$$

This concludes the proof. □

Lemma 8. *We assume that conditions $(\mathcal{A}P_1)$ and $(\mathcal{A}P_3)$ are satisfied. Then, the sequence of matrices*

$$C_i = C(t_{i+1}, t_i) = \Phi(t_{i+1}, t_i) + \left(\int_{t_i}^{t_{i+1}} \Phi(t_{i+1}, s)\mathcal{B}(s)ds\right), \quad i \in \mathbb{Z}$$

is almost periodic.

Proof. From Lemma 7(h), it is known that $\Phi_i = \Phi(t_{i+1}, t_i)$ is almost periodic. Additionally, from Corollary 2, it follows that $\tau \in T(\mathcal{A}, \varepsilon) \cap T(\mathcal{B}, \varepsilon) \cap T(\{t_{i,p}\}, \varepsilon)$ is relatively dense. For $\tau \in T(\mathcal{A}, \varepsilon) \cap T(\mathcal{B}, \varepsilon) \cap T(\{t_{i,p}\}, \varepsilon)$, using Lemma 7($i$) and ($\mathcal{A}P_3$), we have

$$
\begin{aligned}
|C_{i+p} - C_i| &= \left| \int_{t_{i+p}}^{t_{i+p+1}} \Phi(t_{i+p+1}, s)\mathcal{B}(s)ds - \int_{t_i}^{t_{i+1}} \Phi(t_{i+1}, s)\mathcal{B}(s)ds \right| \\
&\le \left| \int_{t_i}^{t_{i+1}} \Phi(t_{i+p+1}, s+\tau)\mathcal{B}(s+\tau)ds - \int_{t_i}^{t_{i+1}} \Phi(t_{i+1}, s)\mathcal{B}(s)ds \right| \\
&\quad + \left| \int_{t_{i+p}}^{t_{i+p+1}} \Phi(t_{i+p+1}, s)\mathcal{B}(s)ds - \int_{t_i+\tau}^{t_{i+p+1}} \Phi(t_{i+p+1}, s)\mathcal{B}(s)ds \right| \\
&\quad + \left| \int_{t_i+\tau}^{t_{i+p+1}} \Phi(t_{i+p+1}, s)\mathcal{B}(s)ds - \int_{t_i}^{t_{i+1}} \Phi(t_{i+p+1}, s+\tau)\mathcal{B}(s+\tau)ds \right| \\
&\le \int_{t_i}^{t_{i+1}} |\Phi(t_{i+p+1}, s+\tau)\mathcal{B}(s+\tau) - \Phi(t_{i+1}, s)\mathcal{B}(s)|ds + \left| \int_{t_{i+p}}^{t_i+\tau} \Phi(t_{i+p+1}, s)\mathcal{B}(s)ds \right| \\
&\quad + \left| \int_{t_i+\tau}^{t_{i+p+1}} \Phi(t_{i+p+1}, s)\mathcal{B}(s)ds - \int_{t_i+\tau}^{t_{i+1}+\tau} \Phi(t_{i+p+1}, s)\mathcal{B}(s)ds \right| \\
&\le \varepsilon\rho\Big(1 + [|A|_\infty + (t_{i+1} - t_i)]|\mathcal{B}|_\infty e^{|\mathcal{A}|_\infty (t_{i+1} - t_i)}\Big)(t_{i+1} - t_i) + \varepsilon|\mathcal{B}|_\infty \rho + \varepsilon|\mathcal{B}|_\infty \rho \\
&\le \varepsilon\rho\Big\{\Big(1 + [|\mathcal{A}|_\infty + (t_{i+1} - t_i)]|\mathcal{B}|_\infty e^{|\mathcal{A}|_\infty (t_{i+1} - t_i)}\Big)(t_{i+1} - t_i) + 2|\mathcal{B}|_\infty\Big\}.
\end{aligned}
$$

This completes the proof. □

Lemma 9. *For $i, p, k \in \mathbb{Z}$,*

$$
\overleftarrow{\prod}_{j=k+p}^{i+p-1} C_j - \overleftarrow{\prod}_{j=k}^{i-1} C_j = \sum_{j=k}^{i-1} \left[\left(\overleftarrow{\prod}_{r=j+1}^{i-1} C_r \right) (C_{j+p} - C_j) \left(\overleftarrow{\prod}_{r=k}^{j-1} C_{r+p} \right) \right].
$$

Proof. We prove this statement by induction.

For $i = k+1$, the case is trivial. If $i = k+2$, we have

$$
\begin{aligned}
\overleftarrow{\prod}_{j=k+p}^{k+p+1} C_j - \overleftarrow{\prod}_{j=k}^{k+1} C_j &= C_{k+p+1}C_{k+p} - C_{k+1}C_k \\
&= \Big(C_{k+p+1} - C_{k+1}\Big)C_{k+p} + C_{k+1}\Big(C_{k+p} - C_k\Big) \\
&= \sum_{j=k}^{k+1} \left[\left(\overleftarrow{\prod}_{r=j+1}^{k+1} C_r \right) (C_{j+p} - C_j) \left(\overleftarrow{\prod}_{r=k}^{j-1} C_{r+p} \right) \right].
\end{aligned}
$$

For $i = k+3$, we obtain

$$
\begin{aligned}
\overleftarrow{\prod}_{j=k+p}^{k+p+2} C_j - \overleftarrow{\prod}_{j=k}^{k+2} C_j &= C_{k+p+2}C_{k+p+1}C_{k+p} - C_{k+2}C_{k+1}C_k \\
&= \Big(C_{k+p+2} - C_{k+2}\Big)C_{k+p+1}C_{k+p} + C_{k+2}\Big(C_{k+p+1} - C_{k+1}\Big)C_{k+p} \\
&\quad + C_{k+2}C_{k+1}\Big(C_{k+p} - C_k\Big) \\
&= \sum_{j=k}^{k+2} \left[\left(\overleftarrow{\prod}_{r=j+1}^{j=k+2} C_r \right) (C_{j+p} - C_j) \left(\overleftarrow{\prod}_{r=k}^{j-1} C_{r+p} \right) \right].
\end{aligned}
$$

Now, we assume that the following statement holds:

$$
\overleftarrow{\prod}_{j=k+p}^{i+p-2} C_j - \overleftarrow{\prod}_{j=k}^{i-2} C_j = \sum_{j=k}^{i-2} \left[\left(\overleftarrow{\prod}_{r=j+1}^{i-2} C_r \right) (C_{j+p} - C_j) \left(\overleftarrow{\prod}_{r=k}^{j-1} C_{r+p} \right) \right].
$$

We must show that

$$\overleftarrow{\prod}_{j=k+p}^{i+p-1} C_j - \overleftarrow{\prod}_{j=k}^{i-1} C_j = \sum_{j=k}^{i-1} \left[\left(\overleftarrow{\prod}_{r=j+1}^{i-1} C_r \right) (C_{j+p} - C_j) \left(\overleftarrow{\prod}_{r=k}^{j-1} C_{r+p} \right) \right].$$

Indeed, we find that

$$\begin{aligned}
&\overleftarrow{\prod}_{j=k+p}^{i+p-1} C_j - \overleftarrow{\prod}_{j=k}^{i-1} C_j = \overleftarrow{\prod}_{j=k+p}^{i+p-1} C_j - C_{i-1} \overleftarrow{\prod}_{j=k+p}^{i+p-2} C_j + C_{i-1} \overleftarrow{\prod}_{j=k+p}^{i+p-2} C_j - \overleftarrow{\prod}_{j=k}^{i-1} C_j \\
&= \overleftarrow{\prod}_{j=k}^{i-1} C_{j+p} - C_{i-1} \overleftarrow{\prod}_{j=k}^{i-2} C_{j+p} + C_{i-1} \overleftarrow{\prod}_{j=k}^{i-2} C_{j+p} - \overleftarrow{\prod}_{j=k}^{i-1} C_j \\
&= (C_{i+p-1} - C_{i-1}) \left(\overleftarrow{\prod}_{j=k}^{i-2} C_{j+p} \right) + C_{i-1} \left(\overleftarrow{\prod}_{j=k}^{i-2} C_{j+p} - \overleftarrow{\prod}_{j=k}^{i-2} C_j \right) \\
&= (C_{i+p-1} - C_{i-1}) \left(\overleftarrow{\prod}_{j=k}^{i-2} C_{j+p} \right) + C_{i-1} \sum_{j=k}^{i-2} \left[\left(\overleftarrow{\prod}_{r=j+1}^{i-2} C_r \right) (C_{j+p} - C_j) \left(\overleftarrow{\prod}_{r=k}^{j-1} C_{r+p} \right) \right] \\
&= (C_{i+p-1} - C_{i-1}) \left(\overleftarrow{\prod}_{j=k}^{i-2} C_{j+p} \right) + \sum_{j=k}^{i-2} \left[\left(\overleftarrow{\prod}_{r=j+1}^{i-1} C_r \right) (C_{j+p} - C_j) \left(\overleftarrow{\prod}_{r=k}^{j-1} C_{r+p} \right) \right] \\
&= \sum_{j=k}^{i-1} \left[\left(\overleftarrow{\prod}_{r=j+1}^{i-1} C_r \right) (C_{j+p} - C_j) \left(\overleftarrow{\prod}_{r=k}^{j-1} C_{r+p} \right) \right].
\end{aligned}$$

Thus, the Lemma is established by induction. □

Corollary 3. *If $\tau \in T(\mathcal{A}, \varepsilon) \cap T(\mathcal{B}, \varepsilon) \cap T(\{t_{i,p}\}, \varepsilon)$, then there exists $\varrho \in \mathbb{R}^+$ such that*

$$\left| \overleftarrow{\prod}_{j=k+p}^{i+p-1} C_j - \overleftarrow{\prod}_{j=k}^{i-1} C_j \right| \le \varepsilon\varrho.$$

Proof. For Lemma 9, we provide the following estimate:

$$\begin{aligned}
\left| \overleftarrow{\prod}_{j=k+p}^{i+p-1} C_j - \overleftarrow{\prod}_{j=k}^{i-1} C_j \right| &= \left| \sum_{j=k}^{i-1} \left[\left(\overleftarrow{\prod}_{r=j+1}^{i-1} C_r \right) (C_{j+p} - C_j) \left(\overleftarrow{\prod}_{r=k}^{j-1} C_{r+p} \right) \right] \right| \\
&\le \sum_{j=k}^{i-1} \left[\left| \left(\overleftarrow{\prod}_{r=j+1}^{i-1} C_r \right) \right| |C_{j+p} - C_j| \left| \left(\overleftarrow{\prod}_{r=k}^{j-1} C_{r+p} \right) \right| \right] \\
&\le \varepsilon \left(\sum_{j=k}^{i-1} \left[\left| \left(\overleftarrow{\prod}_{r=j+1}^{i-1} C_r \right) \right| \left| \left(\overleftarrow{\prod}_{r=k}^{j-1} C_{r+p} \right) \right| \right] \right) \\
&\le \varepsilon\varrho.
\end{aligned}$$

□

Lemma 10. *We assume that conditions $(\mathcal{A}P_1)$ and $(\mathcal{A}P_3)$ are fulfilled. Then, the matrix function*

$$C_i(t) = C(t, t_i) = \Phi(t, t_i) + \left(\int_{t_i}^{t} \Phi(t, s)\mathcal{B}(s) ds \right), \quad i \in \mathbb{Z}$$

is almost periodic.

Proof. We let $\tau \in T(\mathcal{A}, \varepsilon) \cap T(\mathcal{B}, \varepsilon) \cap T(\{t_{i,p}\}, \varepsilon)$. We proceed in a manner similar to Lemma 8, since $C(t, t_i) = \Phi(t, t_i) + \int_{t_i}^{t} \Phi(t, s)\mathcal{B}(s)ds$. Therefore,

$$\begin{aligned} &C(t+\tau, t_{i+p}) - C(t, t_i) \\ &= \Phi(t+\tau, t_{i+p}) + \int_{t_{i+p}}^{t+\tau} \Phi(t+\tau, s)\mathcal{B}(s)ds - \Phi(t, t_i) - \int_{t_i}^{t} \Phi(t, s)\mathcal{B}(s)ds \\ &= [\Phi(t+\tau, t_{i+p}) - \Phi(t, t_i)] + \int_{t_{i+p}-\tau}^{t_i} \Phi(t+\tau, s+\tau)\mathcal{B}(s+\tau)ds \\ &\qquad + \int_{t_i}^{t} (\Phi(t+\tau, s+\tau)\mathcal{B}(s+\tau) - \Phi(t, s)\mathcal{B}(s))ds. \end{aligned}$$

Using Lemma 7(*d*,*e*), and $(\mathcal{A}P_3)$, it follows that $C_i(t)$ is almost periodic. □

Lemma 11. *We assume that conditions* $(\mathcal{A}P_1)$ *and* $(\mathcal{A}P_3)$ *are fulfilled. If* $t \in I_i$*, then the matrix function*

$$\Psi_k(t) := \Psi(t, t_k) = C(t, t_i)\left(\overleftarrow{\prod}_{j=k}^{i-1} C_j\right), \quad k \in \mathbb{Z}$$

is almost periodic.

Proof. We let $\tau \in T(\mathcal{A}, \varepsilon) \cap T(\mathcal{B}, \varepsilon) \cap T(\{t_{i,p}\}, \varepsilon)$. We fix $t \in \mathbb{R}$ and let $t \in I_i$. Then,

$$\begin{aligned} \left|\Psi_{k+p}(t+\tau) - \Psi_k(t)\right| &= \left|C(t+\tau, t_{i+p})\left(\overleftarrow{\prod}_{j=k+p}^{i+p-1} C_j\right) - C(t, t_i)\left(\overleftarrow{\prod}_{j=k}^{i-1} C_j\right)\right| \\ \leq |C(t+\tau, t_{i+p}) - C(t, t_i)| &\left|\overleftarrow{\prod}_{j=k+p}^{i+p-1} C_j\right| + |C(t, t_i)|\left|\overleftarrow{\prod}_{j=k+p}^{i+p-1} C_j - \overleftarrow{\prod}_{j=k}^{i-1} C_j\right|. \end{aligned}$$

By applying Lemma 10 and Corollary 3, we establish that $\Psi_k(t)$ is an almost periodic function, thus concluding the proof. □

Lemma 12. *We assume that conditions* $(\mathcal{A}P_1)$, $(\mathcal{A}P_2)$, *and* $(\mathcal{A}P_3)$ *are fulfilled. If* $t \in I_i$ *and* $s \in I_k$*, then the matrix function*

$$\Psi_{k+1}(t)\Phi(t_{k+1}, s)f(s), \quad k \in \mathbb{Z},$$

is double almost periodic.

Proof. We let $\tau \in T(\mathcal{A}, \varepsilon) \cap T(\mathcal{B}, \varepsilon) \cap T(f, \varepsilon) \cap T(\{t_{i,p}\}, \varepsilon)$, and fix $s, t \in \mathbb{R}$ with $s \in I_k$ and $t \in I_i$. Then,

$$\begin{aligned} &\left|\Psi_{k+p+1}(t+\tau)\Phi(t_{k+p+1}, s+\tau)f(s+\tau) - \Psi_{k+1}(t)\Phi(t_{k+1}, s)f(s)\right| \\ &= \Bigg|C(t+\tau, t_{i+p})\left(\overleftarrow{\prod}_{j=k+p+1}^{i+p-1} C_j\right)\Phi(t_{k+p+1}, s+\tau)f(s+\tau) \\ &\qquad - C(t, t_i)\left(\overleftarrow{\prod}_{j=k+1}^{i-1} C_j\right)\Phi(t_{k+1}, s)f(s)\Bigg| \\ &\leq \left|C(t+\tau, t_{i+p})\left(\overleftarrow{\prod}_{j=k+p+1}^{i+p-1} C_j\right)\right|\left|\Phi(t_{k+p+1}, s+\tau)f(s+\tau) - \Phi(t_{k+1}, s)f(s)\right| \\ &\qquad + \left|C(t+\tau, t_{i+p})\left(\overleftarrow{\prod}_{j=k+p+1}^{i+p-1} C_j\right) - C(t, t_i)\left(\overleftarrow{\prod}_{j=k+1}^{i-1} C_j\right)\right|\left|\Phi(t_{k+1}, s)f(s)\right|. \end{aligned}$$

By applying Lemma 7(*i*) and Lemma 11, the result follows. This completes the proof. □

Lemma 13. *We assume that conditions ($\mathcal{A}P_1$), ($\mathcal{A}P_2$), and ($\mathcal{A}P_3$) are fulfilled. If $t \in I_i$, then the matrix function*

$$\int_{t_k}^{t_{k+1}} \Psi(t, t_{k+1})\Phi(t_{k+1}, s)f(s)ds$$

is almost periodic.

Proof. We let $\tau \in T(\mathcal{A}, \varepsilon) \cap T(\mathcal{B}, \varepsilon) \cap T(f, \varepsilon) \cap T(\{t_{i,p}\}, \varepsilon)$. Fix $t \in \mathbb{R}$, where $t \in I_i$. Then,

$$\begin{aligned}
&\left| \int_{t_{k+p}}^{t_{k+p+1}} \Psi(t+\tau, t_{k+p+1})\Phi(t_{k+p+1}, s)f(s)ds - \int_{t_k}^{t_{k+1}} \Psi(t, t_{k+1})\Phi(t_{k+1}, s)f(s)ds \right| \\
&= \left| \int_{t_{k+p}-\tau}^{t_{k+p+1}-\tau} \Psi(t+\tau, t_{k+p+1})\Phi(t_{k+p+1}, s+\tau)f(s+\tau)ds \right. \\
&\qquad \left. - \int_{t_k}^{t_{k+1}} \Psi(t, t_{k+1})\Phi(t_{k+1}, s)f(s)ds \right| \\
&\leq \left| \int_{t_{k+p}-\tau}^{t_{k+p+1}-\tau} \Psi(t+\tau, t_{k+p+1})\Phi(t_{k+p+1}, s+\tau)f(s+\tau)ds \right. \\
&\qquad \left. - \int_{t_{k+p}-\tau}^{t_{k+p+1}-\tau} \Psi(t, t_{k+1})\Phi(t_{k+1}, s)f(s)ds \right| \\
&\quad + \left| \int_{t_{k+p}-\tau}^{t_{k+p+1}-\tau} \Psi(t, t_{k+1})\Phi(t_{k+1}, s)f(s)ds - \int_{t_k}^{t_{k+p+1}-\tau} \Psi(t, t_{k+1})\Phi(t_{k+1}, s)f(s)ds \right| \\
&\qquad + \left| \int_{t_k}^{t_{k+p+1}-\tau} \Psi(t, t_{k+1})\Phi(t_{k+1}, s)f(s)ds - \int_{t_k}^{t_{k+1}} \Psi(t, t_{k+1})\Phi(t_{k+1}, s)f(s)ds \right| \\
&\leq \left| \int_{t_{k+p}-\tau}^{t_{k+p+1}-\tau} \Big[\Psi(t+\tau, t_{k+p+1})\Phi(t_{k+p+1}, s+\tau)f(s+\tau) - \Psi(t, t_{k+1})\Phi(t_{k+1}, s)f(s) \Big] ds \right| \\
&\quad + \left| \int_{t_k}^{t_{k+p}-\tau} \Psi(t, t_{k+1})\Phi(t_{k+1}, s)f(s)ds \right| + \left| \int_{t_{k+1}}^{t_{k+p+1}-\tau} \Psi(t, t_{k+1})\Phi(t_{k+1}, s)f(s)ds \right|
\end{aligned}$$

and the result follows by using **(**$\mathcal{A}P_3$**)** and Lemma 12. □

The following is similar.

Lemma 14. *We assume that conditions* **(**$\mathcal{A}P_1$**)**, **(**$\mathcal{A}P_2$**)**, *and* **(**$\mathcal{A}P_3$**)** *are fulfilled. If $t \in I_i$, then the matrix function*

$$f_i(t) := f(t, t_i) = \int_{t_i}^{t} \Phi(t, s)f(s)ds$$

is almost periodic.

4. Almost Periodic Solutions

In this section, criteria are established for the existence and stability of almost periodic solutions in both the linear inhomogeneous DEGPCD (2) and the quasilinear DEGPCD (3).

4.1. Almost Periodic Solutions in the Linear Inhomogeneous DEGPCD

By utilizing Proposition 2, we can prove the following theorem in a manner similar to [44], Proposition 11.

Theorem 3. *We assume that conditions ($\mathcal{A}P_1$), ($\mathcal{A}P_2$), ($\mathcal{A}P_3$), and ($\mathcal{D}$) are fulfilled. Then, inhomogeneous difference Equation (6) admits a unique almost periodic solution. Furthermore, for all $k \in \mathbb{Z}$, the following inequality is satisfied:*

$$|\widetilde{y}(k)| \le K\left(\frac{1}{1-\mu_1} + \frac{1}{\mu_2 - 1}\right)\sup_{k\in\mathbb{Z}}\left|\widetilde{f}(k)\right|.$$

The following theorem addresses the existence of a unique almost periodic solution for linear inhomogeneous DEGPCD system (2). We examine an initial condition where solution values are assessed solely at points in the sequence $\varsigma \in \Theta$. However, it is clear that the proof is similar for $\varsigma \in \mathbb{R}$.

Theorem 4. *We assume that conditions ($\mathcal{A}P_1$), ($\mathcal{A}P_2$), ($\mathcal{A}P_3$), and ($\mathcal{D}$) are fulfilled. Then, the linear inhomogeneous DEGPCD system (2) has a unique almost periodic solution.*

Proof. Let us assume without loss of generality that $\varsigma = t_i$, where $i \in \mathbb{Z}$. By employing the variation of Constants Formula (21), we know that the solution to the linear inhomogeneous DEGPCD (2) on the interval I_i satisfies

$$y(t) = \Psi(t, t_i)y(t_i) + \int_{t_i}^{t} G(t,s)f(s)ds.$$

Given that the sequence $\{t_{i,p}\}$ is equipotentially almost periodic, a relatively dense set of common ε-almost periods exists, where $\left|t_{i,p} - \tau\right| < \varepsilon$. From Corollary 2, it follows that $\tau \in T(\mathcal{A}, \varepsilon) \cap T(\mathcal{B}, \varepsilon) \cap T(f, \varepsilon) \cap T(\{t_{i,p}\}, \varepsilon)$, which is relatively dense.

Furthermore, it is not difficult to see that the sequence $\{\widetilde{y}(i)\} = \{y(t_i)\}$ satisfies difference Equation (6). According to Theorem 3, inhomogeneous difference Equation (6) admits a unique almost periodic sequence solution $\widetilde{y}_0(i) = y_0(t_i)$, and

$$|\widetilde{y}_0(i)| \le K\left(\frac{1}{1-\mu_1} + \frac{1}{\mu_2 - 1}\right)\sup_{i\in\mathbb{Z}}\left|\widetilde{f}(i)\right|, \quad i \in \mathbb{Z}.$$

Now, we aim to demonstrate that the solution $\overline{y}(t)$ with $\overline{y}(t_i) = y_0(t_i)$ for $i \in \mathbb{Z}$ is an almost periodic solution of the linear inhomogeneous DEGPCD (2).

For $\tau \in T(\mathcal{A}, \varepsilon) \cap T(\mathcal{B}, \varepsilon) \cap T(f, \varepsilon) \cap T(\{t_{i,p}\}, \varepsilon) \cap T(y_0, \varepsilon)$, we have

$$\begin{aligned}
&\overline{y}(t+\tau, \overline{y}(t_{i+p})) - \overline{y}(t, \overline{y}(t_i)) \\
&= \Psi(t+\tau, t_{i+p})\overline{y}(t_{i+p}) + \int_{t_{i+p}}^{t+\tau} G(t+\tau, s)f(s)ds - \Psi(t, t_i)\overline{y}(t_i) - \int_{t_i}^{t} G(t,s)f(s)ds \\
&= \Psi(t+\tau, t_{i+p})\overline{y}(t_{i+p}) + \sum_{j=i+p}^{i(t)+p-1} \Psi(t+\tau, t_{j+1}) \int_{t_j}^{t_{j+1}} \Phi(t_{j+1}, s)f(s)ds \\
&\quad + \int_{t_{i(t)+p}}^{t+\tau} \Phi(t,s)f(s)ds - \Psi(t,t_i)\overline{y}(t_i) - \sum_{j=i}^{i(t)-1} \Psi(t, t_{j+1}) \int_{t_j}^{t_{j+1}} \Phi(t_{j+1}, s)f(s)ds \\
&\quad - \int_{t_{i(t)}}^{t} \Phi(t,s)f(s)ds \\
&= \Psi(t+\tau, t_{i+p})\overline{y}(t_{i+p}) - \Psi(t, t_i)\overline{y}(t_i) \\
&\quad + \sum_{j=i+p}^{i(t)+p-1} \Psi(t+\tau, t_{j+1}) \int_{t_j}^{t_{j+1}} \Phi(t_{j+1}, s)f(s)ds - \sum_{j=i}^{i(t)-1} \Psi(t, t_{j+1}) \int_{t_j}^{t_{j+1}} \Phi(t_{j+1}, s)f(s)ds \\
&\quad + \int_{t_{i(t)+p}}^{t+\tau} \Phi(t+\tau, s)f(s)ds - \int_{t_{i(t)}}^{t} \Phi(t,s)f(s)ds
\end{aligned}$$

$$
\begin{aligned}
&= \Big[\Psi(t+\tau, t_{i+p}) - \Psi(t,t_i)\Big]\bar{y}(t_{i+p}) + \Psi(t,t_i)\left[\bar{y}(t_{i+p}) - \bar{y}(t_i)\right] \\
&\quad + \sum_{j=i+p}^{i(t)+p-1} \Psi(t+\tau, t_{j+1}) \int_{t_j}^{t_{j+1}} \Phi(t_{j+1},s)f(s)ds - \sum_{j=i}^{i(t)-1} \Psi(t,t_{j+1}) \int_{t_j}^{t_{j+1}} \Phi(t_{j+1},s)f(s)ds \\
&\quad + \int_{t_{i(t)+p}}^{t+\tau} \Phi(t+\tau,s)f(s)ds - \int_{t_{i(t)}}^{t} \Phi(t,s)f(s)ds \\
&= \Big[\Psi(t+\tau, t_{i+p}) - \Psi(t,t_i)\Big]\bar{y}(t_{i+p}) + \Psi(t,t_i)\left[\bar{y}(t_{i+p}) - \bar{y}(t_i)\right] \\
&\quad + \sum_{j=i}^{i(t)-1} \left[\Psi(t+\tau, t_{j+p+1}) \int_{t_{j+p}}^{t_{j+p+1}} \Phi(t_{j+p+1},s)f(s)ds - \Psi(t,t_{j+1}) \int_{t_j}^{t_{j+1}} \Phi(t_{j+1},s)f(s)ds\right] \\
&\quad + \int_{t_{i(t)+p}}^{t+\tau} \Phi(t+\tau,s)f(s)ds - \int_{t_{i(t)}}^{t} \Phi(t,s)f(s)ds.
\end{aligned}
$$

Using Lemmas 11, 13, and 14, it follows that

$$
\Psi(t,t_i), \quad \int_{t_j}^{t_{j+1}} \Psi(t,t_j)\Phi(t_j,s)f(s)ds, \quad \text{and} \quad \int_{t_{i(t)}}^{t} \Phi(t,s)f(s)ds
$$

are almost periodic functions. By Theorem 3, the sequence $\{\bar{y}(t_i)\}$ is almost periodic. Consequently, there exists a positive constant $\mathcal{C} \in \mathbb{R}^+$ such that

$$
\left|\overline{y}(t+\tau, t_{i+p}, \overline{y}(t_{i+p})) - \overline{y}(t, t_i, \overline{y}(t_i))\right| \le \varepsilon\mathcal{C}.
$$

From the definition of $\overline{y}$, it is evident that $\overline{y}(t)$ is almost periodic. If $y(t)$ represents another almost periodic solution of the linear inhomogeneous DEGPCD (2), it follows from Lemma 6 that $\{y(t_i)\}$ is an almost periodic sequence solution for the inhomogeneous difference Equation (6). From the uniqueness, we conclude that $y(t_i) = \overline{y}(t_i)$ for $\{t_i\} \in \Theta$. This implies $y(t) = \overline{y}(t)$ for $t \in \mathbb{R}$. Hence, the linear inhomogeneous DEGPCD (2) has a unique almost periodic solution. This concludes the proof of Theorem 4. □

Remark 4. *If we take the deviating argument $\beta(t) = [t]$, specifically $t_k = k$ for $k \in \mathbb{Z}$, and assume that the linear difference Equation (8) exhibits an exponential dichotomy on $\mathbb{Z}$, then Theorem 4 simplifies to the findings established by R. Yuan et al. in [44], Theorem 1.*

4.2. Almost Periodic Solutions in the Quasilinear DEGPCD

In this subsection, we assume that $g(t,\cdot,\cdot)$ is almost periodic. We let $\Im \subseteq \mathbb{R}^{2q}$ be defined as

$$
T(g,\Im,\varepsilon) = \{\tau \in \mathbb{R} |\, |g(t+\tau, z_1, z_2) - g(\tau, z_1, z_2)| < \varepsilon, \text{ for } (t, z_1, z_2) \in \mathbb{R} \times \Im\}
$$

which is a relatively dense set in $\mathbb{R}$ for all $\varepsilon > 0$.

For Corollary 2, we have the following result:

Lemma 15. *If the conditions $(\mathcal{A}P_1)$, $(\mathcal{A}P_3)$, and $(\mathcal{A}P_4)$ hold, and if $\{z(t_i)\}_{i\in\mathbb{Z}}$, $\{t_i\} \in \Theta$ is an almost periodic sequence, then the set $T(z,\varepsilon) \cap \mathcal{T}(A,\varepsilon) \cap \mathcal{T}(B,\varepsilon) \cap \mathcal{T}(g,\Im,\varepsilon) \cap \mathcal{T}(\{t_{i,p}\},\varepsilon)$ is relatively dense, where $\Im \subseteq \mathbb{R}^{2q}$ is a compact subset.*

Proposition 3. *We assume that conditions $(\mathcal{A}P_3)$, $(\mathcal{A}P_4)$, and $(\mathcal{L}_g)$ are fulfilled. If $z : \mathbb{R} \to \mathbb{R}^q$ is an almost periodic function, then the function $g(t, z(t), z(\beta(t)))$ is also almost periodic.*

Proof. We let $\varepsilon > 0$ and $\tau \in T(z,\varepsilon) \cap T(g,\Im,\varepsilon) \cap T(\{t_{i,p}\},\varepsilon)$. Since $z(t)$ is almost periodic, there exists a compact set $\Im \subseteq \mathbb{R}^{2q}$ such that $(z(t), z(\beta(t)) \subseteq \Im$ for all $t \in \mathbb{R}$. Thus, we have

$$\begin{aligned}
&\big|g\big(t+\tau, z(t+\tau), z(\beta(t+\tau))\big) - g\big(t, z(t), z(\beta(t))\big)\big| \\
&\leq \big|g\big(t+\tau, z(t+\tau), z(\beta(t+\tau))\big) - g(t, z(t+\tau), z(\beta(t+\tau)))\big| \\
&\qquad + \big|g\big(t, z(t+\tau), z(\beta(t+\tau))\big) - g\big(t, z(t), z(\beta(t))\big)\big| \\
&\leq \varepsilon + \mathcal{L}_1^g |z(t+\tau) - z(t)| + \mathcal{L}_2^g \Big| z(t_{i(t)} + \tau) - z(t_{i(t)+p}) \Big| \\
&\leq \varepsilon(1 + \mathcal{L}_1^g + \mathcal{L}_2^g).
\end{aligned}$$

Therefore, the combined almost periodicity of g and z confirms the almost periodicity of $g(t, z(t), z(\beta(t)))$. □

Theorem 5. *We assume that conditions ($\mathcal{AP}_1$), ($\mathcal{AP}_3$), ($\mathcal{AP}_4$), ($\mathcal{L}_g$), ($\mathcal{I}$), (E), and ($\mathcal{D}$) are satisfied. If*

$$\rho\theta\Big(\mathcal{L}_1^g + \mathcal{L}_2^g\Big)\left(\frac{\alpha K}{1-\mu_1} + \frac{\alpha K}{\mu_2 - 1} + 1\right) < 1 \tag{41}$$

holds, where $\rho = \sup_{t,s \in I_i,\ \forall i \in \mathbb{Z}} |\Phi(t,s)|$, $\theta = \sup_{n \in \mathbb{Z}} (t_{n+1} - t_n)$, α *is defined in* (35), *and* K, μ_1, μ_2 *are defined in* (9), *then the quasilinear DEGPCD* (3) *has a unique almost periodic solution.*

Proof. We let $\mathcal{AP}(\mathbb{R}, \mathbb{R}^q)$ represent the Banach space of all almost periodic functions with the supremum norm: $\|\phi\| = \sup_{t \in \mathbb{R}} |\phi(t)|$. For any $\phi \in \mathcal{AP}(\mathbb{R}, \mathbb{R}^q)$, by Proposition 3, we have $g(t, \phi(t), \phi(\beta(t))) \in \mathcal{AP}(\mathbb{R}, \mathbb{R}^q)$. By applying Theorem 4 and Proposition 3, the following DEGPCD,

$$z'(t) = \mathcal{A}(t)z(t) + \mathcal{B}(t)z(\beta(t)) + g(t, \phi(t), \phi(\beta(t)))$$

has a unique almost periodic solution $z_\phi \in \mathcal{AP}(\mathbb{R}, \mathbb{R}^q)$.

Moreover, for $t \in [t_n, t_{n+1})$, the solution z_ϕ satisfies

$$z_\phi(t) = \Psi(t, t_n) z_\phi(t_n) + \int_{t_n}^{t} G(t,s) g(s, \phi(s), \phi(\beta(s))) ds, \tag{42}$$

where $z_\phi(t_n) = \widetilde{z}_\phi(n)$ represents the unique discrete almost periodic solution of the corresponding difference equation

$$\widetilde{z}_\phi(n+1) = C(n)\widetilde{z}_\phi(n) + \widetilde{g}_\phi(n),\ n \in \mathbb{Z}, \tag{43}$$

where

$$C(n) = C_n(t_{n+1}, t_n) \qquad \text{and} \qquad \widetilde{g}_\phi(n) = \int_{t_n}^{t_{n+1}} \Phi(t_{n+1}, u) g(u, \phi(u), \phi(t_n)) du.$$

From Theorem 3, the unique discrete almost periodic solution $z_\phi(t_n) = \widetilde{z}_\phi(n)$ satisfies the following estimate:

$$\big|z_\phi(t_n)\big| \leq K\left(\frac{1}{1-\mu_1} + \frac{1}{\mu_2 - 1}\right) \sup_{n \in \mathbb{Z}} \big|\widetilde{g}_\phi(n)\big|. \tag{44}$$

Now, we define the operator $\mathcal{T} : \mathcal{AP}(\mathbb{R}, \mathbb{R}^q) \to \mathcal{AP}(\mathbb{R}, \mathbb{R}^q)$ by

$$(\mathcal{T}\phi)(t) = z_\phi(t).$$

From Theorem 4, this operator is well defined since for each $\phi \in \mathcal{AP}(\mathbb{R}, \mathbb{R}^q)$, $\mathcal{T}\phi$ is the unique almost periodic solution of the DEGPCD (42).

For arbitrary $\psi, \phi \in \mathcal{AP}(\mathbb{R}, \mathbb{R}^q)$, using Estimate (44), we obtain

$$|(\mathcal{T}\psi)(t_n) - (\mathcal{T}\phi)(t_n)| \leq K\left(\frac{1}{1-\mu_1} + \frac{1}{\mu_2 - 1}\right)\sup_{n\in\mathbb{Z}}|\mathcal{H}(t_n)|,$$

where

$$\begin{aligned} |\mathcal{H}(t_n)| &= \left|\int_{t_n}^{t_{n+1}} \Phi(t_{n+1}, s)[g(s, \psi(s), \psi(\beta(s))) - g(s, \phi(s), \phi(\beta(s)))]ds\right| \\ &\leq \int_{t_n}^{t_{n+1}} \rho\mathcal{L}_1^g|\psi(s) - \phi(s)| + \rho\mathcal{L}_2^g|\psi(\beta(s)) - \phi(\beta(s))|ds \\ &\leq \rho\theta\left(\mathcal{L}_1^g + \mathcal{L}_2^g\right)|\psi - \phi|_\infty. \end{aligned}$$

Using the variation of Constants Formula (42), we derive

$$\begin{aligned} (\mathcal{T}\phi)(t) - (\mathcal{T}\psi)(t) = \Psi(t, t_n)[(\mathcal{T}\phi)(t_n) - (\mathcal{T}\psi)(t_n)] \\ + \int_{t_n}^{t} \Phi(t, s)[g(s, \psi(s), \psi(t_n)) - g(s, \phi(s), \phi(t_n))]ds, \end{aligned}$$

for $t_n \leq t < t_{n+1}$. Therefore, we have

$$\begin{aligned} |(\mathcal{T}\phi)(t) - (\mathcal{T}\psi)(t)| &\leq |\Psi(t, t_n)||(\mathcal{T}\phi)(t_n) - (\mathcal{T}\psi)(t_n)| \\ &\quad + \int_{t_n}^{t} |\Phi(t, s)|\left[\mathcal{L}_1^g|\psi(s) - \phi(s)| + \mathcal{L}_2^g|\psi(t_n) - \phi(t_n)|\right]ds \\ &\leq \alpha K\left(\frac{1}{1-\mu_1} + \frac{1}{\mu_2 - 1}\right)\rho\theta\left(\mathcal{L}_1^g + \mathcal{L}_2^g\right)|\psi - \phi|_\infty \\ &\quad + \rho\theta\left(\mathcal{L}_1^g + \mathcal{L}_2^g\right)|\psi - \phi|_\infty \\ &\leq \rho\theta\left(\mathcal{L}_1^g + \mathcal{L}_2^g\right)\left(\frac{\alpha K}{1-\mu_1} + \frac{\alpha K}{\mu_2 - 1} + 1\right)|\psi - \phi|_\infty. \end{aligned}$$

Thus, by Condition (41), the operator $\mathcal{T} : \mathcal{AP}(\mathbb{R}, \mathbb{R}^q) \to \mathcal{AP}(\mathbb{R}, \mathbb{R}^q)$ is contracting mapping. By the Banach fixed-point theorem, $\mathcal{T}$ admits a unique fixed point $\phi \in \mathcal{AP}(\mathbb{R}, \mathbb{R}^q)$. This completes the proof. □

Remark 5. *If linear difference Equation (8) exhibits an exponential dichotomy on $\mathbb{Z}$, then Condition (41) can be reduced to*

$$\rho\theta\left(\mathcal{L}_1^g + \mathcal{L}_2^g\right)\left(\frac{(\alpha K + 1)e^\sigma + \alpha K + 1}{e^\sigma - 1}\right) < 1,$$

guaranteeing that quasilinear DEGPCD Equation (3) has a unique almost periodic solution. Additionally, using the deviating argument $\beta(t) = [t]$, where $t_k = k$ for $k \in \mathbb{Z}$, Theorem 5 under this specific delay argument coincides with the results provided by R. Yuan et al. in [44], Theorem 2.

4.3. Exponential Stability of Almost Periodic Solutions

In the subsection, we explore the global exponential stability of almost periodic solutions for both the linear inhomogeneous DEGPCD (2) and the quasilinear DEGPCD (3).

The following assertions can be proven using Representation (17) in a manner analogous to the theorems for ordinary differential equations, as discussed in [60,61].

Theorem 6. *We assume that condition $(\mathcal{I})$ holds and that $|C(t_{i+1}, t_i)| \leq 1$, for $i \in \mathbb{Z}$. Then, the zero solution of the linear DEGPCD (1) is stable.*

Theorem 7. *We assume that condition ($\mathcal{I}$) holds, and there exists a nonnegative constant $\kappa < 1$ such that $|C(t_{i+1}, t_i)| \leq \kappa$ for all $i \in \mathbb{Z}$. Then, the zero solution of the linear DEGPCD (1) is asymptotically stable.*

Theorem 8. *We assume that condition ($\mathcal{I}$) holds and the linear difference Equation (8) is exponentially stable. Then, the zero solution of the linear DEGPCD (1) is globally exponentially stable.*

Proof. We let $x(t)$ be arbitrary solution of the linear DEGPCD (1) with the initial condition $x(\varsigma) = x_0$, and let $x^*(t)$ be another solution of the same DEGPCD with initial condition $x^*(\varsigma) = x_0^*$. From (17), we obtain

$$\begin{aligned} x(t) - x^*(t) &= \Psi(t, \varsigma)(x_0 - x_0^*) \\ &= \Psi(t, t_{i(t)}) \cdot \Psi(t_{i(t)}, \varsigma) \cdot (x_0 - x_0^*), \quad t \geq \varsigma. \end{aligned}$$

Applying (10) and (35), we have

$$|x(t) - x^*(t)| \leq \alpha K \mu_1^{i(t)-i(\varsigma)} |x_0 - x_0^*|,$$

where α, K, and μ_1 are positive constants, with $0 < \mu_1 < 1$.

Since $\mu_1 < 1$, it follows that $|x(t) - x^*(t)| \to 0$ as $t \to \infty$. Thus, the zero solution of the linear DEGPCD (1) demonstrates global exponential stability. This concludes the proof. □

It is important to note that, by Theorem 8, the exponential stability of the linear difference Equation (8) directly implies the exponential stability of the linear DEGPCD (1), with $\mu_1 = e^{-\sigma}$, where $\sigma > 0$.

Theorem 9. *Under the conditions of Theorem 4, the unique almost periodic solution of the linear inhomogeneous DEGPCD (2) is globally exponentially stable.*

Proof. We let $y(t)$ be an arbitrary solution of the linear inhomogeneous DEGPCD (2) with initial condition $y(\varsigma) = y_0$, and let $y^*(t)$ be the unique almost periodic solution of the same DEGPCD with initial condition $y^*(\varsigma) = y_0^*$. From (21), we have

$$y(t) - y^*(t) = \Psi(t, \varsigma)(y_0 - y_0^*), \quad t \geq \varsigma.$$

Applying the same technique as in the proof of Theorem 8, we can conclude that the unique almost periodic solution of the linear inhomogeneous DEGPCD (2) is globally exponentially stable. □

The following result establishes the exponential stability of the unique almost periodic solution of the quasilinear DEGPCD (3).

Theorem 10. *Under the conditions of Theorem 5, the unique almost periodic solution of the quasilinear DEGPCD (3) is globally exponentially stable.*

Proof. We let $z(t)$ be an arbitrary solution of the quasilinear DEGPCD (3) with the initial condition $z(\varsigma) = z_0$, and let $z^*(t)$ be the unique almost periodic solution of the quasilinear DEGPCD (3) with the initial condition $z^*(\varsigma) = z_0^*$. From (28), we have

$$\begin{aligned} z(t) - z^*(t) &= \Psi(t, \varsigma)(z_0 - z_0^*) \\ &\quad + \int_{\beta(\varsigma)}^{t} G(t, s)\big[g(s, z(s), z(\beta(s))) - g(s, z^*(s), z^*(\beta(s)))\big] ds, \quad t \geq \varsigma. \end{aligned}$$

From Conditions ($\mathcal{L}_g$), **(**$\mathcal{D}$**)**, and (9), (27) and (35), it follows that

$$
\begin{aligned}
&|z(t)-z^*(t)| \\
&\le |\Psi(t,\varsigma)||z_0-z_0^*| + \int_{\beta(\varsigma)}^{t} |G(t,s)|\Big[\mathcal{L}_1^g|z(s)-z^*(s)| + \mathcal{L}_2^g|z(\beta(s))-z^*(\beta(s))|\Big]ds \\
&\le |\Psi(t,\varsigma)||z_0-z_0^*| + \int_{t_{i(\varsigma)}}^{\varsigma} \Big|\Psi(t,t_{i(\varsigma)+1})\Phi(t_{i(\varsigma)+1},s) - \Psi(t,\varsigma)\Phi(\varsigma,s)\Big| \\
&\qquad\qquad \times \Big[\mathcal{L}_1^g|z(s)-z^*(s)| + \mathcal{L}_2^g|z(\beta(s))-z^*(\beta(s))|\Big]ds \\
&\quad + \Big|\Psi(t,t_{i(\varsigma)+1})\Big| \int_{\varsigma}^{t_{i(\varsigma)+1}} \Big|\Phi(t_{i(\varsigma)+1},s)\Big|\Big[\mathcal{L}_1^g|z(s)-z^*(s)| + \mathcal{L}_2^g|z(\beta(s))-z^*(\beta(s))|\Big]ds \\
&\quad + \sum_{k=i(\varsigma)+1}^{i(t)-1} \Big|\Psi(t,t_{i(\varsigma)+1})\Big| \int_{t_k}^{t_{k+1}} |\Phi(t_{k+1},s)|\Big[\mathcal{L}_1^g|z(s)-z^*(s)| + \mathcal{L}_2^g|z(\beta(s))-z^*(\beta(s))|\Big]ds \\
&\quad + \int_{t_{i(t)}}^{t} |\Phi(t,s)|\Big[\mathcal{L}_1^g|z(s)-z^*(s)| + \mathcal{L}_2^g|z(\beta(s))-z^*(\beta(s))|\Big]ds \\
&\le \alpha K \mu_1^{i(t)-i(\varsigma)}|z_0-z_0^*| \\
&\quad + \alpha K \mu_1^{i(t)-i(\varsigma)} \int_{t_{i(\varsigma)}}^{\varsigma} \rho\Big[\mathcal{L}_1^g|z(s)-z^*(s)| + \mathcal{L}_2^g|z(\beta(s))-z^*(\beta(s))|\Big]ds \\
&\quad + \sum_{k=i(\varsigma)}^{i(t)-1} \alpha K \mu_1^{i(t)-k} \int_{t_k}^{t_{k+1}} \rho\Big[\mathcal{L}_1^g|z(s)-z^*(s)| + \mathcal{L}_2^g|z(\beta(s))-z^*(\beta(s))|\Big]ds \\
&\quad + \int_{t_{i(t)}}^{t} \rho\Big[\mathcal{L}_1^g|z(s)-z^*(s)| + \mathcal{L}_2^g|z(\beta(s))-z^*(\beta(s))|\Big]ds.
\end{aligned}
$$

By using the DEGPCD's Gronwall Inequality (Lemma 2), we obtain

$$
\begin{aligned}
|z(t)-z^*(t)| &\le \alpha K \mu_1^{i(t)-i(\varsigma)}|z_0-z_0^*| \\
&\qquad \times \exp\Bigg(\alpha K \mu_1^{i(t)-i(\varsigma)} \int_{t_{i(\varsigma)}}^{\varsigma} \rho(\mathcal{L}_1^g+\mathcal{L}_2^g)ds \\
&\qquad\qquad + \sum_{k=i(\varsigma)}^{i(t)-1} \alpha K \mu_1^{i(t)-k} \int_{t_k}^{t_{k+1}} \rho(\mathcal{L}_1^g+\mathcal{L}_2^g)ds + \int_{t_{i(t)}}^{t} \rho(\mathcal{L}_1^g+\mathcal{L}_2^g)ds\Bigg) \\
&\le \alpha K \mu_1^{i(t)-i(\varsigma)}|z_0-z_0^*| \exp\left(\frac{\alpha K\theta\rho\left(\mathcal{L}_1^g+\mathcal{L}_2^g\right)}{1-\mu_1} + \rho(\mathcal{L}_1^g+\mathcal{L}_2^g)\theta\right).
\end{aligned}
$$

Since $\mu_1 < 1$, we can conclude that $|z(t)-z^*(t)| \to 0$ as $t \to \infty$. Therefore, the almost periodic solution of the quasilinear DEGPCD (3) is globally exponentially stable. This completes the proof of the theorem. □

5. Application and Example

In 1976, M. Wazewska and A. Lasota [62] introduced a mathematical model expressed as

$$x'(t) = -\delta x(t) + pe^{-\gamma x(t-\tau)},\ t \ge 0$$

which describes the dynamics of red blood cell survival in animals. In this model, $x(t)$ represents the number of red blood cells at time t, $\delta > 0$ signifies the probability of a red blood cell's death, and p and γ are positive constants associated with the rate of red blood cell production. The parameter τ denotes the time lag required for the production of a new red blood cell.

We analyze the Lasota–Wazewska model with DEGPCD, represented by the following equation:

$$x'(t) = -\delta(t)x(t) + p(t)f(x(\beta(t))),\ t \geq 0, \tag{45}$$

where $x(t)$ represents the population of red blood cells at time t, while $\delta(t)$ and $p(t)$ are positive, nonzero, almost periodic functions. Additionally, $\sigma = \inf_{t\in\mathbb{R}^+} \delta(t) > 0$ and $\bar{P} := \max_{t\in\mathbb{R}^+} |P(t)|$. The function f is a positive γ-Lipschitz function, meaning that there exists a constant $\gamma > 0$ such that

$$|f(u) - f(v))| \leq \gamma|u - v|, \quad u, v \in \mathbb{R}^+.$$

Moreover, the sequence $\{t_{i,p}\}$ is equipotentially almost periodic.

The objective of this section is to demonstrate that the Lasota–Wazewska model with DEGPCD possesses a unique almost periodic solution, provided that the parameter $\gamma > 0$ is sufficiently small.

We let $\phi(t)$ be a real almost periodic function and we consider the following DEGPCD:

$$x'(t) = -\delta(t)x(t) + p(t)f(\phi(\beta(t))),\ t \geq 0. \tag{46}$$

In this interval $[t_i, t_{i+1})$, for $i \in \mathbb{N}$ with $t_1 = 0$, the solution to the Lasota–Wazewska model with DEGPCD (46) is given by

$$x(t) = e^{-\int_{t_i}^{t} \delta(s)ds} x(t_i) + f(\phi(t_i)) \int_{t_i}^{t} e^{-\int_s^t \delta(\kappa)d\kappa} p(s)ds.$$

By the continuity of the solution, as $t \to, t_{i+1}$, we obtain the following difference equation:

$$\widetilde{x}(i+1) = \widetilde{C}(i)\widetilde{x}(i) + \widetilde{g}(i), \tag{47}$$

where $\{\widetilde{x}(i)\}_{i\in\mathbb{N}} = \{x(t_i)\}_{i\in\mathbb{N}}$,

$$\widetilde{C}(i) := C(t_{i+1}, t_i) = e^{-\int_{t_i}^{t_{i+1}} \delta(s)ds},$$

and

$$\widetilde{g}(i) = f(t_i, \phi(t_i)) := f(\phi(t_i)) \int_{t_i}^{t_{i+1}} e^{-\int_s^{t_{i+1}} \delta(\kappa)d\kappa} p(s)ds.$$

It is readily observed that since the function f is continuous, the composite function $f(\phi(t_i))$ is a discrete almost periodic function. Furthermore, conditions $(\mathcal{A}P_1)$ and $(\mathcal{A}P_3)$ are satisfied. Therefore, by Lemma 8, the sequence $\{C(i)\}_{i\in\mathbb{N}}$ is almost periodic.

In this case, the linear difference equation associated with (47) exhibits exponential dichotomy. Consequently, its bounded solution is given by

$$\widetilde{x}_\phi(i) = \sum_{h=-\infty}^{i} G(i, h+1)\widetilde{g}(h),$$

where $G(i, h+1)$ represents the Green's function, defined as

$$G(i, h+1) := \prod_{k=h+1}^{i} \widetilde{C}(k) = \prod_{k=h+1}^{i} e^{-\int_{t_k}^{t_{k+1}} \delta(s)ds} = e^{-\int_{t_h}^{t_{i+1}} \delta(s)ds}.$$

By Lemma 7(g), it is clear that $G(i, h+1)$ is almost periodic. According to Theorem 3, $\widetilde{x}_\phi(i)$ is also almost periodic. Moreover, we have

$$|\tilde{x}_\phi(i)| \leq \frac{1}{1 - e^{-\sigma}} \sup_{i\in\mathbb{Z}} |\tilde{g}(i)|.$$

By Theorem 4, we can therefore conclude that the linear inhomogeneous DEGPCD (46) possesses a unique almost periodic solution. If we impose the condition

$$\gamma < \frac{1 - e^{-\sigma}}{\bar{P}\theta(2 - e^{-\sigma})}, \tag{48}$$

where $\theta = \sup_{n\in\mathbb{Z}} (t_{n+1} - t_n)$, then by Theorem 5, the Lasota–Wazewska model with DEGPCD (45) has an almost periodic solution.

The concluding result of this section pertains to the specific case where $f(y) = e^{-\gamma y}$ with $\gamma > 0$ in the Lasota–Wazewska model with DEGPCD (45).

Corollary 4. *We let $\delta(t)$ and $p(t)$ be positive, nonzero, almost periodic functions. Moreover, we assume that the sequence $\{t_{i,p}\}$ is equipotentially almost periodic and that γ is sufficiently small to satisfy condition (48). Under these conditions, the Lasota–Wazewska model with DEGPCD*

$$x'(t) = -\delta(t)x(t) + p(t)e^{-\gamma x(\beta(t))}, \quad t \geq 0, \tag{49}$$

admits a unique almost periodic solution.

Now, we present an example based on the Lasota–Wazewska model with DEGPCD, demonstrating the practical applicability and usefulness of our results.

We let $\delta(t) = 0.1 + |\cos(t)|$, $p(t) = 0.1 + |\sin(\sqrt{3}t)|$, and $\beta : \mathbb{R} \to \mathbb{R}$ be a step function defined by $\beta(t) = t_i$ for $t \in I_i = [t_i, t_{i+1})$, where $t_0 = 0$, $t_k = k + 0.25(\sin(k) - \cos(\sqrt{3}k))$, with $k \in \mathbb{N}$, and the initial condition $x(0) = 0.8$. It is easy to verify that $\theta \approx 2.7369$, $\sigma = 0.1$, and $\bar{P} = 1.1$. Therefore,

$$\gamma < \frac{1 - e^{-\sigma}}{\bar{P}\theta(2 - e^{-\sigma})} \approx 0.02886.$$

Hence, if $\gamma < 0.02886$, the Lasota–Wazewska model with DEGPCD, using the aforementioned functions, admits a unique almost periodic solution, as established by Corollary 4.

Figures 1 and 2 illustrate the simulation results, showcasing the existence of a unique almost periodic solution for the DEPCAG with varying values of γ.

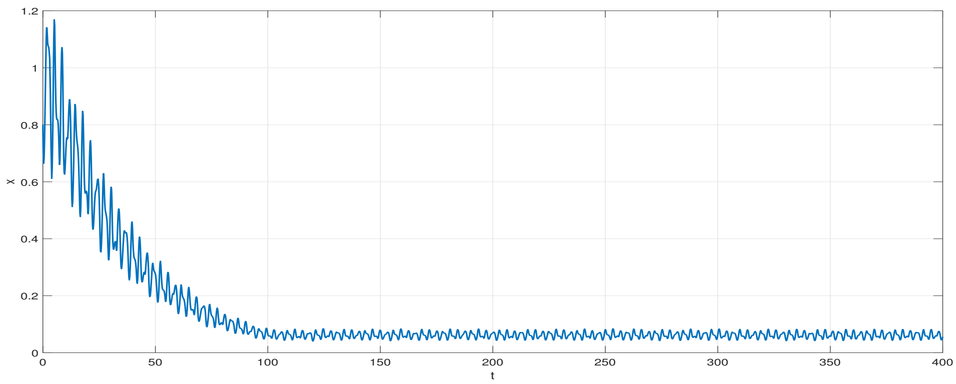

Figure 1. The almost periodic solution of the Lasota–Wazewska model with DEGPCD for $\gamma = 0.028$.

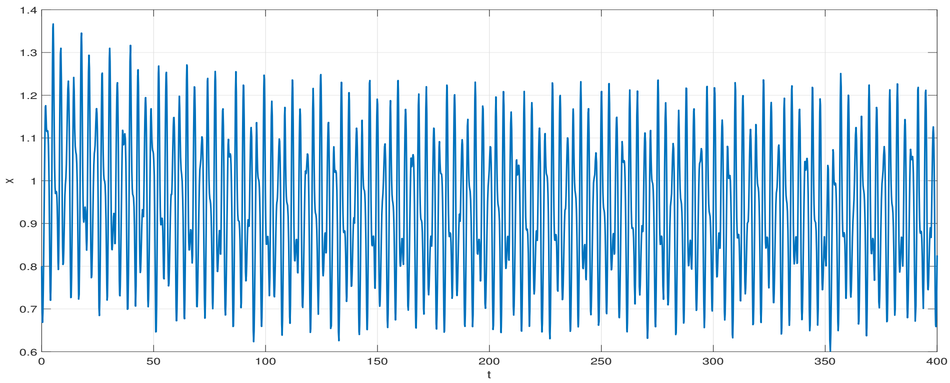

Figure 2. The almost periodic solution of the Lasota–Wazewska model with DEGPCD for $\gamma = 0.001$.

6. Conclusions and Perspectives

In this paper, we investigated the existence of a unique, globally exponentially stable almost periodic solution for differential equations with piecewise constant delay of a generalized type. Through the use of an equivalent integral equation, Cauchy and Green-type matrices, a Gronwall-type inequality for DEGPCDs, the properties of (μ_1, μ_2)-exponential dichotomy, and the Banach fixed-point theorem, we developed novel sufficient conditions that ensure the existence, uniqueness, and global exponential stability of almost periodic solutions for both linear inhomogeneous and quasilinear DEGPCD systems. Given the strongly discrete nature of these equations, the proposed criteria for existence and stability theorems were tested by analyzing the properties of (μ_1, μ_2)-exponential dichotomy in difference equations. As a result of this approach, it is unnecessary to apply the Razumikhin technique or construct a Lyapunov function, which has been commonly used in previous studies. Moreover, the results obtained are novel and serve to recover, extend, and improve upon prior literature, including the findings in [44].

In the following, several open research questions are suggested as potential directions for future investigation:

1. Stochastic Perturbations in Generalized Piecewise Constant Delay Systems: Future work could explore the impact of stochastic perturbations on differential equations with generalized piecewise constant delays. Examining whether stochastic influences alter the conditions for the existence and stability of almost periodic solutions would yield valuable insights. Additionally, formulating stability criteria for such stochastic systems represents an important area of research.
2. (μ_1, μ_2)-Exponential Dichotomy in High-Dimensional Systems: Expanding the current results to encompass high-dimensional difference equations and investigating the behavior of (μ_1, μ_2)-exponential dichotomy within these systems is a promising research avenue. This extension could broaden the theoretical framework and enhance its applicability to more intricate and complex systems.
3. Practical Applications of Generalized Piecewise Constant Delay Systems: Another promising research direction is the application of generalized piecewise constant delay systems to practical domains, including control systems, biological models, and dynamic systems in economics. Empirically analyzing real-world data to assess the efficacy and stability of the proposed models and criteria would significantly enhance their practical relevance.
4. Existence and Stability of Almost Periodic Solutions in Hybrid Systems: Further research could delve into the existence and stability of almost periodic solutions in hybrid systems that integrate piecewise constant delays with other system types. This could involve a detailed analysis of varying dynamic behaviors and delay structures, potentially expanding the scope of current theoretical findings.

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