

## Article

# The Witten Deformation of the Non-Minimal de Rham–Hodge Operator and Noncommutative Residue on Manifolds with Boundary

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**Abstract:** Under the announcement by Alain Connes that the Wodzicki residue of the inverse square of the Dirac operator is proportional to the Einstein–Hilbert action of general relativity, we derive the Lichnerowicz-type formula for the Witten deformation of the non-minimal de Rham–Hodge operator and the gravitational action in the case of  $n$ -dimensional compact manifolds without boundary. Finally, we present the proof of the Kastler–Kalau–Walze-type theorem for the Witten deformation of the non-minimal de Rham–Hodge operator on four- and six-dimensional oriented compact manifolds with boundary.

**Keywords:** the Lichnerowicz type formula; the Witten deformation; the non-minimal de Rham–Hodge operator; the Kastler–Kalau–Walze-type theorem

**MSC:** 53C40; 53C42



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## 1. Introduction

Until now, many geometers have studied noncommutative residues. In [1,2], the authors found that noncommutative residues are of great importance in the study of noncommutative geometry. Wodzicki [2] first introduced the concept of the noncommutative residue in the context of higher-dimensional manifolds, namely, the noncommutative residue is a trace over the algebra of all classical pseudodifferential operators on a closed compact manifold. However, this trace does not extend the usual trace. The noncommutative residue is also known as the Wodzicki residue. Let  $\Phi : \Sigma \rightarrow R^d$  denote the Riemannian surface, where  $\Phi = (\phi_1, \dots, \phi_d)$  is a smooth embedding and  $g = \sum_{ij} \eta_{ij} d\phi^i \otimes d\phi^j$  denotes the metric on the Riemannian surface. Then the Polyakov action is defined by  $I := \frac{1}{2\pi} \int_M \eta_{ij} d\phi^i \otimes d\phi^j$ . In [3], Connes used the noncommutative residue to derive a conformal four-dimensional Polyakov action analogy. Connes showed us that the noncommutative residue on a compact manifold  $M$  coincided with the Dixmier trace (see §7.5 in [4]) on pseudodifferential operators of order  $-\dim M$  in [5]. And Connes claimed the noncommutative residue of the square of the inverse of the Dirac operator was proportional to the Einstein–Hilbert action. That is,  $\text{Wres}(D^{-2}) = c_0 \int_M s \, \text{dvol}_M$ , where  $c_0$  is a constant and  $s$  denotes the scalar curvature. Kastler [6] provided a brute-force proof of this theorem. Kalau and Walze simultaneously proved this theorem using the normal coordinates system in [7]. Therefore, we call it the Kastler–Kalau–Walze theorem. Ackermann proved that the Wodzicki residue of the square of the inverse of the Dirac operator  $\text{Wres}(D^{-2})$  is essentially the second coefficient of the heat kernel expansion of  $D^2$  in [8].

On the other hand, Wang generalized Connes’ results to manifolds with boundary in [9,10], and proved the Kastler–Kalau–Walze-type theorem for the Dirac operator and the

signature operator on lower-dimensional manifolds with boundary [11]. In [11,12], Wang computed  $\widetilde{\text{Wres}}[\pi^+ D^{-1} \circ \pi^+ D^{-1}]$  and  $\widetilde{\text{Wres}}[\pi^+ D^{-2} \circ \pi^+ D^{-2}]$ , where the two operators are symmetric; in these cases, the boundary term vanished. But for  $\widetilde{\text{Wres}}[\pi^+ D^{-1} \circ \pi^+ D^{-3}]$ , Wang discovered a non-vanishing boundary term [13], and provided a theoretical explanation for gravitational action on the boundary. In other words, Wang offers a method to study the Kastler–Kalau–Walze-type theorem for manifolds with boundary. In [14], López et al. introduced an elliptic differential operator known as the Novikov operator. In [15], Wei and Wang proved the Kastler–Kalau–Walze-type theorem for modified Novikov operators on compact manifolds. In [11], the leading symbol of the Dirac operator  $D$  was  $\sqrt{-1}c(\xi)$ . To explore a leading symbol that is not  $\sqrt{-1}c(\xi)$ , we consider the operator  $\sqrt{-1}\hat{c}(V)(d + \delta)$ , which is motivated by the sub-signature operator in [16]. In [17], Wang J and Wang Y proved various Kastler–Kalau–Walze-type theorems associated with non-minimal de Rham–Hodge operators on compact manifolds with boundary. Motivated by [16,17], we introduce the Witten deformation of the non-minimal de Rham–Hodge operator  $\tilde{D}_V = ad + b\delta + \hat{c}(V)$ , and Laplacian operators  $\tilde{D}_V^2 = [ad + b\delta + \hat{c}(V)]^2$ , where  $a$  and  $b$  are constants and  $V = \sum_{i=1}^n V_i e_i$  is a vector field. When  $a = b = 1$ ,  $\tilde{D}_V$  corresponds to the Witten deformation, and when  $\hat{c}(V) = 0$ ,  $\tilde{D}_V$  represents the non-minimal de Rham–Hodge operator. Our main results are as follows:

**Theorem 1.** *If  $M$  is an  $n$ -dimensional compact oriented manifold without boundary, and  $n$  is even, then we obtain the following equality:*

$$\text{Wres}(\tilde{D}_V^2)^{-\frac{n-2}{2}} = \frac{(n-2)(4\pi)^{\frac{n}{2}}}{(\frac{n}{2}-1)!} \int_M 2^n \left( \frac{2-3ab}{12}s + \frac{6ab-a^2-b^2}{4ab}|V|^2 \right) d\text{Vol}_M,$$

where  $s$  denotes the scalar curvature,  $\text{Vol}_M$  denotes the volume of  $M$ , and the definition of  $V$  is defined by (4).

**Theorem 2.** *Let  $M$  be a 4-dimensional oriented compact manifold with boundary  $\partial M$  and the metric  $g^{TM}$ , then we obtain the Kastler–Kalau–Walze-type theorem of the operator  $\tilde{D}_V$ :*

$$\widetilde{\text{Wres}}[\pi^+ \tilde{D}_V^{-1} \circ \pi^+ (\tilde{D}_V)^{-1}] = 32\pi^2 \int_M 16 \left( \frac{2-3ab}{12}s + \frac{6ab-a^2-b^2}{4ab}|V|^2 \right) d\text{Vol}_M.$$

**Theorem 3.** *Let  $M$  be a 6-dimensional oriented compact manifold with boundary  $\partial M$  and the metric  $g^{TM}$ , then we obtain the Kastler–Kalau–Walze-type theorem of the operator  $\tilde{D}_V$ :*

$$\begin{aligned} \widetilde{\text{Wres}}[\pi^+ \tilde{D}_V^{-1} \circ \pi^+ (\tilde{D}_V)^{-3}] &= 128\pi^3 \int_M 64 \left( \frac{2-3ab}{12}s + \frac{6ab-a^2-b^2}{4ab}|V|^2 \right) d\text{Vol}_M \\ &\quad + \int_{\partial M} \left( \frac{60i-545}{16a^2b^2}h'(0) + \frac{(a-b)(37-6i)}{2a^3b^3}V_n \right) \pi\Omega_4 d\text{Vol}_{\partial M}, \end{aligned}$$

where  $\text{Vol}_{\partial M}$  denotes the volume of  $\partial M$ ,  $\Omega_4$  denotes the canonical volume of  $S^4$ , the definition of  $h$  is defined by (44), and  $V_n = g(V, \partial x_n)$  on the boundary  $\partial M$ .

The framework of this paper is organized as follows. First, in Section 2, we define the Witten deformation of the non-minimal de Rham–Hodge operator and the associated Lichnerowicz formulas for this deformation. In Section 3, we study the symbols of the Witten deformation of the non-minimal de Rham–Hodge operator, and we prove the Kastler–Kalau–Walze-type theorem for four-dimensional manifolds with boundary associated with this Witten deformation. In Section 4, the Kastler–Kalau–Walze-type theorem for six-dimensional manifolds with boundary associated with the Witten deformation of the non-minimal de Rham–Hodge operator is proved.

## 2. The Witten Deformation of the Non-Minimal de Rham–Hodge Operator and Its Lichnerowicz Formula

First, we introduce some notation relevant to the Witten deformation of the non-minimal de Rham–Hodge operator. Let  $M$  be an  $n$ -dimensional ( $n \geq 4$ ) oriented compact Riemannian manifold with a Riemannian metric  $g^{TM}$ .

Let  $\nabla^L$  denote the Levi–Civita connection with respect to  $g^{TM}$ . In the fixed orthonormal frame  $\{e_1, \dots, e_n\}$ , the connection matrix  $(\omega_{s,t})$  is defined by the following:

$$\nabla^L(e_1, \dots, e_n) = (e_1, \dots, e_n)(\omega_{s,t}). \quad (1)$$

Let  $\epsilon(e_j^*)$ ,  $\iota(e_j)$  denote the exterior and interior multiplications, respectively, where  $e_j^* = g^{TM}(e_j, \cdot)$ , and let  $c(e_j)$  denote the Clifford action,

We note the following:

$$\hat{c}(e_j) = \epsilon(e_j^*) + \iota(e_j); \quad c(e_j) = \epsilon(e_j^*) - \iota(e_j), \quad (2)$$

which satisfies the following:

$$\begin{aligned} \hat{c}(e_i)\hat{c}(e_j) + \hat{c}(e_j)\hat{c}(e_i) &= 2g^{TM}(e_i, e_j); \\ c(e_i)c(e_j) + c(e_j)c(e_i) &= -2g^{TM}(e_i, e_j); \\ c(e_i)\hat{c}(e_j) + \hat{c}(e_j)c(e_i) &= 0. \end{aligned} \quad (3)$$

The de Rham derivative,  $d$ , is an elliptic differential operator on  $C^\infty(M; \wedge^* T^*M)$  and the de Rham co-derivative  $\delta = d^*$ . Let  $\tilde{c}(e_j) = ae(e_j^*) - b\iota(e_j)$ , where  $a$  and  $b$  are constants. Then, the Witten deformation of the non-minimal de Rham–Hodge operator is defined as follows:

$$\begin{aligned} \tilde{D}_V &= ad + b\delta + \hat{c}(V) \\ &= \sum_{i=1}^n \tilde{c}(e_i) \left( e_i + \frac{1}{4} \sum_{s,t} \omega_{s,t}(e_i) [\hat{c}(e_s)\hat{c}(e_t) - c(e_s)c(e_t)] \right) + \hat{c}(V), \end{aligned} \quad (4)$$

where  $V = \sum_{i=1}^n V_i e_i$  is a vector field.

Next, we present the following theorem concerning the Lichnerowicz formula:

**Theorem 4.** *The following equality holds:*

$$\begin{aligned} \tilde{D}_V^2 &= -g^{ij}(\nabla_{\partial_i} \nabla_{\partial_j} - \nabla_{\nabla_{\partial_i}^L \partial_j}) + \frac{1}{4ab} \sum_{j=1}^n [\tilde{c}(e_j)\hat{c}(V) + \hat{c}(V)\tilde{c}(e_j)][\tilde{c}(e_j)\hat{c}(V) + \hat{c}(V)\tilde{c}(e_j)] \\ &\quad - \frac{1}{2} \sum_{j=1}^n [\hat{c}(\nabla_{e_j}^L V)\tilde{c}(e_j) - \tilde{c}(e_j)\hat{c}(\nabla_{e_j}^L V)] - \frac{ab}{8} \sum_{ijkl} R_{ijkl} \hat{c}(e_i)\hat{c}(e_j)c(e_k)c(e_l) + \frac{ab}{4}s - |V|^2, \end{aligned} \quad (5)$$

where  $s$  denotes the scalar curvature.

**Proof.** Let  $M$  be a smooth, compact, oriented Riemannian  $n$ -dimensional manifold without boundary, and let  $N$  be a vector bundle on  $M$ . If  $P$  is a differential Laplace-type operator, then locally it has the following form:

$$P = -(g^{ij}\partial_i\partial_j + A^i\partial_i + B), \quad (6)$$

where  $\partial_i$  is a natural local frame on  $TM$ ,  $(g^{ij})_{1 \leq i,j \leq n}$  denotes the inverse matrix associated with the metric matrix  $(g_{ij})_{1 \leq i,j \leq n}$  on  $M$ , and  $A^i$  and  $B$  are smooth sections of  $\text{End}(N)$

on  $M$  (endomorphism). If a Laplace-type operator,  $P$ , satisfies (6), then there is a unique connection  $\nabla$  on  $N$  and a unique endomorphism  $E$ , such that we have the following:

$$P = -\left[g^{ij}(\nabla_{\partial_i}\nabla_{\partial_j} - \nabla_{\nabla_{\partial_i}^L\partial_j}) + E\right], \quad (7)$$

where  $\nabla^L$  denotes the Levi-Civita connection on  $M$ . Moreover, with local frames of  $T^*M$  and  $N$ ,  $\nabla_{\partial_i} = \partial_i + \omega_i$  and  $E$  are related to  $g^{ij}$ ,  $A^i$ , and  $B$  through the following:

$$\omega_i = \frac{1}{2}g_{ij}(A^i + g^{kl}\Gamma_{kl}^j \mathbf{d}), \quad (8)$$

$$E = B - g^{ij}(\partial_i(\omega_j) + \omega_i\omega_j - \omega_k\Gamma_{ij}^k), \quad (9)$$

where  $\Gamma_{kl}^j$  denotes the Christoffel coefficient of  $\nabla^L$ .

Let  $g^{ij} = g(dx_i, dx_j)$ ,  $\xi = \sum_j \xi_j dx_j$  and  $\nabla_{\partial_i}^L \partial_j = \sum_k \Gamma_{ij}^k \partial_k$ ; we have the following:

$$\begin{aligned} \sigma_i &= -\frac{1}{4} \sum_{s,t} \omega_{s,t}(e_i) c(e_s) c(e_t); \quad a_i = \frac{1}{4} \sum_{s,t} \omega_{s,t}(e_i) \widehat{c}(e_s) \widehat{c}(e_t); \\ \tilde{\xi}^j &= g^{ij} \xi_i; \quad \Gamma^k = g^{ij} \Gamma_{ij}^k; \quad \sigma^j = g^{ij} \sigma_i; \quad a^j = g^{ij} a_i. \end{aligned} \quad (10)$$

Then, the operators  $D_V$  can be written as follows:

$$\tilde{D}_V = \sum_{i=1}^n \tilde{c}(e_i) [e_i + a_i + \sigma_i] + \widehat{c}(V). \quad (11)$$

By [8,18], we have the following:

$$(d + \delta)^2 = -\Delta_0 - \frac{1}{8} \sum_{ijkl} R_{ijkl} \widehat{c}(e_i) \widehat{c}(e_j) c(e_k) c(e_l) + \frac{1}{4} s; \quad (12)$$

$$-\Delta_0 = \Delta = -g^{ij}(\nabla_i^L \nabla_j^L - \Gamma_{ij}^k \nabla_k^L). \quad (13)$$

By (6) and (7), we have the following:

$$\begin{aligned} \tilde{D}_V^2 &= [ad + b\delta + \widehat{c}(V)]^2 \\ &= ab \left\{ -\sum_{ij} g^{ij} [\partial_i \partial_j + 2\sigma_i \partial_j + 2a_i \partial_j - \Gamma_{ij}^k \partial_k + (\partial_i \sigma_j) + (\partial_i a_j) + \sigma_i \sigma_j + \sigma_i a_j + a_i \sigma_j + a_i a_j \right. \\ &\quad \left. - \Gamma_{ij}^k \sigma_k - \Gamma_{ij}^k a_k] + \frac{1}{ab} \sum_j [\widehat{c}(e_j) (\sum_l \langle e_j, dx_l \rangle \partial_{x_l} + \sigma_j + a_j) \widehat{c}(V) + \widehat{c}(V) \widehat{c}(e_j) (\sum_l \langle e_j, dx_l \rangle \partial_{x_l} \right. \\ &\quad \left. + \sigma_j + a_j)] - \frac{1}{8} \sum_{ijkl} R_{ijkl} \widehat{c}(e_i) \widehat{c}(e_j) c(e_k) c(e_l) + \frac{1}{4} s + \frac{1}{ab} |V|^2 \right\} \\ &:= ab \overline{D}_V^2. \end{aligned} \quad (14)$$

By (10)–(14), we have the following:

$$(\omega_i)_{\overline{D}_V^2} = \sigma_i + a_i - \frac{1}{2ab} \widehat{c}(V) \widehat{c}(e_i) - \frac{1}{2ab} \widehat{c}(e_i) \widehat{c}(V), \quad (15)$$

$$\begin{aligned}
E_{\bar{D}_V^2} &= \sum_{ij} (\sigma^i \sigma_j + \sigma^j a_j + a^i \sigma_j + a^j a_j) + \frac{1}{8} \sum_{ijkl} R_{ijkl} \hat{c}(e_i) \hat{c}(e_j) c(e_k) c(e_l) - \frac{1}{4} s + \frac{1}{ab} |V|^2 \\
&\quad - \frac{1}{4a^2 b^2} \sum_{ij} g^{ij} [\hat{c}(e_i) \hat{c}(V) + \hat{c}(V) \hat{c}(e_i)] [\hat{c}(e_j) \hat{c}(V) + \hat{c}(V) \hat{c}(e_j)] - \frac{1}{ab} \Gamma^k [\hat{c}(e_k) \hat{c}(V) + \hat{c}(V) \hat{c}(e_k)] \\
&\quad - \frac{1}{ab} \sum_{jl} \hat{c}(e_j) \langle e_j, dx_l \rangle \partial_{x_l} (\hat{c}(V)) + \frac{1}{2ab} \partial^j [\hat{c}(e_j) \hat{c}(V) + \hat{c}(V) \hat{c}(e_j)].
\end{aligned} \tag{16}$$

Since  $E$  is globally defined on  $M$ , taking normal coordinates at  $x_0$ , we have the following:

$\sigma^i(x_0) = 0$ ,  $a^i(x_0) = 0$ ,  $\partial^j[\hat{c}(e_j)](x_0) = 0$ ,  $\Gamma^k(x_0) = 0$ ,  $g^{ij}(x_0) = \delta_l^j$ , then

$$\begin{aligned}
E_{\bar{D}_V^2}(x_0) &= -\frac{1}{4a^2 b^2} \sum_{j=1}^n [\hat{c}(e_j) \hat{c}(V) + \hat{c}(V) \hat{c}(e_j)] [\hat{c}(e_j) \hat{c}(V) + \hat{c}(V) \hat{c}(e_j)] + \frac{1}{8} \sum_{ijkl} R_{ijkl} \hat{c}(e_i) \hat{c}(e_j) c(e_k) c(e_l) \\
&\quad + \frac{1}{2ab} \sum_{j=1}^n [\hat{c}(\nabla_{e_j}^L V) \hat{c}(e_j) - \hat{c}(e_j) \hat{c}(\nabla_{e_j}^L V)] - \frac{1}{4} s + \frac{1}{ab} |V|^2,
\end{aligned} \tag{17}$$

$$\begin{aligned}
E_{\bar{D}_V^2}(x_0) &= ab E_{\bar{D}_V^2}(x_0) \\
&= -\frac{1}{4ab} \sum_{j=1}^n [\hat{c}(e_j) \hat{c}(V) + \hat{c}(V) \hat{c}(e_j)] [\hat{c}(e_j) \hat{c}(V) + \hat{c}(V) \hat{c}(e_j)] + \frac{ab}{8} \sum_{ijkl} R_{ijkl} \hat{c}(e_i) \hat{c}(e_j) c(e_k) c(e_l) \\
&\quad + \frac{1}{2} \sum_{j=1}^n [\hat{c}(\nabla_{e_j}^L V) \hat{c}(e_j) - \hat{c}(e_j) \hat{c}(\nabla_{e_j}^L V)] - \frac{ab}{4} s + |V|^2,
\end{aligned} \tag{18}$$

then, by (7), we obtain Theorem 4.  $\square$

From [8], for an  $n$ -dimensional manifold where  $n$  is even, we know that the noncommutative residue of a generalized Laplacian  $\bar{\Delta}$  is expressed as follows:

$$(n-2)\Phi_2(\bar{\Delta}) = (4\pi)^{-\frac{n}{2}} \Gamma\left(\frac{n}{2}\right) \text{Wres}(\bar{\Delta}^{-\frac{n}{2}+1}), \tag{19}$$

where  $\Phi_2(\bar{\Delta})$  denotes the integral over the diagonal part of the second coefficient of the heat kernel expansion of  $\bar{\Delta}$ . Now, let  $\bar{\Delta} = \tilde{D}_V^2$ ; since  $\tilde{D}_V^2$  is a generalized Laplacian, we can suppose that  $\tilde{D}_V^2 = \bar{\Delta} - E_{\bar{D}_V^2}$ , and  $\text{tr}$  denotes the shorthand of trace. Then by (19), for an  $n$ -dimensional manifold, where  $n$  is even, we have the following:

$$\text{Wres}(\tilde{D}_V^2)^{-\frac{n-2}{2}} = \frac{(n-2)(4\pi)^{\frac{n}{2}}}{(\frac{n}{2}-1)!} \int_M \text{tr}\left(\frac{1}{6}s + E_{\bar{D}_V^2}\right) d\text{Vol}_M, \tag{20}$$

where  $\text{Wres}$  denotes the noncommutative residue.

Next, we need to compute  $\text{tr}(E_{\bar{D}_V^2})$ . Obviously, we have the following:

(1)

$$\text{tr}\left(-\frac{ab}{4}s + |V|^2\right) = \left(-\frac{ab}{4}s + |V|^2\right) \text{tr}[\text{id}]. \tag{21}$$

(2)

$$\text{tr} \sum_{ijkl} [R_{ijkl} \hat{c}(e_i) \hat{c}(e_j) c(e_k) c(e_l)] = 0. \tag{22}$$

(3)

$$\operatorname{tr} \sum_{j=1}^n [\widehat{c}(\nabla_{e_j}^L V) \widetilde{c}(e_j) - \widetilde{c}(e_j) \widehat{c}(\nabla_{e_j}^L V)] = 0. \quad (23)$$

(4) Note that we have the following:

$$\begin{aligned} & \operatorname{tr} \sum_{j=1}^n [\widetilde{c}(e_j) \widehat{c}(V) + \widehat{c}(V) \widetilde{c}(e_j)] [\widetilde{c}(e_j) \widehat{c}(V) + \widehat{c}(V) \widetilde{c}(e_j)] \\ &= \operatorname{tr} \sum_{j=1}^n [\widetilde{c}(e_j) \widehat{c}(V) \widetilde{c}(e_j) \widehat{c}(V)] + \operatorname{tr} \sum_{j=1}^n [\widehat{c}(V) \widetilde{c}(e_j) \widetilde{c}(e_j) \widehat{c}(V)] \\ &+ \operatorname{tr} \sum_{j=1}^n [\widetilde{c}(e_j) \widehat{c}(V) \widehat{c}(V) \widetilde{c}(e_j)] + \operatorname{tr} \sum_{j=1}^n [\widehat{c}(V) \widetilde{c}(e_j) \widehat{c}(V) \widetilde{c}(e_j)]. \end{aligned} \quad (24)$$

(4-a)

$$\begin{aligned} \operatorname{tr} \sum_{j=1}^n [\widetilde{c}(e_j) \widehat{c}(V) \widetilde{c}(e_j) \widehat{c}(V)] &= a^2 \sum_{jkl=1}^n V_k V_l \operatorname{tr}[\epsilon(e_j) \iota(e_k) \epsilon(e_j) \iota(e_l)] + b^2 \sum_{jkl=1}^n V_k V_l \operatorname{tr}[\iota(e_j) \epsilon(e_k) \iota(e_j) \epsilon(e_l)]; \\ &- ab \sum_{jkl=1}^n V_k V_l \operatorname{tr}[\epsilon(e_j) \epsilon(e_k) \iota(e_j) \iota(e_l)] - ab \sum_{jkl=1}^n V_k V_l \operatorname{tr}[\epsilon(e_j) \iota(e_k) \iota(e_j) \epsilon(e_l)]; \\ &- ab \sum_{jkl=1}^n V_k V_l \operatorname{tr}[\iota(e_j) \iota(e_k) \epsilon(e_j) \epsilon(e_l)] - ab \sum_{jkl=1}^n V_k V_l \operatorname{tr}[\iota(e_j) \epsilon(e_k) \epsilon(e_j) \iota(e_l)]. \end{aligned} \quad (25)$$

(4-a-I)

$$\begin{aligned} & a^2 \sum_{jkl=1}^n V_k V_l \operatorname{tr}[\epsilon(e_j) \iota(e_k) \epsilon(e_j) \iota(e_l)] \\ &= a^2 \sum_{jkl=1}^n V_k V_l \delta_{jk} \operatorname{tr}[\epsilon(e_j) \iota(e_l)] - a^2 \sum_{jkl=1}^n V_k V_l \operatorname{tr}[\iota(e_k) \epsilon(e_j) \epsilon(e_j) \iota(e_l)] \\ &= a^2 \sum_{jkl=1}^n V_k V_l \delta_{jk} \operatorname{tr}[\epsilon(e_j) \iota(e_l)] + a^2 \sum_{jkl=1}^n V_k V_l \delta_{jl} \operatorname{tr}[\iota(e_k) \epsilon(e_j)] - a^2 \sum_{jkl=1}^n V_k V_l \operatorname{tr}[\epsilon(e_j) \iota(e_k) \epsilon(e_j) \iota(e_l)], \end{aligned} \quad (26)$$

then, we have the following:

$$\begin{aligned} a^2 \sum_{jkl=1}^n V_k V_l \operatorname{tr}[\epsilon(e_j) \iota(e_k) \epsilon(e_j) \iota(e_l)] &= \frac{a^2}{2} \sum_{jkl=1}^n V_k V_l \delta_{jk} \operatorname{tr}[\epsilon(e_j) \iota(e_l)] + \frac{a^2}{2} \sum_{jkl=1}^n V_k V_l \delta_{jl} \operatorname{tr}[\iota(e_k) \epsilon(e_j)] \\ &= \frac{a^2}{4} \sum_{k=1}^n V_k^2 \operatorname{tr}[\operatorname{id}] + \frac{a^2}{4} \sum_{k=1}^n V_k^2 \operatorname{tr}[\operatorname{id}] \\ &= \frac{a^2}{2} \sum_{k=1}^n V_k^2 \operatorname{tr}[\operatorname{id}] \\ &= \frac{a^2}{2} |V|^2 \operatorname{tr}[\operatorname{id}], \end{aligned} \quad (27)$$

similarly, we have the following:

$$b^2 \sum_{jkl=1}^n V_k V_l \operatorname{tr}[\iota(e_j) \epsilon(e_k) \iota(e_j) \epsilon(e_l)] = \frac{b^2}{2} |V|^2 \operatorname{tr}[\operatorname{id}]. \quad (28)$$

(4-a-II)

$$\begin{aligned}
& -ab \sum_{jkl=1}^n V_k V_l \operatorname{tr}[\epsilon(e_j) \epsilon(e_k) \iota(e_j) \iota(e_l)] \\
& = ab \sum_{jkl=1}^n V_k V_l \operatorname{tr}[\epsilon(e_k) \epsilon(e_j) \iota(e_j) \iota(e_l)] \\
& = ab \sum_{jkl=1}^n V_k V_l \operatorname{tr}[\epsilon(e_k) \iota(e_l)] - ab \sum_{jkl=1}^n V_k V_l \operatorname{tr}[\epsilon(e_k) \iota(e_j) \epsilon(e_j) \iota(e_l)] \\
& = ab \sum_{jkl=1}^n V_k V_l \operatorname{tr}[\epsilon(e_k) \iota(e_l)] - ab \sum_{jkl=1}^n V_k V_l \delta_{jl} \operatorname{tr}[\epsilon(e_k) \iota(e_j)] + ab \sum_{jkl=1}^n V_k V_l \operatorname{tr}[\epsilon(e_j) \epsilon(e_k) \iota(e_j) \iota(e_l)], \tag{29}
\end{aligned}$$

then, we have the following:

$$\begin{aligned}
-ab \sum_{jkl=1}^n V_k V_l \operatorname{tr}[\epsilon(e_j) \epsilon(e_k) \iota(e_j) \iota(e_l)] & = \frac{ab}{2} \sum_{jkl=1}^n V_k V_l \operatorname{tr}[\epsilon(e_k) \iota(e_l)] - \frac{ab}{2} \sum_{jkl=1}^n V_k V_l \delta_{jl} \operatorname{tr}[\epsilon(e_k) \iota(e_j)] \\
& = \frac{nab}{4} \sum_{k=1}^n V_k^2 \operatorname{tr}[\operatorname{id}] - \frac{ab}{4} \sum_{k=1}^n V_k^2 \operatorname{tr}[\operatorname{id}] \\
& = \frac{nab - ab}{4} |V|^2 \operatorname{tr}[\operatorname{id}]. \tag{30}
\end{aligned}$$

Similarly, we have the following:

$$\begin{aligned}
-ab \sum_{jkl=1}^n V_k V_l \operatorname{tr}[\epsilon(e_j) \iota(e_k) \iota(e_j) \epsilon(e_l)] & = -ab \sum_{jkl=1}^n V_k V_l \operatorname{tr}[\iota(e_j) \iota(e_k) \epsilon(e_j) \epsilon(e_l)] \\
& = -ab \sum_{jkl=1}^n V_k V_l \operatorname{tr}[\iota(e_j) \epsilon(e_k) \epsilon(e_j) \iota(e_l)] \\
& = \frac{nab - ab}{4} |V|^2 \operatorname{tr}[\operatorname{id}]. \tag{31}
\end{aligned}$$

Therefore, we have the following:

$$\operatorname{tr} \sum_{j=1}^n [\tilde{c}(e_j) \tilde{c}(V) \tilde{c}(e_j) \tilde{c}(V)] = \frac{2nab - 2ab + a^2 + b^2}{2} |V|^2 \operatorname{tr}[\operatorname{id}]. \tag{32}$$

(4-b)

$$\begin{aligned}
\operatorname{tr} \sum_{j=1}^n [\tilde{c}(e_j) \tilde{c}(V) \tilde{c}(V) \tilde{c}(e_j)] & = |V|^2 \operatorname{tr} \sum_{j=1}^n [\tilde{c}(e_j) \tilde{c}(e_j)] \\
& = -nab |V|^2 \operatorname{tr}[\operatorname{id}]. \tag{33}
\end{aligned}$$

By

$$\begin{aligned}
\operatorname{tr} \sum_{j=1}^n [\tilde{c}(e_j) \tilde{c}(V) \tilde{c}(e_j) \tilde{c}(V)] & = \operatorname{tr} \sum_{j=1}^n [\tilde{c}(V) \tilde{c}(e_j) \tilde{c}(V) \tilde{c}(e_j)]; \\
\operatorname{tr} \sum_{j=1}^n [\tilde{c}(V) \tilde{c}(e_j) \tilde{c}(e_j) \tilde{c}(V)] & = \operatorname{tr} \sum_{j=1}^n [\tilde{c}(e_j) \tilde{c}(V) \tilde{c}(V) \tilde{c}(e_j)], \tag{34}
\end{aligned}$$

we obtain the following:

$$\mathrm{tr} \sum_{j=1}^n [\tilde{c}(e_j)\tilde{c}(V) + \tilde{c}(V)\tilde{c}(e_j)][\tilde{c}(e_j)\tilde{c}(V) + \tilde{c}(V)\tilde{c}(e_j)] = (a^2 + b^2 - 2ab)|V|^2 \mathrm{tr}[\mathrm{id}]. \quad (35)$$

Therefore, we obtain the following:

$$\mathrm{tr}(E_{\tilde{D}_V^2}) = \left( -\frac{ab}{4}s + \frac{6ab - a^2 - b^2}{4ab}|V|^2 \right) \mathrm{tr}[\mathrm{id}]. \quad (36)$$

Then, by summing the results from (21)–(36), we obtain  $\mathrm{tr}\left(\frac{1}{6}s + E_{\tilde{D}_V^2}\right)$ . Next, substituting this result into (20), we obtain the following theorem, which provides a general result concerning the noncommutative residue of the non-minimal de Rham–Hodge operator on even-dimensional manifolds without boundary.

**Theorem 5.** *If  $M$  is an  $n$ -dimensional compact oriented manifold without boundary, and  $n$  is even, then we obtain the following equality:*

$$\mathrm{Wres}(\tilde{D}_V^2)^{-\frac{n-2}{2}} = \frac{(n-2)(4\pi)^{\frac{n}{2}}}{(\frac{n}{2}-1)!} \int_M 2^n \left( \frac{2-3ab}{12}s + \frac{6ab-a^2-b^2}{4ab}|V|^2 \right) d\mathrm{Vol}_M. \quad (37)$$

### 3. A Kastler–Kalau–Walze-Type Theorem for 4-Dimensional Manifolds with Boundary

In this section, we prove the Kastler–Kalau–Walze-type theorem for 4-dimensional oriented compact manifolds with boundary. For more basic facts and formulas concerning Boutet de Monvel’s calculus, as well as the definition of the noncommutative residue for manifolds with boundary, which will be used in the following, see Section 2 in [11].

Let  $M$  be an  $n$ -dimensional compact oriented manifold with boundary  $\partial M$ . Denote by  $\mathcal{B}$  the Boutet de Monvel algebra. We recall the main theorem presented in [11,19].

**Theorem 6 ([19]).** *(Fedosov–Golse–Leichtnam–Schrohe) Let  $M$  and  $\partial M$  be connected, with  $\dim M = n \geq 3$ . Let  $\tilde{S}$  (resp.  $\tilde{S}'$ ) denote the unit sphere around  $\xi$  (resp.  $\xi'$ ), and let  $\sigma(\xi)$  (resp.  $\sigma(\xi')$ ) be the corresponding canonical  $n-1$  (resp.  $(n-2)$ ) volume form. Set  $\tilde{A} = \begin{pmatrix} \pi^+ P + G & K \\ T & \tilde{S} \end{pmatrix} \in \mathcal{B}$ , and denote by  $p$ ,  $b$ , and  $s$  the local symbols of  $P$ ,  $G$ , and  $\tilde{S}$ , respectively. We define the following:*

$$\begin{aligned} \widetilde{\mathrm{Wres}}(\tilde{A}) &= \int_X \int_{\tilde{S}} \mathrm{tr}_E[p_{-n}(x, \xi)] \sigma(\xi) dx \\ &\quad + 2\pi \int_{\partial X} \int_{\tilde{S}'} \{ \mathrm{tr}_E[(\mathrm{tr} b_{-n})(x', \xi')] + \mathrm{tr}_F[s_{1-n}(x', \xi')] \} \sigma(\xi') dx', \end{aligned} \quad (38)$$

where  $\widetilde{\mathrm{Wres}}$  denotes the noncommutative residue of an operator in the Boutet de Monvel algebra. Then, a)  $\widetilde{\mathrm{Wres}}([\tilde{A}, B]) = 0$ , for any  $\tilde{A}, B \in \mathcal{B}$ , and b) it is the unique continuous trace on  $\mathcal{B}/\mathcal{B}^{-\infty}$ .

**Definition 1 ([11]).** *Lower-dimensional volumes of spin manifolds with boundary are defined by the following:*

$$\mathrm{Vol}_n^{(p_1, p_2)} M := \widetilde{\mathrm{Wres}}[\pi^+ D^{-p_1} \circ \pi^+ D^{-p_2}], \quad (39)$$

Firstly, by Theorem 6, we know that the noncommutative residue of the Witten deformation of the non-minimal de Rham–Hodge operator on manifolds with boundary can be divided into two parts: the interior term and the boundary term. This is expressed by the following equality:

$$\widetilde{\text{Wres}}[\pi^+ \tilde{D}_V^{-p_1} \circ \pi^+ \tilde{D}_V^{-p_2}] = \int_M \int_{|\xi'|=1} \text{trace}_{\wedge^* T^* M \otimes \mathbb{C}} [\sigma_{-n}(\tilde{D}_V^{-p_1-p_2})] \sigma(\xi) dx + \int_{\partial M} \Phi. \quad (40)$$

From [9], we obtain the boundary term of the pseudodifferential operators  $A_1$  and  $A_2$  on manifolds with boundary. If  $A_1$  and  $A_2$  have symbols  $a_1(\eta_n) = a_1(x, x_n, \xi, \eta_n)$  and  $a_2(\xi_n) = a_2(x, x_n, \xi, \xi_n)$ , respectively, we obtain the following equality:

$$\begin{aligned} \Phi &= \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{\infty} \frac{(-i)^{j+k+1}}{(j+k+1)!} \times \text{trace}_{\wedge^* T^* M \otimes \mathbb{C}} [\partial_{x_n}^j \partial_{\xi_n}^k a_{1(r)}^+(x', 0, \xi', \xi_n) \\ &\quad \times \partial_{\xi_n}^{j+1} \partial_{x_n}^k a_{2(l)}(x', 0, \xi', \xi_n)]_{-n} d\xi_n \sigma(\xi') dx' \end{aligned} \quad (41)$$

$$\begin{aligned} &= \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{\infty} \frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!} \times \text{trace}_{\wedge^* T^* M \otimes \mathbb{C}} [\partial_{x_n}^j \partial_{\xi'}^{\alpha} \partial_{\xi_n}^k a_{1(r)}^+(x', 0, \xi', \xi_n) \\ &\quad \times \partial_{x'}^{\alpha} \partial_{\xi_n}^{j+1} \partial_{x_n}^k a_{2(l)}(x', 0, \xi', \xi_n)] d\xi_n \sigma(\xi') dx', \end{aligned} \quad (42)$$

where  $a_{1(r)}^+(a_{2(l)})$  denotes the order  $r(l)$  symbol of  $a_1^+(a_2)$ .

Therefore, we replace  $a_{1(r)}^+$  and  $a_{2(l)}$  with  $\sigma_r^+(\tilde{D}_V^{-p_1})$  and  $\sigma_l(\tilde{D}_V^{-p_2})$ , and we obtain the boundary term of the Witten deformation of the non-minimal de Rham–Hodge operator on manifolds with boundary.

$$\begin{aligned} \Phi &= \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{\infty} \sum \frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!} \times \text{trace}_{\wedge^* T^* M \otimes \mathbb{C}} [\partial_{x_n}^j \partial_{\xi'}^{\alpha} \partial_{\xi_n}^k \sigma_r^+(\tilde{D}_V^{-p_1})(x', 0, \xi', \xi_n) \\ &\quad \times \partial_{x'}^{\alpha} \partial_{\xi_n}^{j+1} \partial_{x_n}^k \sigma_l(\tilde{D}_V^{-p_2})(x', 0, \xi', \xi_n)] d\xi_n \sigma(\xi') dx', \end{aligned} \quad (43)$$

where the sum is taken over  $r + l - k - |\alpha| - j - 1 = -n$ ,  $r \leq -p_1$ ,  $l \leq -p_2$ .

Since  $[\sigma_{-n}(\tilde{D}_V^{-p_1-p_2})]|_M$  has the same expression as  $\sigma_{-n}(\tilde{D}_V^{-p_1-p_2})$  in the case of manifolds without boundary, so locally, we can compute the first term by [6,7,11,20].

For any fixed point  $x_0 \in \partial M$ , we choose the normal coordinates  $U$  of  $x_0$  within  $\partial M$  (not in  $M$ ), and compute  $\Phi(x_0)$  using the coordinates in  $\tilde{U} = U \times [0, 1] \subset M$ , along with the metric  $\frac{1}{h(x_n)} g^{\partial M} + dx_n^2$ . The dual metric of  $g^{TM}$  on  $\tilde{U}$  is  $h(x_n) g^{\partial M} + dx_n^2$ . Write  $g_{ij}^{TM} = g^{TM}(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ ;  $g_{TM}^{ij} = g^{TM}(dx_i, dx_j)$ , then we have the following:

$$[g_{ij}^{TM}] = \begin{bmatrix} \frac{1}{h(x_n)} [g_{ij}^{\partial M}] & 0 \\ 0 & 1 \end{bmatrix}; \quad [g_{TM}^{ij}] = \begin{bmatrix} h(x_n) [g_{\partial M}^{ij}] & 0 \\ 0 & 1 \end{bmatrix}, \quad (44)$$

and

$$\partial_{x_n} g_{ij}^{\partial M}(x_0) = 0, 1 \leq i, j \leq n-1; \quad g_{ij}^{TM}(x_0) = \delta_{ij}. \quad (45)$$

From [11], we can obtain the following three lemmas:

**Lemma 1 ([11]).** *With the metric  $g^{TM}$  on  $M$  near the boundary, we have the following:*

$$\partial_{x_j} (|\xi|_{g^{TM}}^2)(x_0) = \begin{cases} 0, & \text{if } j < n, \\ h'(0) |\xi'|_{g^{\partial M}}^2, & \text{if } j = n, \end{cases} \quad (46)$$

$$\partial_{x_j} [c(\xi)](x_0) = \begin{cases} 0, & \text{if } j < n, \\ \partial_{x_n} [c(\xi')](x_0), & \text{if } j = n, \end{cases} \quad (47)$$

where  $\xi = \xi' + \xi_n dx_n$ .

**Lemma 2** ([11]). With the metric  $g^{TM}$  on  $M$  near the boundary, we have the following:

$$\omega_{s,t}(e_i)(x_0) = \begin{cases} \omega_{n,i}(e_i)(x_0) = \frac{1}{2}h'(0), & \text{if } s = n, t = i, i < n, \\ \omega_{i,n}(e_i)(x_0) = -\frac{1}{2}h'(0), & \text{if } s = i, t = n, i < n, \\ \omega_{s,t}(e_i)(x_0) = 0, & \text{other cases,} \end{cases} \quad (48)$$

where  $(\omega_{s,t})$  denotes the connection matrix of the Levi–Civita connection,  $\nabla^L$ .

**Lemma 3** ([11]). When  $i < n$ , then we have the following:

$$\Gamma_{st}^k(x_0) = \begin{cases} \Gamma_{ii}^n(x_0) = \frac{1}{2}h'(0), & \text{if } s = t = i, k = n, \\ \Gamma_{ni}^i(x_0) = -\frac{1}{2}h'(0), & \text{if } s = n, t = i, k = i, \\ \Gamma_{in}^i(x_0) = -\frac{1}{2}h'(0), & \text{if } s = i, t = n, k = i, \end{cases} \quad (49)$$

in other cases,  $\Gamma_{st}^i(x_0) = 0$ .

By (40) and (43), we first compute the following:

$$\widetilde{\text{Wres}}[\pi^+ \tilde{D}_V^{-1} \circ \pi^+ \tilde{D}_V^{-1}] = \int_M \int_{|\xi'|=1} \text{trace}_{\wedge^* T^* M \otimes \mathbb{C}} [\sigma_{-4}(\tilde{D}_V^{-2})] \sigma(\xi) dx + \int_{\partial M} \Phi, \quad (50)$$

where

$$\begin{aligned} \Phi &= \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{\infty} \sum \frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!} \times \text{trace}_{\wedge^* T^* M \otimes \mathbb{C}} [\partial_{x_n}^j \partial_{\xi'}^\alpha \partial_{\xi_n}^k \sigma_r^+(\tilde{D}_V^{-1})(x', 0, \xi', \xi_n) \\ &\times \partial_{x'}^\alpha \partial_{\xi_n}^{j+1} \partial_{x_n}^k \sigma_l(\tilde{D}_V^{-1})(x', 0, \xi', \xi_n)] d\xi_n \sigma(\xi') dx', \end{aligned} \quad (51)$$

and the sum is taken over  $r + l - k - j - |\alpha| = -3$ ,  $r \leq -1$ ,  $l \leq -1$ .

By Theorem 5, we can compute the interior of  $\widetilde{\text{Wres}}[\pi^+ \tilde{D}_V^{-1} \circ \pi^+ \tilde{D}_V^{-1}]$ ; thus, we have the following:

$$\int_M \int_{|\xi'|=1} \text{trace}_{\wedge^* T^* M} [\sigma_{-4}(\tilde{D}_V^{-2})] \sigma(\xi) dx = 32\pi^2 \int_M \left( \frac{2-3ab}{12} s + \frac{6ab-a^2-b^2}{4ab} |V|^2 \right) d\text{Vol}_M. \quad (52)$$

Now, we need to compute  $\int_{\partial M} \Phi$ . By the definition of the Witten deformation of the non-minimal de Rham–Hodge operator  $\tilde{D}_V$  in (4), some symbols of positive and negative orders for the Witten deformation of the non-minimal de Rham–Hodge operator  $\tilde{D}_V$  are given in the following lemmas.

**Lemma 4.** Certain symbols of positive order for the Witten deformation of the non-minimal de Rham–Hodge operator  $\tilde{D}_V$  are as follows:

$$\begin{aligned} \sigma_1(\tilde{D}_V) &= i\tilde{c}(\xi); \\ \sigma_0(\tilde{D}_V) &= \frac{1}{4} \sum_{i,s,t} \omega_{s,t}(e_i) \tilde{c}(e_i) [\tilde{c}(e_s) \tilde{c}(e_t) - c(e_s) c(e_t)] + \tilde{c}(V), \end{aligned} \quad (53)$$

where  $\xi = \sum_{i=1}^n \xi_i dx_i$  denotes the cotangent vector.

We write the following:

$$D_x^\alpha = (-i)^{|\alpha|} \partial_x^\alpha; \sigma(\tilde{D}_V) = p_1 + p_0; \sigma(\tilde{D}_V)^{-1} = \sum_{j=1}^{\infty} q_{-j}. \quad (54)$$

By the composition formula of pseudodifferential operators, we have the following:

$$\begin{aligned} 1 &= \sigma(\tilde{D}_V \circ \tilde{D}_V^{-1}) = \sum_{\alpha} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} [\sigma(\tilde{D}_V)] D_x^{\alpha} [\sigma(\tilde{D}_V^{-1})] \\ &= (p_1 + p_0)(q_{-1} + q_{-2} + q_{-3} + \cdots) \\ &\quad + \sum_j (\partial_{\xi_j} p_1 + \partial_{\xi_j} p_0) (D_{x_j} q_{-1} + D_{x_j} q_{-2} + D_{x_j} q_{-3} + \cdots) \\ &= p_1 q_{-1} + (p_1 q_{-2} + p_0 q_{-1} + \sum_j \partial_{\xi_j} p_1 D_{x_j} q_{-1}) + \cdots, \end{aligned} \quad (55)$$

so, we have the following:

$$q_{-1} = p_1^{-1}; \quad q_{-2} = -p_1^{-1} [p_0 p_1^{-1} + \sum_j \partial_{\xi_j} p_1 D_{x_j} (p_1^{-1})]. \quad (56)$$

**Lemma 5.** Let  $\tilde{D}_V$  act on  $C^{\infty}(\Lambda^*(T^*M))$ . Then, some symbols of negative order for the Witten deformation of the non-minimal de Rham–Hodge operator  $\tilde{D}_V^{-1}$  are as follows:

$$\begin{aligned} \sigma_{-1}(\tilde{D}_V^{-1}) &= \frac{i\tilde{c}(\xi)}{ab|\xi|^2}; \\ \sigma_{-2}(\tilde{D}_V^{-1}) &= \frac{\tilde{c}(\xi)\sigma_0(\tilde{D}_V)\tilde{c}(\xi)}{a^2b^2|\xi|^4} + \frac{\tilde{c}(\xi)}{a^2b^2|\xi|^6} \sum_j c(dx_j) \left[ \partial_{x_j}(\tilde{c}(\xi))|\xi|^2 - \tilde{c}(\xi)\partial_{x_j}(|\xi|^2) \right]. \end{aligned} \quad (57)$$

**Theorem 7.** Let  $M$  be a 4-dimensional oriented compact manifold with boundary  $\partial M$  and the metric  $g^{TM}$ , then we obtain the Kastler–Kalau–Walze-type theorem of the operator  $\tilde{D}_V$ . We have the following:

$$\widetilde{\text{Wres}}[\pi^+ \tilde{D}_V^{-1} \circ \pi^+ \tilde{D}_V^{-1}] = 32\pi^2 \int_M 16 \left( \frac{2-3ab}{12} s + \frac{6ab-a^2-b^2}{4ab} |V|^2 \right) d\text{Vol}_M. \quad (58)$$

**Proof.** When  $n = 4$ , then  $\text{tr}_{\wedge^* T^*M}[\text{id}] = \dim(\wedge^*(\mathbb{R}^4)) = 16$ . Considering that the sum is taken over conditions where  $r + l - k - j - |\alpha| = -3$ ,  $r \leq -1$ ,  $l \leq -1$ , then we have the following five cases:

case (a) (I)  $r = -1$ ,  $l = -1$ ,  $k = j = 0$ ,  $|\alpha| = 1$ .

By (51), we obtain the following:

$$\Phi_1 = - \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{tr}[\partial_{\xi'}^{\alpha} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_V^{-1}) \times \partial_{x'}^{\alpha} \partial_{\xi_n} \sigma_{-1}(\tilde{D}_V^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \quad (59)$$

By Lemma 1, for  $i < n$ , we have the following:

$$\partial_{x_i} \left( \frac{i\tilde{c}(\xi)}{ab|\xi|^2} \right) (x_0) = \frac{i\partial_{x_i}[\tilde{c}(\xi)]}{ab|\xi|^2} (x_0) - \frac{i\tilde{c}(\xi)\partial_{x_i}(|\xi|^2)}{ab|\xi|^4} (x_0) = 0. \quad (60)$$

Therefore, we have the following:

$$\Phi_1 = 0. \quad (61)$$

case (a) (II)  $r = -1$ ,  $l = -1$ ,  $k = |\alpha| = 0$ ,  $j = 1$ .

By (51), we obtain the following:

$$\Phi_2 = -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_V^{-1}) \times \partial_{\xi_n}^2 \sigma_{-1}(\tilde{D}_V^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \quad (62)$$

By Lemma 5, we have the following:

$$\partial_{\xi_n}^2 \sigma_{-1}(\tilde{D}_V^{-1})(x_0) = \frac{i}{ab} \left[ \frac{a(6\xi_n^2 - 2)\epsilon(\xi')}{|\xi|^3} - \frac{b(6\xi_n^2 - 2)\iota(\xi')}{|\xi|^3} + \frac{a(2\xi_n^3 - 6\xi_n)\epsilon(dx_n)}{|\xi|^3} - \frac{b(2\xi_n^3 - 6\xi_n)\iota(dx_n)}{|\xi|^3} \right]; \quad (63)$$

$$\partial_{x_n} \sigma_{-1}(\tilde{D}_V^{-1})(x_0) = \frac{i\partial_{x_n}[\tilde{c}(\xi')]}{ab|\xi|^2}(x_0) - \frac{i\tilde{c}(\xi')h'(0)}{ab|\xi|^4} - \frac{i\xi_n \tilde{c}(dx_n)h'(0)}{ab|\xi|^4}. \quad (64)$$

By the Cauchy integral formula, we have the following:

$$\begin{aligned} & \pi_{\xi_n}^+ \partial_{x_n} \sigma_{-1}(\tilde{D}_V^{-1})(x_0)|_{|\xi'|=1} \\ &= \frac{\partial_{x_n}[\epsilon(\xi')](x_0)}{2b(\xi_n - i)} - \frac{\partial_{x_n}[\iota(\xi')](x_0)}{2a(\xi_n - i)} + \frac{h'(0)}{ab} \left[ \frac{a(2i - \xi_n)\epsilon(\xi')}{4(\xi_n - i)^2} - \frac{b(2i - \xi_n)\epsilon(\xi')}{4(\xi_n - i)^2} - \frac{a\epsilon(dx_n)}{4(\xi_n - i)^2} + \frac{b\iota(dx_n)}{4(\xi_n - i)^2} \right]. \end{aligned} \quad (65)$$

By the relation of the Clifford action and the property  $\text{tr}ab = \text{tr}ba$ , we have the following equalities:

$$\begin{aligned} & \text{tr}[\epsilon(\xi')\iota(\xi')](x_0)|_{|\xi'|=1} = \text{tr}[\epsilon(dx_n)\iota(dx_n)](x_0)|_{|\xi'|=1} = 8; \\ & \text{tr}[\epsilon(\xi')\epsilon(\xi')](x_0)|_{|\xi'|=1} = \text{tr}[\iota(\xi')\iota(\xi')](x_0)|_{|\xi'|=1} = 0; \\ & \text{tr}[\epsilon(dx_n)\epsilon(dx_n)](x_0)|_{|\xi'|=1} = \text{tr}[\iota(dx_n)\iota(dx_n)](x_0)|_{|\xi'|=1} = 0; \\ & \text{tr}[\partial_{x_n}[\iota(\xi')]\epsilon(\xi')](x_0)|_{|\xi'|=1} = 4h'(0); \quad \text{tr}[\partial_{x_n}[\epsilon(\xi')]\iota(\xi')](x_0)|_{|\xi'|=1} = 4h'(0). \end{aligned} \quad (66)$$

By (63) and (66), we have the following:

$$\text{trace}[\partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_V^{-1}) \times \partial_{\xi_n}^2 \sigma_{-1}(\tilde{D}_V^{-1})](x_0) = \frac{8h'(0)(i\xi_n^3 + 3\xi_n^2 - 3i\xi_n - 1)}{ab(\xi_n - i)^5(\xi_n + i)^3}. \quad (67)$$

Then, we have the following:

$$\begin{aligned} \Phi_2 &= -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_V^{-1}) \times \partial_{\xi_n}^2 \sigma_{-1}(\tilde{D}_V^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &= -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{8h'(0)(i\xi_n^3 + 3\xi_n^2 - 3i\xi_n - 1)}{ab(\xi_n - i)^5(\xi_n + i)^3} d\xi_n \sigma(\xi') dx' \\ &= -\frac{4}{ab} h'(0) \Omega_3 \int_{\Gamma^+} \left[ \frac{i\xi_n^3 + 3\xi_n^2 - 3i\xi_n - 1}{(\xi_n - i)^5(\xi_n + i)^3} \right] d\xi_n dx' \\ &= -\frac{4}{ab} h'(0) \Omega_3 \frac{2\pi i}{4!} \left[ \frac{i\xi_n^3 + 3\xi_n^2 - 3i\xi_n - 1}{(\xi_n + i)^3} \right]^{(4)} \Big|_{\xi_n=i} dx' \\ &= -\frac{3}{2ab} h'(0) \pi \Omega_3 dx', \end{aligned} \quad (68)$$

where  $\Omega_3$  denotes the canonical volume of  $S^2$ .

case (a) (III)  $r = -1$ ,  $l = -1$ ,  $j = |\alpha| = 0$ ,  $k = 1$ .

By (51), we obtain the following:

$$\Phi_3 = -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_V^{-1}) \times \partial_{\xi_n} \partial_{x_n} \sigma_{-1}(\tilde{D}_V^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \quad (69)$$

By Lemma 5, we have the following:

$$\begin{aligned} & \partial_{\xi_n} \partial_{x_n} \sigma_{-1}(\tilde{D}_V^{-1})(x_0)|_{|\xi'|=1} \\ &= \frac{ih'(0)}{ab} \left[ \frac{4a\tilde{\xi}_n}{|\tilde{\xi}|^6} \epsilon(\xi') - \frac{4b\tilde{\xi}_n}{|\tilde{\xi}|^6} \iota(\xi') - \frac{a(|\tilde{\xi}|^2 - 4\tilde{\xi}_n^2)}{|\tilde{\xi}|^6} \epsilon(dx_n) + \frac{b(|\tilde{\xi}|^2 - 4\tilde{\xi}_n^2)}{|\tilde{\xi}|^6} \iota(dx_n) \right] \\ & - \frac{2ai\tilde{\xi}_n}{ab|\tilde{\xi}|^4} \partial_{\xi_n} [\epsilon(\xi')](x_0) + \frac{2bi\tilde{\xi}_n}{ab|\tilde{\xi}|^4} \partial_{\xi_n} [\iota(\xi')](x_0), \end{aligned} \quad (70)$$

$$\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_V^{-1})(x_0)|_{|\xi'|=1} = -\frac{[a\epsilon(\xi') - b\iota(\xi')] + i[a\epsilon(dx_n) - b\iota(dx_n)]}{2ab(\xi_n - i)^2}. \quad (71)$$

By (66), (70) and (71), we have the following:

$$\text{trace}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_V^{-1}) \times \partial_{\xi_n} \partial_{x_n} \sigma_{-1}(\tilde{D}_V^{-1})](x_0) = \frac{8h'(0)(-i\tilde{\xi}_n^3 - 3\tilde{\xi}_n^2 + 3i\tilde{\xi}_n + 1)}{ab(\xi_n - i)^5(\xi_n + i)^3}. \quad (72)$$

Then, we have the following:

$$\begin{aligned} \Phi_3 &= -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_V^{-1}) \times \partial_{\xi_n} \partial_{x_n} \sigma_{-1}(\tilde{D}_V^{-1})](x_0) d\tilde{\xi}_n \sigma(\xi') dx' \\ &= -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{8h'(0)(-i\tilde{\xi}_n^3 - 3\tilde{\xi}_n^2 + 3i\tilde{\xi}_n + 1)}{ab(\xi_n - i)^5(\xi_n + i)^3} d\tilde{\xi}_n \sigma(\xi') dx' \\ &= -\frac{4h'(0)}{ab} \Omega_3 \int_{\Gamma^+} \frac{-i\tilde{\xi}_n^3 - 3\tilde{\xi}_n^2 + 3i\tilde{\xi}_n + 1}{(\xi_n - i)^5(\xi_n + i)^3} d\tilde{\xi}_n dx' \\ &= -\frac{4h'(0)}{ab} \Omega_3 \frac{2\pi i}{4!} \left[ \frac{-i\tilde{\xi}_n^3 - 3\tilde{\xi}_n^2 + 3i\tilde{\xi}_n + 1}{(\xi_n + i)^3} \right]^{(4)} \Big|_{\xi_n=i} dx' \\ &= \frac{3h'(0)}{2ab} \pi \Omega_3 dx'. \end{aligned} \quad (73)$$

case (b)  $r = -2$ ,  $l = -1$ ,  $k = j = |\alpha| = 0$ .

By (51), we obtain the following:

$$\Phi_4 = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ \sigma_{-2}(\tilde{D}_V^{-1}) \times \partial_{\xi_n} \sigma_{-1}(\tilde{D}_V^{-1})](x_0) d\tilde{\xi}_n \sigma(\xi') dx'. \quad (74)$$

By Lemma 5, we have the following:

$$\partial_{\xi_n} \sigma_{-1}(\tilde{D}_V^{-1}) = \frac{i}{ab} \left[ \frac{-2a\tilde{\xi}_n \epsilon(\xi')}{(1 + \tilde{\xi}_n^2)^2} + \frac{2b\tilde{\xi}_n \iota(\xi')}{(1 + \tilde{\xi}_n^2)^2} + \frac{a(1 - \tilde{\xi}_n^2) \epsilon(dx_n)}{(1 + \tilde{\xi}_n^2)^2} - \frac{b(1 - \tilde{\xi}_n^2) \iota(dx_n)}{(1 + \tilde{\xi}_n^2)^2} \right]. \quad (75)$$

We denote  $\sigma_0(\tilde{D}_V)$  by  $\tilde{p}_0$ , then we have the following:

$$\sigma_{-2}(\tilde{D}_V^{-1})(x_0) = \frac{\tilde{c}(\xi) \tilde{p}_0(x_0) \tilde{c}(\xi)}{a^2 b^2 |\tilde{\xi}|^4} + \frac{\tilde{c}(\xi)}{a^2 b^2 |\tilde{\xi}|^6} \tilde{c}(dx_n) [\partial_{x_n} [\tilde{c}(\xi')](x_0) |\tilde{\xi}|^2 - \tilde{c}(\xi) h'(0) |\tilde{\xi}|_{\partial M}^2]. \quad (76)$$

By the computations, we have the following:

$$\begin{aligned} & \pi_{\xi_n}^+ \sigma_{-2}(\tilde{D}_V^{-1})(x_0)|_{|\xi'|=1} \\ &= \pi_{\xi_n}^+ \left[ \frac{\tilde{c}(\xi) \tilde{p}_0(x_0) \tilde{c}(\xi) + \tilde{c}(\xi) \tilde{c}(dx_n) \partial_{x_n} [\tilde{c}(\xi')](x_0)}{a^2 b^2 (1 + \tilde{\xi}_n^2)^2} \right] - h'(0) \pi_{\xi_n}^+ \left[ \frac{\tilde{c}(\xi) \tilde{c}(dx_n) \tilde{c}(\xi)}{a^2 b^2 (1 + \tilde{\xi}_n^2)^3} \right] := \tilde{N}_1 - \tilde{N}_2, \end{aligned} \quad (77)$$

where we have the following:

$$\begin{aligned} \tilde{N}_1 = & \frac{-1}{4a^2b^2(\xi_n - i)^2} \left[ (2 + i\xi_n)\tilde{c}(\xi')\tilde{p}_0(x_0)\tilde{c}(\xi') + i\xi_n\tilde{c}(dx_n)\tilde{p}_0(x_0)\tilde{c}(dx_n) + i\xi_n\tilde{c}(dx_n)\tilde{p}_0(x_0)\tilde{c}(\xi') \right. \\ & \left. + i\xi_n\tilde{c}(\xi')\tilde{p}_0(x_0)\tilde{c}(dx_n) \right] - \frac{(2 + i\xi_n)}{4a^2b^2(\xi_n - i)^2} \tilde{c}(\xi')\tilde{c}(dx_n)\partial_{x_n}[\tilde{c}(\xi')](x_0) - \frac{i\partial_{x_n}[\tilde{c}(\xi')](x_0)}{4ab(\xi_n - i)^2}, \end{aligned} \quad (78)$$

and

$$\tilde{N}_2 = \frac{h'(0)}{b} \left[ \frac{(i\xi_n + 3)\epsilon(\xi')}{8(\xi_n - i)^3} - \frac{(i\xi_n^2 + 3\xi_n - 4i)\epsilon(dx_n)}{8(\xi_n - i)^3} \right] + \frac{h'(0)}{a} \left[ \frac{(i\xi_n^2 + 3\xi_n - 4i)\iota(dx_n)}{8(\xi_n - i)^3} - \frac{(i\xi_n + 3)\iota(\xi')}{8(\xi_n - i)^3} \right]. \quad (79)$$

Then, by the relation of the Clifford action and  $\text{tr}ab = \text{tr}ba$ , we have the following equalities:

$$\begin{aligned} \text{tr}[\tilde{c}(\xi')\tilde{p}_0\tilde{c}(\xi')\epsilon(\xi')] &= 8b^2 \sum_{i=1}^{n-1} V_i \xi_i; \quad \text{tr}[\tilde{c}(\xi')\tilde{p}_0\tilde{c}(\xi')\epsilon(dx_n)] = 6ab^2h'(0) + 8abV_n; \\ \text{tr}[\tilde{c}(dx_n)\tilde{p}_0\tilde{c}(dx_n)\iota(\xi')] &= 8ab \sum_{i=1}^{n-1} V_i \xi_i; \quad \text{tr}[\tilde{c}(dx_n)\tilde{p}_0\tilde{c}(dx_n)\iota(dx_n)] = 6a^2bh'(0) + 8a^2V_n; \\ \text{tr}[\tilde{c}(dx_n)\tilde{p}_0\tilde{c}(\xi')\epsilon(\xi')] &= -2ab^2h'(0) + 4b(b-a)V_n; \quad \text{tr}[\tilde{c}(dx_n)\tilde{p}_0\tilde{c}(\xi')\iota(\xi')] = 10a^2bh'(0) + 4a(b-a)V_n; \\ \text{tr}[\tilde{c}(dx_n)\tilde{p}_0\tilde{c}(\xi')\epsilon(dx_n)] &= -8ab \sum_{i=1}^{n-1} V_i \xi_i; \quad \text{tr}[\tilde{c}(dx_n)\tilde{p}_0\tilde{c}(\xi')\iota(dx_n)] = (16a^2 - 8ab) \sum_{i=1}^{n-1} V_i \xi_i; \\ \text{tr}[\tilde{c}(\xi')\tilde{p}_0\tilde{c}(dx_n)\epsilon(\xi')] &= -10ab^2h'(0) + 4b(b-a)V_n; \quad \text{tr}[\tilde{c}(\xi')\tilde{p}_0\tilde{c}(dx_n)\iota(\xi')] = 2a^2bh'(0) - 4a(b-a)V_n; \\ \text{tr}[\tilde{c}(\xi')\tilde{p}_0\tilde{c}(dx_n)\epsilon(dx_n)] &= (-10ab + 8b^2) \sum_{i=1}^{n-1} V_i \xi_i; \quad \text{tr}[\tilde{c}(\xi')\tilde{p}_0\tilde{c}(dx_n)\iota(dx_n)] = (16a^2 + 8ab + 8a) \sum_{i=1}^{n-1} V_i \xi_i; \\ \text{tr}[\epsilon(\xi')\epsilon(dx_n)\partial_{x_n}[\iota(\xi')](x_0)\epsilon(\xi')] &= 0; \quad \text{tr}[\epsilon(\xi')\epsilon(dx_n)\partial_{x_n}[\iota(\xi')](x_0)\iota(\xi')] = 0; \\ \text{tr}[\iota(\xi')\iota(dx_n)\partial_{x_n}[\epsilon(\xi')](x_0)\epsilon(\xi')] &= 0; \quad \text{tr}[\iota(\xi')\iota(dx_n)\partial_{x_n}[\epsilon(\xi')](x_0)\iota(\xi')] = 0. \end{aligned} \quad (80)$$

We note that  $i < n$ ,  $\int_{|\xi'|=1} \xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2d+1}}\sigma(\xi') = 0$ ; thus, we omit some items that have no contribution for computing case (b). Then, by (75), (78) and (80), we have the following:

$$\text{tr}[\tilde{N}_1 \times \partial_{\xi_n}\sigma_{-1}(\tilde{D}_V^{-1})]_{|\xi'|=1} = \frac{h'(0)(2\xi_n^3 + 6\xi_n + 2i\xi_n^2 - 2i)}{ab(\xi_n - i)^4(\xi_n + i)^2} + \frac{(b-a)(-4i\xi_n^2 - 8\xi_n + 4i)}{a^2b^2(\xi_n - i)^4(\xi_n + i)^2} V_n. \quad (81)$$

By (75), (79) and (80), we have the following:

$$\text{tr}[\tilde{N}_2 \times \partial_{\xi_n}\sigma_{-1}(\tilde{D}_V^{-1})]_{|\xi'|=1} = \frac{2h'(0)(-7\xi_n^2 + 9i\xi_n - 3i\xi_n^3 + \xi_n^4 + 4)}{ab(\xi_n - i)^5(\xi_n + i)^2}. \quad (82)$$

Then, we have the following:

$$\begin{aligned}
& -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\tilde{N}_1 \times \partial_{\xi_n} \sigma_{-1}(\tilde{D}_V^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\
& = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{h'(0)(2\tilde{\xi}_n^3 + 6\tilde{\xi}_n + 2i\tilde{\xi}_n^2 - 2i)}{ab(\tilde{\xi}_n - i)^4(\tilde{\xi}_n + i)^2} d\xi_n \sigma(\xi') dx' \\
& -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} + \frac{(b-a)(-4i\tilde{\xi}_n^2 - 8\tilde{\xi}_n + 4i)}{a^2b^2(\tilde{\xi}_n - i)^4(\tilde{\xi}_n + i)^2} V_n d\xi_n \sigma(\xi') dx' \\
& = -i \frac{h'(0)}{ab} \frac{2\pi i}{3!} \Omega_3 \left[ \frac{2\tilde{\xi}_n^3 + 6\tilde{\xi}_n + 2i\tilde{\xi}_n^2 - 2i}{(\tilde{\xi}_n + i)^2} \right]^{(3)} \Big|_{\xi_n=i} dx' \\
& -i \frac{(b-a)V_n}{a^2b^2} \frac{2\pi i}{3!} \Omega_3 \left[ \frac{-4i\tilde{\xi}_n^2 - 8\tilde{\xi}_n + 4i}{(\tilde{\xi}_n + i)^2} \right]^{(3)} \Big|_{\xi_n=i} dx' \\
& = \left( \frac{3}{2ab} h'(0) + \frac{2(a-b)}{a^2b^2} V_n \right) \pi \Omega_3 dx'. \tag{83}
\end{aligned}$$

$$\begin{aligned}
& -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\tilde{N}_2 \times \partial_{\xi_n} \sigma_{-1}(\tilde{D}_V^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\
& = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{2h'(0)(-7\tilde{\xi}_n^2 + 9i\tilde{\xi}_n - 3i\tilde{\xi}_n^3 + \tilde{\xi}_n^4 + 4)}{ab(\tilde{\xi}_n - i)^5(\tilde{\xi}_n + i)^2} d\xi_n \sigma(\xi') dx' \\
& = -i \frac{2h'(0)}{ab} \frac{2\pi i}{4!} \Omega_3 \left[ \frac{-7\tilde{\xi}_n^2 + 9i\tilde{\xi}_n - 3i\tilde{\xi}_n^3 + \tilde{\xi}_n^4 + 4}{(\tilde{\xi}_n + i)^2} \right]^{(4)} \Big|_{\xi_n=i} dx' \\
& = -\frac{3}{ab} h'(0) \pi \Omega_3 dx'. \tag{84}
\end{aligned}$$

By (83) and (84), we have the following:

$$\begin{aligned}
\Phi_4 & = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[(\tilde{N}_1 - \tilde{N}_2) \times \partial_{\xi_n} \sigma_{-1}(\tilde{D}_V^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\
& = \left( \frac{9}{2ab} h'(0) + \frac{2(a-b)}{a^2b^2} V_n \right) \pi \Omega_3 dx'. \tag{85}
\end{aligned}$$

case (c)  $r = -1$ ,  $l = -2$ ,  $k = j = |\alpha| = 0$ .

By (51), we obtain the following:

$$\Phi_5 = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_V^{-1}) \times \partial_{\xi_n} \sigma_{-2}(\tilde{D}_V^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \tag{86}$$

By Lemma 5, we have the following:

$$\pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_V^{-1})|_{|\xi'|=1} = \frac{\epsilon(\xi')}{2b(\xi_n - i)} - \frac{\iota(\xi')}{2a(\xi_n - i)} + \frac{i\epsilon(dx_n)}{2b(\xi_n - i)} - \frac{i\iota(dx_n)}{2a(\xi_n - i)}. \tag{87}$$

Then, we have the following:

$$\begin{aligned}
& \partial_{\xi_n} \sigma_{-2}(\tilde{D}_V^{-1})(x_0)|_{|\xi'|=1} \\
&= \frac{2(\xi_n - \xi_n^3)}{a^2 b^2 (1 + \xi_n^2)^3} \tilde{c}(dx_n) \tilde{p}_0 \tilde{c}(dx_n) + \frac{1 - \xi_n^3}{a^2 b^2 (1 + \xi_n^2)^3} \tilde{c}(dx_n) \tilde{p}_0 \tilde{c}(\xi') + \frac{1 - \xi_n^3}{a^2 b^2 (1 + \xi_n^2)^3} \tilde{c}(\xi') \tilde{p}_0 \tilde{c}(dx_n) \\
&- \frac{4\xi_n}{a^2 b^2 (1 + \xi_n^2)^3} \tilde{c}(\xi') \tilde{p}_0 \tilde{c}(\xi') + \frac{3(\xi_n^2 - 1)}{a^2 b^2 (1 + \xi_n^2)^3} \partial_{x_n} [\tilde{c}(\xi')] - \frac{4\xi_n}{a^2 b^2 (1 + \xi_n^2)^3} \tilde{c}(\xi') \tilde{c}(dx_n) \partial_{x_n} [\tilde{c}(\xi')] \\
&+ \frac{2h'(0)}{ab(1 + \xi_n^2)^3} \tilde{c}(\xi') + \frac{2h'(0)\xi_n}{ab(1 + \xi_n^2)^3} \tilde{c}(dx_n) + \frac{6h'(0)\xi_n}{a^2 b^2 (1 + \xi_n^2)^4} [ab(1 - \xi_n^2) \tilde{c}(dx_n) - 2ab\xi_n \tilde{c}(\xi')]. \quad (88)
\end{aligned}$$

When  $i < n$ ,  $\int_{|\xi'|=1} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_{2d+1}} \sigma(\xi') = 0$ , we omit some items that have no contribution for computing case (c), then by (87) and (88), we have the following:

$$\begin{aligned}
& \text{tr}[\pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_V^{-1}) \times \partial_{\xi_n} \sigma_{-2}(\tilde{D}_V^{-1})] \\
&= \frac{(b-a)(-8i\xi_n^3 - 24\xi_n^2 + 24i\xi_n + 8)}{a^2 b^2 (\xi_n - i)^4 (\xi_n + i)^3} V_n + \frac{96\xi_n^2 - 48i\xi_n + 48i\xi_n^3}{ab(\xi_n - i)^5 (\xi_n + i)^4} h'(0) + \frac{24\xi_n^2 - 36i\xi_n + 12i\xi_n^3 - 24}{ab(\xi_n - i)^4 (\xi_n + i)^3} h'(0). \quad (89)
\end{aligned}$$

Then, we obtain the following:

$$\begin{aligned}
\Phi_5 &= -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_V^{-1}) \times \partial_{\xi_n} \sigma_{-2}(\tilde{D}_V^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\
&= -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{(b-a)(-8i\xi_n^3 - 24\xi_n^2 + 24i\xi_n + 8)}{a^2 b^2 (\xi_n - i)^4 (\xi_n + i)^3} V_n d\xi_n \sigma(\xi') dx' \\
&- i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{h'(0)(96\xi_n^2 - 48i\xi_n + 48i\xi_n^3)}{ab(\xi - i)^5 (\xi + i)^4} d\xi_n \sigma(\xi') dx' \\
&- i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{h'(0)(24\xi_n^2 - 36i\xi_n + 12i\xi_n^3 - 24)}{ab(\xi - i)^4 (\xi + i)^3} d\xi_n \sigma(\xi') dx' \\
&= -\frac{i(b-a)}{a^2 b^2} \Omega_3 \frac{2\pi i}{3!} V_n \left[ \frac{-8i\xi_n^3 - 24\xi_n^2 + 24i\xi_n + 8}{(\xi + i)^3} \right]^{(3)} \Big|_{\xi_n=i} dx' \\
&- \frac{i}{ab} \Omega_3 \frac{2\pi i}{4!} h'(0) \left[ \frac{96\xi_n^2 - 48i\xi_n + 48i\xi_n^3}{(\xi + i)^3} \right]^{(4)} \Big|_{\xi_n=i} dx' \\
&- \frac{i}{ab} \Omega_3 \frac{2\pi i}{3!} h'(0) \left[ \frac{24\xi_n^2 - 44i\xi_n + 12i\xi_n^3 - 16}{(\xi + i)^3} \right]^{(3)} \Big|_{\xi_n=i} dx' \\
&= \left( \frac{2(b-a)}{a^2 b^2} V_n - \frac{9}{2ab} h'(0) \right) \pi \Omega_3 dx'. \quad (90)
\end{aligned}$$

Now,  $\Phi$  denotes the sum of the cases (a), (b), and (c); we obtain the following:

$$\Phi = \sum_{i=1}^5 \Phi_i = 0. \quad (91)$$

Then, by (50)–(52), we obtain Theorem 7.  $\square$

#### 4. A Kastler–Kalu–Walze-Type Theorem for 6-Dimensional Manifolds with Boundary

In this section, we prove the Kastler–Kalu–Walze-type theorem for 6-dimensional manifolds with boundary. Then, by (40) and (43), we first compute the following:

$$\widetilde{\text{Wres}}[\pi^+ \tilde{D}_V^{-1} \circ \pi^+ \tilde{D}_V^{-3}] = \int_M \int_{|\xi|=1} \text{trace}_{\wedge^* T^* M} [\sigma_{-6}(\tilde{D}_V^{-4})] \sigma(\xi) dx + \int_{\partial M} \Psi, \quad (92)$$

where we have the following:

$$\Psi = \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{\infty} \sum \frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!} \times \text{trace}_{\wedge^* T^* M} [\partial_{x_n}^j \partial_{\xi'}^{\alpha} \partial_{\xi_n}^k \sigma_r^+ (\tilde{D}_V^{-1})(x', 0, \xi', \xi_n) \\ \times \partial_{x'}^{\alpha} \partial_{\xi_n}^{j+1} \partial_{x_n}^k \sigma_l (\tilde{D}_V^{-3})(x', 0, \xi', \xi_n)] d\xi_n \sigma(\xi') dx', \quad (93)$$

and the sum is taken over  $r + l - k - j - |\alpha| - 1 = -6$ ,  $r \leq -1$ ,  $l \leq -3$ .

Locally, we can use Theorem 5 to compute the interior term of (92); we have the following:

$$\int_M \int_{|\xi|=1} \text{trace}_{\wedge^* T^* M} [\sigma_{-6}(\tilde{D}_V^{-4})] \sigma(\xi) dx = 128\pi^3 \int_M 64 \left( \frac{2-3ab}{12} s + \frac{6ab-a^2-b^2}{4ab} |V|^2 \right) d\text{Vol}_M. \quad (94)$$

Next, we only need to compute  $\int_{\partial M} \Psi$ . Let us now turn to compute the specification of  $\tilde{D}_V^3$ .

$$\begin{aligned} \tilde{D}_V^3 = ab \Big\{ & \sum_{i=1}^n \tilde{c}(e_i) \langle e_i, dx_l \rangle (-g^{ij} \partial_l \partial_i \partial_j) + \sum_{i=1}^n \tilde{c}(e_i) \langle e_i, dx_l \rangle \left[ -(\partial_l g^{ij}) \partial_i \partial_j - g^{ij} \left( 4(\sigma_i + a_i) \partial_j - 2\Gamma_{ij}^k \partial_k \right) \partial_l \right] \\ & + \sum_{i=1}^n \tilde{c}(e_i) \langle e_i, dx_l \rangle \left[ -2(\partial_l g^{ij}) (\sigma_i + a_i) \partial_j + g^{ij} (\partial_l \Gamma_{ij}^k) \partial_k - 2g^{ij} (\partial_l \sigma_i + \partial_l a_i) \partial_j + (\partial_l g^{ij}) \Gamma_{ij}^k \partial_k \right. \\ & + \frac{1}{ab} \sum_{jk} \left[ \partial_l (\tilde{c}(V) \tilde{c}(e_j) + \tilde{c}(e_j) \tilde{c}(V)) \right] \langle e_j, dx_k \rangle \partial_k + \frac{1}{ab} \sum_{jk} \left( \tilde{c}(V) \tilde{c}(e_j) + \tilde{c}(e_j) \tilde{c}(V) \right) (\partial_l \langle e_j, dx_k \rangle) \partial_k \Big] \\ & + \sum_{i=1}^n \tilde{c}(e_i) \langle e_i, dx_l \rangle \partial_l \left[ -g^{ij} (\partial_i \sigma_j + \partial_i a_j + \sigma_i \sigma_j + \sigma_i a_j + a_i \sigma_j + a_i a_j - \Gamma_{ij}^k \sigma_k - \Gamma_{ij}^k a_k) \right. \\ & + \frac{1}{4} s - \frac{1}{8} \sum_{ijkl} R_{ijkl} \tilde{c}(e_i) \tilde{c}(e_j) c(e_k) c(e_l) + \frac{1}{ab} \sum_j \tilde{c}(e_j) \left( \sum_k \langle e_j, dx_k \rangle \partial_k + \sigma_j + a_j \right) \tilde{c}(V) \\ & + \frac{1}{ab} \tilde{c}(V) \sum_j \tilde{c}(e_j) \left( \sum_k \langle e_j, dx_k \rangle \partial_k + \sigma_j + a_j \right) + \frac{1}{ab} |V|^2 \Big] + \sum_{i=1}^n \tilde{c}(e_i) (\sigma_i + a_i + \tilde{c}(V)) (-g^{ij} \partial_i \partial_j) \\ & + \sum_{i=1}^n \tilde{c}(e_i) \langle e_i, dx_l \rangle \left( \frac{1}{ab} \sum_{jk} \left( \tilde{c}(V) \tilde{c}(e_j) + \tilde{c}(e_j) \tilde{c}(V) \right) \times \langle e_i, dx_k \rangle \right) \partial_l \partial_k + (\sigma_i + a_i + \tilde{c}(V)) \\ & \left[ -\sum_{ij} g^{ij} (2\sigma_i \partial_j + 2a_i \partial_j - \Gamma_{ij}^k \partial_k + \partial_i \sigma_j + \partial_i a_j + \sigma_i \sigma_j + \sigma_i a_j + a_i \sigma_j + a_i a_j - \Gamma_{ij}^k \sigma_k - \Gamma_{ij}^k a_k) \right. \\ & + \sum_{ij} (\tilde{c}(e_i) \tilde{c}(V) + \tilde{c}(V) \tilde{c}(e_i)) \partial_j + \sum_{i,j} (\tilde{c}(V) \tilde{c}(e_j) \partial_i + \tilde{c}(V) \tilde{c}(e_j) a_i + \tilde{c}(e_j) \partial_i (\tilde{c}(V)) + \tilde{c}(e_j) \sigma_i \tilde{c}(V) \\ & \left. + \tilde{c}(e_j) a_i \tilde{c}(V)) + \frac{1}{4} s + \frac{1}{ab} |V|^2 - \frac{1}{8} \sum_{ijkl} R_{ijkl} \tilde{c}(e_i) \tilde{c}(e_j) c(\tilde{e}_k) c(\tilde{e}_l) \Big] \Big\}. \quad (95) \end{aligned}$$

Then, by the definition of the Witten deformation of the non-minimal de Rham–Hodge operator  $\tilde{D}_V^3$  in (95), we specify certain symbols of positive and negative order for  $\tilde{D}_V^3$  in the following lemmas:

**Lemma 6.** *Some symbols of positive order for the Witten deformation of the non-minimal de Rham–Hodge operator  $\tilde{D}_V^3$  are as follows:*

$$\begin{aligned}
\sigma_2(\tilde{D}_V^3) &= ab \sum_{i,j,l} \tilde{c}(dx_l) \partial_l (g^{ij}) \tilde{\zeta}_i \tilde{\zeta}_j + ab \tilde{c}(\tilde{\zeta}) (4\sigma^k + 4a^k - 2\Gamma^k) \tilde{\zeta}_k - 2[\tilde{c}(\tilde{\zeta}) \tilde{c}(V) \tilde{c}(\tilde{\zeta}) + ab|\tilde{\zeta}|^2 \tilde{c}(V)] \\
&\quad + \frac{1}{4} ab |\tilde{\zeta}|^2 \sum_{s,t,l} \omega_{s,t}(e_l) [\tilde{c}(e_l) \tilde{c}(e_s) \tilde{c}(e_t) - \tilde{c}(e_l) c(e_s) c(e_t)] + ab |\tilde{\zeta}|^2 \tilde{c}(V); \\
\sigma_3(\tilde{D}_V^3) &= iab \tilde{c}(\tilde{\zeta}) |\tilde{\zeta}|^2.
\end{aligned} \tag{96}$$

By the composition formula of pseudodifferential operators, we obtain the following equality:

$$1 = \sigma(\tilde{D}_V^3 \circ \tilde{D}_V^{-3}) = \sum_{\alpha} \frac{1}{\alpha!} \partial_{\tilde{\zeta}}^{\alpha} [\sigma(\tilde{D}_V^3)] D_x^{\alpha} [\sigma(\tilde{D}_V^{-3})]. \tag{97}$$

Furthermore, substituting the symbols of positive order for the Witten deformation of the non-minimal de Rham–Hodge operator  $\tilde{D}_V^3$  into the above equality (97), we obtain the following lemma, which provides the negative order for the Witten deformation of the non-minimal de Rham–Hodge operator  $\tilde{D}_V^{-3}$ .

**Lemma 7.** *Some symbols of negative order for the Witten deformation of the non-minimal de Rham–Hodge operator  $\tilde{D}_V^{-3}$  are as follows:*

$$\begin{aligned}
\sigma_{-3}(\tilde{D}_V^{-3}) &= \frac{i\tilde{c}(\tilde{\zeta})}{a^2 b^2 |\tilde{\zeta}|^4}; \\
\sigma_{-4}(\tilde{D}_V^{-3}) &= \frac{\tilde{c}(\tilde{\zeta}) \sigma_2(\tilde{D}_V^3) \tilde{c}(\tilde{\zeta})}{a^4 b^4 |\tilde{\zeta}|^8} + \frac{\tilde{c}(\tilde{\zeta})}{a^3 b^3 |\tilde{\zeta}|^{10}} \sum_j \left( c(dx_j) |\tilde{\zeta}|^2 + 2\tilde{\zeta}_j \tilde{c}(\tilde{\zeta}) \right) \left( \partial_{x_j} [\tilde{c}(\tilde{\zeta})] |\tilde{\zeta}|^2 - 2\tilde{c}(\tilde{\zeta}) \partial_{x_j} [|\tilde{\zeta}|^2] \right).
\end{aligned} \tag{98}$$

**Theorem 8.** *Let  $M$  be a 6-dimensional oriented compact manifold with boundary  $\partial M$  and the metric  $g^{TM}$ ; then, we obtain the Kastler–Kalau–Walze-type theorem of the operator  $\tilde{D}_V$ :*

$$\begin{aligned}
\widetilde{\text{Wres}}[\pi^+ \tilde{D}_V^{-1} \circ \pi^+ (\tilde{D}_V)^{-3}] &= 128\pi^3 \int_M 64 \left( \frac{2-3ab}{12} s + \frac{6ab-a^2-b^2}{4ab} |V|^2 \right) d\text{Vol}_M \\
&\quad + \int_{\partial M} \left( \frac{60i-545}{16a^2b^2} h'(0) + \frac{(a-b)(37-6i)}{2a^3b^3} V_n \right) \pi \Omega_4 d\text{Vol}_{\partial M}.
\end{aligned} \tag{99}$$

**Proof.** Now, we can compute  $\Psi$ . Since  $n = 6$ , we have  $\text{tr}_{\wedge^* T^* M} [\text{id}] = 64$ . Since the sum is taken over  $r + l - k - j - |\alpha| - 1 = -6$ , with  $r \leq -1$ ,  $l \leq -3$ , we find that the integral  $\int_{\partial M} \Psi$  is the sum of the following five cases:

case (a) (I)  $r = -1$ ,  $l = -3$ ,  $j = k = 0$ ,  $|\alpha| = 1$ .

By (93), we obtain the following:

$$\Psi_1 = - \int_{|\tilde{\zeta}'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{trace} \left[ \partial_{\tilde{\zeta}'}^{\alpha} \pi_{\tilde{\zeta}_n}^+ \sigma_{-1}(\tilde{D}_V^{-1}) \times \partial_{x'}^{\alpha} \partial_{\tilde{\zeta}_n} \sigma_{-3}(\tilde{D}_V^{-3}) \right] (x_0) d\tilde{\zeta}_n \sigma(\tilde{\zeta}') dx'. \tag{100}$$

By Lemma 7, for  $i < n$ , we have the following:

$$\partial_{x_i} \sigma_{-3}(\tilde{D}_V^{-3})(x_0) = \partial_{x_i} \left( \frac{i\tilde{c}(\tilde{\zeta})}{a^2 b^2 |\tilde{\zeta}|^4} \right) (x_0) = i\partial_{x_i} \left( \frac{i(a\epsilon(\tilde{\zeta}) - b\iota(\tilde{\zeta}))}{a^2 b^2 |\tilde{\zeta}|^4} \right) (x_0) = 0, \tag{101}$$

so  $\Psi_1 = 0$ .

case (a) (II)  $r = -1$ ,  $l = -3$ ,  $|\alpha| = k = 0$ ,  $j = 1$ .

By (93), we have the following:

$$\Psi_2 = - \frac{1}{2} \int_{|\tilde{\zeta}'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[ \partial_{x_n} \pi_{\tilde{\zeta}_n}^+ \sigma_{-1}(\tilde{D}_V^{-1}) \times \partial_{\tilde{\zeta}_n}^2 \sigma_{-3}(\tilde{D}_V^{-3}) \right] (x_0) d\tilde{\zeta}_n \sigma(\tilde{\zeta}') dx'. \tag{102}$$

By Lemma 7 and direct calculations, we have the following:

$$\partial_{\xi_n}^2 \sigma_{-3}(\tilde{D}_V^{-3}) = \frac{i(20\xi_n^2 - 4)\epsilon(\xi')}{ab^2(1 + \xi_n^2)^4} - \frac{i(20\xi_n^2 - 4)\iota(\xi')}{a^2b(1 + \xi_n^2)^4} + \frac{12i(\xi_n^3 - \xi_n)\epsilon(dx_n)}{ab^2(1 + \xi_n^2)^4} - \frac{12i(\xi_n^3 - \xi_n)\iota(dx_n)}{a^2b(1 + \xi_n^2)^4}. \quad (103)$$

By using the relationship of the Clifford action and the property  $\text{tr}ab = \text{tr}ba$ , we can derive the following:

$$\text{tr}[\epsilon(\xi')\partial_{x_n}[\iota(\xi')](x_0)] = \text{tr}[\partial_{x_n}[\epsilon(\xi')](x_0)\iota(\xi')] = 16h'(0); \quad \text{tr}[\epsilon(\xi')\iota(\xi')] = \text{tr}[\epsilon(dx_n)\iota(dx_n)] = 32. \quad (104)$$

By (65), (103) and (104), we obtain the following:

$$\text{trace}\left[\partial_{x_n}\pi_{\xi_n}^+\sigma_{-1}(\tilde{D}_V^{-1}) \times \partial_{\xi_n}^2\sigma_{-3}(\tilde{D}_V^{-3})\right](x_0) = \frac{64h'(0)(-1 - 3i\xi_n + 5\xi_n^2 + 3i\xi_n^3)}{a^2b^2(\xi_n - i)^6(\xi_n + i)^4}. \quad (105)$$

Then, we obtain the following:

$$\begin{aligned} \Psi_2 &= -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{64h'(0)(-1 - 3i\xi_n + 5\xi_n^2 + 3i\xi_n^3)}{a^2b^2(\xi_n - i)^6(\xi_n + i)^4} d\xi_n \sigma(\xi') dx' \\ &= -\frac{32h'(0)}{a^2b^2} \Omega_4 \int_{\Gamma^+} \frac{-1 - 3i\xi_n + 5\xi_n^2 + 3i\xi_n^3}{(\xi_n - i)^6(\xi_n + i)^4} d\xi_n dx' \\ &= -\frac{32h'(0)}{a^2b^2} \Omega_4 \frac{2\pi i}{5!} \left[ \frac{-1 - 3i\xi_n + 5\xi_n^2 + 3i\xi_n^3}{(\xi_n + i)^4} \right]^{(5)} \Big|_{\xi_n=i} dx' \\ &= -\frac{15h'(0)}{2a^2b^2} \pi \Omega_4 dx', \end{aligned} \quad (106)$$

where  $\Omega_4$  denotes the canonical volume of  $S^4$ .

case (a) (III)  $r = -1$ ,  $l = -3$ ,  $|\alpha| = j = 0$ ,  $k = 1$ .

By (93), we have the following:

$$\Psi_3 = -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}\left[\partial_{\xi_n}\pi_{\xi_n}^+\sigma_{-1}(\tilde{D}_V^{-1}) \times \partial_{\xi_n}\partial_{x_n}\sigma_{-3}(\tilde{D}_V^{-3})\right](x_0) d\xi_n \sigma(\xi') dx'. \quad (107)$$

By Lemma 7 and direct calculations, we have the following:

$$\begin{aligned} \partial_{\xi_n}\partial_{x_n}\sigma_{-3}(\tilde{D}_V^{-3}) &= -\frac{2ih'(0)\xi_n\epsilon(\xi')}{ab^2(1 + \xi_n^2)^3} + \frac{2ih'(0)\xi_n\iota(\xi')}{ab^2(1 + \xi_n^2)^3} + \frac{12ih'(0)\xi_n\epsilon(\xi')}{ab^2(1 + \xi_n^2)^4} - \frac{12ih'(0)\xi_n\iota(\xi')}{ab^2(1 + \xi_n^2)^4} \\ &\quad - \frac{ih'(0)(2 - 10\xi_n^2)\epsilon(dx_n)}{ab^2(1 + \xi_n^2)^4} + \frac{ih'(0)(2 - 10\xi_n^2)\iota(dx_n)}{ab^2(1 + \xi_n^2)^4}. \end{aligned} \quad (108)$$

Combining (65) and (108), we have the following:

$$\text{trace}\left[\partial_{\xi_n}\pi_{\xi_n}^+\sigma_{-1}(\tilde{D}_V^{-1}) \times \partial_{\xi_n}\partial_{x_n}\sigma_{-3}(\tilde{D}_V^{-3})\right](x_0)|_{|\xi'|=1} = \frac{64h'(0)(1 - i\xi_n^3 + 5i\xi_n - 5\xi_n^2)}{a^2b^2(\xi_n - i)^6(\xi_n + i)^4}. \quad (109)$$

Then, we have the following:

$$\begin{aligned} \Psi_3 &= -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{64h'(0)(1 - i\xi_n^3 + 5i\xi_n - 5\xi_n^2)}{a^2b^2(\xi_n - i)^6(\xi_n + i)^4} d\xi_n \sigma(\xi') dx' \\ &= -\frac{32h'(0)}{a^2b^2} \Omega_4 \int_{\Gamma^+} \frac{1 - i\xi_n^3 + 5i\xi_n - 5\xi_n^2}{(\xi_n - i)^6(\xi_n + i)^4} d\xi_n dx' \\ &= -\frac{32h'(0)}{a^2b^2} \Omega_4 \frac{2\pi i}{5!} \left[ \frac{1 - i\xi_n^3 + 5i\xi_n - 5\xi_n^2}{(\xi_n + i)^4} \right]^{(5)} \Big|_{\xi_n=i} dx' \\ &= \frac{25h'(0)}{2a^2b^2} \pi \Omega_4 dx'. \end{aligned} \quad (110)$$

case (b)  $r = -1$ ,  $l = -4$ ,  $|\alpha| = j = k = 0$ .

By (93), we have the following:

$$\begin{aligned}\Psi_4 &= -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[ \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_V^{-1}) \times \partial_{\xi_n} \sigma_{-4}(\tilde{D}_V^{-3}) \right] (x_0) d\xi_n \sigma(\xi') dx' \\ &= i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} [\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}_V^{-1}) \times \sigma_{-4}(\tilde{D}_V^{-3})] (x_0) d\xi_n \sigma(\xi') dx'.\end{aligned}\quad (111)$$

In the normal coordinates at  $g^{ij}(x_0) = \delta_i^j$  and  $\partial_{x_j}(g^{\alpha\beta})(x_0) = 0$ , if  $j < n$ ;  $\partial_{x_j}(g^{\alpha\beta})(x_0) = h'(0)\delta_\beta^\alpha$ , if  $j = n$ . So, by Lemma A.2 in [11], we have  $\Gamma^n(x_0) = \frac{5}{2}h'(0)$  and  $\Gamma^k(x_0) = 0$  for  $k < n$ . By the definitions of  $\sigma^k$  and Lemma 2.3 in [11], we have  $\sigma^n(x_0) = 0$  and  $\sigma^k = \frac{1}{4}h'(0)c(e_k)c(e_n)$  for  $k < n$ . By Lemma 7, we obtain the following:

$$\begin{aligned}\sigma_2(\tilde{D}_V^3)(x_0)|_{|\xi'|=1} &= \frac{1}{a^4 b^4 |\xi|^8} \tilde{c}(\xi) \left( abh'(0)\tilde{c}(\xi) \sum_{k < n} \xi_k c(e_k)c(e_n) - abh'(0)\tilde{c}(\xi) \sum_{k < n} \xi_k \hat{c}(e_k)\hat{c}(e_n) + 5a^2 b^2 h'(0)\xi_n |\xi|^2 \right. \\ &+ \frac{5}{4} abh'(0)|\xi|^2 \tilde{c}(e_i)\hat{c}(dx_n)\hat{c}(e_i) - \frac{1}{4} abh'(0)|\xi|^2 \tilde{c}(dx_n) - 2\tilde{c}(\xi)\hat{c}(V)\tilde{c}(\xi) - ab|\xi|^2 \hat{c}(V) \Big) \tilde{c}(\xi) \\ &+ \frac{\tilde{c}(\xi)}{a^3 b^3 |\xi|^{10}} \left( |\xi|^4 \tilde{c}(dx_n) \partial_{x_n} [\tilde{c}(\xi)] + 2|\xi|^2 \xi_n \tilde{c}(\xi) \partial_{x_n} [\tilde{c}(\xi)] - 2h'(0)|\xi|^2 \tilde{c}(dx_n)\tilde{c}(\xi) + 4ab|\xi|^2 \xi_n h'(0) \right).\end{aligned}\quad (112)$$

By (71) and (112), we have the following:

$$\sigma_{-4}(\tilde{D}_V^{-3})(x_0)|_{|\xi'|=1} = \tilde{M}_1 + \tilde{M}_2 + \tilde{M}_3, \quad (113)$$

where we have the following:

$$\begin{aligned}\tilde{M}_1 &= -\frac{1}{a^2 b^2 |\xi|^6} h'(0)c(\xi')c(dx_n)\tilde{c}(\xi') - \frac{1}{a^2 b^2 |\xi|^6} h'(0)\xi_n c(\xi')c(dx_n)\tilde{c}(dx_n) + \frac{1}{a^2 b^2 |\xi|^6} h'(0)\tilde{c}(\xi')\hat{c}(dx_n)\tilde{c}(dx_n) \\ &+ \frac{1}{a^2 b^2 |\xi|^6} h'(0)\xi_n \hat{c}(\xi')\hat{c}(dx_n)\tilde{c}(dx_n) + \frac{5}{a^2 b^2 |\xi|^6} h'(0)\xi_n \tilde{c}(\xi') + \frac{5}{a^2 b^2 |\xi|^6} h'(0)\xi_n^2 \tilde{c}(dx_n),\end{aligned}\quad (114)$$

$$\begin{aligned}\tilde{M}_2 &= \frac{5}{a^3 b^3 |\xi|^6} h'(0)\tilde{c}(\xi')c(e_i)\hat{c}(dx_n)\hat{c}(e_i)\tilde{c}(\xi') + \frac{5\xi_n}{a^3 b^3 |\xi|^6} h'(0)\tilde{c}(\xi')c(e_i)\hat{c}(dx_n)\hat{c}(e_i)\tilde{c}(dx_n) \\ &+ \frac{5\xi_n}{a^3 b^3 |\xi|^6} h'(0)\tilde{c}(dx_n)c(e_i)\hat{c}(dx_n)\hat{c}(e_i)\tilde{c}(\xi') + \frac{5\xi_n^2}{a^3 b^3 |\xi|^6} h'(0)\tilde{c}(dx_n)c(e_i)\hat{c}(dx_n)\hat{c}(e_i)\tilde{c}(dx_n) \\ &- \frac{1}{a^3 b^3 |\xi|^6} h'(0)\tilde{c}(\xi')\tilde{c}(dx_n)\tilde{c}(\xi') - \frac{\xi_n}{a^3 b^3 |\xi|^6} h'(0)\tilde{c}(\xi')\tilde{c}(dx_n)\tilde{c}(dx_n) - \frac{\xi_n}{a^3 b^3 |\xi|^6} h'(0)\tilde{c}(dx_n)\tilde{c}(dx_n)\tilde{c}(\xi') \\ &- \frac{\xi_n^2}{a^3 b^3 |\xi|^6} h'(0)\tilde{c}(dx_n)\tilde{c}(dx_n)\tilde{c}(dx_n) - \frac{2}{a^2 b^2 |\xi|^4} \hat{c}(V) - \frac{1}{a^3 b^3 |\xi|^6} \tilde{c}(\xi')\hat{c}(V)\tilde{c}(\xi') \\ &- \frac{\xi_n}{a^3 b^3 |\xi|^6} \tilde{c}(dx_n)\hat{c}(V)\tilde{c}(dx_n) - \frac{\xi_n}{a^3 b^3 |\xi|^6} \tilde{c}(dx_n)\hat{c}(V)\tilde{c}(\xi') - \frac{\xi_n^2}{a^3 b^3 |\xi|^6} \tilde{c}(dx_n)\hat{c}(V)\tilde{c}(dx_n)\end{aligned}\quad (115)$$

and

$$\tilde{M}_3 = \frac{1}{a^3 b^3 |\xi|^6} \tilde{c}(\xi)\tilde{c}(dx_n)\partial_{x_n} [\tilde{c}(\xi)] - \frac{2\xi_n}{a^2 b^2 |\xi|^6} \partial_{x_n} [\tilde{c}(\xi)] - \frac{2}{a^3 b^3 |\xi|^8} \tilde{c}(\xi)\tilde{c}(dx_n)\tilde{c}(\xi) + \frac{4\xi_n h'(0)}{a^2 b^2 |\xi|^8} \tilde{c}(\xi). \quad (116)$$

By a direct calculation and the relation of the Clifford action and  $\text{tr}ab = \text{tr}ba$ , we have the following equalities:

$$\begin{aligned}
& \text{tr}[\epsilon(\xi')c(\xi')c(dx_n)\tilde{c}(\xi)] = 0; \quad \text{tr}[\epsilon(\xi')c(\xi')c(dx_n)\tilde{c}(dx_n)] = 16(a+b); \\
& \text{tr}[\epsilon(\xi')c(\xi')\tilde{c}(dx_n)\tilde{c}(dx_n)] = 16(a-b); \quad \text{tr}[\epsilon(\xi')c(\xi')] = -32b; \\
& \text{tr}[\epsilon(\xi')\tilde{c}(\xi)\tilde{c}(dx_n)\partial_{x_n}[\tilde{c}(\xi)]] = 32ab^2h'(0)\xi_n; \quad \text{tr}[\epsilon(\xi')\partial_{x_n}[\tilde{c}(\xi)]] = -16bh'(0); \\
& \text{tr}[\epsilon(dx_n)\tilde{c}(\xi)\tilde{c}(dx_n)\partial_{x_n}[\tilde{c}(\xi)]] = -16ab^2h'(0) + ab^2h'(0)\xi_n^2; \\
& \text{tr}[\epsilon(\xi')\tilde{c}(\xi)\tilde{c}(e_i)\tilde{c}(dx_n)\tilde{c}(e_i)\tilde{c}(\xi)] = 0; \quad \text{tr}[\epsilon(dx_n)\tilde{c}(V)] = 32V_n; \\
& \text{tr}[\epsilon(\xi')\tilde{c}(\xi)\tilde{c}(V)\tilde{c}(\xi)] = 32(b^2 - ab)V_n\xi_n; \quad \text{tr}[\epsilon(dx_n)\tilde{c}(\xi)\tilde{c}(V)\tilde{c}(\xi)] = 32abV_n + 32a^2V_n\xi_n.
\end{aligned} \tag{117}$$

Then, we have the following:

$$\begin{aligned}
& \text{tr}[\partial_{\xi_n}\pi_{\xi_n}^+\sigma_{-1}(\tilde{D}_V^{-1}) \times \tilde{M}_1](x_0)|_{|\xi'|=1} = \frac{32h'(0)(5i\xi_n^2 + 6\xi_n - i)}{a^2b^2(\xi_n - i)^5(\xi_n + i)^3}; \\
& \text{tr}[\partial_{\xi_n}\pi_{\xi_n}^+\sigma_{-1}(\tilde{D}_V^{-1}) \times \tilde{M}_2](x_0)|_{|\xi'|=1} = \frac{8h'(0)(i\xi_n^2 + 2 - i)}{a^2b^2(\xi_n - i)^5(\xi_n + i)^3} + \frac{16(a-b)(2i\xi_n^2 - 2\xi_n + \xi_n^2 + 1 + 2i)V_n}{a^3b^3(\xi_n - i)^5(\xi_n + i)^3}; \\
& \text{tr}[\partial_{\xi_n}\pi_{\xi_n}^+\sigma_{-1}(\tilde{D}_V^{-1}) \times \tilde{M}_3](x_0)|_{|\xi'|=1} = \frac{16h'(0)(-3i\xi_n^2 - 4\xi_n + i)}{a^2b^2(\xi_n - i)^5(\xi_n + i)^3} + \frac{4h'(0)(43i\xi_n^2 + 64\xi_n - 16i)}{a^2b^2(\xi_n - i)^6(\xi_n + i)^4}.
\end{aligned} \tag{118}$$

Then, we have the following:

$$\begin{aligned}
\Psi_4 &= i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{\xi_n}\pi_{\xi_n}^+\sigma_{-1}(\tilde{D}_V^{-1}) \times (\tilde{M}_1 + \tilde{M}_2 + \tilde{M}_3)](x_0)d\xi_n\sigma(\xi')dx' \\
&= \left( \frac{60i - 625}{16a^2b^2}h'(0) + \frac{(a-b)(21-6i)}{2a^3b^3}V_n \right) \pi\Omega_4 dx'.
\end{aligned} \tag{119}$$

case (c)  $r = -2$ ,  $l = -3$ ,  $|\alpha| = j = k = 0$ .

By (93), we have the following:

$$\Psi_5 = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+\sigma_{-2}(\tilde{D}_V^{-1}) \times \partial_{\xi_n}\sigma_{-3}(\tilde{D}_V^{-3})](x_0)d\xi_n\sigma(\xi')dx'. \tag{120}$$

By Lemma 7, we have the following:

$$\partial_{\xi_n}\sigma_{-3}(\tilde{D}_V^{-3}) = -\frac{4i\xi_n\epsilon(\xi')}{ab^2(1+\xi_n^2)^3} + \frac{4i\xi_n\iota(\xi')}{a^2b(1+\xi_n^2)^3} + \frac{i(1-3\xi_n^2)\epsilon(dx_n)}{ab^2(1+\xi_n^2)^3} - \frac{i(1-3\xi_n^2)\iota(dx_n)}{a^2b(1+\xi_n^2)^3}. \tag{121}$$

By (78) and (79), we have the following:

$$\begin{aligned}
& \text{tr}[\tilde{N}_1 \times \sigma_{-3}(\tilde{D}_V^{-3})]|_{|\xi'|=1} = \frac{8h'(0)(3\xi_n^3 + 3i\xi_n^2 + 7\xi_n - i)}{a^2b^2(\xi_n - i)^5(\xi_n + i)^3} + \frac{16(b-a)(-3i\xi_n^2 - 4\xi_n + i)}{a^3b^3(\xi_n - i)^5(\xi_n + i)^3}V_n; \\
& \text{tr}[\tilde{N}_2 \times \sigma_{-3}(\tilde{D}_V^{-3})]|_{|\xi'|=1} = \frac{8h'(0)(-17\xi_n^2 + 3\xi_n^4 - 9i\xi_n^3 + 15i\xi_n + 4)}{a^2b^2(\xi_n - i)^6(\xi_n + i)^3}.
\end{aligned} \tag{122}$$

Then, we obtain the following:

$$\begin{aligned}\Psi_5 &= -ih'(0) \int_{|\zeta'|=1} \int_{-\infty}^{+\infty} \frac{8h'(0)(3\zeta_n^3 + 3i\zeta_n^2 + 7\zeta_n - i)}{a^2b^2(\zeta_n - i)^5(\zeta_n + i)^3} d\zeta_n \sigma(\zeta') dx' \\ &\quad - ih'(0) \int_{|\zeta'|=1} \int_{-\infty}^{+\infty} \frac{16(b-a)(-3i\zeta_n^2 - 4\zeta_n + i)}{a^3b^3(\zeta_n - i)^5(\zeta_n + i)^3} V_n d\zeta_n \sigma(\zeta') dx' \\ &\quad - ih'(0) \int_{|\zeta'|=1} \int_{-\infty}^{+\infty} \frac{8h'(0)(-17\zeta_n^2 + 3\zeta_n^4 - 9i\zeta_n^3 + 15i\zeta_n + 4)}{a^2b^2(\zeta_n - i)^6(\zeta_n + i)^3} d\zeta_n \sigma(\zeta') dx' \\ &= \left( \frac{32}{a^2b^2} h'(0) + \frac{8(a-b)}{a^3b^3} V_n \right) \pi \Omega_4 dx'.\end{aligned}\quad (123)$$

Now,  $\Psi$  denotes the sum of the cases (a), (b), and (c), then we have the following:

$$\Psi = \sum_{i=1}^5 \Psi_i = \left( \frac{60i - 545}{16a^2b^2} h'(0) + \frac{(a-b)(37-6i)}{2a^3b^3} V_n \right) \pi \Omega_4 dx'. \quad (124)$$

Then, by (92)–(94) and (124), we obtain Theorem 8.  $\square$

## 5. Conclusions

The primary objective of this paper is the Witten deformation of the non-minimal de Rham–Hodge operator, particularly its application to the Kastler–Kalau–Walze-type theorem on low-dimensional manifolds with boundary. First, we present a general result concerning the noncommutative residue of the non-minimal de Rham–Hodge operator on even-dimensional manifolds without boundary, as detailed in Theorem 5. This addresses the interior term in even-dimensional manifolds with boundary. Secondly, on four-dimensional manifolds with boundary, we obtain the vanishing boundary terms, as shown in Theorem 7. Lastly, in order to obtain the boundary term that does not disappear, we usually consider the case of asymmetry, that is, calculations of  $\widetilde{\text{Wres}}[\pi^+ \tilde{D}_V^{-1} \circ \pi^+ (\tilde{D}_V)^{-3}]$  on six-dimensional manifolds with boundary. As shown in Theorem 8, we obtain the non-vanishing boundary terms.

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