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Weighted Statistical Convergence and Cluster Points: The Fibonacci Sequence-Based Approach Using Modulus Functions

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Abstract: In this paper, the Fibonacci sequence, renowned for its significance across various fields, its ability to illuminate numerical concepts, and its role in uncovering patterns in mathematics and nature, forms the foundation of this research. This study introduces innovative concepts of weighted density, weighted statistical summability, weighted statistical convergence, and weighted statistical Cauchy, uniquely defined via the Fibonacci sequence and modulus functions. Key theorems, relationships, examples, and properties substantiate these novel principles, advancing our comprehension of sequence behavior. Additionally, we extend the notion of statistical cluster points within a broader framework, surpassing traditional definitions and offering deeper insights into convergence in a statistical context. Our findings in this paper open avenues for new applications and further exploration in various mathematical fields.

Keywords: Fibonacci sequence; modulus function; weighted Fibonacci statistical convergence; statistical Cauchy; statistical cluster point

MSC: 11B39; 40A05; 40A35

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1. Introduction

The concept of convergence plays a pivotal role, providing foundational understanding in various branches of mathematics. The classical notions of convergence have been thoroughly explored and widely applied across a multitude of mathematical areas. However, these classical methods often fail to capture the more complex behaviors of sequences in mathematics. To address these limitations, the concept of statistical convergence was proposed. This idea was first independently put forward by Fast [1] and Steinhaus [2] in close succession. The idea behind statistical convergence can also be traced back to earlier work, specifically in [3], where the notion of almost convergence was discussed. This earlier concept has since been shown to be equivalent to what is now known as statistical convergence. Let $\Xi \subset \mathbb{N}$ (the set of natural numbers). Then, the natural density of Ξ is denoted as $\delta(\Xi)$ and is formally defined by

$$\delta(\Xi) = \lim_{\nu \to \infty} \frac{1}{\nu} |\{\lambda \leq \nu : \lambda \in \Xi\}|,$$

if the limit exists, in which $|\{\lambda \leq \nu : \lambda \in \Xi\}|$ represents the count of elements in Ξ that are less than or equal to ν (see [4]).

A sequence $\mathscr{T} = (\mathscr{T}_k)$ is defined to be statistically convergent to $\mathscr{T}_0$ if the set $\{k \in \mathbb{N} : |\mathscr{T}_k - \mathscr{T}_0| \geq \varrho\}$ has a natural density zero for all $\varrho > 0$, that is,

$$\lim_{\nu\to\infty} \frac{1}{\nu} |\{k \leq \nu : |\mathscr{T}_k - \mathscr{T}_0| \geq \varrho\}| = 0.$$

Maddox [5] broadened the scope of this concept to include sequences within Hausdorff locally convex topological vector spaces. Kolk [6] initiated the examination of its relevance to Banach space theory. In [7], the researchers identified a notable link between statistical convergence and various classical properties. They specifically characterized Banach spaces with separable duals in a unique way that is not possible through conventional convergence methods. Recent studies on this concept can be found in references [8–10].

Karakaya and Chishti [11] initially defined weighted statistical convergence, and Mursaleen et al. [12] later refined the concept. Recently, Ghosal [13] updated and clarified the framework of weighted statistical convergence.

Let $\mu = (\mu_k)$ be a sequence of non-negative numbers such that $\mathcal{M}_\nu = \sum_{k=0}^{\nu} \mu_k \to \infty$ as $\nu \to \infty$ and $\mu_0 > 0$.

Let $\Xi \in \mathbb{N}$. A weighted density of Ξ is defined by

$$\delta_{\overline{N}}(\Xi) = \lim_{\nu\to\infty} \frac{1}{\mathcal{M}_\nu} |\{\lambda \leq \mathcal{M}_\nu : \lambda \in \Xi\}|,$$

provided the limit exists.

A sequence $\mathscr{T} = (\mathscr{T}_k)$ is defined as weighted statistically convergent to $\mathscr{T}_0$ if for every $\varrho > 0$,

$$\lim_{\nu\to\infty} \frac{1}{\mathcal{M}_\nu} |\{k \leq \mathcal{M}_\nu : \mu_k |\mathscr{T}_k - \mathscr{T}_0| \geq \varrho\}| = 0.$$

A Fibonacci sequence, as described in [14], is a sequence in which each term results from adding the two previous terms and follows the recurrence relation $f_\nu = f_{\nu-1} + f_{\nu-2}$ for $\nu \geq 2$. Consequently, the sequence begins as follows:

$$(f_\nu) = (0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots).$$

Fibonacci numbers possess several fundamental characteristics, including

$$\sum_{k=0}^{\nu} f_k = f_{\nu+2} - 1,\ \nu \in \mathbb{N},$$

$$\sum_k \frac{1}{f_k} \text{ converges},$$

$$f_{\nu-1} f_{\nu+1} - f_\nu^2 = (-1)^{\nu+1}, \nu \geq 1 \text{ (Cassinis formula)},$$

$$\lim_{\nu\to\infty} \frac{f_{\nu+1}}{f_\nu} = \frac{\sqrt{5}+1}{2} = \rho \text{ (Golden ratio)}.$$

Kara and Basarir [15] were the pioneers in incorporating the Fibonacci sequence into sequence space theory. Basarir et al. [16] developed the Fibonacci difference sequence spaces, denoted as $c_0\left(\widehat{\mathcal{F}}\right)$ and $c\left(\widehat{\mathcal{F}}\right)$, in which c_0 and c denote the paces of null and convergent sequences, respectively, i.e.,

$$c_0\left(\widehat{\mathcal{F}}\right) = \left\{\mathscr{T} = (\mathscr{T}_k) : \lim_{k\to\infty} \widehat{\mathcal{F}}_k(\mathscr{T}) = 0\right\}$$

and

$$c\left(\widehat{\mathcal{F}}\right) = \left\{ \mathscr{T} = (\mathscr{T}_k) : \lim_{k\to\infty} \widehat{\mathcal{F}}_k(\mathscr{T}) = \mathscr{T}_0 \text{ for some number } \mathscr{T}_0 \right\},$$

in which $\widehat{\mathcal{F}_k}(\mathscr{T})$ represents the $\widehat{\mathcal{F}}-$transform of $(\mathscr{T}_k)$, defined as

$$\widehat{\mathcal{F}_k}(\mathscr{T}) = \begin{cases} \frac{f_0}{f_1}\mathscr{T}_0 = \mathscr{T}_0, & k = 0, \\ \\ \frac{f_k}{f_{k+1}}\mathscr{T}_k - \frac{f_{k+1}}{f_k}\mathscr{T}_{k-1}, & k \geq 1. \end{cases}$$

Further details and uses concerning the use of the Fibonacci sequence are available in [17–20].

The concept of a modulus function was first outlined in [21]. A function $\Phi : [0, \infty) \to [0, \infty)$ is considered a modulus function (or simply, a modulus) if it satisfies the following criteria:

1. $\Phi(x) = 0 \Leftrightarrow x = 0$;
2. $\Phi(x_1 + x_2) \leq \Phi(x_1) + \Phi(x_2)$ for every $x_1, x_2 \in [0, \infty)$;
3. Φ is continuous from the right at 0;
4. Φ is increasing.

This type of function can exhibit either unbounded or bounded. For example, $\Phi(x) = x^q$, $q \in (0, 1]$, exemplifies an unbounded modulus, while $\Phi(x) = \frac{x}{x+1}$ is an instance of a bounded modulus. We refer to $\mathfrak{F}_{(un.)}$ and $\mathfrak{F}_{(boun.)}$ as the collections of all unbounded and bounded modulus functions, respectively, in this study. In the context of sequence spaces, modulus functions serve as a crucial tool for defining and analyzing various types of convergence and summability. Various contributors have formulated and established different sequence spaces through the use of modulus functions, leveraging them to introduce and develop a diverse array of sequence spaces, thereby making significant contributions to the field (for an example, see [22–26]).

The primary results of this study include the development of novel definitions for weighted Φ-Fibonacci statistical summability, weighted Φ-Fibonacci statistical convergence, and weighted Φ-Fibonacci statistical Cauchy. Furthermore, we extend the concepts of weighted Fibonacci Φ-statistical limit points and weighted Fibonacci Φ-statistical cluster points. In the context of this research, the motivation for using the Fibonacci structure lies in its particular appropriateness for extending classical concepts of convergence and summability. By leveraging its recursive and weighted properties, we introduce novel definitions that offer a richer perspective on sequence behavior. This approach not only generalizes traditional convergence methods but also bridges them with naturally occurring patterns, making this study more intuitive and widely applicable. Thus, the Fibonacci sequence serves as a natural and mathematically elegant basis for advancing the theoretical framework of weighted statistical convergence and summability.

This paper is organized as follows: In Section 2, we present the main definitions, foundational concepts, and theorems that form the basis of our study. Section 3 introduces and examines the concepts of weighted Fibonacci Φ-statistical convergence and weighted Fibonacci Φ-statistical cluster points, including a detailed analysis of their properties and implications. Finally, the conclusion summarizes the key results, highlights the contributions of this work, and suggests potential directions for future research in related fields.

2. Main Section

Definition 1. *Let $\Phi \in \mathfrak{F}_{(un.)}$ and $\Xi \in \mathbb{N}$. A weighted $\Phi-$density of Ξ is defined by*

$$\delta_N^{\Phi}(\Xi) = \lim_{\nu\to\infty} \frac{1}{\Phi(\mathcal{M}_\nu)} \Phi(|\{\lambda \leq \mathcal{M}_\nu : \lambda \in \Xi\}|),$$

provided the limit exists.

Definition 2. *Let $\Phi \in \mathfrak{F}_{(un.)}$. For a sequence $\mathscr{T} = (\mathscr{T}_k)$, we set*

$$\mathfrak{t}_\nu\left[\widehat{\mathcal{F}}(\mathscr{T})\right] = \frac{1}{\mathcal{M}_\nu}\sum_{k=0}^{\nu}\mu_k\widehat{\mathcal{F}_k}(\mathscr{T}),\ n = 0, 1, 2, \ldots.$$

Then, it is said that the sequence $\mathscr{T}$ is weighted $\Phi-$Fibonacci statistically summable based on the sequence (μ_k) (or concisely, $\left[\widehat{\mathcal{F}}S^{\Phi}_{\mathfrak{t}_k}(\overline{\mathrm{N}}, \mu_k)\right]-$summable) to $\mathscr{T}_0$ if

$$\lim_{\nu\to\infty}\frac{1}{\Phi(\nu)}\Phi\left(\left|\left\{k \le \nu : \left|\mathfrak{t}_k\left[\widehat{\mathcal{F}}(\mathscr{T})\right] - \mathscr{T}_0\right| \ge \varrho\right\}\right|\right) = 0.$$

In this context, we state $\left[\widehat{\mathcal{F}}S^{\Phi}_{\mathfrak{t}_k}(\overline{\mathrm{N}}, \mu_k)\right] - \lim \mathscr{T}_k = \mathscr{T}_0$. Throughout this study, the class of all $\left[\widehat{\mathcal{F}}S^{\Phi}_{\mathfrak{t}_k}(\overline{\mathrm{N}}, \mu_k)\right]-$summable sequences is denoted by $\left[\widehat{\mathcal{F}}S^{\Phi}_{\mathfrak{t}_k}(\overline{\mathrm{N}}, \mu_k)\right]$, that is,

$$\left[\widehat{\mathcal{F}}S^{\Phi}_{\mathfrak{t}_k}(\overline{\mathrm{N}}, \mu_k)\right] = \left\{\mathscr{T} : \left[\widehat{\mathcal{F}}S^{\Phi}_{\mathfrak{t}_k}(\overline{\mathrm{N}}, \mu_k)\right] - \lim \mathscr{T}_k = \mathscr{T}_0 \text{ for some number } \mathscr{T}_0\right\}.$$

Definition 3. *A sequence $\mathscr{T} = (\mathscr{T}_k)$ is said to be weighted Fibonacci convergent (in short, $\left[\widehat{\mathcal{F}}(\overline{\mathrm{N}}, \mu_k)\right]-$convergent) to $\mathscr{T}_0$ if $\lim\limits_{k\to\infty}\mu_k\left|\widehat{\mathcal{F}_k}(\mathscr{T}) - \mathscr{T}_0\right| = 0$.*

Definition 4. *Let $\Phi \in \mathfrak{F}_{(un.)}$. Then, a sequence $\mathscr{T} = (\mathscr{T}_k)$ is defined to be weighted $\Phi-$Fibonacci statistically convergent (or concisely, $\left[\widehat{\mathcal{F}}S^{\Phi}(\overline{\mathrm{N}}, \mu_k)\right]-$convergent) to*
$\mathscr{T}_0$ if $\delta^{\Phi}_{\overline{N}}\left(\left\{k \in \mathbb{N} : \mu_k\left|\widehat{\mathcal{F}_k}(\mathscr{T}) - \mathscr{T}_0\right| \ge \varrho\right\}\right) = 0$ for every $\varrho > 0$, i.e.,

$$\lim_{\nu\to\infty}\frac{1}{\Phi(\mathcal{M}_\nu)}\Phi\left(\left|\left\{k \le \mathcal{M}_\nu : \mu_k\left|\widehat{\mathcal{F}_k}(\mathscr{T}) - \mathscr{T}_0\right| \ge \varrho\right\}\right|\right) = 0.$$

In this instance, we state $\left[\widehat{\mathcal{F}}S^{\Phi}(\overline{\mathrm{N}}, \mu_k)\right] - \lim \mathscr{T}_k = \mathscr{T}_0$. The set of all $\left[\widehat{\mathcal{F}}S^{\Phi}(\overline{\mathrm{N}}, \mu_k)\right]-$convergent sequences will be denoted by $\left[\widehat{\mathcal{F}}S^{\Phi}(\overline{\mathrm{N}}, \mu_k)\right]$, that is,

$$\left[\widehat{\mathcal{F}}S^{\Phi}(\overline{\mathrm{N}}, \mu_k)\right] = \left\{\mathscr{T} : \left[\widehat{\mathcal{F}}S^{\Phi}(\overline{\mathrm{N}}, \mu_k)\right] - \lim \mathscr{T}_k = \mathscr{T}_0 \text{ for some number } \mathscr{T}_0\right\}.$$

Theorem 1. *Let $\Phi \in \mathfrak{F}_{(un.)}$ and $\mathscr{T} = (\mathscr{T}_k)$ be a sequence. If $\mathscr{T}$ is $\left[\widehat{\mathcal{F}}S^{\Phi}(\overline{\mathrm{N}}, \mu_k)\right]-$convergent to $\mathscr{T}_0$ and there exists $U > 0$ such that $\mu_k\left|\widehat{\mathcal{F}_k}(\mathscr{T}) - \mathscr{T}_0\right| \le U$ for all $k \in \mathbb{N}$, then $\mathscr{T}$ is $\left[\widehat{\mathcal{F}}S^{\Phi}_{\mathfrak{t}_k}(\overline{\mathrm{N}}, \mu_k)\right]-$summable to $\mathscr{T}_0$; however, the converse is not generally correct.*

Proof. Define the sets $\Xi(\mathcal{M}_\nu, \varrho) = \left\{k \le \mathcal{M}_\nu : \mu_k\left|\widehat{\mathcal{F}_k}(\mathscr{T}) - \mathscr{T}_0\right| \ge \varrho\right\}$ and $\Xi^c(\mathcal{M}_\nu, \varrho) = \left\{k \le \mathcal{M}_\nu : \mu_k\left|\widehat{\mathcal{F}_k}(\mathscr{T}) - \mathscr{T}_0\right| < \varrho\right\}$. Since $\mathscr{T}$ is $\left[\widehat{\mathcal{F}}S^{\Phi}(\overline{\mathrm{N}}, \mu_k)\right]-$convergent to $\mathscr{T}_0$,

$$\delta^{\Phi}_{\overline{N}}\left(\left\{k \le \mathcal{M}_\nu : \mu_k\left|\widehat{\mathcal{F}_k}(\mathscr{T}) - \mathscr{T}_0\right| \ge \varrho\right\}\right) = 0.$$

So, we have

$$
\begin{aligned}
\left|\mathrm{t}_k\left[\widehat{\mathcal{F}}(\mathscr{T})\right]-\mathscr{T}_0\right| &= \left|\frac{1}{\mathcal{M}_\nu}\sum_{k=1}^{\nu}\mu_k\widehat{\mathcal{F}_k}(\mathscr{T})-\mathscr{T}_0\right| \\
&= \left|\frac{1}{\mathcal{M}_\nu}\sum_{\substack{k=1\\ k\in\Xi(\mathcal{M}_\nu,\varrho)}}^{\nu}\mu_k\left(\widehat{\mathcal{F}_k}(\mathscr{T})-\mathscr{T}_0\right)+\frac{1}{\mathcal{M}_\nu}\sum_{\substack{k=1\\ k\in\Xi^c(\mathcal{M}_\nu,\varrho)}}^{\nu}\mu_k\left(\widehat{\mathcal{F}_k}(\mathscr{T})-\mathscr{T}_0\right)\right| \\
&\le \left|\frac{1}{\mathcal{M}_\nu}\sum_{\substack{k=1\\ k\in\Xi(\mathcal{M}_\nu,\varrho)}}^{\nu}\mu_k\left(\widehat{\mathcal{F}_k}(\mathscr{T})-\mathscr{T}_0\right)\right|+\left|\frac{1}{\mathcal{M}_\nu}\sum_{\substack{k=1\\ k\in\Xi^c(\mathcal{M}_\nu,\varrho)}}^{\nu}\mu_k\left(\widehat{\mathcal{F}_k}(\mathscr{T})-\mathscr{T}_0\right)\right| \\
&\le \frac{1}{\mathcal{M}_\nu}\sum_{\substack{k=1\\ k\in\Xi(\mathcal{M}_\nu,\varrho)}}^{\nu}\mu_k\left|\widehat{\mathcal{F}_k}(\mathscr{T})-\mathscr{T}_0\right|+\frac{1}{\mathcal{M}_\nu}\sum_{\substack{k=1\\ k\in\Xi^c(\mathcal{M}_\nu,\varrho)}}^{\nu}\mu_k\left|\widehat{\mathcal{F}_k}(\mathscr{T})-\mathscr{T}_0\right| \\
&\le \frac{U}{\mathcal{M}_\nu}\left|\left\{k\le\mathcal{M}_\nu:\mu_k\left|\widehat{\mathcal{F}_k}(\mathscr{T})-\mathscr{T}_0\right|\ge\varrho\right\}\right|+\frac{1}{\mathcal{M}_\nu}\sum_{\substack{k=1\\ k\in\Xi^c(\mathcal{M}_\nu,\varrho)}}^{\nu}\varrho \\
&= \frac{U}{\mathcal{M}_\nu}\left|\left\{k\le\mathcal{M}_\nu:\mu_k\left|\widehat{\mathcal{F}_k}(\mathscr{T})-\mathscr{T}_0\right|\ge\varrho\right\}\right| \\
&\quad+\frac{\varrho}{\mathcal{M}_\nu}\left|\left\{k\le\mathcal{M}_\nu:\mu_k\left|\widehat{\mathcal{F}_k}(\mathscr{T})-\mathscr{T}_0\right|<\varrho\right\}\right|\to 0+\varrho
\end{aligned}
$$

as $\nu\to\infty$. That is, $\mathrm{t}_k\left[\widehat{\mathcal{F}}(\mathscr{T})\right]\to\mathscr{T}_0$. So, for any $\varrho>0$, there exists $n_0\in\mathbb{N}$ such that $\left|\mathrm{t}_k\left[\widehat{\mathcal{F}}(\mathscr{T})\right]-\mathscr{T}_0\right|<\varrho$ for $k\ge n_0$. It follows that the set $\left\{k\in\mathbb{N}:\left|\mathrm{t}_k\left[\widehat{\mathcal{F}}(\mathscr{T})\right]-\mathscr{T}_0\right|\ge\varrho\right\}$ is finite and so that

$$\lim_{\nu\to\infty}\frac{1}{\Phi(\nu)}\Phi\left(\left|\left\{k\le\nu:\left|\mathrm{t}_k\left[\widehat{\mathcal{F}}(\mathscr{T})\right]-\mathscr{T}_0\right|\ge\varrho\right\}\right|\right)=0.$$

Therefore, $\mathscr{T}$ is $\left[\widehat{\mathcal{F}}S^{\Phi}_{t_k}(\overline{N},\mu_k)\right]-$summable to $\mathscr{T}_0$. To illustrate the converse, we present the following example: $\square$

Example 1. *Define the sequence* $\mathscr{T}=(\mathscr{T}_k)$ *by*

$$\widehat{\mathcal{F}_k}(\mathscr{T})=\begin{cases}1 & \text{if } k=u^2-u,\ u^2-u+1,\dots,u^2-1;\\[2ex] -u & \text{if } k=u^2,\ u=2,3,4,\dots;\\[2ex] 0 & \text{otherwise.}\end{cases}$$

Then,

$$\mathrm{t}_\nu\left[\widehat{\mathcal{F}}(\mathscr{T})\right]=\frac{1}{1+\nu}\sum_{k=0}^{\nu}\widehat{\mathcal{F}_k}(\mathscr{T})=\begin{cases}\frac{r+1}{\nu+1} & \text{if } r=0,1,\dots,u-1;\ \nu=u^2-u+r;\ u=2,3,4,\dots;\\[3ex] 0 & \text{otherwise.}\end{cases}$$

Taking $\Phi(x)=x$, *then* $\lim\limits_{\nu\to\infty}\mathrm{t}_\nu\left[\widehat{\mathcal{F}}(\mathscr{T})\right]=0$ *and thus* $\left[\widehat{\mathcal{F}}S^{\Phi}_{\mathrm{t}_k}(\overline{N},\mu_k)\right]-\lim\mathscr{T}_k=0$, *i.e.,* $\mathscr{T}$ *is* $\left[\widehat{\mathcal{F}}S^{\Phi}_{t_k}(\overline{N},\mu_k)\right]-$*summable to* 0. *However,* $\mathscr{T}$ *is not* $\left[\widehat{\mathcal{F}}S^{\Phi}(\overline{N},\mu_k)\right]-$*convergent to 0.*

The following theorem provides a weighted $\Phi-$Fibonacci statistical formulation of the well-established APO property [27].

Theorem 2. *Let $\Phi \in \mathfrak{F}_{(un.)}$ and $(\Xi_i) \subset \mathcal{P}(\mathbb{N})$ be a sequence of sets such that $\delta_N^{\Phi}(\Xi_i) = 0$ for any $i \in \mathbb{N}$. Then, there is a sequence $(\Upsilon_i) \subset \mathcal{P}(\mathbb{N})$ such that $\Xi_i \backslash \Upsilon_i$ is finite and $\Upsilon_i \subset \Xi_i$ for every $i \in \mathbb{N}$, and $\delta_N^{\Phi}\left(\bigcup_{i\in\mathbb{N}} \Upsilon_i\right) = 0$.*

Proof. Consider the sequence of sets $\left(\Xi_i^{'}\right)$ given by $\Xi_1^{'} = \Xi_1, \Xi_2^{'} = \Xi_2 \backslash \Xi_1, \Xi_3^{'} = \Xi_3 \backslash (\Xi_1 \cup \Xi_2)$, $\ldots$. It is clear that these sets are mutually disjoint. We observe that $\delta_N^{\Phi}\left(\bigcup_{i=1}^{j} \Xi_i^{'}\right) = 0$ for every $j \in \mathbb{N}$. Therefore, there is a strictly increasing sequence (k_j) of natural numbers such that

$$\frac{1}{\Phi(\mathcal{M}_\nu)}\Phi\left(\sum_{i=1}^{\mathcal{M}_\nu}\sum_{m=1}^{j}\chi_{\Xi_m^{'}}(i)\right) \leq \frac{1}{j}$$

whenever $\mathcal{M}_\nu \geq k_j$. For every $\mathcal{M}_\nu \geq k_1$, let $q_\nu \in \mathbb{N}$ such that $k_{q_\nu} \leq \mathcal{M}_\nu < k_{q_\nu+1}$, it is obvious that $q_\nu \to \infty$ as $\nu \to \infty$. For every $m \in \mathbb{N}$, we define $\Upsilon_m^{'} = \Xi_m^{'} \backslash \{1, 2, \ldots, k_m\}$. Let us take $\Upsilon = \bigcup_{i\in\mathbb{N}} \Upsilon_i^{'}$. Then, we have

$$\begin{aligned}
\lim_\nu \sup \frac{1}{\Phi(\mathcal{M}_\nu)}\Phi\left(\sum_{i=1}^{\mathcal{M}_\nu}\chi_{\Upsilon}(i)\right) &= \lim_\nu \sup \frac{1}{\Phi(\mathcal{M}_\nu)}\Phi\left(\sum_{i=1}^{\mathcal{M}_\nu}\sum_{m=1}^{\infty}\chi_{\Upsilon_m^{'}}(i)\right)\\
&= \lim_\nu \sup \frac{1}{\Phi(\mathcal{M}_\nu)}\Phi\left(\sum_{i=1}^{\mathcal{M}_\nu}\sum_{m=1}^{q_\nu}\chi_{\Upsilon_m^{'}}(i)\right)\\
&\leq \lim_\nu \sup \frac{1}{\Phi(\mathcal{M}_\nu)}\Phi\left(\sum_{i=1}^{\mathcal{M}_\nu}\sum_{m=1}^{q_\nu}\chi_{\Xi_m^{'}}(i)\right)\\
&\leq \lim_\nu \sup \frac{1}{q_\nu} = 0,
\end{aligned}$$

where the second equality arises from the condition that if $m > q_\nu$, it follows that $\min \Upsilon_m^{'} > k_m \geq k_{q_\nu+1} > \nu$. So, $\chi_{\Upsilon_m^{'}}(i) = 0$ if $i \leq \mathcal{M}_\nu$ and so that $\sum_{m=q_\nu+1}^{\infty}\chi_{\Upsilon_m^{'}}(i)$. To summarize, we deduce that

$$\lim_\nu \frac{1}{\Phi(\mathcal{M}_\nu)}\Phi\left(\sum_{i=1}^{\mathcal{M}_\nu}\chi_{\Upsilon}(i)\right) = 0,$$

i.e., $\delta_N^{\Phi}(\Upsilon) = 0$. Now, take $\Upsilon_1^{'} = \Upsilon_1$, $\Upsilon_1^{'} \cup \Upsilon_2^{'} = \Upsilon_2$, $\Upsilon_1^{'} \cup \Upsilon_2^{'} \cup \Upsilon_3^{'} = \Upsilon_3, \ldots$. Therefore, $\delta_N^{\Phi}\left(\bigcup_{i\in\mathbb{N}} \Upsilon_i\right) = 0$ and it is clear that the family (Υ_i) fulfills all the requested property. □

Theorem 3. *Let $\Phi, \Psi \in \mathfrak{F}_{(un.)}$ and $\mathscr{T} = (\mathscr{T}_k)$ be a sequence.*

1. *If*

$$\lim_{x\to\infty}\frac{\Phi(x)}{\Psi(x)} > 0, \tag{1}$$

and $\mathscr{T}$ is $\left[\widehat{\mathcal{F}}S^{\Psi}(\overline{N}, \mu_k)\right]-$convergent, then $\mathscr{T}$ is $\left[\widehat{\mathcal{F}}S^{\Phi}(\overline{N}, \mu_k)\right]-$convergent and the inclusion may be strict, that is, $\left[\widehat{\mathcal{F}}S^{\Psi}(\overline{N}, \mu_k)\right] \subsetneqq \left[\widehat{\mathcal{F}}S^{\Phi}(\overline{N}, \mu_k)\right]$.

2. *If*

$$0 < \lim_{x\to\infty}\frac{\Phi(x)}{\Psi(x)} = \beta < \infty, \tag{2}$$

then $\mathscr{T}$ *is* $\left[\widehat{\mathcal{F}S}^{\Psi}(\overline{\mathrm{N}},\mu_k)\right]$*−convergent if and only if it is* $\left[\widehat{\mathcal{F}S}^{\Phi}(\overline{\mathrm{N}},\mu_k)\right]$*−convergent, that is,* $\left[\widehat{\mathcal{F}S}^{\Psi}(\overline{\mathrm{N}},\mu_k)\right]=\left[\widehat{\mathcal{F}S}^{\Phi}(\overline{\mathrm{N}},\mu_k)\right]$.

Proof. Part (1). Suppose that (1) holds and $\mathscr{T}$ is $\left[\widehat{\mathcal{F}S}^{\Psi}(\overline{\mathrm{N}},\mu_k)\right]$−convergent to $\mathscr{T}_0$. For any $\varrho>0$, there exists $n_0\in\mathbb{R}$ such that

$$(\alpha-\varrho)\,\Psi(x)<\Phi(x)<(\alpha+\varrho)\,\Psi(x)$$

for $x>n_0$ (we may choose $\varrho>0$ so small that $\alpha-\varrho>0$). So, we obtain the inequality $\Phi(x)<2\alpha\Psi(x)$ if $x>n_0$. Now, we get

$$\Psi\left(\left|\left\{k\le\mathcal{M}_\nu:\mu_k\left|\widehat{\mathcal{F}_k}(\mathscr{T})-\mathscr{T}_0\right|\ge\varrho\right\}\right|\right)\ge\frac{1}{2\alpha}\Phi\left(\left|\left\{k\le\mathcal{M}_\nu:\mu_k\left|\widehat{\mathcal{F}_k}(\mathscr{T})-\mathscr{T}_0\right|\ge\varrho\right\}\right|\right)$$

or

$$\frac{\Psi\left(\left|\left\{k\le\mathcal{M}_\nu:\mu_k\left|\widehat{\mathcal{F}_k}(\mathscr{T})-\mathscr{T}_0\right|\ge\varrho\right\}\right|\right)}{\Psi(\mathcal{M}_\nu)}\ge\frac{1}{2\alpha}\frac{\Phi\left(\left|\left\{k\le\mathcal{M}_\nu:\mu_k\left|\widehat{\mathcal{F}_k}(\mathscr{T})-\mathscr{T}_0\right|\ge\varrho\right\}\right|\right)}{\Phi(\mathcal{M}_\nu)}\frac{\Phi(\mathcal{M}_\nu)}{\Psi(\mathcal{M}_\nu)},\quad(3)$$

if $\left|\left\{k\le\mathcal{M}_\nu:\mu_k\left|\widehat{\mathcal{F}_k}(\mathscr{T})-\mathscr{T}_0\right|\ge\varrho\right\}\right|>n_0$ (so $\nu>n_0$). Taking the limits as $\nu\to\infty$ on both sides of the inequality (3), we get

$$\lim_{\nu\to\infty}\frac{1}{\Phi(\mathcal{M}_\nu)}\Phi\left(\left|\left\{k\le\mathcal{M}_\nu:\mu_k\left|\widehat{\mathcal{F}_k}(\mathscr{T})-\mathscr{T}_0\right|\ge\varrho\right\}\right|\right)=0.$$

Therefore, $\mathscr{T}$ is $\left[\widehat{\mathcal{F}S}^{\Phi}(\overline{\mathrm{N}},\mu_k)\right]$−convergent to $\mathscr{T}_0$. To show that

$$\left[\widehat{\mathcal{F}S}^{\Psi}(\overline{\mathrm{N}},\mu_k)\right]\subsetneq\left[\widehat{\mathcal{F}S}^{\Phi}(\overline{\mathrm{N}},\mu_k)\right].$$

Let us consider the sequence $\mathscr{T}$ defined by

$$\widehat{\mathcal{F}_k}(\mathscr{T})=\begin{cases}\frac{2k}{3k+1}, & \text{if } k\ne\nu^2,\\ k, & \text{otherwise.}\end{cases}\quad\nu\in\mathbb{N}.$$

Then, $\mathscr{T}$ is $\left[\widehat{\mathcal{F}S}^{\Phi}(\overline{\mathrm{N}},\mu_k)\right]$−convergent but it is not $\left[\widehat{\mathcal{F}S}^{\Psi}(\overline{\mathrm{N}},\mu_k)\right]$−convergent if we take $\Psi(x)=\log(x+1)$ and $\Phi(x)=\sqrt[3]{x}$.

Part (2). Given any $\varrho>0$. Then, we have the following equality:

$$\frac{\Psi\left(\left|\left\{k\le\mathcal{M}_\nu:\mu_k\left|\widehat{\mathcal{F}_k}(\mathscr{T})-\mathscr{T}_0\right|\ge\varrho\right\}\right|\right)}{\Psi(\mathcal{M}_\nu)}=\frac{\Psi\left(\left|\left\{k\le\mathcal{M}_\nu:\mu_k\left|\widehat{\mathcal{F}_k}(\mathscr{T})-\mathscr{T}_0\right|\ge\varrho\right\}\right|\right)}{\Phi\left(\left|\left\{k\le\mathcal{M}_\nu:\mu_k\left|\widehat{\mathcal{F}_k}(\mathscr{T})-\mathscr{T}_0\right|\ge\varrho\right\}\right|\right)}\cdot\frac{\Phi\left(\left|\left\{k\le\mathcal{M}_\nu:\mu_k\left|\widehat{\mathcal{F}_k}(\mathscr{T})-\mathscr{T}_0\right|\ge\varrho\right\}\right|\right)}{\Phi(\mathcal{M}_\nu)}\cdot\frac{\Phi(\mathcal{M}_\nu)}{\Psi(\mathcal{M}_\nu)}.\quad(4)$$

Since (2) holds, then $\lim\limits_{x\to\infty}\frac{\Psi(x)}{\Phi(x)}=\frac{1}{\beta}$. Using this fact from (4), we have

$$\lim_{\nu\to\infty}\frac{\Phi\left(\left|\left\{k\leq\mathcal{M}_\nu:\mu_k\left|\widehat{\mathcal{F}}_k(\mathscr{T})-\mathscr{T}_0\right|\geq\varrho\right\}\right|\right)}{\Phi(\mathcal{M}_\nu)}=0$$
$$\Leftrightarrow\lim_{\nu\to\infty}\frac{\Psi\left(\left|\left\{k\leq\mathcal{M}_\nu:\mu_k\left|\widehat{\mathcal{F}}_k(\mathscr{T})-\mathscr{T}_0\right|\geq\varrho\right\}\right|\right)}{\Psi(\mathcal{M}_\nu)}=0. \tag{5}$$

Therefore, $\mathscr{T}$ is $\left[\widehat{\mathcal{F}}S^{\Psi}(\overline{\mathrm{N}},\mu_k)\right]-$convergent if and only if it is $\left[\widehat{\mathcal{F}}S^{\Phi}(\overline{\mathrm{N}},\mu_k)\right]-$ convergent and so that $\left[\widehat{\mathcal{F}}S^{\Psi}(\overline{\mathrm{N}},\mu_k)\right]=\left[\widehat{\mathcal{F}}S^{\Phi}(\overline{\mathrm{N}},\mu_k)\right]$. □

Theorem 4. *Let $\Phi\in\mathfrak{F}_{(un.)}$ and $\mathscr{T}=(\mathscr{T}_k)$ be a sequence.*

1. *If $\mathscr{T}$ is $\left[\widehat{\mathcal{F}}S^{\Phi}(\overline{\mathrm{N}},\mu_k)\right]-$convergent, then it is $\left[\widehat{\mathcal{F}}S(\overline{\mathrm{N}},\mu_k)\right]-$ convergent.*
2. *If $\lim\limits_{x\to\infty}\frac{\Phi(x)}{x}$, then a sequence $\mathscr{T}$ is $\left[\widehat{\mathcal{F}}S^{\Phi}(\overline{\mathrm{N}},\mu_k)\right]-$ convergent if and only if it is $\left[\widehat{\mathcal{F}}S(\overline{\mathrm{N}},\mu_k)\right]-$ convergent.*

Proof. The proof of part (1) is derived directly from the first part of the proof of Theorem 3 by considering the specific case where $\Psi(x)=x$. Likewise, the proof of part (2) is derived by utilizing the second part of the proof of Theorem 3 with the assumption that $\Psi(x)=x$. □

Definition 5. *Let $\Phi\in\mathfrak{F}_{(un.)}$. Then, a sequence $\mathscr{T}=(\mathscr{T}_k)$ is defined to be weighted $\Phi-$Fibonacci statistically Cauchy (or briefly, $\left[\widehat{\mathcal{F}}S^{\Phi}(\overline{\mathrm{N}},\mu_k)\right]-$Cauchy) if there exists $J\in\mathbb{N}$ such that set $\left\{k\in\mathbb{N}:\mu_k\left|\widehat{\mathcal{F}}_k(\mathscr{T})-\widehat{\mathcal{F}}_J(\mathscr{T})\right|\geq\varrho\right\}$ has $\varphi-$weighted density zero for every $\varrho>0$, i.e.,*

$$\lim_{\nu\to\infty}\frac{1}{\Phi(\mathcal{M}_\nu)}\Phi\left(\left|\left\{k\leq\mathcal{M}_\nu:\mu_k\left|\widehat{\mathcal{F}}_k(\mathscr{T})-\widehat{\mathcal{F}}_J(\mathscr{T})\right|\geq\varrho\right\}\right|\right)=0.$$

The set of all $\left[\widehat{\mathcal{F}}S^{\Phi}(\overline{\mathrm{N}},\mu_k)\right]-$Cauchy sequences will be represented by $\left[\widehat{\mathcal{F}}SC^{\Phi}(\overline{\mathrm{N}},\mu_k)\right]$. That is,

$$\begin{aligned}&\left[\widehat{\mathcal{F}}SC^{\Phi}(\overline{\mathrm{N}},\mu_k)\right]\\&=\left\{\mathscr{T}:\exists J\in\mathbb{N},\ \lim_{\nu\to\infty}\frac{1}{\Phi(\mathcal{M}_\nu)}\Phi\left(\left|\left\{k\leq\mathcal{M}_\nu:\mu_k\left|\widehat{\mathcal{F}}_k(\mathscr{T})-\widehat{\mathcal{F}}_J(\mathscr{T})\right|\geq\varrho\right\}\right|\right)=0\right\}.\end{aligned}$$

Definition 6. *Let $\Phi\in\mathfrak{F}_{(un.)}$, Ξ be an infinite subset of $\mathbb{N}$ and $\mathscr{T}=(\mathscr{T}_k)$ be a sequence.*

1. *If $\delta_{\overline{N}}(\Xi)=0$, then $\mathscr{T}=(\mathscr{T}_k)_{k\in\Xi}$ is defined to be a weighted thin subsequence of $\mathscr{T}$; otherwise, it is defined to be a weighted non-thin subsequence of $\mathscr{T}$.*
2. *If $\delta_{\overline{N}}^{\Phi}(\Xi)=0$, then $\mathscr{T}=(\mathscr{T}_k)_{k\in\Xi}$ is defined to be a $\Phi-$weighted thin subsequence of $\mathscr{T}$; otherwise, it is defined to be a $\Phi-$weighted non-thin subsequence of $\mathscr{T}$.*

Theorem 5. *Let $\Phi\in\mathfrak{F}_{(un.)}$ and $\mathscr{T}=(\mathscr{T}_k)$ be a $\left[\widehat{\mathcal{F}}S^{\Phi}(\overline{\mathrm{N}},\mu_k)\right]-$ Cauchy sequence which has a $\Phi-$ weighted convergent non-thin subsequence. Then, $\mathscr{T}$ is $\left[\widehat{\mathcal{F}}S^{\Phi}(\overline{\mathrm{N}},\mu_k)\right]-$ convergent.*

Proof. Let Ξ be the set of indices of the weighted $\Phi-$non-thin subsequence of $\mathscr{T}$. Since $\mathscr{T}$ is a $\left[\widehat{\mathcal{F}}S^{\Phi}(\overline{\mathrm{N}},\mu_k)\right]-$Cauchy, for each $\varrho>0$, there exists $J\in\mathbb{N}$ such that

$$\delta_{\overline{N}}^{\Phi}\left(\left\{k\in\mathbb{N}:\mu_k\left|\widehat{\mathcal{F}}_k(\mathscr{T})-\widehat{\mathcal{F}}_J(\mathscr{T})\right|\geq\frac{\varrho}{3}\right\}\right),$$

that is,

$$\lim_{\nu\to\infty}\frac{1}{\Phi(\mathcal{M}_\nu)}\Phi\Big(\Big|\Big\{k\leq\mathcal{M}_\nu:\mu_k\Big|\widehat{\mathcal{F}}_k(\mathscr{T})-\widehat{\mathcal{F}}_J(\mathscr{T})\Big|\geq\frac{\varrho}{3}\Big\}\Big|\Big)=0.$$

We set $\Xi^*=\Big\{k\in\mathbb{N}:\mu_k\Big|\widehat{\mathcal{F}}_k(\mathscr{T})-\widehat{\mathcal{F}}_J(\mathscr{T})\Big|\geq\frac{\varrho}{3}\Big\}$. Then, $(\mathbb{N}\backslash\Xi^*)\cap\Xi$ in infinite; otherwise, we can write $\Xi=(\Xi\cap\Xi^*)\cup((\mathbb{N}\backslash\Xi^*)\cap\Xi)$. Since $(\Xi\cap\Xi^*)\subset\Xi^*$, then we have $\delta_{\overline{N}}^{\Phi}(\Xi\cap\Xi^*)\leq\delta_{\overline{N}}^{\Phi}(\Xi^*)$. This means $\delta_{\overline{N}}^{\Phi}(\Xi\cap\Xi^*)=0$ since $\delta_{\overline{N}}^{\Phi}(\Xi^*)=0$. And, $\delta_{\overline{N}}^{\Phi}((\mathbb{N}\backslash\Xi^*)\cap\Xi)\neq0$ because Ξ would have zero weighted $\Phi-$density, a contradiction. So, for some $i\in(\mathbb{N}\backslash\Xi^*)\cap\Xi$, $\mu_i\Big|\widehat{\mathcal{F}}_i(\mathscr{T})-\mathscr{T}_0\Big|\leq\frac{\varrho}{3}$ and $\mu_i\Big|\widehat{\mathcal{F}}_i(\mathscr{T})-\widehat{\mathcal{F}}_J(\mathscr{T})\Big|\leq\frac{\varrho}{3}$. Since

$$\Big|\widehat{\mathcal{F}}_J(\mathscr{T})-\mathscr{T}_0\Big|\leq\Big|\widehat{\mathcal{F}}_J(\mathscr{T})-\widehat{\mathcal{F}}_i(\mathscr{T})\Big|+\Big|\widehat{\mathcal{F}}_i(\mathscr{T})-\mathscr{T}_0\Big|,$$

we have $\Big\{k\in\mathbb{N}:\mu_k\Big|\widehat{\mathcal{F}}_k(\mathscr{T})-\mathscr{T}_0\Big|>\varrho\Big\}\subset\Big\{k\in\mathbb{N}:\mu_k\Big|\widehat{\mathcal{F}}_k(\mathscr{T})-\widehat{\mathcal{F}}_J(\mathscr{T})\Big|>\frac{\varrho}{3}\Big\}$. This implies that

$$\delta_{\overline{N}}^{\Phi}\Big(\Big\{k\in\mathbb{N}:\mu_k\Big|\widehat{\mathcal{F}}_k(\mathscr{T})-\mathscr{T}_0\Big|>\varrho\Big\}\Big)\leq\delta_{\overline{N}}^{\Phi}\Big(\Big\{k\in\mathbb{N}:\mu_k\Big|\widehat{\mathcal{F}}_k(\mathscr{T})-\widehat{\mathcal{F}}_J(\mathscr{T})\Big|>\frac{\varrho}{3}\Big\}\Big).$$

Since $\delta_{\overline{N}}^{\Phi}\Big(\Big\{k\in\mathbb{N}:\mu_k\Big|\widehat{\mathcal{F}}_k(\mathscr{T})-\widehat{\mathcal{F}}_J(\mathscr{T})\Big|>\frac{\varrho}{3}\Big\}\Big)$, then $\delta_{\overline{N}}^{\Phi}\Big(\Big\{k\in\mathbb{N}:\mu_k\Big|\widehat{\mathcal{F}}_k(\mathscr{T})-\mathscr{T}_0\Big|>\varrho\Big\}\Big)=0$ and the sequence $\mathscr{T}$ is $\Big[\widehat{\mathcal{F}}S^{\Phi}(\overline{N},\mu_k)\Big]-$convergent to $\mathscr{T}_0$. □

Theorem 6. *Let $\Phi\in\mathfrak{F}_{(un.)}$ and $\mathscr{T}=(\mathscr{T}_k)$ be a sequence. If there exists an $\Big[\widehat{\mathcal{F}}S^{\Phi}(\overline{N},\mu_k)\Big]-$convergent sequence $\zeta=(\zeta_k)$ such that $\delta_{\overline{N}}^{\Phi}\Big(\Big\{k\in\mathbb{N}:\widehat{\mathcal{F}}_k(\mathscr{T})\neq\widehat{\mathcal{F}}_k(\zeta)\Big\}\Big)=0$. Then, $\mathscr{T}$ is $\Big[\widehat{\mathcal{F}}S^{\Phi}(\overline{N},\mu_k)\Big]-$ convergent.*

Proof. Suppose that ζ is $\Big[\widehat{\mathcal{F}}S^{\Phi}(\overline{N},\mu_k)\Big]-$convergent to $\mathscr{T}_0$. For any $\varrho>0$,

$$\Big\{k\in\mathbb{N}:\mu_k\Big|\widehat{\mathcal{F}}_k(\mathscr{T})-\mathscr{T}_0\Big|>\varrho\Big\}\subset\Big\{k\in\mathbb{N}:\widehat{\mathcal{F}}_k(\mathscr{T})\neq\widehat{\mathcal{F}}_k(\zeta)\Big\}\cup\Big\{k\in\mathbb{N}:\mu_k\Big|\widehat{\mathcal{F}}_k(\mathscr{T})-\mathscr{T}_0\Big|<\varrho\Big\}.$$

Since ζ is $\Big[\widehat{\mathcal{F}}S^{\Phi}(\overline{N},\mu_k)\Big]-$convergent to $\mathscr{T}_0$, this set comprises a fixed number of integers, which we denote as $e=e(\varrho)$. Since $\Phi\in\mathfrak{F}_{(un.)}$, then

$$\begin{aligned}&\frac{1}{\Phi(\mathcal{M}_\nu)}\Phi\Big(\Big|\Big\{k\leq\mathcal{M}_\nu:\mu_k\Big|\widehat{\mathcal{F}}_k(\mathscr{T})-\mathscr{T}_0\Big|>\varrho\Big\}\Big|\Big)\\&\qquad\leq\frac{1}{\Phi(\mathcal{M}_\nu)}\Phi\Big(\Big|\Big\{k\leq\mathcal{M}_\nu:\widehat{\mathcal{F}}_k(\mathscr{T})\neq\widehat{\mathcal{F}}_k(\zeta)\Big\}\Big|\Big)+\frac{\Phi(e)}{\Phi(\mathcal{M}_\nu)}.\end{aligned}\tag{6}$$

Since $\frac{1}{\Phi(\mathcal{M}_\nu)}\Phi\Big(\Big|\Big\{k\leq\mathcal{M}_\nu:\widehat{\mathcal{F}}_k(\mathscr{T})\neq\widehat{\mathcal{F}}_k(\zeta)\Big\}\Big|\Big)\to0$ and $\frac{\Phi(e)}{\Phi(\mathcal{M}_\nu)}\to0$ and $\nu\to\infty$, then $\mathscr{T}$ is $\Big[\widehat{\mathcal{F}}S^{\Phi}(\overline{N},\mu_k)\Big]-$ convergent to $\mathscr{T}_0$. □

Theorem 7. *Let $\Phi,\Psi\in\mathfrak{F}_{(un.)}$.*

1. *If (1) holds, then a sequence $\mathscr{T}=(\mathscr{T}_k)$ is $\Big[\widehat{\mathcal{F}}S^{\Psi}(\overline{N},\mu_k)\Big]-$ Cauchy if it is $\Big[\widehat{\mathcal{F}}S^{\Phi}(\overline{N},\mu_k)\Big]-$Cauchy.*
2. *If (2) holds, then a sequence $\mathscr{T}=(\mathscr{T}_k)$ is $\Big[\widehat{\mathcal{F}}S^{\Psi}(\overline{N},\mu_k)\Big]-$ Cauchy if and only if it is $\Big[\widehat{\mathcal{F}}S^{\Phi}(\overline{N},\mu_k)\Big]-$ Cauchy.*

Proof. The proof adopts the same techniques as shown in Theorem 2. □

Based on Theorem 7, we derive the following result.

Corollary 1. *Let* $\Phi \in \mathfrak{F}_{(un.)}$.

1. *If* $\mathscr{T}$ *is* $\left[\widehat{\mathcal{F}}S^{\Phi}\left(\overline{\mathrm{N}}, \mu_k\right)\right]-$ *Cauchy, then it is* $\left[\widehat{\mathcal{F}}S\left(\overline{\mathrm{N}}, \mu_k\right)\right]-$ *Cauchy.*
2. *If* $\lim\limits_{x\to\infty} \frac{\Phi(x)}{x}$, *then a sequence* $\mathscr{T}$ *is* $\left[\widehat{\mathcal{F}}S^{\Phi}\left(\overline{\mathrm{N}}, \mu_k\right)\right]-$ *Cauchy if and only if it is* $\left[\widehat{\mathcal{F}}S\left(\overline{\mathrm{N}}, \mu_k\right)\right]-$ *Cauchy.*

3. Weighted Fibonacci $\Phi-$Statistical Cluster Points

Definition 7. *Let* $\Phi \in \mathfrak{F}_{(un.)}$ *and* $\mathscr{T} = (\mathscr{T}_k)$ *be a sequence.*

1. *A number* $\mathscr{T}_0$ *is defined to be a weighted Fibonacci limit point of* $\mathscr{T}$ *if there exists a subsequence of* $\mathscr{T}$ *which is weighted Fibonacci convergent to* $\mathscr{T}_0$. *In this context,* $\widehat{\mathcal{F}}\left[L_{\overline{N}}(\mathscr{T})\right]$ *represents the set of all weighted Fibonacci limit points of* $\mathscr{T}$.
2. *A number* $\mathscr{T}_0$ *is defined to be a weighted Fibonacci statistical limit point of* $\mathscr{T}$ *if there exists a weighted non-thin subsequence of* $\mathscr{T}$ *which is weighted Fibonacci convergent to* $\mathscr{T}_0$. $\widehat{\mathcal{F}}S\left[\Lambda_{\overline{N}}(\mathscr{T})\right]$ *represents the set of all weighted statistical limit points of* $\mathscr{T}$.
3. *A number* $\mathscr{T}_0$ *is defined to be a weighted Fibonacci* $\Phi-$*statistical limit point of* $\mathscr{T}$ *if there exists a weighted* $\Phi-$*non-thin subsequence of* $\mathscr{T}$ *which is weighted Fibonacci convergent to* $\mathscr{T}_0$. $\widehat{\mathcal{F}}S\left[\Lambda^{\Phi}_{\overline{N}}(\mathscr{T})\right]$ *represents the set of all weighted* $\Phi-$*statistical limit points of* $\mathscr{T}$.
4. *A number* $\mathscr{T}_0$ *is defined to be a weighted Fibonacci statistical cluster point of* $\mathscr{T}$ *if for every* $\varrho > 0$, $\delta_{\overline{N}}\left(\left\{k \in \mathbb{N} : \mu_k\left|\widehat{\mathcal{F}_k}(\mathscr{T}) - \mathscr{T}_0\right| < \varrho\right\}\right) \neq 0$. $\widehat{\mathcal{F}}S\left[\Gamma_{\overline{N}}(\mathscr{T})\right]$ *represents the set of all weighted Fibonacci statistical cluster points of* $\mathscr{T}$.
5. *A number* $\mathscr{T}_0$ *is defined to be a weighted Fibonacci* $\Phi-$*statistical cluster point of* $\mathscr{T}$ *if for every* $\varrho > 0$, $\delta^{\Phi}_{\overline{N}}\left(\left\{k \in \mathbb{N} : \mu_k\left|\widehat{\mathcal{F}_k}(\mathscr{T}) - \mathscr{T}_0\right| < \varrho\right\}\right) \neq 0$. $\widehat{\mathcal{F}}S\left[\Gamma^{\Phi}_{\overline{N}}(\mathscr{T})\right]$ *represents the set of all weighted Fibonacci* $\Phi-$*statistical cluster points of* $\mathscr{T}$.

Theorem 8. *Let* $\Phi \in \mathfrak{F}_{(un.)}$ *and* $\mathscr{T} = (\mathscr{T}_k)$ *be a sequence.*

1. $\widehat{\mathcal{F}}S\left[\Lambda_{\overline{N}}(\mathscr{T})\right] \subset \widehat{\mathcal{F}}S\left[\Lambda^{\Phi}_{\overline{N}}(\mathscr{T})\right]$.
2. $\widehat{\mathcal{F}}S\left[\Gamma_{\overline{N}}(\mathscr{T})\right] \subset \widehat{\mathcal{F}}S\left[\Gamma^{\Phi}_{\overline{N}}(\mathscr{T})\right]$.
3. $\widehat{\mathcal{F}}S\left[\Lambda^{\Phi}_{\overline{N}}(\mathscr{T})\right] \subset \widehat{\mathcal{F}}S\left[\Gamma^{\Phi}_{\overline{N}}(\mathscr{T})\right]$.
4. $\widehat{\mathcal{F}}S\left[\Lambda_{\overline{N}}(\mathscr{T})\right] \subset \widehat{\mathcal{F}}S\left[\Gamma_{\overline{N}}(\mathscr{T})\right]$.
5. $\widehat{\mathcal{F}}S\left[\Gamma^{\Phi}_{\overline{N}}(\mathscr{T})\right] \subset \widehat{\mathcal{F}}\left[L_{\overline{N}}(\mathscr{T})\right]$.

Proof. Since $\left[\widehat{\mathcal{F}}S^{\Phi}\left(\overline{N}, \mu_k\right)\right]-$convergence implies $\left[\widehat{\mathcal{F}}S\left(\overline{N}, \mu_k\right)\right]-$convergence by the first part of Theorem 4, then the proofs of part (1) and part (2) follow immediately.

(3) To prove $\widehat{\mathcal{F}}S\left[\Lambda^{\Phi}_{\overline{N}}(\mathscr{T})\right] \subset \widehat{\mathcal{F}}S\left[\Gamma^{\Phi}_{\overline{N}}(\mathscr{T})\right]$, let $\mathscr{T}_0 \in \widehat{\mathcal{F}}S\left[\Lambda^{\Phi}_{\overline{N}}(\mathscr{T})\right]$. Then, there exists an infinite $\Upsilon \subset \mathbb{N}$ such that $\delta^{\Phi}_{\overline{N}}(\Upsilon) \neq 0$ and $\lim\limits_{k\in\Upsilon} \mu_k|\mathscr{T}_k - \mathscr{T}_0| = 0$. For any $\varrho > 0$, the set $\Xi(\varrho) = \left\{k \in \mathbb{N} : \mu_k\left|\widehat{\mathcal{F}_k}(\mathscr{T}) - \mathscr{T}_0\right| \geq \varrho\right\}$ is finite. This implies that

$$\delta^{\Phi}_{\overline{N}}(\Upsilon\backslash\Xi(\varrho)) \geq \delta^{\Phi}_{\overline{N}}(\Upsilon) - \delta^{\Phi}_{\overline{N}}(\Xi(\varrho)) = \delta^{\Phi}_{\overline{N}}(\Upsilon) \neq 0.$$

Since Φ is increasing and $\Upsilon\backslash\Xi(\varrho) \subset \left\{k \in \mathbb{N} : \mu_k\left|\widehat{\mathcal{F}_k}(\mathscr{T}) - \mathscr{T}_0\right| < \varrho\right\}$, then

$$\delta^{\Phi}_{\overline{N}}\left(\left\{k \in \mathbb{N} : \mu_k\left|\widehat{\mathcal{F}_k}(\mathscr{T}) - \mathscr{T}_0\right| < \varrho\right\}\right) \geq \delta^{\Phi}_{\overline{N}}(\Upsilon\backslash\Xi(\varrho)) \neq 0.$$

Thus, $\mathscr{T}_0 \in \widehat{\mathcal{F}}S\left[\Gamma^{\Phi}_{\overline{N}}(\mathscr{T})\right]$.

(4) The proof of this part is obtained in the case when $\Phi(x) = x$ from part (3).

(5) To show that $\widehat{\mathcal{F}S}\Big[\Gamma^{\Phi}_{\overline{N}}(\mathscr{T})\Big] \subset \widehat{\mathcal{F}}\Big[L_{\overline{N}}(\mathscr{T})\Big]$. Let $\mathscr{T}_0 \in \widehat{\mathcal{F}S}\Big[\Gamma^{\Phi}_{\overline{N}}(\mathscr{T})\Big]$. So, for every $i \in \mathbb{N}$,

$$\delta^{\Phi}_{\overline{N}}\left(\left\{k \in \mathbb{N} : \mu_k\left|\widehat{\mathcal{F}_k}(\mathscr{T}) - \mathscr{T}_0\right| < \frac{1}{i}\right\}\right) \neq 0.$$

Consider $\Xi(i) = \left\{k \in \mathbb{N} : \mu_k\left|\widehat{\mathcal{F}_k}(\mathscr{T}) - \mathscr{T}_0\right| < \frac{1}{i}\right\}$ which is an infinite set of natural numbers and $\Xi(i+1) \subset \Xi(i)$ for each $i \in \mathbb{N}$. Now, we can take an increasing sequence $k_1 < k_2 < \ldots$ with $k_i \in \Xi(i)$. If $\nu \in \mathbb{N}$ and $\nu \geq i$, then

$$\mu_{k_\nu}\left|\widehat{\mathcal{F}_{k_\nu}}(\mathscr{T}) - \mathscr{T}_0\right| < \frac{1}{\nu} \leq \frac{1}{i}.$$

So, $(\mathscr{T}_{k_\nu})_\nu$ is a weighted Fibonacci convergent subsequence of $\mathscr{T}$. Therefore, $\widehat{\mathcal{F}S}\Big[\Gamma^{\Phi}_{\overline{N}}(\mathscr{T})\Big] \subset \widehat{\mathcal{F}}\Big[L_{\overline{N}}(\mathscr{T})\Big]$. □

Example 2. *Consider the set* $Y = \{1, 4, 9, 16, \ldots\}$, $(\mathcal{M}_\nu) = (1)$, $Y(\mathcal{M}_\nu) = \{\lambda \leq \mathcal{M}_\nu : \lambda \in Y\}$ *and* $\Phi(x) = \log(x+1)$. *Then,* $\widehat{\mathcal{F}S}\Big[\Lambda_{\overline{N}}\Big(\chi_{\mathbb{N}\setminus Y(\mathcal{M}_\nu)}\Big)\Big] = \{1\}$ *and* $\widehat{\mathcal{F}S}\Big[\Lambda^{\Phi}_{\overline{N}}\Big(\chi_{\mathbb{N}\setminus Y(\mathcal{M}_\nu)}\Big)\Big] = \{0, 1\}$ *since* $\delta_{\overline{N}}(Y) = 0$, $\delta^{\Phi}_{\overline{N}}(Y) = \frac{1}{2}$ *and* $\delta^{\Phi}_{\overline{N}}(\mathbb{N}\setminus Y) = 1$. *Additionally, we have*

$$\widehat{\mathcal{F}S}\Big[\Gamma_{\overline{N}}\Big(\chi_{\mathbb{N}\setminus Y(\mathcal{M}_\nu)}\Big)\Big] = \{1\} \quad \textit{and} \quad \widehat{\mathcal{F}S}\Big[\Gamma^{\Phi}_{\overline{N}}\Big(\chi_{\mathbb{N}\setminus Y(\mathcal{M}_\nu)}\Big)\Big] = \{0, 1\}.$$

That is,

$$\widehat{\mathcal{F}S}\Big[\Lambda_{\overline{N}}\Big(\chi_{\mathbb{N}\setminus Y(\mathcal{M}_\nu)}\Big)\Big] \subsetneqq \widehat{\mathcal{F}S}\Big[\Lambda^{\Phi}_{\overline{N}}\Big(\chi_{\mathbb{N}\setminus Y(\mathcal{M}_\nu)}\Big)\Big]$$

and

$$\widehat{\mathcal{F}S}\Big[\Gamma_{\overline{N}}\Big(\chi_{\mathbb{N}\setminus Y(\mathcal{M}_\nu)}\Big)\Big] \subsetneqq \widehat{\mathcal{F}S}\Big[\Gamma^{\Phi}_{\overline{N}}\Big(\chi_{\mathbb{N}\setminus Y(\mathcal{M}_\nu)}\Big)\Big].$$

Let $\Xi = \mathbb{N}$ *such that*

$$|\Xi(\mathcal{M}_\nu)| = \left\lfloor \mathcal{M}_\nu^{\frac{1}{\sqrt{\log(\nu+1)}}} \right\rfloor$$

and let (q_ν) *be a sequence such that* $q_\nu \in \mathbb{Q}$ *for all* $\nu \in \mathbb{N}$, *which is defined by*

$$\mathscr{T}_\nu = \begin{cases} q_\nu, & \textit{if } \nu \in \Xi, \\ \nu, & \textit{if } \nu \in \mathbb{N}\setminus\Xi\,. \end{cases}$$

Since $\delta^{\Phi}_{\overline{N}}(\Xi) = 0$, *then* $\widehat{\mathcal{F}S}\Big[\Gamma^{\Phi}_{\overline{N}}(\mathscr{T})\Big] = \varnothing$ *while the fact that* (q_ν) *is dense in* $\mathbb{R}$ *implies that* $\widehat{\mathcal{F}S}\Big[L_{\overline{N}}(\mathscr{T})\Big] = \mathbb{R}$.

The subsequent below of relations, which is derived from the results of Theorem 8 and Example 2, provides a detailed examination and illustration of the relationships between the various sets discussed.

Corollary 2. *Let* $\Phi \in \mathfrak{F}_{(un.)}$ *and* $\mathscr{T} = (\mathscr{T}_k)$ *be a sequence. Then,*

$$\begin{array}{ccccc} \widehat{\mathcal{F}S}\Big[\Lambda_{\overline{N}}(\mathscr{T})\Big] & \subsetneqq & \widehat{\mathcal{F}S}\Big[\Lambda^{\Phi}_{\overline{N}}(\mathscr{T})\Big] & & \\ \not\shortparallel\cap & & \not\shortparallel\cap & & \\ \widehat{\mathcal{F}S}\Big[\Gamma_{\overline{N}}(\mathscr{T})\Big] & \subsetneqq & \widehat{\mathcal{F}S}\Big[\Gamma^{\Phi}_{\overline{N}}(\mathscr{T})\Big] & \subsetneqq & \widehat{\mathcal{F}}\Big[L_{\overline{N}}(\mathscr{T})\Big]. \end{array}$$

Theorem 9. *Let $\Phi \in \mathfrak{F}_{(un.)}$ and $\mathscr{T} = (\mathscr{T}_k)$ be a sequence. Then,*

$$\widehat{\mathcal{F}}\Big[L_{\overline{N}}(\mathscr{T})\Big] = \bigcup\Big\{\widehat{\mathcal{F}}S\Big[\Gamma^{\Phi}_{N}(\mathscr{T})\Big] : \textit{for all } \Phi \in \mathfrak{F}_{(un.)}\Big\}.$$

Proof. Since $\widehat{\mathcal{F}}S\Big[\Gamma^{\Phi}_{N}(\mathscr{T})\Big] \subset \widehat{\mathcal{F}}\Big[L_{\overline{N}}(\mathscr{T})\Big]$ for every modulus Φ by Theorem 8, it remains to show that

$$\widehat{\mathcal{F}}\Big[L_{\overline{N}}(\mathscr{T})\Big] = \bigcup\Big\{\widehat{\mathcal{F}}S\Big[\Gamma^{\Phi}_{N}(\mathscr{T})\Big] : \text{for all } \Phi \in \mathfrak{F}_{(un.)}\Big\}.$$

Let $\mathscr{T}_0 \in \widehat{\mathcal{F}}\Big[L_{\overline{N}}(\mathscr{T})\Big]$. Assume that $\mathscr{T}_0 \notin \widehat{\mathcal{F}}S\Big[\Gamma^{\Phi}_{N}(\mathscr{T})\Big]$ for every $\Phi \in \mathfrak{F}_{(un.)}$. Then, there is $\varrho_{\Phi} > 0$ such that $\delta^{\Phi}_{N}\Big(\Big\{k \in \mathbb{N} : \mu_k\Big|\widehat{\mathcal{F}_k}(\mathscr{T}) - \mathscr{T}_0\Big| < \varrho_{\Phi}\Big\}\Big) = 0$ for every $\Phi \in \mathfrak{F}_{(un.)}$. From another perspective, since $\mathscr{T}_0 \in \widehat{\mathcal{F}}\Big[L_{\overline{N}}(\mathscr{T})\Big]$, there is an infinite subset Ξ of $\mathbb{N}$ such that $\lim\limits_{k\in\Xi} \mu_k|\mathscr{T}_k - \mathscr{T}_0| = 0$. This means that for every $\varrho > 0$, the set $\Upsilon_{\varrho} = \Big\{k \in \Xi : \mu_k\Big|\widehat{\mathcal{F}_k}(\mathscr{T}) - \mathscr{T}_0\Big| \geq \varrho\Big\}$ is finite.

Next, we need to show that there exists a function Ψ such that $\delta^{\Psi}_{N}(\Xi) = 1$. Define a new modulus function $\Psi' : \mathbb{N} \to \mathbb{N}$ such that $\Psi'(1) = 1$, $\Psi'(2) = \min\{\nu : |\Xi_{\mathcal{M}_\nu}| = 2\}$, and for $k \geq 2$, $\Psi'(k+1) = \max\Big\{\min\Big\{\nu : |\Xi_{\mathcal{M}_\nu}| = 1 + \Psi'(k)\Big\}, 2\Psi'(k) - \Psi'(k-1)\Big\}$, where $|\Xi_{\mathcal{M}_\nu}| = |\{\lambda \leq \mathcal{M}_\nu : \lambda \in \Xi\}|$. According to the construction, Ψ' is increasing and

$$\Big|\Xi_{\Psi'(k+1)}\Big| = \Big|\Big\{\lambda \leq \Psi'(k+1) : \lambda \in \Xi\Big\}\Big| \geq 1 + \Psi'(k).$$

Proceed to define a function $\Psi : [0,\infty) \to [0,\infty)$ by $\Psi(0) = 0$. Moreover, for $\nu \in \mathbb{N}$, we set $\Psi\Big(\Psi'(\nu)\Big) = \nu$. So, for $k > \Psi'(2)$, there is $\nu \in \mathbb{N}$ with $\Psi'(\mathcal{M}_\nu + 1) \leq k \leq \Psi'(\mathcal{M}_\nu + 2)$. This implies that

$$\frac{\Psi(|\Xi_{\mathcal{M}_\nu}|)}{\Psi(\mathcal{M}_\nu)} \geq \frac{\Psi\Big(\Big|\Xi_{\Psi'(\mathcal{M}_\nu+1)}\Big|\Big)}{\Psi(\Psi'(\mathcal{M}_\nu+2))} \geq \frac{\Psi\Big(\Psi'(\mathcal{M}_\nu)+1\Big)}{\Psi(\Psi'(\mathcal{M}_\nu+2))} \geq \frac{\Psi\Big(\Psi'(\mathcal{M}_\nu)\Big)}{\Psi(\Psi'(\mathcal{M}_\nu+2))} = \frac{\mathcal{M}_\nu}{\mathcal{M}_\nu+2} \to 1$$

as $\nu \to \infty$. It fulfills that if the set $\Xi \subset \mathbb{N}$ is infinite, we can choose $\Psi \in \mathfrak{F}_{(un.)}$ such that $\delta^{\Psi}_{N}(\Xi) = 1$. Therefore, $\delta^{\Psi}_{N}(\Xi \backslash \Upsilon_{\varrho_\Psi}) = 1$ and

$$\Xi\backslash\Upsilon_{\varrho_\Psi} = \Big\{\nu \in \Xi : \mathcal{M}_\nu\Big|\widehat{\mathcal{F}_\nu}(\mathscr{T}) - \mathscr{T}_0\Big| < \varrho_\Psi\Big\} \subset \Big\{\nu \in \mathbb{N} : \mathcal{M}_\nu\Big|\widehat{\mathcal{F}_\nu}(\mathscr{T}) - \mathscr{T}_0\Big| < \varrho_\Psi\Big\}.$$

This leads to a contradiction since $\delta^{\Psi}_{N}\Big(\Big\{\nu \in \mathbb{N} : \mathcal{M}_\nu\Big|\widehat{\mathcal{F}_\nu}(\mathscr{T}) - \mathscr{T}_0\Big| < \varrho_\Psi\Big\}\Big) = 0$. □

Theorem 10. *Let $\Phi \in \mathfrak{F}_{(un.)}$, and let $\mathscr{T} = (\mathscr{T}_k)$ and $\zeta = (\zeta_k)$ be two sequences of numbers.*

If $\delta^{\Phi}_{N}\Big(\Big\{\nu \in \mathbb{N} : \widehat{\mathcal{F}_\nu}(\mathscr{T}) \neq \widehat{\mathcal{F}_\nu}(\zeta)\Big\}\Big) = 0$, then $\widehat{\mathcal{F}}S\Big[\Lambda^{\Phi}_{N}(\mathscr{T})\Big] = \widehat{\mathcal{F}}S\Big[\Lambda^{\Phi}_{N}(\zeta)\Big]$ and $\widehat{\mathcal{F}}S\Big[\Gamma^{\Phi}_{N}(\mathscr{T})\Big] = \widehat{\mathcal{F}}S\Big[\Gamma^{\Phi}_{N}(\zeta)\Big]$.

Proof. Let $\mathscr{T}_0 \in \widehat{\mathcal{F}}S\Big[\Lambda^{\Phi}_{N}(\mathscr{T})\Big]$. Then, there exists an infinite $\Upsilon \subset \mathbb{N}$ such that $\delta^{\Phi}_{N}(\Upsilon) \neq 0$ and $\lim\limits_{k\in\Upsilon} \mu_k|\mathscr{T}_k - \mathscr{T}_0| = 0$. Let us take $\Xi = \Big\{\nu \in \mathbb{N} : \widehat{\mathcal{F}_\nu}(\mathscr{T}) \neq \widehat{\mathcal{F}_\nu}(\zeta)\Big\}$, where $\delta^{\Phi}_{N}(\Xi) = 0$. Consider a subsequence $(\zeta_k)_{k\in\Upsilon\backslash\Xi'}$ of ζ which is weighted Fibonacci convergent to $\mathscr{T}_0$ and is weighted $\Phi-$non-thin. Indeed, if $\delta^{\Phi}_{N}(\Upsilon\backslash\Xi) = 0$, then

$$\delta^{\Phi}_{N}(\Xi \cup \Upsilon) = \delta^{\Phi}_{N}(\Xi \cup (\Upsilon\backslash\Xi)) \leq \delta^{\Phi}_{N}(\Xi) + \delta^{\Phi}_{N}(\Upsilon\backslash\Xi) = 0.$$

However, $\Upsilon \subset \Xi \cup \Upsilon$ and $\delta_{\overline{N}}^{\Phi}(\Upsilon) \neq 0$. It means that $\mathscr{T}_0 \in \widehat{\mathcal{F}S}\left[\Lambda_{\overline{N}}^{\Phi}(\zeta)\right]$ and the other inclusion can be derived similarly by applying symmetry. Let $\mathscr{T}_0 \in \widehat{\mathcal{F}S}\left[\Gamma_{\overline{N}}^{\Phi}(\mathscr{T})\right]$. Then, for each $\varrho > 0$,

$$\delta_{\overline{N}}^{\Phi}\left(\left\{\nu \in \mathbb{N} : \mathcal{M}_\nu\left|\widehat{\mathcal{F}_\nu}(\mathscr{T}) - \mathscr{T}_0\right| < \varrho\right\}\right) \neq 0.$$

Consider $\Upsilon_\varrho = \left\{\nu \in \mathbb{N} : \mathcal{M}_\nu\left|\widehat{\mathcal{F}_\nu}(\mathscr{T}) - \mathscr{T}_0\right| < \varrho\right\}$ and $\Theta_\varrho = \left\{\nu \in \mathbb{N} : \mathcal{M}_\nu\left|\widehat{\mathcal{F}_\nu}(\zeta) - \mathscr{T}_0\right| < \varrho\right\}$. Now, we have $\Upsilon_\varrho \backslash \Xi \subset \Theta_\varrho$ and so that

$$\delta_{\overline{N}}^{\Phi}(\Theta_\varrho) \geq \delta_{\overline{N}}^{\Phi}(\Upsilon_\varrho \backslash \Xi) \geq \delta_{\overline{N}}^{\Phi}(\Upsilon_\varrho) - \delta_{\overline{N}}^{\Phi}(\Xi) = \delta_{\overline{N}}^{\Phi}(\Upsilon_\varrho) \neq 0.$$

Thus, $\mathscr{T}_0 \in \widehat{\mathcal{F}S}\left[\Gamma_{\overline{N}}^{\Phi}(\zeta)\right]$ and the corresponding inclusion can be obtained through a symmetrical argument. $\square$

Theorem 11. *Let $\Phi \in \mathfrak{F}_{(un.)}$ and $\mathscr{T} = (\mathscr{T}_k)$ be a sequence of numbers. Then, there exists a sequence $\zeta = (\zeta_k)$ such that $\widehat{\mathcal{F}}\left[L_{\overline{N}}(\zeta)\right] = \widehat{\mathcal{F}S}\left[\Gamma_{\overline{N}}^{\Phi}(\mathscr{T})\right]$, and the terms of ζ are the same as the terms of $\mathscr{T}$, except on a set of weighted $\Phi-$density zero.*

Proof. Let $u \in \widehat{\mathcal{F}}\left[L_{\overline{N}}(\mathscr{T})\right] \backslash \widehat{\mathcal{F}S}\left[\Gamma_{\overline{N}}^{\Phi}(\mathscr{T})\right]$ (since $\widehat{\mathcal{F}S}\left[\Gamma_{\overline{N}}^{\Phi}(\mathscr{T})\right] \subset \widehat{\mathcal{F}}\left[L_{\overline{N}}(\mathscr{T})\right]$ by Theorem 7, then the other cases are trivial), then there exists $\varrho_u > 0$ such that

$$\delta_{\overline{N}}^{\Phi}\left(\left\{\nu \in \mathbb{N} : \mathcal{M}_\nu\left|\widehat{\mathcal{F}_\nu}(\mathscr{T}) - u\right| < \varrho_u\right\}\right) = 0.$$

We have that $\widehat{\mathcal{F}}\left[L_{\overline{N}}(\mathscr{T})\right] \backslash \widehat{\mathcal{F}S}\left[\Gamma_{\overline{N}}^{\Phi}(\mathscr{T})\right]$ is separable and

$$\widehat{\mathcal{F}}\left[L_{\overline{N}}(\mathscr{T})\right] \backslash \widehat{\mathcal{F}S}\left[\Gamma_{\overline{N}}^{\Phi}(\mathscr{T})\right] = \left(\bigcup_{u \in \widehat{\mathcal{F}}\left[L_{\overline{N}}(\mathscr{T})\right] \backslash \widehat{\mathcal{F}S}\left[\Gamma_{\overline{N}}^{\Phi}(\mathscr{T})\right]} \{y \in \mathbb{R} : \mu_k|u - y| < \varrho \text{ for } k \in \mathbb{N}\}\right).$$

By the Lindelöf property, there exists $(u_k)_k \subset \widehat{\mathcal{F}}\left[L_{\overline{N}}(\mathscr{T})\right] \backslash \widehat{\mathcal{F}S}\left[\Gamma_{\overline{N}}^{\Phi}(\mathscr{T})\right]$ such that

$$\widehat{\mathcal{F}}\left[L_{\overline{N}}(\mathscr{T})\right] \backslash \widehat{\mathcal{F}S}\left[\Gamma_{\overline{N}}^{\Phi}(\mathscr{T})\right] = \left(\bigcup_{k \in \mathbb{N}} \left\{y \in \mathbb{R} : \mu_k\left|\widehat{\mathcal{F}_k}(u) - y\right| < \varrho_{u_k}\right\}\right).$$

For every $k \in \mathbb{N}$, let $\Xi_k = \left\{\nu \in \mathbb{N} : \mathcal{M}_\nu\left|\widehat{\mathcal{F}_\nu}(x) - \widehat{\mathcal{F}_k}(u)\right| < \varrho_{u_k}\right\}$ with $\delta_{\overline{N}}^{\Phi}(\Xi_k) = 0$. By Theorem 2, there exists $(\Upsilon_k) \subset \mathcal{P}(\mathbb{N})$ such that $\Xi_k \backslash \Upsilon_k$ is finite for each $k \in \mathbb{N}$ and $\delta_{\overline{N}}^{\Phi}(\Upsilon) = 0$, where $\Upsilon = \bigcup_{k \in \mathbb{N}} \Upsilon_k$. Let us write $\mathbb{N} \backslash \Upsilon = \{t_1, t_2, \ldots\}$ with $t_1 < t_2 < \ldots$ and define $\zeta = (\zeta_k)$ by

$$\zeta_\nu = \begin{cases} \mathscr{T}_{t_\nu}, & \text{if } \nu \in \Upsilon, \\ \\ \mathscr{T}_\nu, & \text{if } \nu \in \mathbb{N} \backslash \Upsilon. \end{cases}$$

Let $w \in \widehat{\mathcal{F}}\left[L_{\overline{N}}(\zeta)\right] \subset \widehat{\mathcal{F}}\left[L_{\overline{N}}(\mathscr{T})\right]$. Assume that $w \notin \widehat{\mathcal{F}S}\left[\Gamma_{\overline{N}}^{\Phi}(\mathscr{T})\right]$, then there exists $r \in \mathbb{N}$ such that $w \in \left\{y \in \mathbb{R} : \mu_r\left|\widehat{\mathcal{F}_r}(u) - y\right| < \varrho_{u_r}\right\}$. So, there exists an infinite $\Theta \subset \mathbb{N} \backslash \Upsilon$ such that

$$(\mathscr{T}_k)_{k \in \Theta} \subset \left\{y \in \mathbb{R} : \mu_r\left|\widehat{\mathcal{F}_r}(u) - y\right| < \varrho_{u_r}\right\}.$$

We have that $\Theta \subset \Xi_m$ and $\Theta \subset \Xi_m \cap \Theta \subset \Xi_m \backslash \Upsilon$ is finite. This is a contradiction. Thus, $w \in \widehat{\mathcal{F}S}\left[\Gamma^{\Phi}_{N}(\mathscr{T})\right]$ and so that $w \in \widehat{\mathcal{F}}\left[L_{\overline{N}}(\zeta)\right] \subset \widehat{\mathcal{F}S}\left[\Gamma^{\Phi}_{N}(\mathscr{T})\right]$. The reverse inclusion is a consequence of Theorem 10. □

4. Conclusions

This research introduces the notions of weighted Φ-Fibonacci statistical summability, weighted Φ-Fibonacci statistical convergence, and weighted Φ-Fibonacci statistical Cauchy, along with the broader concepts of weighted Fibonacci Φ-statistical limit points and weighted Fibonacci Φ-statistical cluster points. These new definitions provide a more generalized framework than previously established concepts.

This paper aims to offer meaningful insights that can drive future research in related fields. For example, the concept of double sequences examined in [28] can be extended to introduce notions of weighted Φ-Fibonacci statistical summability, weighted Fibonacci statistical convergence, and weighted Φ-Fibonacci statistical Cauchy for double sequences. Such extensions not only enhance existing theoretical frameworks but also present new directions for research and application within mathematical analysis.

Furthermore, the integration of these concepts into Korovkin-type theorems, as discussed in [29–33], offers potential for practical applications in computational mathematics. Specifically, the use of Fibonacci-weighted approaches could aid in solving large structured linear systems more efficiently, presenting an exciting direction for further exploration. Future work may also explore modifications and extensions of these methods to other types of sequences, such as lacunary or strongly summable sequences, or examine the application of modulus functions in new mathematical areas, including sequence space analysis and functional analysis. These possibilities underscore the versatility and wide-ranging potential of the methods introduced in this paper.

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References

1. Fast, H. Sur la convergence statistique. *Colloq. Math.* **1951**, *2*, 241–244. [CrossRef]
2. Steinhaus, H. Sur la convergence ordinaire et la convergence asymptotique. *Colloq. Math.* **1951**, *2*, 73–74.
3. Zygmund, A. *Trigonometric Series*; Reprint of the 1979 Edition; Cambridge Mathematical Library, Cambridge University Press: Cambridge, UK, 1988; Volume I, II.
4. Šalát, T. On statistically convergent sequences of real numbers. *Math. Slovaca* **1980**, *30*, 139–150.
5. Maddox, I.J. Statistical convergence in a locally convex space. In *Mathematical Proceedings of the Cambridge Philosophical Society*; Cambridge University Press: Cambridge, UK, 1988; pp. 141–145.
6. Kolk, E. The statistical convergence in Banach spaces. *Acta Comment. Univ. Tartu.* **1991**, *928*, 41–52.
7. Connor, J.; Ganichev, M.; Kadets, V. A characterization of Banach spaces with separable duals via weak statistical convergence. *J. Math. Anal. Appl.* **2000**, *244*, 251–261. [CrossRef]
8. Totur, Ü.; Çanak, I. Some Tauberian Theorems for the Weighted Mean Method of Summability of Double Sequences. *Ukr. Math. J.* **2024**, *75*, 1453–1472. [CrossRef]
9. Khana, V.A.; Rahamana, S.A.; Hazarikab, B. On Deferred *I*-Statistical Rough Convergence of Difference Sequences in Intuitionistic Fuzzy Normed Spaces. *Filomat* **2024**, *38*, 6333–6354.
10. Ibrahim, I.S.; Yousif, M.A.; Mohammed, P.O.; Baleanu, D.; Zeeshan, A.; Abdelwahed, M. Bessel Statistical Convergence: New Concepts and Applications in Sequence Theory. *PLoS ONE* **2024**, *19*, e0313273. [CrossRef]
11. Karakaya, V.; Chishti, T.A. Weighted statistical convergence. *Iran. J. Sci. Technol. Trans. A Sci.* **2009**, *33*, 219–223.

12. Mursaleen, M.; Karakaya, V.; Ertürk, M.; Gürsoy, F. Weighted statistical convergence and its application to Korovkin type approximation theorem. *Appl. Math. Comput.* **2012**, *218*, 9132–9137. [CrossRef]
13. Ghosal, S. Weighted statistical convergence of order α and its applications. *J. Egypt. Math. Soc.* **2016**, *24*, 60–67. [CrossRef]
14. Koshy, T. *Fibonacci and Lucas Numbers with Applications*; John Wiley & Sons: Hoboken, NJ, USA, 2019; Volume 2.
15. Kara, E.E.; Basarir, M. An application of Fibonacci numbers into infinite Toeplitz matrices. *Casp. J. Math. Sci.* **2012**, *1*, 43–47.
16. Basarir, M.; Basar, F.; Kara, E.E. On the spaces of Fibonacci difference absolutely p-summable, null and convergent sequences. *Sarajevo J. Math.* **2016**, *12*, 2. [CrossRef]
17. Candan, M. Some characteristics of matrix operators on generalized Fibonacci weighted difference sequence space. *Symmetry* **2022**, *14*, 1283. [CrossRef]
18. Demiriz, S.; Kara, E.E.; Basarir, M. On the Fibonacci almost convergent sequence space and Fibonacci core. *Kyungpook Math. J.* **2015**, *55*, 355–372. [CrossRef]
19. Ibrahim, I.S.; Listán-García, M.C.; Colak, R. A new notion of convergence defined by weak Fibonacci lacunary statistical convergence in normed spaces. *J. Appl. Anal.* **2024**. [CrossRef]
20. Ibrahim, I.S.; Listán-García, M.C. A New Notion of Convergence Defined by The Fibonacci Sequence: A Novel Framework and Its Tauberian Conditions. *Mathematics* **2024**, *12*, 2718. [CrossRef]
21. Nakano, H. Concave modulars. *J. Math. Soc. Jpn.* **1953**, *5*, 29–49. [CrossRef]
22. Listán-García, M.C. On uniform f-statistical convergence of sequences of functions. *Quaest. Math.* **2023**, *46*, 1643–1651. [CrossRef]
23. León-Saavedra, F.; Listán-García, M.C.; Romero de la Rosa, M.P. On statistical convergence and strong Cesàro convergence by moduli for double sequences. *J. Inequal. Appl.* **2022**, *2022*, 62. [CrossRef]
24. Torgut, B.; Altin, Y. f-Statistical Convergence of Double Sequences of Order α. *Proc. Natl. Acad. Sci. India Sect. A Phys. Sci.* **2020**, *90*, 803–808. [CrossRef]
25. Kama, R. Spaces of vector sequences defined by the f-statistical convergence and some characterizations of normed spaces. *Rev. R. Acad. Cienc. Exactas Fis. Nat. A Mat.* **2020**, *114*, 74. [CrossRef]
26. Listán-García, M.C. f-Statistical convergence, completeness and f-cluster points. *Bull. Belg. Math. Soc.-Simon Stevin* **2016**, *23*, 235–245. [CrossRef]
27. Freedman, A.; Sember, J. Densities and summability. *Pac. J. Math.* **1981**, *95*, 293–305. [CrossRef]
28. Pringsheim, A. Zur theorie der zweifach unendlichen Zahlenfolgen. *Math. Ann.* **1900**, *53*, 289–321. [CrossRef]
29. Serra, S. A Korovkin-type theory for finite Toeplitz operators via matrix algebras. *Numer. Math.* **1999**, *82*, 117–142. [CrossRef]
30. Capizzano, S.S. A Korovkin-Based Approximation of Multilevel Toeplitz Matrices (With Rectangular Unstructured Blocks) via Multilevel Trigonometric Matrix Spaces. *SIAM J. Numer. Anal.* **1999**, *36*, 1831–1857. [CrossRef]
31. Rao, N.; Ayman-Mursaleen, M.; Aslan, R. A Note on a General Sequence of λ-Szász Kantorovich Type Operators. *Comput. Appl. Math.* **2024**, *43*, 428. [CrossRef]
32. Rao, N.; Yadav, A.K.; Mursaleen, M.; Sinha, B.K.; Jha, N.K. Szász-Beta Operators via Hermite Polynomial. *J. King Saud Univ.-Sci.* **2024**, *36*, 103–120. [CrossRef]
33. Ayman-Mursaleen, M.; Heshamuddin, M.; Rao, N.; Sinha, B.K.; Yadav, A.K. Hermite Polynomials Linking Szász–Durrmeyer Operators. *Comput. Appl. Math.* **2024**, *43*, 223. [CrossRef]