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Multi-Sided Delayed Impulsive Feedback Method for Controlling Chaos and Reducing Loss Ratio in Switched Arrival System with Switching Time Greater than 0

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Abstract: The switched arrival system is a typical hybrid system that is commonly used to simulate industrial control systems. The corresponding mathematical model and switching time are described. In order to be closer to the actual industrial control systems, the switching time is changed from 0 to greater than 0. In this case, the system not only generates chaos but also system losses. For this purpose, firstly, the causes of system losses are analyzed. Secondly, the Poincare section is selected to define the control target—periodic orbits. And then, the delayed impulsive feedback method is improved for the system at a switching time greater than 0, and extended to each boundary in order to enhance the control effect. This not only controls chaos in the system but also reduces the loss ratio and detects periodic orbits. Finally, numerical simulations of the system orbits and loss ratio with and without implementing control are compared. The possible intervals for the optimal control coefficient under the same initial conditions are detected. Period-1 orbits are detected at switching times greater than 0, and the stability of system operation is verified.

Keywords: switched arrival system; switching time; controlling chaos; reducing loss ratio; detecting periodic orbit; system stability

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1. Introduction

Hybrid dynamical [1–3] systems are a class of dynamic systems consisting of the interaction of continuous and discrete subsystems. Most complex industrial control systems contain dynamic evolutionary processes at the physical level described by continuous variables and coordinated processes of discrete symbol manipulation. Therefore, there is a broad industrial background for their research. Switched flow systems (SFSs) [1] are typical hybrid systems that are divided into two categories: switched arrival systems (SASs) [2,3] and switched service systems (SSSs). However, their geometrical properties are very different, with switched arrival systems generating chaos and switched service systems ultimately generating periodicity. Chaos [4–6] is a type of motion that cannot be precisely repeated and appears to be random. So far, the singularity and complexity of chaotic systems have not been thoroughly understood. Because of the uncertainty of chaos, it should be suppressed in most cases. A switched arrival system generates chaos in some cases, and the system itself has periodic orbits. The first chaos control method, known as the OGY method proposed by Ott et al. [4], stabilizes unstable periodic orbits using a

small discontinuous parameter perturbation. Pyragas [7,8] proposed a delayed feedback control method for stabilizing inherent unstable periodic orbits [9,10] in chaotic systems. Ushio et al. [11] proposed a method that changes the limited continuous processing time. Rem et al. [12] discussed the possibility of controlling the system to a periodic orbit by introducing a reduced production regime into the system scheduling. These methods, however, require a given orbit as the reference. Therefore, Tian et al. [13–15] proposed a method of stabilizing the system at an unstable (long) periodic orbit embedded in the system. This method, called the delayed impulsive feedback method, is a combination of both impulse control [16,17] and delayed feedback control [18,19]. As one of the new trilogy of contemporary intelligent technologies, neural networks have been widely applied in the fields of control and optimization. Tan et al. [20] proposed a new method for chaotic control with a feed-forward backpropagating neural network based on the OGY method. By combining input–output data obtained from the perturbation parameter model with a nonlinear learning algorithm, neural networks are trained to generate small disturbance control, and then to suppress chaotic behavior. The unstable periodic orbit embedded in the chaotic attractor is derived and directed to a stable fixed point. Related neural networks, such as grid multibutterfly memristive neural networks, diversified butterfly attractors of memristive HNNs, etc. [21,22], are successively applied in practical production and life. All of these will bring new directions for the development of chaos control. In this article, to contribute to this field, we choose the delayed impulsive feedback method and extend the control to each side, a method called multi-sided [23] delayed impulsive feedback control (M-SDIFC). However, this method is based on the case where the switching time is 0. In practical industrial control, the switching time [24–26] is generally greater than 0. In this case, the system not only generates chaos but also system losses. For these reasons, in this article, the M-SDIFC is improved so that it can control chaos and reduce the loss ratio of the system, as well as detect periodic orbits.

2. Mathematical Model of Switched Arrival System

The switched arrival system is divided into n buffers and one server. Figure 1 shows a switched arrival system with three buffers and one server. The buffers are used to simulate the processes of industrial production. The server is used to simulate the process of supplying materials to the buffers. The dynamic process of the system is that buffer i is output from the system at rate $\rho_i \geq 0$, and for a selected buffer, the server injects it at a rate of unit 1. We assume $\sum_{i=1}^n \rho_i = 1$ to stabilize the system's dynamic equilibrium. Let the mixed state of the system be $\{x(t), q(t)\}$, $x_i(t)$ be the value of buffer $i \in \{1, 2, \dots, n\}$, and $q(t) \in \{1, 2, \dots, n\}$ be the number of the buffer selected for injection by the server.

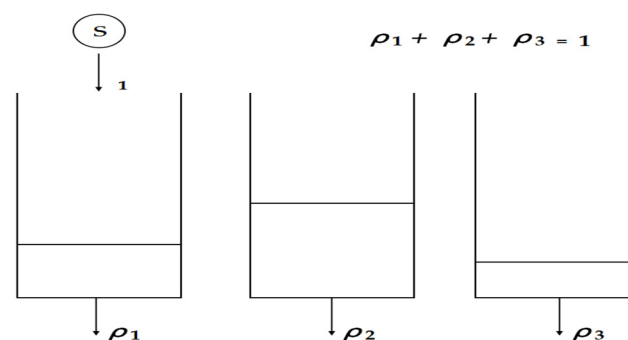


Figure 1. The switched arrival system has three buffers and one server. At this moment, the server is injecting material into the first buffer at a rate of unit 1. The sum of the rate of the three buffers' output is $\rho_1 + \rho_2 + \rho_3 = 1$.

Server switching method: When the value of buffer i is 0, the server instantly comes to this buffer to inject material. Let $\{t_n\}$ be the time sequence of the server switching between buffers. According to the server switching method, the following equation is obtained:

$$t_{n+1} - t_n = \min_{i \neq q(t_n)} \{x_i(t_n)/\rho_i\} \quad (1)$$

When $t \in [t_n, t_{n+1})$, the server is injecting material into buffer $q(t_n)$. The value of buffer i is Equation (2).

$$x_i(t) = \begin{cases} x_i(t_n) + (1 - \rho_i)(t - t_n) & i = q(t_n) \\ x_i(t_n) - \rho_i(t - t_n) & i \neq q(t_n) \end{cases} \quad (2)$$

When $t = t_{n+1}$, the number of the next buffer selected by the server is

$$q(t_{n+1}) = \arg \min_{i \neq q(t_n)} x_i(t_n)/\rho_i \quad (3)$$

Let $\sum_{i=1}^n x_i(0) = 1$, i.e., the sum of value for all buffers is unit 1. The system is in dynamic equilibrium during operation, and $\sum_{i=1}^n x_i(t) = 1 (t \geq 0)$. Let the region of run of the system orbits be $S = \{x | \sum_{i=1}^n x_i = 1\}$ and $S_i = \{x \in S | x_i = 0\}$ be the boundary of the region in which the system orbits run. When $n = 3$, the boundaries of the region in which the system orbits run are three sides of a triangle. When buffer i is empty, it corresponds to the system orbits just running to S_i . The server is immediately switched to buffer i for injecting material.

The time it takes for a server to switch from one buffer to another is called switching time. The process of the server switching between buffers is called a switching event. The above switching time is negligible, i.e., the switching time is 0. In actual industrial production control, the process of the server switching between buffers takes time, and this is also unavoidable. In order to be closer to the reality of industrial production control, for the following discussion, we set the system's switching time to $\lambda > 0$. For the convenience of exposition and numerical simulation, without loss of generality, $n = 3$ is taken as an example.

3. Chaos and System Loss with Switching Time Greater than 0

If the output rate of the buffers is 0 for a certain time, the overall output rate of the system is less than 1. In this case, the productivity of the system is reduced, and the system incurs losses. The switching time is known to be $\lambda > 0$, and $\{t_n\}$ is the time sequence in which the server is switching between buffers. The system incurs loss in two cases: Firstly, according to the switching method in an SAS, when $t = t_n$, some buffer in the system is emptied. The server does not come to this buffer immediately, but continues to inject material into the previously selected buffer. The server does not come to the empty buffer until $t = t_n + \lambda$. When $t \in [t_n, t_n + \lambda)$, there is an empty buffer in the system. The empty buffer has an output rate of 0, resulting in a decrease in the overall output rate of the system. Thus, the system incurs losses. Secondly, when one buffer is already emptied, another buffer is also emptied. Assuming another buffer is emptied at $t = t_n + \gamma (0 < \gamma < \lambda)$, when $t = t_n + \lambda$, the server comes to the first emptied buffer, and when $t = t_n + \lambda + \gamma$, the server comes to the second emptied buffer. During $t \in [t_n, t_n + \gamma)$ and $t \in [t_n + \lambda, t_n + \lambda + \gamma)$, there is one empty buffer in the system, while during $t \in [t_n + \gamma, t_n + \lambda)$, there are two empty buffers. Since the output rate of empty buffers is 0, the system incurs losses. To keep the system in dynamic equilibrium, the injection rate of the server is reduced to equal

the output rate of the system during the time above. The mathematical expression for the overall state of the system is as follows:

$$x(t) = \sum_{i=1}^{n=3} x_i(t) = \begin{cases} 1 - \rho_i & t \in [t_n, t_n + \gamma) \cup [t_n + \lambda, t_n + \lambda + \gamma) \\ 1 - \rho_i - \rho_j & t \in [t_n + \gamma, t_n + \lambda) \\ 1 & t \notin [t_n, t_n + \lambda + \gamma) \end{cases} \quad (4)$$

$(i, j \in \{1, 2, 3\})$

where i and j denote the number of the buffer emptied first and emptied second at certain times, respectively.

Next, we observe the running image of system orbits. The system orbits slide along the boundary for time γ , reach a vertex, and stay for time $\lambda - \gamma$, then cross this vertex and continue to slide along another boundary for time γ , and finally leave this boundary.

Example: let $x = [0 \ 0.25 \ 0.75]^T$, $\lambda = 0.3$, $\rho_i = 1/3 (i = 1, 2, 3)$. It can be seen that the system orbits slide on the boundaries and lie at the vertexes of the triangle region in Figure 2, and the output rate of the system changes with time in Figure 3. When there are one empty buffer and two empty buffers, the output rate of the system is reduced to $2/3$ and $1/3$, respectively. The system incurs losses.

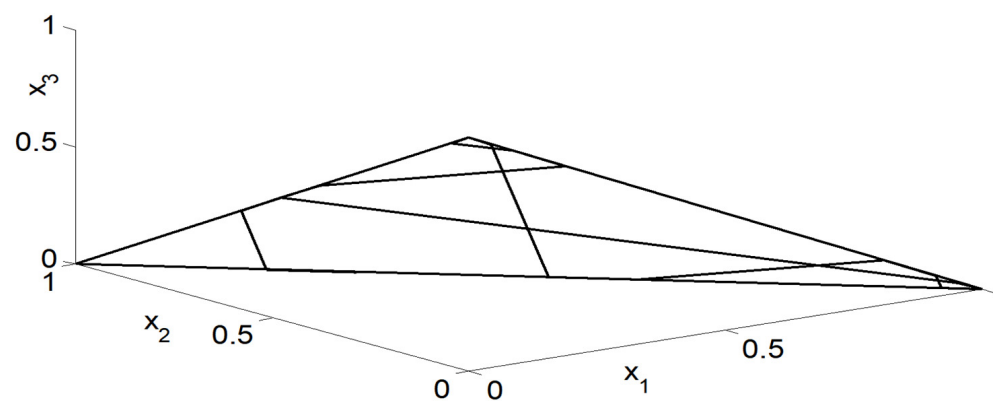


Figure 2. When switching time $\lambda = 0.3$, the system orbits exhibit random and unpredictable motion in the triangle region. This indicates that the system operation is unstable and disordered.

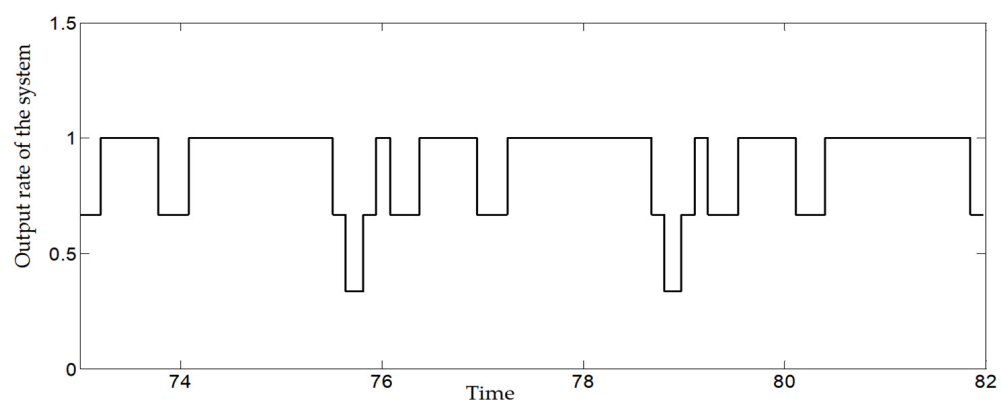


Figure 3. When switching time $\lambda = 0.3$, the output rate of the system constantly varies among $1/3$, $2/3$, and 1 . The output rate of the system is less than 1 at many times, even $1/3$. This indicates that the system has one or two empty buffers at these times above. The system incurs losses.

4. Multi-Sided Delayed Impulsive Feedback Control with Switching Time Greater than 0

4.1. Control Target—Periodic Orbits

We use the Poincare section to define the periodic orbits. One of the boundaries of the region where the system orbits run is selected as the Poincare section. The specific position of the system orbits reaching this section at different times is used to study the running pattern of the system orbits.

Let $\{\theta_{n_i}\} (n_i = 1, 2, \dots; i = 1, 2, 3)$ be the time sequence of the n_i th arrival of the system orbits $x(t)$ at boundary S_i . If there is

$$x(\theta_{n_i}) = x(\theta_{n_i+m}), \quad (5)$$

then it is a period- m orbit defined by the boundary S_i , denoted as

$$\text{per}(x(t)|S_i) = m. \quad (6)$$

But the period sizes of the obtained periodic orbits may be different if different boundaries are selected as the Poincare section. Therefore, Equation (7) is further defined as the period sizes of periodic orbits, denoted as period- m orbits.

$$\max_{i=1,2,3} \{\text{per}(x(t)|S_i)\} = m \quad (7)$$

4.2. Multi-Sided Delayed Impulsive Feedback Control

This method is a combination of impulsive control and time-delayed feedback control. Its basic idea is as follows: when the orbits reach a predetermined boundary, control is implemented. The sizes of the control quantity are related to the target periodic orbits, causing the system orbits to converge towards the target periodic orbits. The mathematical expression of the method is

$$x(\theta_{km+1}^+) = (I - K)x(\theta_{km+1}) + Kx(\theta_{(k-1)m+1}^+). \quad (8)$$

$x(t)$ is the state of the system. θ_{km+1} and θ_{km+1}^+ are the start and end times of k th control, respectively. $K = \alpha I$ is the matrix of control coefficient, and $\alpha (0 < \alpha < 1)$ is the control coefficient that exists and can be set. I is the unit matrix. m is the period size of the target periodic orbit. The advantage of the method is that it does not require the specific location of the target periodic orbits, but instead detects the specific location of the periodic orbits. To enhance the control effect, the control method is extended to each boundary. That is to say, the multi-sided delayed impulsive feedback control is discussed in the following content.

Let $x_i(t) (i = 1, 2, 3)$ be the value of buffer i . The server injects the selected buffer at a unit rate 1. The output rate of buffers is $\rho_i > 0 (\sum_{i=1}^3 \rho_i = 1)$. Impulsive control is the instantaneous application of change for the system. But it is impossible or difficult to operate in practice. Thus, the control time is changed from instantaneous to $[\theta_{k_i m+1}, \theta_{k_i m+1}^+]$. $\theta_{k_i m+1}$ and $\theta_{k_i m+1}^+$ are the control start and end times, respectively, when the orbits reach the boundary S_i for the k_i th. The period- m orbit is used as the control target, and $\alpha (0 < \alpha < 1)$ is the control coefficient that exists and can be set. Since control is implemented at each boundary, the expression of this method is

$$\begin{aligned}
 \begin{pmatrix} \Delta x(k_1) & \Delta x(k_2) & \Delta x(k_3) \end{pmatrix} &= \begin{pmatrix} \Delta x_1(k_1) & \Delta x_1(k_2) & \Delta x_1(k_3) \\ \Delta x_2(k_1) & \Delta x_2(k_2) & \Delta x_2(k_3) \\ \Delta x_3(k_1) & \Delta x_3(k_2) & \Delta x_3(k_3) \end{pmatrix} \\
 &= \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix} \begin{pmatrix} x_1(\theta_{(k_1-1)m+1}^+) - x_1(\theta_{k_1m+1}) & x_1(\theta_{(k_2-1)m+1}^+) - x_1(\theta_{k_2m+1}) & x_1(\theta_{(k_3-1)m+1}^+) - x_1(\theta_{k_3m+1}) \\ x_2(\theta_{(k_1-1)m+1}^+) - x_2(\theta_{k_1m+1}) & x_2(\theta_{(k_2-1)m+1}^+) - x_2(\theta_{k_2m+1}) & x_2(\theta_{(k_3-1)m+1}^+) - x_2(\theta_{k_3m+1}) \\ x_3(\theta_{(k_1-1)m+1}^+) - x_3(\theta_{k_1m+1}) & x_3(\theta_{(k_2-1)m+1}^+) - x_3(\theta_{k_2m+1}) & x_3(\theta_{(k_3-1)m+1}^+) - x_3(\theta_{k_3m+1}) \end{pmatrix} \quad (9)
 \end{aligned}$$

$\Delta x(k_i) = [\Delta x_1(k_i) \ \Delta x_2(k_i) \ \Delta x_3(k_i)]^T$ represents the value of system change after the k_i th control at boundary S_i . Let $K = \alpha I$ ($0 < \alpha < 1$) be the matrix of control coefficient, where I is the unit matrix. Equation (9) is rewritten as

$$x(\theta_{k_i m+1}^+) = (I - K)x(\theta_{k_i m+1}) + Kx(\theta_{(k_i-1)m+1}^+) \quad (i = 1, 2, 3). \quad (10)$$

The following Theorem 1 can demonstrate the existence and effectiveness of control matrices K .

Theorem 1. *If there is a matrix K , such that*

$$\max_{i=1,2,3} \{\rho(A_i - KA_i + K)\} < 1, \quad (11)$$

then the equations

$$\lim_{k_i \rightarrow +\infty} (x(\theta_{k_i m+1}) - \bar{x}^i) = 0 \quad (12)$$

and

$$\lim_{k_i \rightarrow +\infty} (x(\theta_{k_i m+1}^+) - x(\theta_{k_i m+1})) = 0 \quad (13)$$

hold simultaneously in the neighborhood of $\bar{x}^i$.

Here, $\bar{x}^i$ is the m -periodic orbit defined on the boundary S_i as a fixed point on the Poincare map, A_i is described in Appendix A, and the symbol ρ represents the spectral radius of the matrix. Equation (12) represents the convergence of the system orbits to the target periodic orbits. Equation (13) represents the control gradually disappearing. Equations (12) and (13) hold simultaneously, indicating that the selected coefficient K is valid.

Theorem 1 theoretically explains the existence of the control coefficient matrix K , but it cannot be directly calculated from the theorem. Nevertheless, this theorem lays a theoretical foundation for the feasibility of the control method.

For the proof, see Appendix A.

Next, we introduce the specific implementation method when the control is extended to each boundary. Each control is divided into three stages according to time: early stage $t = \theta_{k_i m+1}$, middle stage $t \in (\theta_{k_i m+1}, \theta_{k_i m+1}^+)$, and late stage $t = \theta_{k_i m+1}^+$.

Early stage $t = \theta_{k_i m+1}$: At this moment, the control starts and the system makes the following two adjustments.

- i. Because $t \in [\theta_{k_i m+1}, \theta_{k_i m+1}^+]$, $x_i(t) = 0$ holds. It can be assumed that the output rate of buffer i is adjusted from ρ_i to 0. At the same time, in order to maintain the dynamic balance of the system, the injection rate of the server is adjusted from unit 1 to $1 - \rho_i$.
- ii. According to the switched method, the server should have switched to the buffer i at this moment. Due to the application of delayed impulsive feedback control, the server

only chooses between the two buffers at this moment, and the switched method at this moment is

$$q(\theta_{k_i m+1}) = \begin{cases} \omega \in \{1, 2, 3\} \setminus \{i\} & \Delta x_\omega(k_i) > 0 \\ i & \text{else} \end{cases}. \quad (14)$$

where $q(\theta_{k_i m+1})$ denotes the sequence number of the buffer selected by the server. Notice that when $t \in [\theta_{k_i m+1}, \theta_{k_i m+1}^+]$, $x_i(t) \equiv 0$. So, we have $\Delta x_a(k_i) + \Delta x_b(k_i) = 0$, $a, b \in \{1, 2, 3\} \setminus \{i\}$, and when $\Delta x_a(k_i) = \Delta x_b(k_i) = 0$, it means that the orbits go into periodic motion at this moment.

Middle stage $t \in (\theta_{k_i m+1}, \theta_{k_i m+1}^+)$: According to Equation (14), the conversion of the amount between the two working buffers is completed by adjusting the size of $|\Delta x_{q(\theta_{k_i m+1})}|$. The end time of k th control can be obtained from the previous assumptions as follows:

$$\theta_{k_i m+1}^+ = \theta_{k_i m+1} + |\Delta x_{q(\theta_{k_i m+1})}| / (1 - \rho_i - \rho_{q(\theta_{k_i m+1})}). \quad (15)$$

At this moment, the server is injecting into buffer $q(\theta_{k_i m+1})$, and the orbits slide along boundary S_i . The state of $x(t)$ is

$$x_j(t) = \begin{cases} x_j(\theta_{k_i m+1}) + (1 - \rho_i - \rho_{q(\theta_{k_i m+1})})(t - \theta_{k_i m+1}) & j = q(\theta_{k_i m+1}) \\ 0 & j = i \\ x_j(\theta_{k_i m+1}) - (1 - \rho_i - \rho_{q(\theta_{k_i m+1})})(t - \theta_{k_i m+1}) & \text{else} \end{cases} \quad (16)$$

$(i, j \in \{1, 2, 3\})$

Late stage $t = \theta_{k_i m+1}^+$: At this moment, the control ends. Corresponding to the previous stages i and ii, the system makes the following two adjustments:

- i. The injection rate of server goes back to the original unit 1.
- ii. The server is switched to the empty buffer, i.e., the buffer with number i .

By substituting Equations (14) and (15) into Equation (16) and organizing them, Equation (9) can be obtained, indicating the theoretical effectiveness of this method.

Here, we would like to point out the main difference between this method and event-triggered intermittent control [27]. Firstly, in terms of control time, event-triggered intermittent control is only implemented when specific events occur. Control actions are not implemented at each time step, but rather when events are detected, such as errors exceeding a certain threshold. The delayed impulse feedback control delays the execution of control actions after an event or disturbance occurs, and the control actions are implemented at each time step. Secondly, in terms of feedback type, event-triggered intermittent control is usually relatively stable, but there may still be intermittent situations depending on the event situation. The delayed impulse feedback control involves discrete correction pulses. Finally, in terms of control objectives, event-triggered intermittent control is only applied when necessary, minimizing resource usage while maintaining performance. The delayed impulsive feedback method is typically used for system stability or performance optimization. In addition, there are certain differences between the two in terms of usage scope and other aspects.

4.3. Improvement of the Method for Switching Time Greater than 0

When the switching time is greater than 0, the system incurs losses. Therefore, it is feasible to reduce the loss ratio by decreasing the number of switching events to control system

chaos. The control time is reselected to avoid switching events during the implementation control. The period-1 orbit is selected as the control target, i.e., $m = 1$.

$\{\theta_{n_i}\} (n_i = 1, 2, \dots; i = 1, 2, 3)$ are the time sequences of the arrival of system orbits at the boundary S_i . When $t = \theta_{n_i}$ the buffer i is empty, the server is ending the injection for buffer $v \in \{1, 2, 3\} \setminus \{i\}$ and is about to come to buffer i for injection. Assuming that control is implemented on the boundary S_i , the previous $k_i - 1$ control times are known to be $[\theta_1, \theta_1^+], [\theta_2, \theta_2^+] \cdots [\theta_{k_i-1}, \theta_{k_i-1}^+]$. The start time of k_i th control is θ_{k_i} , which is selected from the sequence $\{\theta_{n_i}\}$ under the following condition:

$$x_v(\theta_{k_i-1}^+) - x_v(\theta_{k_i}) > 0 \quad (17)$$

Clearly, $\{\theta_{k_i}\}$ is a subsequence selected from $\{\theta_{n_i}\}$, i.e., there is $\{\theta_{k_i}\} \subset \{\theta_{n_i}\}$, and the value of the k_i th control is

$$\Delta x_v(k_i) = \alpha(x_v(\theta_{k_i-1}^+) - x_v(\theta_{k_i})) > 0. \quad (18)$$

Therefore, the end time of the k_i th control is

$$\theta_{k_i}^+ = \theta_{k_i} + |\Delta x_v(k_i)| / (1 - \rho_i - \rho_v). \quad (19)$$

When $t \in [\theta_{k_i}, \theta_{k_i}^+)$, buffer i is empty and its output speed is reduced from ρ_i to 0. To keep dynamic equilibrium, the injection rate of the server reduces from unit 1 to $1 - \rho_i$ and the server continues to inject $|\Delta x_v(k_i)|$ into buffer v . When $t = \theta_{k_i}^+$, the injection rate of the server is changed back to unit 1, and the server comes to buffer i for injection.

The change in control time not only prevents the occurrence of switching events during the implementation control, but also does not affect the effectiveness of the method. In other words, this improved method not only controls chaos but also reduces the loss ratio and detects periodic orbits.

5. Numerical Verification

5.1. Controlling Chaos and Comparison of Loss Ratio

In the actual control of industrial production, the output speed of the buffer is generally the same, i.e., $\rho_i = 1/3$. The period-1 orbit is used as the control target. Considering the image simulation effect, let $n = 3$, and Matlab2023 software is used for numerical validation.

Let the initial value of the system $x = [0 \ 0.2 \ 0.8]^T$ and control coefficient $\alpha = 0.95$. Figures 4 and 5 show the images of the system with and without control at switching times $\lambda = 0.2$ and $\lambda = 0.28$, respectively. The period-1 orbit is used as the control target to simulate the running of system orbits with and without control. It is clear that the implementation of control not only eliminates chaos, but also stabilizes the system orbits to a period-1 orbit.

Let the initial values of the system $x = [0 \ 0.25 \ 0.75]^T$ and control coefficient $\alpha = 0.96$. The loss ratio of the system with and without control is compared when the switching time takes different values in Figure 6. Let the switching time $\lambda = 0.2$ and the control coefficient $\alpha = 0.97$. The loss ratio of the system with and without control is compared when the system takes different initial values in Table 1. It can be seen that the loss ratio of the system is reduced after implementing control.

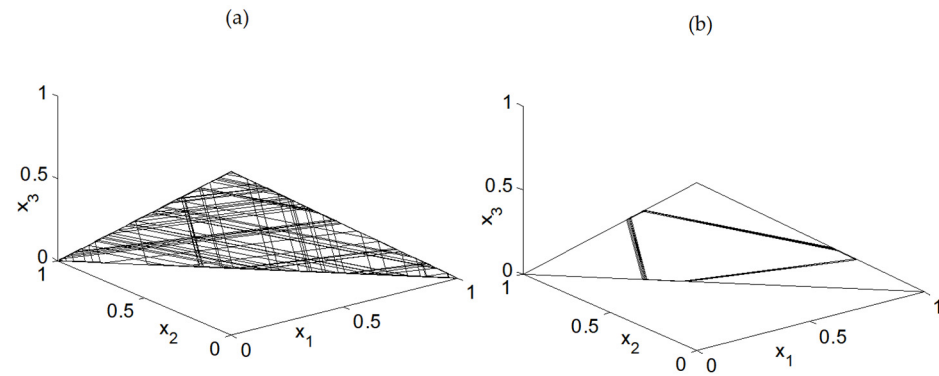


Figure 4. When switching time $\lambda = 0.2$, (a) illustrates that if no control is implemented, the system orbit is random and disordered, i.e., there is chaos in the system. (b) illustrates that if control is implemented, the system orbit is periodic and ordered, i.e., chaos is suppressed in the system.

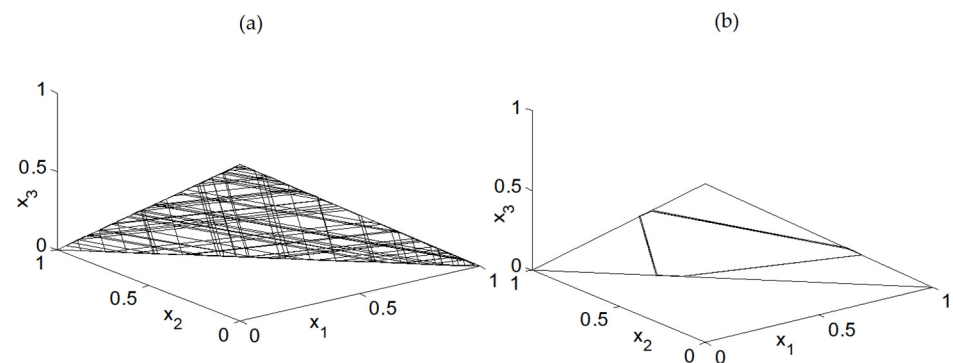


Figure 5. When switching time $\lambda = 0.28$, (a) illustrates that if no control is implemented, the system orbit is random and disordered, i.e., there is chaos in the system. (b) illustrates that if control is implemented, the system orbit is periodic and ordered, i.e., chaos is suppressed in the system.

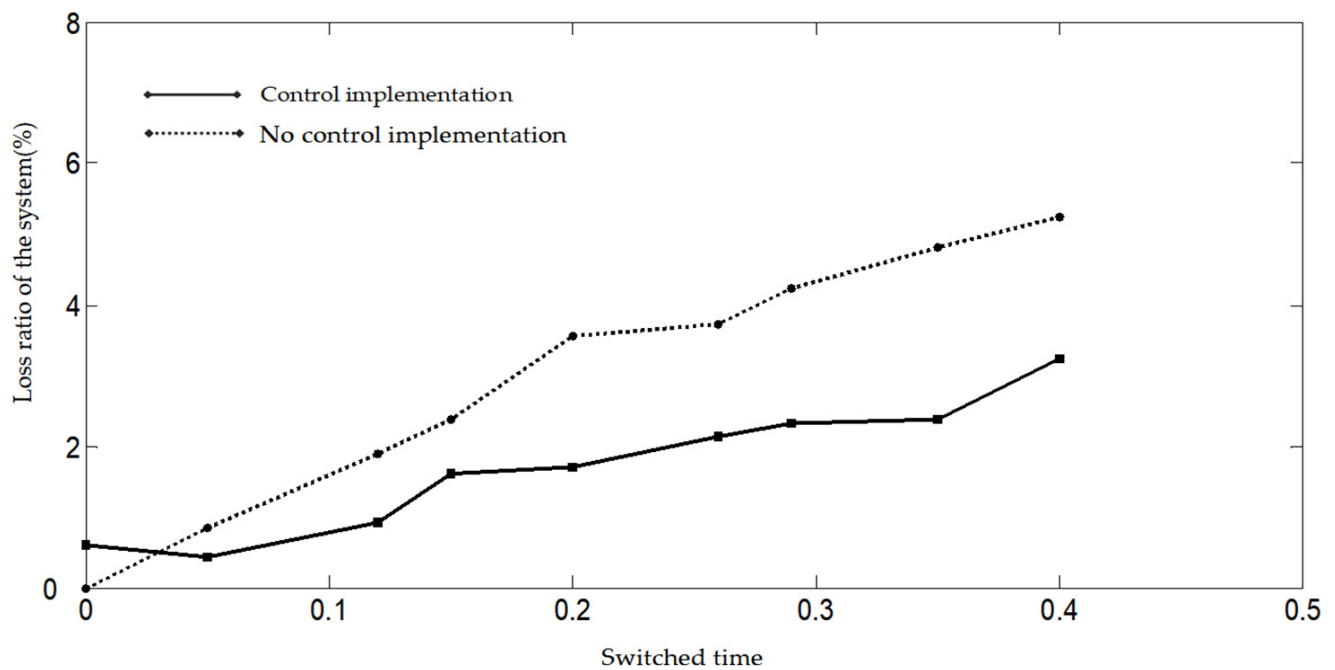


Figure 6. When the switching times are different, the loss ratio of the system is compared with and without implementing control. It can be seen that the loss ratio of the system is reduced.

Table 1. When the initial values of the system are different, the loss ratio of the system is compared with and without implementing control. It can be seen that the loss ratio of the system reduced by more than 7% on average.

$\lambda = 0.2$	Loss Ratio with Implementing Control	Loss Ratio Without Implementing Control
$x = [0 \ 0.1 \ 0.9]^T$	3.65%	10.30%
$x = [0 \ 0.2 \ 0.8]^T$	3.22%	10.25%
$x = [0 \ 0.3 \ 0.7]^T$	3.90%	10.65%
$x = [0 \ 0.4 \ 0.6]^T$	4.12%	10.70%
$x = [0 \ 0.5 \ 0.5]^T$	3.02%	10.75%

Note that, when $t \in [\theta_{k_i}, \theta_{k_i}^+]$, the control is implemented on the boundary S_i . The buffer i is empty and the system still incurs losses. However, this comparison is based on the situation where the system orbits have been stabilized to the periodic orbits. The appropriate coefficients must be chosen to ensure that the method is successfully implemented and does not take too long to complete the control.

Next, we attempt to explore the optimal control coefficient α when the initial conditions $x(t)$ and λ are fixed. Let the initial value of the system $x = [0 \ 0.35 \ 0.65]^T$ and switching time $\lambda = 0.15$, and the initial value of the system $x = [0 \ 0.45 \ 0.55]^T$ and switching time $\lambda = 0.1$. Figure 7a,b show scatter plots of the loss ratio under different initial conditions.

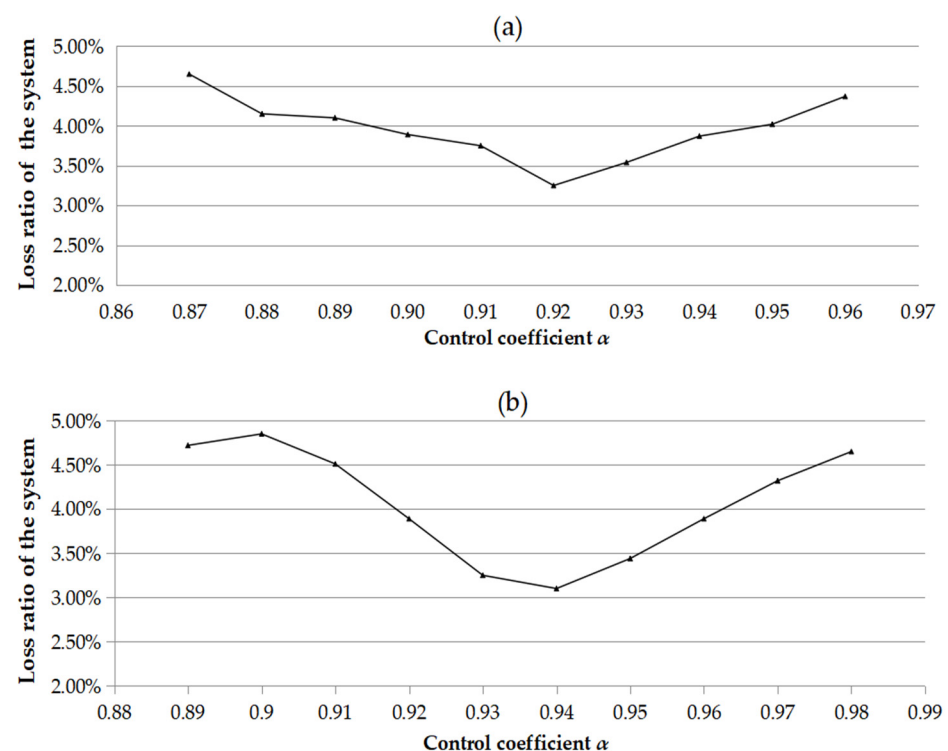


Figure 7. From the trend of the scatter points in (a), it can be seen that the optimal control coefficient is likely to be located in the interval around 0.92. From the trend of the scattered points in (b), it can be seen that the optimal control coefficient is likely to be located in the interval around 0.94.

5.2. Detection of Period-1 Orbits and Stabilization of System

Let the initial values of the system $x = [0 \ 0.25 \ 0.75]^T$, control coefficient $\alpha = 0.94$, and switching time $\lambda = 0.25$. Figure 8a shows that the system orbits run without control. Figure 9a shows the detected period-1 orbit. Note that when the switching time is greater than 0, the sliding process and sliding time of the periodic orbit on the boundaries cor-

respond to the switching event and switching time, respectively. The output rate of the system is simulated with control in Figure 9b and without control in Figure 8b. It can be seen that the output rate of the system is periodic and stable, that is, the system goes into periodic operation.

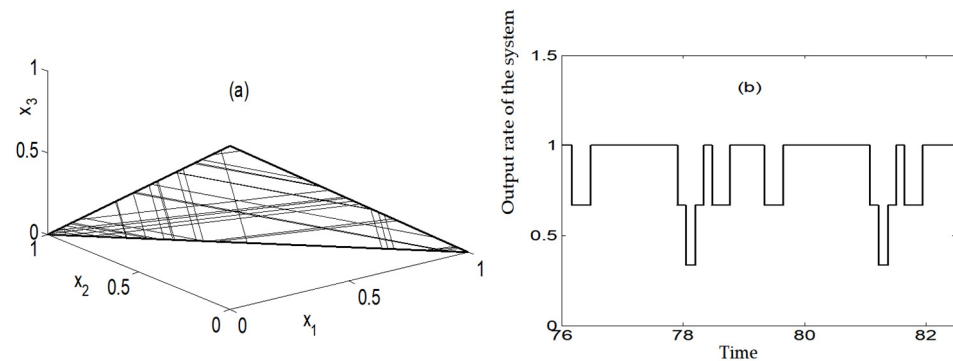


Figure 8. (a) illustrates that if no control is implemented, the system orbits are random, and of course periodic orbits cannot be detected. The system is in a state of disorder. (b) illustrates that the output rate of the system constantly varies and the system output is unstable. When the output rate of the system is $1/3$ or $2/3$, it indicates that the system has two or one empty buffer, respectively. The operation of the system is unstable.

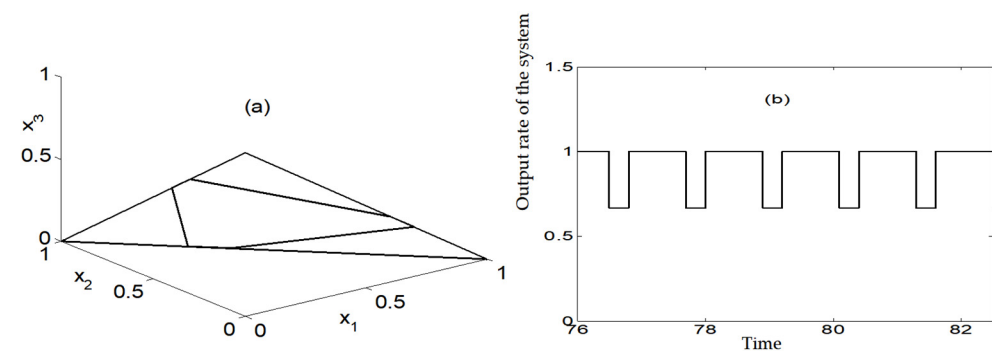


Figure 9. (a) illustrates that if control is implemented, the system orbit is stabilized to a periodic orbit, i.e., a period-1 orbit is detected. The system is in a state of order. (b) illustrates that the output rate of the system is stable and periodic. When the output rate of the system is $2/3$, it indicates that the system has one empty buffer and does not have two empty buffers. The time of the output rate of the system being $2/3$ exactly matches the switching time. The operation of the system is stable. The process of the system orbits sliding along the boundaries in (a) is equivalent to the process of the output rate of the system being $2/3$ in (b).

6. Conclusions

In order to be closer to actual industrial production, we changed the switching time in the SAS to be greater than 0. This resulted in system loss in addition to chaos. Here, we chose the delayed impulsive feedback method. The advantage of this method is that it does not require the specific location of the target periodic orbits in advance, but can detect periodic orbits. Starting from analyzing the causes of system losses, the delayed impulsive feedback method is improved and extended to control each side. Three main achievements have been made in suppressing chaos, reducing the loss ratio of the system, and detecting its periodic orbits. The operation of the system also exhibits periodicity and reliability after control implementation. The feasibility and effectiveness of the method and the veracity of the results are verified at the numerical simulation level. So, this method improves industrial productivity and has broad effectiveness and replicability. The next step is to focus on selecting the optimal coefficient matrix K for SASs with different initial

conditions, such as different values of $x(t)$ and λ , in order to achieve faster convergence and better control.

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Description of Mathematical Symbols

$\text{Max } f$	The maximum value of the function f .
$\text{argmin } f(x)$	The value of x when the function f takes the minimum value.
$\sigma(A)$	The spectrum of the matrix A .
$\rho(A)$	The spectral radius of the matrix A .
$\det(A)$	The determinant of the matrix A .
δx	The increment of the variable x .
$\frac{\partial f}{\partial x}$	The partial derivative of function f with respect to x .

Appendix A. Proof of Theorem 1

According to the Poincare map, the equation

$$x(\theta_{(k_i+1)m+1}) = \phi_i(x(\theta_{k_i m+1}^+))$$

can be obtained, where ϕ_i is the Poincare map under S_i . Equation (10) is rewritten as

$$\begin{aligned} x(\theta_{k_i m+1}^+) &= (I - K)x(\theta_{k_i m+1}) + Kx(\theta_{(k_i-1)m+1}^+) \\ &= (I - K)\phi_i(x(\theta_{(k_i-1)m+1}^+)) + Kx(\theta_{(k_i-1)m+1}^+) \quad (i = 1, 2, 3) \end{aligned}$$

Let $x_{k_i} = x(\theta_{k_i m+1}^+)$ and $x_{k_i-1} = x(\theta_{(k_i-1)m+1}^+)$, then there are

$$x_{k_i} = (I - K)\phi_i(x_{k_i-1}) + Kx_{k_i-1}.$$

Let $\delta x_{k_i} = x_{k_i} - \bar{x}^i$; the linearized system of this equation in a certain neighborhood of $\bar{x}^i$ is

$$\delta x_{k_i} = (A_i - KA_i + K)\delta x_{k_i-1},$$

where $A_i = \frac{\partial \phi_i(x)}{\partial x}|_{x = \bar{x}^i}$ ($i = 1, 2, 3$). Given that Equation (11) holds, we can obtain

$$\lim_{k_i \rightarrow +\infty} (x(\theta_{k_i m+1}^+) - \bar{x}^i) = 0.$$

It is known that $x(\theta_{(k_i+1)m+1}) = \phi_i(x(\theta_{k_i m+1}^+))$. By taking the limits on both sides of this equation simultaneously, we can obtain

$$\lim_{k_i \rightarrow +\infty} (x(\theta_{(k_i+1)m+1})) = \lim_{k_i \rightarrow +\infty} (\phi_i(x(\theta_{k_i m+1}^+))).$$

Let its limit be $\tilde{x}^i$; by taking the limits on both sides of Equation (10) simultaneously, we can obtain

$$\lim_{k_i \rightarrow +\infty} x(\theta_{k_i m+1}^+) = (I - K) \lim_{k_i \rightarrow +\infty} x(\theta_{k_i m+1}) + K \lim_{k_i \rightarrow +\infty} x(\theta_{(k_i-1)m+1}^+).$$

By substituting $\lim_{k_i \rightarrow +\infty} x(\theta_{k_i m+1}^+) = \bar{x}^i$ and $\lim_{k_i \rightarrow +\infty} x(\theta_{k_i m+1}) = \tilde{x}^i$ into the above equation, we can obtain

$$(K - I)(\bar{x}^i - \tilde{x}^i) = 0.$$

Given that Equation (9) holds, we have $\det(I - A_i + K(A_i - I)) \neq 0$, i.e.,

$$\det(K - I) \neq 0.$$

Therefore, $K - I$ is a non-singular matrix, so $\bar{x}^i = \tilde{x}^i$ holds, i.e., Equation (13) holds. Knowing already that $\lim_{k_i \rightarrow +\infty} (x(\theta_{k_i m+1}^+) - \bar{x}^i) = 0$, it is easy to obtain that Equation (12) holds. In other words, Theorem 1 holds. $\square$

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