mathematics

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Revisiting the Group Classification of the General Nonlinear Heat Equation $u_t = \left(K(u)\, u_x\right)_x$

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Abstract: Group classification is a powerful tool for identifying and selecting the free elements—functions or parameters—in partial differential equations (PDEs) that maximize the symmetry properties of the model. In this paper, we revisit the group classification of the general nonlinear heat (or diffusion) equation $u_t = \left(K(u)\, u_x\right)_x$, where $K(u)$ is a non-constant function of the dependent variable. We present the group classification framework, derive the determining equations for the coefficients of the infinitesimal generators of the admitted symmetry groups, and systematically solve for admissible forms of $K(u)$. Our analysis recovers the classical results of Ovsyannikov and Bluman and provides additional clarity and detailed derivations. The classification yields multiple cases, each corresponding to a specific form of $K(u)$, and reveals the dimension of the associated Lie symmetry algebra.

Keywords: group classification; nonlinear heat equation; Lie symmetries; diffusion models

MSC: 35B06; 35K55; 58J70

1. Introduction

Differential equations that model various phenomena frequently incorporate free elements—either parameters or functions—to capture diverse physical behaviors. When dealing with such equations that allow for flexibility in the selection of these parameters and functions, it is crucial to choose those elements that enhance the model's symmetry. This is because the simplicity and tractability of the model are often closely tied to its symmetries. Group classification offers a structured approach for selecting functions and parameters that guarantee the model possesses a specific symmetry group or, ideally, the most extensive symmetry group.

The origins of group classification methods can be traced back to the pioneering work of Sophus Lie, with his first paper on the subject being [1]. The modern formulation of the group classification problem for PDEs was introduced by Ovsyannikov in [2].

Since Ovsyannikov's seminal work, numerous contributions have advanced the theory and practice of group classification. Bluman and Kumei [3] demonstrated the power of Lie–Bäcklund invariance by showing that a nonlinear diffusion equation admits an infinite number of one-parameter Lie–Bäcklund groups under a specific conductivity condition. Subsequently, Yung et al. [4] and Torrisi et al. [5] further applied group analysis to nonlinear diffusion–convection and diffusion equations, respectively, obtaining comprehensive classifications of functional forms that allow for the construction of exact invariant solutions. Bruzón et al. [6] focused on nonlinear dispersive equations, identifying both classical and potential symmetries and establishing wide classes of physical solutions. In parallel, Freire [7] and Nadjafikhah et al. [8] concentrated on Burgers' equation, investigating its

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Academic Editor: Paolo Mercorelli

Received: 15 February 2025
Revised: 5 March 2025
Accepted: 6 March 2025
Published: 9 March 2025

Citation: Sinkala, W. Revisiting the Group Classification of the General Nonlinear Heat Equation $u_t = \left(K(u)\, u_x\right)_x$. *Mathematics* **2025**, *13*, 911. https://doi.org/10.3390/math13060911

Lie symmetries and equivalence algebras to extend group classification results. Another important strand of research involves applying symmetry techniques to financial models; Sinkala et al. [9] conducted a group classification of a general bond-pricing partial differential equation, highlighting parameter values that admit nontrivial Lie algebras and justifying classical interest rate models. More recently, Arif et al. [10] analyzed a $(2+1)$-dimensional nonlinear damped Klein–Gordon Fock equation, successfully deriving multiple invariant and traveling wave solutions through classical Lie methods. Finally, Zhdanov et al. [11] offered a broader perspective on the group classification of general evolution equations, discussing local and quasilocal symmetries that enrich the overall understanding of PDE classification and solution structures. Furthermore, group classification has been extended to systems of PDEs, as demonstrated by Mogorosi and Muatjetjeja [12] in their study of a generalized coupled hyperbolic Lane–Emden system.

In recent years, nonlinear diffusion processes described by the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[K(u)\frac{\partial u}{\partial x}\right], \tag{1}$$

with a variable conductivity $K(u)$, have emerged in problems related to plasma and solid-state physics. The form of $K(u)$ depends on the specific physical system under consideration.

The group classification of this equation was first performed by Ovsyannikov [2], who derived the results by treating the PDE as a system of PDEs: $v = K(u)u_x$, $v_x = u_t$. Bluman [13] later revisited the problem, producing a classification without considering the system approach (see also [14]).

The group classification of the nonlinear heat equation represents a seminal contribution to the group classification methods and remains a cornerstone in the study of invariance properties of PDEs. Its importance lies not only in the instructive insights it provides but also in its relevance to other PDEs with structural similarities. The approaches developed for the nonlinear heat equation can be adapted to the group classification of such equations.

Building on the foundational studies, in this work, we re-examine the group classification of the nonlinear heat equation, provide more detailed derivations of the key steps, and recover the various admissible forms of $K(u)$ obtained in earlier studies.

Let

$$X = \xi(t,x,u)\,\frac{\partial}{\partial x} + \tau(t,x,u)\,\frac{\partial}{\partial t} + \eta(t,x,u)\,\frac{\partial}{\partial u} \tag{2}$$

be the infinitesimal generator of the Lie group of point transformations of (1). The invariance condition dictates that

$$X^{(2)}\left[u_t - (K(u)u_x)_x\right]\big|_{(1)} = 0, \tag{3}$$

where $X^{(2)}$ is the second prolongation of X. Its explicit form

$$X^{(2)} = X + \eta^t\frac{\partial}{\partial u_t} + \eta^x\frac{\partial}{\partial u_x} + \eta^{xx}\frac{\partial}{\partial u_{xx}} + \eta^{xt}\frac{\partial}{\partial u_{xt}} + \eta^{tt}\frac{\partial}{\partial u_{tt}}$$

can be found in [15].

If the derivative eliminated from Equation (3) is u_{xx}, then the invariance condition results in the following polynomial in u_t, u_x, and u_{tx}:

$$\begin{aligned}
P(u_t, u_x, u_{tx}) &= 2\,u_t\,u_x\big(\xi_u + \tau_x\,K'(u) + K(u)\,\tau_{xu}\big) \\
&+ u_t\,(u_x)^2\big(\tau_u\,K'(u) + K(u)\,\tau_{uu}\big) \\
&+ \eta_t - K(u)\,\eta_{xx} + u_{tx}\,\big(2\,K(u)\,\tau_x\big) + 2\,u_x\,u_{tx}\,K(u)\,\tau_u \\
&+ (u_x)^2\big(2\,K(u)\,\xi_{xu} - \eta_u\,K'(u) + \tfrac{\eta\,(K'(u))^2}{K(u)} - K(u)\,\eta_{uu} - \eta\,K''(u)\big) \\
&+ (u_x)^3\big(K(u)\,\xi_{uu} - \xi_u\,K'(u)\big) + u_t\big(K(u)\,\tau_{xx} - \tau_t + 2\,\xi_x - \tfrac{\eta\,K'(u)}{K(u)}\big) \\
&+ u_x\big(K(u)\,\xi_{xx} - \xi_t - 2\,\eta_x\,K'(u) - 2\,K(u)\,\eta_{xu}\big) = 0. \qquad (4)
\end{aligned}$$

Equation (4) must be satisfied for arbitrary values of x, t, u, u_t, u_x, and u_{xt}. Consequently, the coefficients of each term in the polynomial must independently vanish, resulting in the following determining equations:

$$\eta_t - K(u)\,\eta_{xx} = 0, \qquad (5)$$
$$K(u)\,\tau_x = 0, \qquad (6)$$
$$K(u)\,\xi_{xx} - \xi_t - 2\,\eta_x\,K'(u) - 2\,K(u)\,\eta_{xu} = 0, \qquad (7)$$
$$K(u)\,\tau_u = 0, \qquad (8)$$
$$2\,K(u)\,\xi_{xu} - \eta_u\,K'(u) + \frac{\eta\,\big(K'(u)\big)^2}{K(u)} - K(u)\,\eta_{uu} - \eta\,K''(u) = 0, \qquad (9)$$
$$K(u)\,\xi_{uu} - \xi_u\,K'(u) = 0, \qquad (10)$$
$$K(u)\,\tau_{xx} - \tau_t + 2\,\xi_x - \frac{\eta\,K'(u)}{K(u)} = 0, \qquad (11)$$
$$\xi_u + \tau_x\,K'(u) + K(u)\,\tau_{xu} = 0, \qquad (12)$$
$$\tau_u\,K'(u) + K(u)\,\tau_{uu} = 0. \qquad (13)$$

From Equations (6), (8) and (12), we deduce

$$\tau = \alpha(t), \qquad (14)$$

and

$$\xi = \beta(t, x), \qquad (15)$$

where α and β are arbitrary functions. Substituting these into Equations (5)–(13), Equation (11) simplifies to

$$2\beta'(t, x) - \alpha_{tt} - \frac{\eta K'(u)}{K(u)} = 0. \qquad (16)$$

Solving for η, we obtain

$$\eta = \frac{K(u)}{K'(u)}(2\beta_x - \alpha_{tt}). \qquad (17)$$

The remaining Equations (5), (7) and (9) reduce to

$$\frac{K(u)}{K'(u)}(2\beta_{tx} - \alpha_{tt} - 2K(u)\beta_{xxx}) = 0, \tag{18}$$

$$\left[4K''(u)\left(\frac{K(u)}{K'(u)}\right)^2 - 7\,K(u)\right]\beta_{xx} - \beta_t = 0, \tag{19}$$

$$K(u)\left(\frac{K(u)}{K'(u)}\right)''(\alpha_t - 2\,\beta_x) = 0. \tag{20}$$

The solutions to Equations (18)–(20) depend on whether $K(u)$ is arbitrary or takes a specific form.

Case 1: Arbitrary $K(u)$.

When $K(u)$ is arbitrary, satisfying Equation (19) requires that $\beta_{xx} = 0$. This condition leads to

$$\beta(t,x) = \gamma(t) + x\omega(t), \tag{21}$$

where $\gamma(t)$ and $\omega(t)$ are arbitrary functions of t. Equations (5) and (7) now reduce to

$$2\omega_t - \alpha_{tt} = 0, \quad \text{and} \quad \gamma_t + x\omega_t = 0, \tag{22}$$

respectively. Solving these equations, we obtain

$$\gamma(t) = k_1, \quad \omega(t) = k_2, \quad \alpha(t) = k_3 + k_4 t, \tag{23}$$

where k_1, k_2, and k_3 are arbitrary constants.

The remaining equation, (9), reduces to

$$(2k_2 - k_4)K(u)\left(\frac{K(u)}{K'(u)}\right)'' = 0. \tag{24}$$

From (24), we conclude $k_4 = 2k_2$, and thus

$$\tau = 2k_2 t + k_3, \quad \xi = k_1 + k_2 x, \quad \eta = 0. \tag{25}$$

Thus, for arbitrary $K(u)$, the general nonlinear heat conduction equation admits a three-parameter Lie group of point transformations, with infinitesimal generators given by

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}. \tag{26}$$

Remark: The infinitesimal generators in (26) constitute the principal Lie algebra of Equation (1). Additional symmetries of Equation (1) may emerge depending on specific forms of $K(u)$ and these are examined in Cases 2 and 3 below.

Case 2: $K(u) = \lambda(u + \kappa)^\nu$

From (20), we equate the coefficient of "$\alpha_t - 2\beta_x$" to zero. This leads to solving the equation:

$$K(u)\left(\frac{K(u)}{K'(u)}\right)'' = 0, \tag{27}$$

The general solution of (27), if $K(u)$ is not constant, is given by

$$K(u) = \lambda(u + \kappa)^\nu, \tag{28}$$

with the limiting case $K(u) = \lambda e^{\nu u}$, where λ, κ, and ν are arbitrary constants.

Substituting (28) into (19), we find

$$\beta(t,x) = k_1 + k_2 x, \quad \alpha(t) = k_3 + k_4 t. \tag{29}$$

This results in a four-parameter Lie group of point transformations with its infinitesimal generators given by those in (26) and

$$X_4 = x\frac{\partial}{\partial x} + 2\left(\frac{u+\kappa}{\nu}\right)\frac{\partial}{\partial u}. \tag{30}$$

In the limiting case, where $K(u) = \lambda e^{\nu u}$, the infinitesimal generator (30) becomes

$$X_4 = x\frac{\partial}{\partial x} + \frac{2}{\nu}\frac{\partial}{\partial u}. \tag{31}$$

Case 3: $K(u) = \frac{\lambda}{(\kappa+u)^{4/3}}$

From (19), we set the coefficient of β_{xx} equal to zero and obtain

$$K(u) = \frac{\lambda}{(\kappa+u)^{4/3}}, \tag{32}$$

where λ and κ are arbitrary constants.

Substituting this result into (19), we find

$$\beta(t,x) = \gamma(x), \tag{33}$$

where $\gamma(x)$ is an arbitrary function. This reduces the problem to solving only Equation (18), which simplifies to

$$\alpha_{tt} + 2K(u)\gamma_{xxx} = 0. \tag{34}$$

It follows from (34) that

$$\alpha(t) = k_1 + k_2 t \tag{35}$$

$$\gamma(x) = k_3 + x k_4 + k_5 x^2, \tag{36}$$

where k_1, k_2, k_3, k_4, and k_5 are arbitrary constants.

Thus, in this case, the PDE (1) admits a five-parameter Lie group of point transformations with its infinitesimal generators given by those in (26) and

$$X_4 = 2x\frac{\partial}{\partial x} - 3(u+\kappa)\frac{\partial}{\partial u}, \tag{37}$$

$$X_5 = x^2\frac{\partial}{\partial x} - 3x(u+\kappa)\frac{\partial}{\partial u}. \tag{38}$$

2. Conclusions

In this paper, we revisited the group classification of the general nonlinear heat equation $u_t = \left(K(u)\,u_x\right)_x$, providing a detailed and systematic presentation of the method. Our analysis recovers the classical symmetry results:

- For an arbitrary $K(u)$, the equation admits a three-parameter Lie group;
- For $K(u) = \lambda(u+\kappa)^{\nu}$ (or its exponential limit $K(u) = \lambda e^{\nu u}$), the symmetry group extends to a four-parameter Lie group;
- For $K(u) = \frac{\lambda}{(\kappa+u)^{4/3}}$, the equation admits a five-parameter Lie group.

Beyond just reproducing these known results, our work offers a clear, step-by-step explanation of the classification process, making it accessible to researchers new to the group classification method. This clarity improves understanding and provides a guide for applying the method to other partial differential equations.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Acknowledgments: The author would like to thank the Directorate of Research Development and Innovation of Walter Sisulu University for their continued support. Additionally, the author is deeply grateful to the anonymous reviewers for their insightful and constructive feedback, which has significantly enhanced the quality of this manuscript.

Conflicts of Interest: The author declares no conflicts of interest.

References

1. Lie, S. On integration of a class of linear differential equations. *Arch. Math.* **1981**, *6*, 328–368. (In German)
2. Ovsyannikov, L.V. Group relations of the nonlinear heat equation. *Dokl. Akad. Nauk SSSR* **1959**, *125*, 492–495.
3. Bluman, G.; Kumei, S. On the remarkable nonlinear diffusion equation $\frac{\partial}{\partial x}\left[a(u+b)-2\frac{\partial u}{\partial x}\right]-\frac{\partial u}{\partial t}=0$. *J. Math. Phys.* **1980**, *21*, 1019–1023. [CrossRef]
4. Yung, C.M.; Verburg, K.; Baveye, P. Group classification and symmetry reductions of the nonlinear diffusion-convection equation $u_t=(D(u)\,u_x)_x-K'(u)\,u_x$. *Int. J. Nonlin. Mech.* **1994**, *29*, 273–278. [CrossRef]
5. Torrisi, M.; Tracina, R.; Valenti, A. A group analysis approach for a nonlinear differential system arising in diffusion phenomena. *J. Math. Phys.* **1996**, *37*, 4758–4767. [CrossRef]
6. Bruzón, M.S.; Gandarias, M.L.; Torrisi, M.; Tracina, R. On some applications of transformation groups to a class of nonlinear dispersive equations. *Nonlinear Anal. Real World Appl.* **2012**, *13*, 1139–1151. [CrossRef]
7. Leite Freire, I. Group classification of Burgers' equations. *arXiv* **2008**, arXiv:0804.3003.
8. Nadjafikhah, M.; Bakhshandeh-Chamazkoti, R. Symmetry group classification for general Burgers' equation. *Commun. Nonlinear Sci. Numer. Simul.* **2010**, *15*, 2303–2310. [CrossRef]
9. Sinkala, W.; Leach, P.G.L.; O'Hara, J.G. Invariance properties of a general bond-pricing equation. *J. Differ. Equ.* **2008**, *244*, 2820–2835. [CrossRef]
10. Arif, F.; Jhangeer, A.; Mahomed, F.M.; Zaman, F.D. Lie group classification and conservation laws of a (2+1)-dimensional nonlinear damped Klein–Gordon Fock equation. *Partial Differ. Equ. Appl. Math.* **2024**, *12*, 100962. [CrossRef]
11. Zhdanov, R.; Lahno, V. Group classification of the general evolution equation: Local and quasilocal symmetries. *Symmetry Integr. Geom.* **2005**, *1*, 9. [CrossRef]
12. Mogorosi, T.E.; Muatjetjeja, B. Group classification of a generalized coupled hyperbolic Lane–Emden system. *Iran. J. Sci. Technol. Trans. A.* **2019**, *43*, 273–278. [CrossRef]
13. Bluman, G.W. Construction of Solutions to Partial Differential Equations by the Use of Transformation Groups. Ph.D. Thesis, California Institute of Technology, Pasadena, CA, USA, 1967.
14. Bluman, G.W.; Cole, J.D. *Similarity Methods for Differential Equations*; Springer: New York, NY, USA, 2012.
15. Bluman, G.W.; Kumei, S. *Symmetries and Differential Equations*; Springer: New York, NY, USA, 1989.