



Article

A Note on New Near-Extremal Type I \mathbb{Z}_4 -Codes of Length 48

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Abstract: The subject of this work is self-dual Type I \mathbb{Z}_4 -codes of length 48. It is known that the usual upper bound for the minimum Euclidean weight for the self-dual \mathbb{Z}_4 -codes cannot be achieved for this length and type of codes; i.e., extremal Type I \mathbb{Z}_4 -codes of length 48 do not exist. We are therefore looking for near-extremal Type I \mathbb{Z}_4 -codes of length 48. The only known Type I near-extremal \mathbb{Z}_4 -code of length 48 was constructed in 2015 by M. Harada. We adapted a known pseudo-random search method and found at least two new near-extremal Type I \mathbb{Z}_4 -codes of length 48. These codes are the first Type I near-extremal \mathbb{Z}_4 -codes with a residue code of minimum weight 8.

Keywords: \mathbb{Z}_4 -code; Euclidean weight; self-dual code; Type I \mathbb{Z}_4 -code; near-extremal \mathbb{Z}_4 -code

MSC: 94B05; 94B65

1. Introduction

In this paper, we study codes over the ring \mathbb{Z}_4 . Interest in such codes was aroused by the discovery of their connection to classes of nonlinear binary codes such as Kerdock and Preparata codes [1]. Further interest developed in particular in the class of self-dual \mathbb{Z}_4 -codes, and all self-dual \mathbb{Z}_4 -codes with lengths up to 20 are fully classified [2–6]. This is motivated by the fact that some nice examples of nonlinear binary codes are generated as Gray images of such codes (for example, the Gray image of octacode is the Nordstrom–Robinson code [7] (p. 474)). And while \mathbb{Z}_4 codes were developed in the 20th century in conjunction with and in comparison to binary codes, some of the more recent research (for example, [8–11]) relates to their technical application. This arouses interest in the construction of self-dual \mathbb{Z}_4 codes, which is a very active field of research.

It is a well-known fact that self-dual \mathbb{Z}_4 -codes have all Euclidean weights divisible by four. Self-dual \mathbb{Z}_4 -codes are divided into Type I and Type II codes, the latter denoting those self-dual \mathbb{Z}_4 -codes in which all codewords have Euclidean weights divisible by eight. In this paper, we restrict our attention to the construction of Type I self-dual \mathbb{Z}_4 -codes, more precisely to the construction of near-extremal \mathbb{Z}_4 -codes of Type I and length 48. To our knowledge, only one such code is known so far (see [12]).

Extremal self-dual \mathbb{Z}_4 -codes are codes that have the largest possible minimum Euclidean weight. Particular attention was paid to codes whose length is divisible by 8. All extremal Type II \mathbb{Z}_4 -codes of length 24 were classified by R. A. L. Betty and A. Munemasa in [13]. Type I extremal \mathbb{Z}_4 -codes of length 24 do not exist (see [12]). All extremal \mathbb{Z}_4 -codes of length 24 are therefore known. This is the largest length for which a classification of extremal \mathbb{Z}_4 -codes is made. There are many known extremal \mathbb{Z}_4 -codes of length 32 and 40.



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To save space, we refer the reader to [14], where new codes of these lengths are constructed and information about known codes is given.

A small number of extremal \mathbb{Z}_4 -codes of length 48 is known. There are only two known extremal Type II \mathbb{Z}_4 -codes of length 48 (see [15,16]). In addition, as with length 24, it is shown in [12] that Type I extremal \mathbb{Z}_4 -codes of this length do not exist. Therefore, it makes sense to search for Type I near-extremal \mathbb{Z}_4 -codes of length 48. The only known Type I near-extremal \mathbb{Z}_4 -code of length 48 was constructed in 2015 by M. Harada (see [12]). In this work, we have constructed 145 Type I near-extremal \mathbb{Z}_4 -codes of length 48, including at least two inequivalent codes that were not known before.

This paper is organized as follows. In the next section, we present the basic facts that are used in our work. This is followed by Section 3, which describes the method we used to construct new Type I near-extremal \mathbb{Z}_4 -codes of length 48. The computational results are then presented in Section 4. In the same section, we describe the properties of the codes obtained and discuss the computational data. Finally, the generator matrices of the new Type I near-extremal \mathbb{Z}_4 -codes of length 48 are given in Appendix A.

2. Preliminaries

We assume that the reader is familiar with the basic facts of coding theory. We refer the reader to [7] in relation to terms not defined in this paper.

A binary linear $[n, k]$ code of length n is a k -dimensional vector subspace of the vector space \mathbb{F}_2^n , where \mathbb{F}_2 is the binary field. For $x \in \mathbb{F}_2^n$, the Hamming weight of x , denoted by $wt(x)$ is the number of non-zero coordinates of x . A binary code C is doubly even if $wt(x)$ is divisible by 4, for all $x \in C$. The number $d = \min\{wt(x) | x \in C, x \neq 0\}$ is called the minimum weight of the binary code C . A binary linear code C of length n , dimension k and minimum weight d is denoted as an $[n, k, d]$ code.

The dual code of the code C of length n is defined as

$$C^\perp = \{x \in \mathbb{F}_2^n | x \cdot y = 0, \forall y \in C\},$$

where $x \cdot y$ denotes the standard inner product in \mathbb{F}_2^n . The code C is self-orthogonal if $C \subseteq C^\perp$. It is a well-known fact that every doubly even binary code is self-orthogonal (see, for example, [7]).

A \mathbb{Z}_4 -code C of length n is the \mathbb{Z}_4 -submodule of \mathbb{Z}_4^n . Two \mathbb{Z}_4 -codes, C and C' of length n , are equivalent if there exists a monomial matrix M with non-zero elements in $\{1, 3\}$ such that $C' = \{vM | v \in C\}$. The elements of a \mathbb{Z}_4 -code C are called codewords of C . The Euclidean weight of the codeword $x \in C$ is defined as $wt_E(x) = n_1(x) + 4n_2(x) + n_3(x)$, where $n_i(x)$ denotes the number of coordinates in x equal to i , for $i = 1, 2, 3$. The minimum Euclidean weight of a \mathbb{Z}_4 -code C is the number $wt_E(C) = \min\{wt_E(x) | x \in C, x \neq 0\}$. The Euclidean weight distribution of the code C is represented by the polynomial $p_C(x) = \sum_{v \in C} x^{wt_E(v)}$. Both $wt_E(C)$ and $p_C(x)$ are invariants for the \mathbb{Z}_4 -codes. The dual code of the \mathbb{Z}_4 -code C of length n is defined as

$$C^\perp = \{x \in \mathbb{Z}_4^n | x \cdot y = 0, \forall y \in C\},$$

where $x \cdot y = x_1y_1 + x_2y_2 + \dots + x_ny_n$ for $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$. The code C is self-orthogonal if $C \subseteq C^\perp$ and self-dual if $C = C^\perp$. Every \mathbb{Z}_4 -code C contains a set of $k_1 + k_2$ codewords $\{c_1, \dots, c_{k_1}, c_{k_1+1}, \dots, c_{k_1+k_2}\}$ such that each codeword in C is uniquely expressible in the form

$$\sum_{i=1}^{k_1} a_i c_i + \sum_{i=k_1+1}^{k_1+k_2} a_i c_i,$$

where, for $1 \leq i \leq k_1$, $a_i \in \mathbb{Z}_4$, and c_i has at least one coordinate equal to 1 or 3, and, for $k_1 + 1 \leq i \leq k_1 + k_2$, $a_i \in \mathbb{Z}_2$, and c_i has all coordinates equal to 0 or 2. It is said that C is of type $4^{k_1}2^{k_2}$. The matrix whose rows are elements of the set $\{c_1, \dots, c_{k_1}, c_{k_1+1}, \dots, c_{k_1+k_2}\}$ is called the generator matrix of a \mathbb{Z}_4 -code C . There are two binary linear codes of length n associated with C . These codes are the residue code, defined as $Res(C) = \{c \pmod{2} \mid c \in C\}$, and the torsion code, defined as $Tor(C) = \{c \in \mathbb{F}_2^n \mid 2c \in C\}$.

Every self-dual \mathbb{Z}_4 -code is equivalent to a code that has a generator matrix of the following form:

$$G' = \begin{bmatrix} F & I_k + 2B \\ 2H & O \end{bmatrix}, \tag{1}$$

where F, B, H are binary matrices, I_k is the $k \times k$ identity matrix and O is a $(n - 2k) \times k$ zero matrix (see [7] (p. 497)). The type of such self-dual \mathbb{Z}_4 -code C is $4^k 2^{n-2k}$. If C has the generator matrix of the form (1), then the generator matrices of its residue code and its torsion code are

$$G'_{Res} = \begin{bmatrix} F & I_k \end{bmatrix}, \tag{2}$$

$$G'_{Tor} = \begin{bmatrix} F & I_k \\ H & O \end{bmatrix}. \tag{3}$$

The following theorem gives a characterization of self-dual \mathbb{Z}_4 -codes (see [6]).

Theorem 1. *Let C be a \mathbb{Z}_4 -code with a generator matrix of the form (1). The code C is self-dual if and only if $Res(C)$ is doubly even, $Res(C) = Tor(C)^\perp$, and B is such that rows of G' are orthogonal.*

In [17], it is shown that the lower diagonal elements of B are uniquely determined by the upper diagonal elements by the following condition:

$$b_{ji} = \begin{cases} b_{ij}, & f_i \cdot f_j \equiv 0 \pmod{4}, \\ b_{ij} + 1, & f_i \cdot f_j \equiv 2 \pmod{4}, \end{cases} \tag{4}$$

where f_s denotes the s -th row vector of the matrix F , $1 \leq i < j \leq k$, $1 \leq s \leq k$. The diagonal elements can be chosen freely, and for different choices of diagonal elements, we obtain monomially equivalent \mathbb{Z}_4 -codes. The previous discussion gives the standard way to construct a self-dual \mathbb{Z}_4 -code starting from any doubly even binary code. Given a doubly even binary code $C^{(1)}$ with a generator matrix of the form (2), one can find a dual code $C^{(2)}$ with a generator matrix of the form (3). Then, it only remains to choose one of the possible $2^{\frac{k(k-1)}{2}}$ suitable choices for the matrix B to obtain a generator matrix of a \mathbb{Z}_4 -code of the form (1).

The upper bounds on the minimum Euclidean weight of self-dual \mathbb{Z}_4 -codes are known (see [7] (p. 495)) and are given in the following theorem.

Theorem 2. *Let C be a self-dual \mathbb{Z}_4 -code of length n . If C is Type II, then the minimum Euclidean weight of C is at most $8 \lfloor \frac{n}{24} \rfloor + 8$. If C is Type I, then the minimum Euclidean weight of C is at most $8 \lfloor \frac{n}{24} \rfloor + 8$ except when $n \equiv 23 \pmod{24}$, in which case the bound is $8 \lfloor \frac{n}{24} \rfloor + 12$. If equality holds in this latter bound, then C is obtained by shortening a Type II code of length $n + 1$.*

A self-dual \mathbb{Z}_4 -code is called extremal if its minimum Euclidean weight meets the bounds from the previous theorem. Let C be a self-dual \mathbb{Z}_4 -code of length n , and let d_E be the upper bound on $wt_E(C)$ from Theorem 2. The code C is a near-extremal \mathbb{Z}_4 -code if $wt_E(C) = d_E - 4$ (see [12]).

In [12], for $k \geq 2$, \mathbb{Z}_k -codes (defined as \mathbb{Z}_k -submodules of \mathbb{Z}_k^n) are studied. In our work, we will need the following fact.

Lemma 1 ([12] (Lemma 2.5, b)). *Let C be a Type I \mathbb{Z}_k -code of length n . If $n = 48$, then $wt_E(C) \leq 5k$.*

We end this section with brief information about Hadamard designs and codes spanned by their incidence matrices. For further background reading on this topic, we refer the reader to [18,19].

An Hadamard matrix of order n is a square matrix H of order n such that every entry of H is in $\{-1, 1\}$, and $HH^T = H^T H = nI_n$. An Hadamard matrix H of order n is normalized (or in standard form) if every entry of the first row and the first column of H is equal to 1. Two Hadamard matrices of order n are equivalent if one can be obtained from another by permuting rows or columns and by the multiplication of any row/column by -1 . It is well established that every Hadamard matrix is equivalent to some normalized Hadamard matrix of the same order. Additionally, every normalized Hadamard matrix H of order n yields a $3 - \left(n, \frac{n}{2}, \frac{n}{4} - 1\right)$ design (see [18]). Such designs have an incidence matrix of the form

$$\left[\begin{array}{c|c} 1 & M \\ \hline 0 & J-M \end{array} \right],$$

where J is the all-ones matrix, and M is the matrix obtained from H by removing its first row and column of ones and interchanging every -1 element with 0. The following corollary gives an important fact about codes generated by these incidence matrices (see [20] (Corollary 3.3)).

Corollary 1. *Let H be a normalized Hadamard matrix of order $n \equiv 0 \pmod{8}$. Then, the incidence matrix of a corresponding $3 - \left(n, \frac{n}{2}, \frac{n}{4} - 1\right)$ design spans a doubly even binary code of length n .*

Therefore, such codes can be used as residue codes of self-dual \mathbb{Z}_4 -codes.

3. Method of Construction

One of the major obstacles in the construction of near-extremal \mathbb{Z}_4 -codes is the large search space (a total of $2^{\frac{k(k-1)}{2}}$ codes) and the time-consuming determination of the minimum Euclidean weight of a code. To tackle this problem, a method for determining the minimum Euclidean weight of a larger number of codes based on the known Euclidean weight distribution of a code was developed in [14]. This construction was used to obtain new extremal \mathbb{Z}_4 -codes of lengths 32 and 40. Here, we give a brief overview of the main results from [14], as we will use the same algorithm to perform an efficient random search for near-extremal \mathbb{Z}_4 -codes of length 48.

Let C be a self-dual \mathbb{Z}_4 -code with the generator matrix $G(B)$ in the form (1), i.e.,

$$G(B) = \begin{bmatrix} F & I_k + 2B \\ 2H & O \end{bmatrix}, \tag{5}$$

where $B = [b_{s,t}]$ is a $k \times k$ binary matrix for which the self-duality condition (4) holds. Let (i, j) , $1 \leq i < j \leq k$, be an upper diagonal position of B such that $b_{ij} = 0$. Let $B' = [b'_{s,t}]$ be a $k \times k$ binary matrix such that

$$b'_{st} = \begin{cases} b_{st}, & (s, t) \neq (i, j) \\ 1, & (s, t) = (i, j), \end{cases} \tag{6}$$

for all $1 \leq s < t \leq k$, and such that the \mathbb{Z}_4 -code C' is generated by $G(B')$, i.e.,

$$G(B') = \begin{bmatrix} F & I_k + 2B' \\ 2H & O \end{bmatrix}, \tag{7}$$

is self-dual (condition (4) holds for B'). We say that B' is the (i, j) -neighbor of B , and that the code C' is the (i, j) -neighbor of the code C .

Let $v \in C$ be of the form

$$v = c_i g_i + c_j g_j + \sum_{\substack{m=1 \\ m \neq i,j}}^k c_m g_m + \sum_{m=k+1}^{n-k} c_m g_m, \tag{8}$$

where $g_s, s \in \{1, 2, \dots, n - k\}$ is the s -th row of the matrix $G(B)$. Let $v' \in C'$ be

$$v' = c_i g'_i + c_j g'_j + \sum_{\substack{m=1 \\ m \neq i,j}}^k c_m g_m + \sum_{m=k+1}^{n-k} c_m g_m, \tag{9}$$

i.e., v' is the codeword in C' generated by the same coefficients c_1, c_2, \dots, c_{n-k} as v .

It was proven in [14] that $wt_E(v') = wt_E(v) + 4r$, where the value of the parameter r is represented in Table 1 for all possible coefficients c_i and c_j . In that table, $|I|$ and $|J|$ represent the number of non-zero off-diagonal entries in columns i and j of the matrix B . The parity of the size of sets marked by $*$ does not impact the value of r . For all combinations of the coefficients c_i and c_j that are not in Table 1, $r = 0$ holds.

Table 1. The value of the parameter r .

$b_{i,j} = b_{j,i}$					$b_{i,j} \neq b_{j,i}$				
c_i	c_j	$ I (\text{Mod } 2)$	$ J (\text{Mod } 2)$	r	c_i	c_j	$ I (\text{Mod } 2)$	$ J (\text{Mod } 2)$	r
0	1,3	0	*	1	0	1,3	0	*	-1
		1	*	-1			1	*	1
1,3	0	*	0	1	1,3	0	*	0	1
		*	1	-1			*	1	-1
2	1,3	0	*	-1	2	1,3	0	*	1
		1	*	1			1	*	-1
1,3	2	*	0	-1	1,3	2	*	0	-1
		*	1	1			*	1	1

The following statement is a direct consequence of Theorem 3.6. from [14].

Corollary 2. Let C be a self-dual \mathbb{Z}_4 -code of length 48 and C' its (i, j) -neighbor. Let $p(x) = \sum_{l=0}^{48} A_{4l} x^{4l}$ and $p'(x) = \sum_{l=0}^{48} A'_{4l} x^{4l}$ be the Euclidean weight enumerators of C and C' , respectively. For $m \in \{0, 4, \dots, 48\}$, let S_m and S'_m denote the sets of codewords of Euclidean weight m from C and C' , respectively. For $l = 1, 2, \dots, 48$, we define the following sets and their cardinality:

$$\begin{aligned} S_{4l}^- &= \{v \in S_{4l} | v' \in S'_{4l-4}\}, & |S_{4l}^-| &= A_{4l}^-, \\ S_{4l}^0 &= \{v \in S_{4l} | v' \in S'_{4l}\}, & |S_{4l}^0| &= A_{4l}^0, \\ S_{4l}^+ &= \{v \in S_{4l} | v' \in S'_{4l+4}\}, & |S_{4l}^+| &= A_{4l}^+, \end{aligned}$$

where for $v \in C$, v' is the codeword from C' generated by the same coefficients as v . Then,

$$A'_4 + A'_8 + \dots + A'_{16} = A_4 + A_8 + \dots + A_{16} - A_{16}^+ + A_{20}^-.$$

Based on this corollary, we use the following pseudo-random search for near-extremal \mathbb{Z}_4 -codes of length 48:

1. Start with an arbitrary matrix B .
2. In each iteration of the algorithm do the following.
 - 2.1 Generate a self-dual \mathbb{Z}_4 -code C with the generator matrix $G(B)$, and evaluate

$$D = |\{v \in C \mid 0 < wt_E(v) < 20\}|,$$
 (this is $A_4 + A_8 + \dots + A_{16}$ from Corollary 2).
 - 2.2 If $D = 0$ then C is near-extremal.
 - 2.3 Determine the sets S_{16} and S_{20} defined in the Corollary 2.
 - 2.4 For every upper diagonal element (i, j) of the matrix B that is equal to 0, determine the (i, j) -neighbor C' . If the near-extremality of the corresponding neighbor is undetermined, from Table 1, evaluate the numbers A_{16}^+, A_{20}^- and $d = D - A_{16}^+ + A_{20}^-$.
 - 2.5 All C' that have $d = 0$ are near-extremal.
 - 2.6 Mark all neighbors of B as checked.
 - 2.7 Repeat the process with the next unchecked and randomly generated matrix B .

4. Computational Results

In this section, we use the described algorithm for the construction of new near-extremal Type I \mathbb{Z}_4 -codes of length 48. The following upper bound for Type I \mathbb{Z}_4 -codes of length 48 is derived from Lemma 1 for the special case of $k = 4$.

Proposition 1. *Let C be a Type I \mathbb{Z}_4 -code of length 48. Then, $wt_E(C) \leq 20$.*

According to Theorem 2, since for $n = 48$ $n \not\equiv 23 \pmod{24}$, the upper bound for $wt_E(C)$ is $8 \lfloor \frac{48}{24} \rfloor + 8 = 24$. Therefore, an extremal Type I \mathbb{Z}_4 -code of length 48 should have $wt_E(C) = 24$. But Proposition 1 shows that this is impossible. Therefore, there are no extremal \mathbb{Z}_4 -codes of Type I of length 48. Furthermore, a self-dual Type I \mathbb{Z}_4 -code C of length 48 is near-extremal if $wt_E(C) = 20$. Moreover, based on the previous discussion, such codes have the largest possible minimum Euclidean weight among Type I \mathbb{Z}_4 -codes of length 48.

The following statement gives the necessary condition on the minimum weight of the torsion code of a near-extremal \mathbb{Z}_4 -code of length 48.

Proposition 2. *If C is a Type I near-extremal \mathbb{Z}_4 -code of length 48, then $Tor(C)$ has a minimum weight $d \geq 5$.*

Proof. Since C is near-extremal, $wt_E(x) \geq 20$ for all $x \in C$. By definition, $x = 2y$, for some $y \in Tor(C) \subset \mathbb{F}_2^{48}$. Therefore, $4wt(y) = wt_E(2y)$, which completes the proof. \square

As described, we start our construction from doubly even binary codes of length 48. To obtain such codes, we used 53 Hadamard matrices, given in [21]. From these matrices, we constructed incidence matrices of the corresponding Hadamard 3 – (48, 24, 11) designs and span doubly even binary codes (see Corollary 1). In that way, we obtained six inequivalent

doubly even binary codes $C'_{48,1}, C'_{48,2}, \dots, C'_{48,6}$. The weight distributions of codes $C'_{48,1}, C'_{48,2}, \dots, C'_{48,6}$ are given in Table 2.

Table 2. Parameters and the weight distribution of binary codes $C'_{48,1}, C'_{48,2}, \dots, C'_{48,6}$.

The Code	$[n, k, d]$	0	4	8	12	16	20	24
$C'_{48,1}$	$[48, 24, 4]$	1	276	10,626	134,596	735,471	1,961,256	11,092,764
$C'_{48,2}$	$[48, 24, 8]$	1	0	276	15,088	542,823	3,979,920	7,701,000
$C'_{48,3}$	$[48, 24, 8]$	1	0	138	16,192	538,959	3,987,648	7,691,340
$C'_{48,4}$	$[48, 24, 12]$	1	0	0	17,296	535,095	3,995,376	7,681,680
$C'_{48,5}$	$[48, 24, 8]$	1	0	138	16,192	538,959	3,987,648	7,691,340
$C'_{48,6}$	$[48, 24, 8]$	1	0	1518	5152	577,599	3,910,368	7,787,940

The code $C'_{48,1}$ has a minimum weight of 4, so this code is not suitable for the construction of near-extremal \mathbb{Z}_4 -codes (see Proposition 2). Since the minimum weights of the codes $C'_{48,2}, C'_{48,3}, \dots, C'_{48,6}$ are greater than 5, these codes are suitable for the construction of near-extremal \mathbb{Z}_4 codes according to Proposition 2. The code $C'_{48,4}$ is the residue code of an already known near-extremal \mathbb{Z}_4 -code, which was constructed by Harada in [12].

We have applied the pseudo-random search procedure described in the previous section to the binary codes $C'_{48,2}, C'_{48,3}, \dots, C'_{48,6}$. The only residue codes that resulted in near-extremal \mathbb{Z}_4 -codes are $C'_{48,4}$ and $C'_{48,6}$. The number of codewords with a Euclidean weight of 20 in all near-extremal Type I \mathbb{Z}_4 -codes constructed from $C'_{48,4}$ is the same as in the known near-extremal \mathbb{Z}_4 -code constructed by Harada. Further, a total of 145 near-extremal Type I \mathbb{Z}_4 -codes are obtained from the code $C'_{48,6}$. Their generator matrices are available at <http://www.math.uniri.hr/~sanjar/structures/> (accessed on 5 February 2025).

Since $C'_{48,6}$ is self-dual, all the obtained near-extremal \mathbb{Z}_4 -codes are of type 4^{24} . At least two of these 145 \mathbb{Z}_4 -codes are non-equivalent. This fact is established by looking at the number of codewords with a Euclidean weight of 20. In these near-extremal Type I \mathbb{Z}_4 -codes, the numbers of codewords with a Euclidean weight of 20 are 307,520 and 308,752. Since the binary codes $C'_{48,4}$ and $C'_{48,6}$ are not equivalent (different minimum weights), the near-extremal Type I \mathbb{Z}_4 -codes obtained from $C'_{48,6}$ are new. As far as we know, these are the first near-extremal Type I \mathbb{Z}_4 -codes constructed from this binary residue code and the first such codes to have the binary residue code with a minimum weight of 8.

The generator matrices of the two new near-extremal Type I \mathbb{Z}_4 -codes of length 48, together with the generator matrix of the binary code $C'_{48,6}$, are given in Appendix A. In Appendix A.2 of Appendix A, only the matrix F is given, since $C'_{48,6}$ is self-dual and has the generator matrix of the form $[F \ I_{24}]$. The matrices B_1 and B_2 from Appendix A.1 are the matrices that give the generator matrices $G(B_1)$ and $G(B_2)$ of the new near-extremal codes. The number of codewords of Euclidean weight 20 is 307,520 for the code generated by the $G(B_1)$ and 308,752 for the code generated by $G(B_2)$.

All the computations in this work were carried out using a computer algebra system MAGMA [22].

Remark 1. All of the constructed near-extremal codes have the minimum Lee weight of 16. The Gray images of the constructed codes are non-linear binary codes of length 96. For more information on Lee weights and Gray images, see [7] (ch. 12.2.). We provide this information as it can be useful in further analysis of the error-correcting performance of the obtained codes.

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Appendix A. Generator Matrices

Appendix A.1. Matrices for New Near-Extremal Codes

$$B_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

References

1. Hammons, A.R.; Kumar, P.V.; Calderbank, A.R.; Sloane, N.J.A.; Solé, P. The \mathbb{Z}_4 -linearity of Kerdock, Preparata, Goethals, and related codes. *IEEE Trans. Inform. Theory* **1994**, *IT-40*, 301–319. [[CrossRef](#)]
2. Betty, R.A.L.; Munemasa, A. Classification of Self-dual Codes of Length 20 over \mathbb{Z}_4 and Length at Most 18 over $\mathbb{F}_2 + u\mathbb{F}_2$. In *Cryptography and Coding—17th IMA International Conference*; Albrecht, M., Ed.; Springer: Cham, Switzerland, 2019; pp. 64–77.
3. Conway, J.H.; Sloane, N.J.A. Self-dual codes over the integers modulo 4. *J. Comb. Theory Ser. A* **1993**, *62*, 30–45. [[CrossRef](#)]
4. Fields, J.; Gaborit, P.; Leon, J.S.; Pless, V. All self-dual \mathbb{Z}_4 codes of length 15 or less are known. *IEEE Trans. Inf. Theory* **1998**, *44*, 311–322. [[CrossRef](#)]
5. Harada, M.; Munemasa, A. On the Classification of Self-dual \mathbb{Z}_k -Codes. In *Cryptography and Coding*; Parker, M.G., Ed.; Springer: Berlin/Heidelberg, Germany, 2009; pp. 78–90.
6. Pless, V.; Leon, J.S.; Fields, J. All \mathbb{Z}_4 Codes of Type II and Length 16 Are Known. *J. Comb. Theory Ser. A* **1997**, *78*, 32–50. [[CrossRef](#)]
7. Huffman, W.C.; Pless, V. *Fundamentals of Error-Correcting Codes*; Cambridge University Press: Cambridge, UK, 2003.
8. Alahmadi, A.N.; Melibari, F.A.; Gupta, M.K. DNA Code Design Based on the Cosets of Codes over \mathbb{Z}_4 . *Mathematics* **2023**, *11*, 4732. [[CrossRef](#)]
9. Benkhouya, R.; Chana, I.; Hadi, Y. Codes over \mathbb{Z}_4 for PAPR and BER reduction in OFDM communication system. In Proceedings of the 3rd International Conference on Smart City Applications (SCA '18), Association for Computing Machinery, Tetouan, Morocco, 10–11 October 2018; pp. 1–7.
10. Feng, B.; Bai, S.S.; Chen, B.Y.; Zhou, X.N. The Constructions of DNA Codes from Linear Self-Dual Codes over \mathbb{Z}_4 . In Proceedings of the International Conference on Computer Information Systems and Industrial Applications (CISIA 2015), Bangkok, Thailand, 28–29 June 2015; pp. 496–498.
11. Rocha, A.S.L.L.; Faria, L.C.B.; Kleinschmidt, J.H.; Palazzo, R.; Silva-Filho, M.C. DNA sequences generated by \mathbb{Z}_4 -linear codes. In Proceedings of the IEEE International Symposium on Information Theory (ISIT 2010), Austin, TX, USA, 13–18 June 2010; pp. 1320–1324.
12. Harada, M. Extremal Type I \mathbb{Z}_k -codes and k -frames of odd unimodular lattices. *IEEE Trans. Inf. Theory* **2015**, *61*, 72–81. [[CrossRef](#)]
13. Betty, R.A.L.; Munemasa, A. Classification of Extremal Type II \mathbb{Z}_4 -codes of Length 24. In *WCC 2022: The Twelfth International Workshop on Coding and Cryptography*; University of Rostock: Rostock, Germany, 2022.
14. Crnković, D.; Mravić, M.; Rukavina, S. Construction of extremal \mathbb{Z}_4 -codes using a neighborhood search algorithm. *Adv. Math. Commun.* **2025**, *19*, 91–108. [[CrossRef](#)]
15. Bonnetcaze, A.; Solé, P.; Bachoc, C.; Mourrain, B. Type II codes over \mathbb{Z}_4 . *IEEE Trans. Inf. Theory* **1997**, *43*, 969–976. [[CrossRef](#)]
16. Harada, M.; Kitazume, M.; Munemasa, A.; Venkov, B. On some self-dual codes and unimodular lattices in dimension 48. *Eur. J. Comb.* **2005**, *26*, 543–557. [[CrossRef](#)]
17. Gaborit, P. Mass formulas for self-dual codes over \mathbb{Z}_4 and $\mathbb{F}_q + u\mathbb{F}_q$ rings. *IEEE Trans. Inf. Theory* **1996**, *42*, 1222–1228. [[CrossRef](#)]
18. Craigen, R.; Kharaghani, H. Hadamard matrices and Hadamard designs. In *Handbook of Combinatorial Designs*; Colbourn, C.J., Dinitz, J.H., Eds.; Chapman & Hall/CRC Press: Boca Raton, FL, USA, 2007; pp. 273–280.
19. Tonchev, V.D. On Pless symmetry codes, ternary QR codes, and related Hadamard matrices and designs. *Des. Codes Cryptogr.* **2022**, *90*, 2753–2762. [[CrossRef](#)]
20. Ban, S.; Crnković, D.; Mravić, M.; Rukavina, S. New extremal Type II \mathbb{Z}_4 -codes of length 32 obtained from Hadamard matrices. *Discret. Math. Algorithms Appl.* **2019**, *11*, 1950057. [[CrossRef](#)]
21. Kotsireas, I.S.; Koukouvinos, C.; Seberry, J. Hadamard ideals and Hadamard matrices with two circulant cores. *Eur. J. Comb.* **2006**, *27*, 658–668. [[CrossRef](#)]
22. Bosma, W.; Cannon, J.; Playoust, C. The Magma algebra system. I. The user language. *J. Symb. Comput.* **1993**, *24*, 235–265. [[CrossRef](#)]

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