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# On New Generalized Mitrinović-Adamović-Type Inequalities

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**Abstract:** In this paper, we establish new generalized Mitrinović-Adamović-type inequalities in a wider range $(0, \pi)$ by using the monotonicity of certain functions. These inequalities contain sharp and tractable bounds for the function $\left(\frac{\sin x}{x}\right)^3$. All the main results are also true in $(-\pi, 0)$ due to the symmetry of the curves involved.

**Keywords:** Mitrinović–Adamović inequality; Bernoulli numbers; Qi's inequality; monotonicity of function; circular function

**MSC:** 26D05; 26D15; 42A10

## 1. Introduction

An obvious relation $\cos x < \frac{\sin x}{x}$, $x \in (0, \pi/2)$ was refined in [1,2] as

$$\cos x < \left(\frac{\sin x}{x}\right)^3, \quad x \in \left(0, \frac{\pi}{2}\right). \tag{1}$$

The inequality (1) is known in the literature as Mitrinović–Adamović inequality. In recent years, some mathematicians tried to refine and extend the inequality (1). For instance, the inequality

$$\cos^4 \frac{x}{2} < \left(\frac{\sin x}{x}\right)^3, \quad x \in \left(0, \frac{\pi}{2}\right)$$

appeared in the references [3–5]. Mortici [6] and Chouikha [7] independently obtained, respectively, the following double inequalities:

$$\cos x + \frac{x^4}{15} - \frac{23x^6}{1890} < \left(\frac{\sin x}{x}\right)^3 < \cos x + \frac{x^4}{15}, \quad x \in \left(0, \frac{\pi}{2}\right)$$

and

$$\cos x + x^3\left(1 - \frac{x^2}{63}\right)\frac{\sin x}{15} < \left(\frac{\sin x}{x}\right)^3 < \cos x + \frac{x^3 \sin x}{15}, \quad x \in \left(0, \frac{\pi}{2}\right).$$

We observe that it is not difficult to show the validity of (1) in an interval $(0, \pi)$. In this direction, Zhu [8] achieved the inequalities

$$\cos x < \left(\frac{\sin x}{x}\right)^3 < \frac{8}{\pi^3} + \left(1 - \frac{8}{\pi^3}\right)\cos x, \quad x \in \left(0, \frac{\pi}{2}\right),$$

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$$\frac{8}{\pi^3} + \left(1 - \frac{8}{\pi^3}\right)\cos x < \left(\frac{\sin x}{x}\right)^3 < \frac{1+\cos x}{2}, \quad x \in \left(\frac{\pi}{2}, \pi\right)$$

and

$$\left(1 - \frac{32}{\pi^5}x^2\right)\cos x + \frac{32}{\pi^5}x^2 < \left(\frac{\sin x}{x}\right)^3 < \left(1 - \frac{2}{15}x^2\right)\cos x + \frac{2}{15}x^2, \quad x \in \left(0, \frac{\pi}{2}\right),$$

$$\left(1 - \frac{1}{2\pi^2}x^2\right)\cos x + \frac{1}{2\pi^2}x^2 < \left(\frac{\sin x}{x}\right)^3 < \left(1 - \frac{32}{\pi^5}x^2\right)\cos x + \frac{32}{\pi^5}x^2, \quad x \in \left(\frac{\pi}{2}, \pi\right).$$

On the other hand, W.-D. Jiang [9] very recently maintained the uniformity and sharpness of the bounds for $\left(\frac{\sin x}{x}\right)^3$ in a wider range $(0, \pi)$ and established a better double inequality

$$\cos x + \frac{21}{5}\frac{x^3 \sin x}{63 + x^2} < \left(\frac{\sin x}{x}\right)^3 < \cos x + \frac{\pi^2}{15}\frac{x^3 \sin x}{\pi^2 - x^2}, \quad x \in (0, \pi) \tag{2}$$

along with

$$\cos x + \frac{x^3 \sin x}{15 + \frac{5}{21}x^2} < \left(\frac{\sin x}{x}\right)^3 < \cos x + \frac{x^3 \sin x}{15 + \frac{\pi^6 - 960}{16\pi^2}x^2}, \quad x \in (0, \pi/2). \tag{3}$$

We also refer the reader to [10–24] and the references therein for more information on this topic.

In this work, by using the monotonicity of certain functions, we establish new generalized Mitrinović–Adamović-type inequalities in a wider range $(0, \pi)$. We obtain sharp bounds for $\left(\frac{\sin x}{x}\right)^3$ in a wider range $(0, \pi)$ and refine double inequalities (2)–(3).

## 2. Lemmas for Bernoulli Numbers

Recall that the Bernoulli numbers $B_n$ can be generated by

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = 1 - \frac{z}{2} + \sum_{k=1}^{\infty} B_{2k}\frac{z^{2k}}{(2k)!}, \quad |z| < 2\pi,$$

and all of the Bernoulli numbers $B_{2k+1}$ for $k \in \mathbb{N}$ equal 0. For more details, we refer the interested readers to the research monograph [25].

To prove our main results, we need the following important lemmas.

**Lemma 1** (see [8,9,25])**.** *Let $B_{2k}$ $(k \in \mathbb{N})$ be the even indexed Bernoulli numbers. Then, for $|x| < \pi$, the following identities hold:*

$$\frac{x}{\sin x} = 1 + \sum_{k=1}^{\infty} \frac{2(2^{2k-1} - 1)|B_{2k}|}{(2k)!}x^{2k},$$

$$\cot x = \frac{1}{x} - \sum_{k=1}^{\infty} \frac{2^{2k}|B_{2k}|}{(2k)!}x^{2k-1},$$

$$\frac{1}{\sin^2 x} = \frac{1}{x^2} + \sum_{k=1}^{\infty} \frac{2^{2k}(2k-1)|B_{2k}|}{(2k)!}x^{2k-2},$$

$$\frac{\cos x}{\sin^2 x} = \frac{1}{x^2} - \sum_{k=1}^{\infty} \frac{2(2k-1)(2^{2k-1} - 1)|B_{2k}|}{(2k)!}x^{2k-2},$$

$$\frac{\cos x}{\sin^3 x} = \frac{1}{x^3} - \sum_{k=2}^{\infty} \frac{2^{2k}(2k-1)(k-1)|B_{2k}|}{(2k)!} x^{2k-3},$$

$$\frac{1}{\sin^3 x} = \frac{1}{x^3} + \frac{1}{2} \sum_{k=2}^{\infty} \frac{(2^{2k}-2)(2k-1)(2k-2)|B_{2k}|}{(2k)!} x^{2k-3} + \frac{1}{2x} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{(2^{2k}-2)|B_{2k}|}{(2k)!} x^{2k-1},$$

$$\frac{\cos x}{\sin^4 x} = -\frac{1}{3x^3} - \frac{1}{6x} - \sum_{k=1}^{\infty} \frac{(2^{2k}-2)(2k-1)}{6 \cdot (2k)!} |B_{2k}| x^{2k-2} - \sum_{k=2}^{\infty} \frac{(2^{2k}-2)(2k-1)(2k-2)(2k-3)}{6 \cdot (2k)!} |B_{2k}| x^{2k-4},$$

*and*

$$\frac{1}{\sin^4 x} = \frac{1}{x^4} + \sum_{k=1}^{\infty} \frac{2^{2k}(2k-1)(2k-2)(2k-3)}{6 \cdot (2k)!} |B_{2k}| x^{2k-4} + \frac{2}{3x^2} + \sum_{k=1}^{\infty} \frac{2^{2k+1}(2k-1)}{3 \cdot (2k)!} |B_{2k}| x^{2k-2}.$$

**Lemma 2** (see [26]). *For $k \in \mathbb{N}$, we have*

$$|B_{2k}| > \frac{2(2k)!}{\pi^{2k}(2^{2k}-1)}, \tag{4}$$

*where $B_{2k}$ are the even indexed Bernoulli numbers.*

The following well-known Qi's inequality for Bernoulli numbers is crucial in this paper.

**Lemma 3** (see [27]). *For $k \in \mathbb{N}$, the even indexed Bernoulli numbers satisfy*

$$\frac{|B_{2k+2}|}{|B_{2k}|} > \frac{2^{2k-1}-1}{2^{2k+1}-1} \frac{(2k+1)(2k+2)}{\pi^2}. \tag{5}$$

We now establish some new auxiliary results for even indexed Bernoulli numbers.

**Lemma 4.** *For $k = 4, 5, 6, \cdots$, we have*

$$|B_{2k}| > \frac{2k}{3(2^{2k}-1)}. \tag{6}$$

**Proof.** It is obvious that

$$3(2k)! > \pi^{2k} \cdot k, \text{ for } k = 4, 5, 6, \cdots.$$

From this, we write

$$\frac{2(2k)!}{\pi^{2k}(2^{2k}-1)} > \frac{2k}{3(2^{2k}-1)}, \text{ for } k = 4, 5, 6, \cdots.$$

Combining this with the inequality (4), we obtain the inequality (6). □

**Lemma 5.** *For $k = 4, 5, 6, \cdots$, it is true that*

$$\frac{|B_{2k+2}|}{|B_{2k}|} > \frac{2(2k-1)(2k+1)(2k+2)}{3(2^{2k+2}-1)}. \tag{7}$$

**Proof.** Since,

$$2^{2k+2} - 1 > 2^{2k+1} - 1$$

and

$$3(2^{2k-1} - 1) > 2\pi^2(2k-1),$$

for $k = 4, 5, 6, \cdots$, we obtain

$$3(2^{2k-1} - 1)(2^{2k+2} - 1) > 2\pi^2(2k-1)(2^{2k+1} - 1),$$

for $k = 4, 5, 6, \cdots$. Equivalently, we write

$$\frac{(2^{2k-1}-1)(2k+1)(2k+2)}{\pi^2(2^{2k+1}-1)} > \frac{2(2k-1)(2k+1)(2k+2)}{3(2^{2k+2}-1)}, \ k = 4, 5, 6, \cdots.$$

Combining this with (5), we obtain the desired inequality (7). □

**Lemma 6.** *Let $k = 3, 4, 5, \cdots$. Then, it holds that*

$$\frac{|B_{2k+2}|}{|B_{2k}|} > \frac{(2k+1)(2k+2)P(k)}{3\left[16k(2^{2k+2}-2)(2k+1) - 14k(2k+1)\cdot 2^{2k+2} - 15\cdot 2^{2k+2}\right]},$$

*where* $P(k) = \left[16(2k-1)(2^{2k-1}-1) + 48(2^{2k-1}-1) - 24(2^{2k}-2) - 7\cdot 2^{2k}(2k-1)\right]$.

**Proof.** In view of Lemma 3, it suffices to prove

$$\frac{(2^{2k-1}-1)}{\pi^2(2^{2k+1}-1)} > \frac{16(2k-1)(2^{2k-1}-1) + 48(2^{2k-1}-1) - 24(2^{2k}-2) - 7\cdot 2^{2k}(2k-1)}{3\left[16k(2k+1)(2^{2k+2}-2) - 14k(2k+1)\cdot 2^{2k+2} - 15\cdot 2^{2k+2}\right]}$$

for $k = 3, 4, 5, \cdots$. Equivalently,

$$\begin{aligned}&\pi^2(2^{2k+1}-1)\left[16(2k-1)(2^{2k-1}-1) + 48(2^{2k-1}-1) - 24(2^{2k}-2) - 7\cdot 2^{2k}(2k-1)\right]\\ &\quad < 3(2^{2k-1}-1)\left[16k(2k+1)(2^{2k+2}-2) - 14k(2k+1)\cdot 2^{2k+2} - 15\cdot 2^{2k+2}\right].\end{aligned}$$

A direct computation gives

$$\begin{aligned}&(4\pi^2\cdot 2^{2k} - 2\pi^2)(2k\cdot 2^{2k} - 2^{2k} - 32k + 16)\\ &< (3\cdot 2^{2k} - 6)(16k^2\cdot 2^{2k} - 64k^2 + 8k\cdot 2^{2k} - 60\cdot 2^{2k} - 32k),\end{aligned}$$

which can be put as

$$\begin{aligned}&\left[(48k^2 + 24k + 4\pi^2)\cdot 2^{2k} + (32\pi^2 k + 360)\right]\cdot 2^{2k} + (384k^2 + 192k + 32\pi^2)\\ &> \left[(8\pi^2 k + 180)\cdot 2^{2k} + 288k^2 + (66\pi^2 + 144k)\right]\cdot 2^{2k} + 64\pi^2 k.\end{aligned}$$

Since

$$384k^2 + 192k + 32\pi^2 > 64\pi^2 k,$$

for $k = 3, 4, 5, \cdots$, we only need to prove that

$$Q(k) > 0, \text{ for } k = 3, 4, 5, \cdots,$$

where

$$Q(k) = (48k^2 + 24k + 4\pi^2 - 8\pi^2 k - 180) \cdot 2^{2k} + (32\pi^2 k + 360 - 288k^2 - 66\pi^2 - 144k).$$

Now, $k \geq 3$ implies

$$\begin{aligned} Q(k) &\geq (48k^2 + 24k + 4\pi^2 - 8\pi^2 k - 180) \cdot 64 + (32\pi^2 k + 360 - 288k^2 - 66\pi^2 - 144k) \\ &= 2784k^2 + 1392k + 190\pi^2 - 480\pi^2 k - 11160 \\ &\geq 2784 \cdot 3k + 1392k + 190\pi^2 - 480\pi^2 k - 11160 \\ &= (9744 - 480\pi^2)k + (190\pi^2 - 11160) \\ &\geq (9744 - 480\pi^2) \cdot 3 + (190\pi^2 - 11160) = (29232 + 190\pi^2) - 12600 > 0. \end{aligned}$$

□

## 3. Some New Generalized Mitrinović–Adamović-Type Inequalities

We are now in a position to state and prove our main results.

**Theorem 1.** *The function*

$$f(x) = \frac{\left(\frac{\sin x}{x}\right)^3 - \cos x}{(1 - \cos x)^2}$$

*is strictly decreasing on* $(0, \pi)$.

**Proof.** The differentiation of $f(x)$ gives

$$\begin{aligned} &(1 - \cos x)^4 f'(x) \\ &\quad = (1 - \cos x)\left[(1 - \cos x)\left(\frac{3\sin^2 x(x\cos x - \sin x)}{x^4} + \sin x\right) - 2\sin x\left(\frac{\sin^3 x - x^3\cos x}{x^3}\right)\right], \end{aligned}$$

and then deduces

$$\begin{aligned} &x^4(1 - \cos x)^3 f'(x) \\ &\quad = (1 - \cos x)(3x\sin^2 x\cos x - 3\sin^3 x + x^4 \sin x) - 2\sin x(x\sin^3 x - x^4\cos x) \\ &\quad = 3x\sin^2 x\cos x - 3\sin^3 x + x^4\sin x - 3x\sin^2 x + x\sin^4 x + 3\sin^3 x\cos x + x^4\sin x\cos x \\ &\quad = \sin^3 x \cdot \phi(x), \end{aligned}$$

where

$$\phi(x) = 3x\cot x - 3 + x^4\frac{1}{\sin^2 x} - 3\frac{x}{\sin x} + x\sin x + 3\cos x + x^4\frac{\cos x}{\sin^2 x}.$$

Using power series expansions in Lemma 1, and known series expansions of $\sin x$ and $\cos x$, we have

$$\begin{aligned}\phi(x) &= 2x^2 - \sum_{k=1}^{\infty}\frac{3\cdot 2^{2k}}{(2k)!}|B_{2k}|x^{2k} + \sum_{k=1}^{\infty}\frac{2^{2k}(2k-1)}{(2k)!}|B_{2k}|x^{2k+2} - \sum_{k=1}^{\infty}\frac{6(2^{2k-1}-1)}{(2k)!}|B_{2k}|x^{2k} \\ &\quad + \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+2} + \sum_{k=1}^{\infty}\frac{3(-1)^k}{(2k)!}x^{2k} - \sum_{k=1}^{\infty}\frac{2(2^{2k-1}-1)(2k-1)}{(2k)!}|B_{2k}|x^{2k+2} \\ &= -\sum_{k=1}^{\infty}\frac{3\cdot 2^{2k+2}}{(2k+2)!}|B_{2k+2}|x^{2k+2} + \sum_{k=1}^{\infty}\frac{2^{2k}(2k-1)}{(2k)!}|B_{2k}|x^{2k+2} \\ &\quad - \sum_{k=1}^{\infty}\frac{6(2^{2k+1}-1)}{(2k+2)!}|B_{2k+2}|x^{2k+2} + \sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+2} + \sum_{k=1}^{\infty}\frac{3(-1)^{k+1}}{(2k+2)!}x^{2k+2} \\ &\quad - \sum_{k=1}^{\infty}\frac{2(2^{2k-1}-1)(2k-1)}{(2k)!}|B_{2k}|x^{2k+2} \\ &:= \sum_{k=1}^{\infty}\frac{a_k}{(2k+2)!}x^{2k+2},\end{aligned}$$

where

$$\begin{aligned}a_k &= 2^{2k}(2k+1)(2k+2)(2k-1)|B_{2k}| - 2(2k+1)(2k+2)(2^{2k-1}-1)(2k-1)|B_{2k}| \\ &\quad - 3\cdot 2^{2k+2}|B_{2k+2}| - 6(2^{2k+1}-1)|B_{2k+2}| + (-1)^k(2k+2) + 3(-1)^{k+1} \\ &= 2(2k-1)(2k+1)(2k+2)|B_{2k}| - 6(2^{2k+2}-1)|B_{2k+2}| + (2k+2)(-1)^k + 3(-1)^{k+1}.\end{aligned}$$

We calculate $a_1 = a_2 = 0$ and $a_3 = -\frac{128}{3}$.

Next, we claim that $a_k < 0$ for $k = 4,5,6,7,8,\cdots$. First, we need to prove $a_k < 0$ for $k = 4,6,8,\cdots$, that is,

$$(2k+2) + 2(2k-1)(2k+1)(2k+2)|B_{2k}| < 3 + 6(2^{2k+2}-1)|B_{2k+2}| \tag{8}$$

for $k = 4,6,8,\cdots$, and then $a_k < 0$ for $k = 5,7,9,\cdots$, that is,

$$(1-2k) + 2(2k-1)(2k+1)(2k+2)|B_{2k}| < 6(2^{2k+2}-1)|B_{2k+2}| \tag{9}$$

for $k = 5,7,9,\cdots$. Now, by Lemma 4 and Lemma 5, we, respectively, have

$$(2k+2) < 3(2^{2k+2}-1)|B_{2k+2}|,\ \ k = 3,4,5,\cdots \tag{10}$$

and

$$2(2k-1)(2k+1)(2k+2)|B_{2k}| < 3(2^{2k+2}-1)|B_{2k+2}|,\ \ k = 4,5,6,\cdots. \tag{11}$$

Combining (10) with (11), we obtain

$$(2k+2) + 2(2k-1)(2k+1)(2k+2)|B_{2k}| < 6(2^{2k+2}-1)|B_{2k+2}|,\ \ k = 4,5,6,\cdots,$$

which implies (8). Also, the inequality (11) implies (9). Thus, $a_k \le 0$ for $k = 1,2,3,4,5,\cdots$. Therefore, we conclude that $\phi(x) < 0,\ \ x \in (0,\pi)$ and consequently $f'(x) < 0,\ \ x \in (0,\pi)$. This shows that $f(x)$ is strictly decreasing on $(0,\pi)$. The proof is completed. □

The following new generalized Mitrinović–Adamović-type inequality can be established by applying Theorem 1.

**Theorem 2.** *The following double inequalities are valid:*

(i) *The double inequality*

$$\cos x + \frac{8}{\pi^3}(1-\cos x)^2 < \left(\frac{\sin x}{x}\right)^3 < \cos x + \frac{4}{15}(1-\cos x)^2, \ \ x \in \left(0, \frac{\pi}{2}\right) \tag{12}$$

*holds with the best possible constants* $\frac{8}{\pi^3}$ *and* $\frac{4}{15}$.

(ii) *The double inequality*

$$\cos x + \frac{1}{4}(1-\cos x)^2 < \left(\frac{\sin x}{x}\right)^3 < \cos x + \frac{4}{15}(1-\cos x)^2, \ \ x \in (0, \pi) \tag{13}$$

*holds with the best possible constants* $\frac{1}{4}$ *and* $\frac{4}{15}$.

**Proof.** Applying Theorem 1, we have

$$f(0^+) > f(x) > f(\pi/2^-),$$

and

$$f(0^+) > f(x) > f(\pi^-).$$

From the limits

$$\lim_{x\to 0^+} f(x) = \frac{4}{15}, \ \lim_{x\to \pi/2^-} f(x) = \frac{8}{\pi^3}, \ \lim_{x\to\pi^-} f(x) = \frac{1}{4},$$

we can show the desired inequalities (12) and (13). □

**Remark 1.** *The left inequality of (13) can be written as*

$$\left(\frac{1+\cos x}{2}\right)^2 < \left(\frac{\sin x}{x}\right)^3, \ \ x \in (0, \pi).$$

*This inequality was first proved by Neuman and Sándor [4] and reappeared in [15]. However, it was shown to be true on* $\left(0, \frac{\pi}{2}\right)$ *only.*

**Remark 2.** *It can be observed that the right inequality of (13) is in fact true for* $x > 0$. *This fact can be verified graphically at https://www.desmos.com/calculator (accessed on 1 March 2025). Let us write*

$$L(x) = \cos x + \frac{21}{5}\frac{x^3 \sin x}{63 + x^2}, \ \ U(x) = \cos x + \frac{\pi^2}{15}\frac{x^3 \sin x}{\pi^2 - x^2}$$

*and*

$$L_1(x) = \cos x + \frac{1}{4}(1-\cos x)^2, \ \ U_1(x) = \cos x + \frac{4}{15}(1-\cos x)^2$$

*for* $x \in (0, \pi)$. *Then, we compare graphically the bounds in (13) with those in (2).*

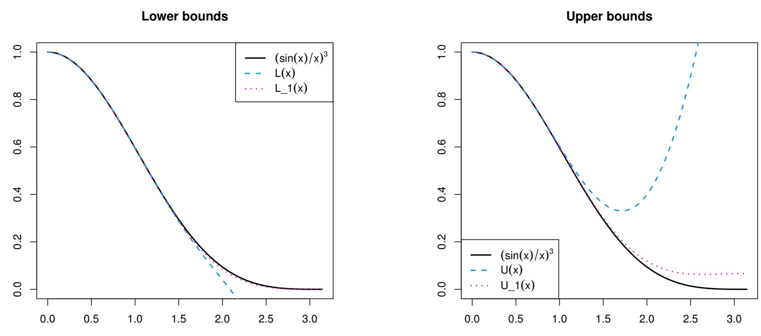
*The numerical calculations and graphs in Figure 1 show that the lower bound* $L_1(x)$ *of* $\left(\frac{\sin x}{x}\right)^3$ *in (13) is sharper than the corresponding lower bound* $L(x)$ *in (2) for* $x > \lambda \approx 1.5301$, *whereas the upper bound* $U_1(x)$ *of* $\left(\frac{\sin x}{x}\right)^3$ *in (13) is uniformly sharper than the corresponding upper bound* $U(x)$ *in (2).*

In what follows, we further refine the bounds of $\left(\frac{\sin x}{x}\right)^3$.

**Theorem 3.** *The function*

$$g(x) = \frac{1}{x^6}\left[\left(\frac{\sin x}{x}\right)^3 - \cos x - \frac{4}{15}(1-\cos x)^2\right]$$

*is strictly increasing on* $(0, \pi)$.




(**a**) Graphs of the lower bounds (**b**) Graphs of the upper bounds

**Figure 1.** Graphs of the lower and upper bounds of $\left(\frac{\sin x}{x}\right)^3$ in (2) and (13) for $x \in (0, \pi)$.

**Proof.** Differentiating $g(x)$ with respect to $x$ yields

$$\begin{aligned}
x^{10}g'(x) &= x\left[3\sin^2 x\cos x + x^3\sin x - 3x^2\cos x - \frac{8}{15}x^3(1-\cos x)\sin x - \frac{12}{15}x^2(1-\cos x)^2\right] \\
&\quad - 9\left[\sin^3 x - x^3\cos x - \frac{4}{15}x^3(1-\cos x)^2\right] \\
&= 3x\sin^2 x\cos x + \frac{7}{15}x^4\sin x - \frac{7}{5}x^3\cos x + \frac{8}{15}x^4\sin x\cos x - \frac{12}{15}x^2 \\
&\quad - \frac{12}{15}x^2\cos^2 x - 9\sin^3 x + 9x^3\cos x + \frac{12}{5}x^3 - \frac{24}{5}x^3\cos x + \frac{12}{5}x^3\cos^2 x \\
&= 3x\sin^2 x\cos x + \frac{7}{15}x^4\sin x + \frac{14}{5}x^3\cos x + \frac{8}{15}x^4\sin x\cos x + \frac{16}{5}x^3 \\
&\quad - \frac{8}{5}x^3\sin^2 x - 9\sin^3 x.
\end{aligned}$$

This can be written as $\frac{x^{10}}{\sin^3 x} = \psi(x)$, where

$$\psi(x) = 3x\cot x + \frac{7}{15}x^4\frac{1}{\sin^2 x} + \frac{14}{5}x^3\frac{\cos x}{\sin^3 x} + \frac{8}{15}x^4\frac{\cos x}{\sin^2 x} + \frac{16}{5}x^3\frac{1}{\sin^3 x} - \frac{8}{5}x^3\frac{1}{\sin x} - 9.$$

Utilizing Lemma 1, we obtain

$$
\begin{aligned}
\psi(x) &= 3-\sum_{k=1}^{\infty}\frac{3\cdot 2^{2k}}{(2k)!}|B_{2k}|x^{2k}+\frac{7}{15}x^2+\sum_{k=1}^{\infty}\frac{7}{15}\frac{2^(2k-1)}{(2k)!}|B_{2k}|x^{2k+2}+\frac{14}{5}\\
&\quad-\sum_{k=2}^{\infty}\frac{14}{5}\frac{2^{2k}(2k-1)(k-1)}{(2k)!}|B_{2k}|x^{2k}+\frac{8}{15}x^2-\sum_{k=1}^{\infty}\frac{8}{15}\frac{2(2k-1)(2^{2k-1}-1)}{(2k)!}|B_{2k}|x^{2k+2}\\
&\quad+\frac{16}{5}+\sum_{k=2}^{\infty}\frac{8}{5}\frac{(2^{2k}-2)(2k-1)(2k-2)}{(2k)!}|B_{2k}|x^{2k}+\frac{8}{5}x^2+\sum_{k=1}^{\infty}\frac{8}{5}\frac{(2^{2k}-2)}{(2k)!}|B_{2k}|x^{2k+2}\\
&\quad-\frac{8}{5}x^2-\sum_{k=1}^{\infty}\frac{16}{5}\frac{(2^{2k-1}-1)}{(2k)!}|B_{2k}|x^{2k+2}-9\\
&= x^2+\sum_{k=1}^{\infty}\left[\frac{8}{5}\frac{(2^{2k}-2)(2k-1)(2k-2)}{(2k)!}-\frac{14}{5}\frac{2^{2k}(2k-1)(k-1)}{(2k)!}-\frac{3\cdot 2^{2k}}{(2k)!}\right]|B_{2k}|x^{2k}\\
&\quad+\sum_{k=1}^{\infty}\left[\frac{7}{15}\frac{2^{2k}(2k-1)}{(2k)!}+\frac{8}{5}\frac{(2^{2k}-2)}{(2k)!}-\frac{16}{5}\frac{(2^{2k-1}-1)}{(2k)!}-\frac{16}{15}\frac{(2k-1)(2^{2k-1}-1)}{(2k)!}\right]|B_{2k}|x^{2k+2}\\
&= x^2+\sum_{k=1}^{\infty}\left[\frac{8(2^{2k}-2)(2k-1)(2k-2)-14\cdot 2^{2k}(2k-1)(k-1)-15\cdot 2^{2k}}{5\cdot(2k)!}\right]|B_{2k}|x^{2k}\\
&\quad+\sum_{k=1}^{\infty}\left[\frac{7\cdot 2^{2k}(2k-1)+24(2^{2k}-2)-48(2^{2k-1}-1)-16(2k-1)(2^{2k-1}-1)}{15\cdot(2k)!}\right]|B_{2k}|x^{2k+2}\\
&= x^2+\sum_{k=0}^{\infty}\left[\frac{16k(2k+1)(2^{2k+2}-2)-14k(2k+1)\cdot 2^{2k+2}-15\cdot 2^{2k+2}}{5\cdot(2k+2)!}\right]|B_{2k+2}|x^{2k+2}\\
&\quad+\sum_{k=1}^{\infty}\left[\frac{7\cdot 2^{2k}(2k-1)+24(2^{2k}-2)-48(2^{2k-1}-1)-16(2k-1)(2^{2k-1}-1)}{15\cdot(2k)!}\right]|B_{2k}|x^{2k+2}\\
&= \sum_{k=3}^{\infty}\frac{1}{5\cdot(2k)!}b_k x^{2k+2}
\end{aligned}
$$

where

$$
\begin{aligned}
b_k = {} & \frac{\left(16k(2^{2k+2}-2)(2k+1)-14k(2k+1)\cdot 2^{2k+2}-15\cdot 2^{2k+2}\right)}{(2k+1)(2k+2)}|B_{2k+2}|\\
&-\frac{\left(16(2k-1)(2^{2k-1}-1)+48(2^{2k-1}-1)-24(2^{2k}-2)-7\cdot 2^{2k}(2k-1)\right)}{3}|B_{2k}|.
\end{aligned}
$$

By Lemma 6, $b_k > 0$. This proves that $\psi(x)$ and hence $g'(x)$ is positive, implying that $g(x)$ is strictly increasing on $(0,\pi)$. The proof is completed. □

**Theorem 4.** *The following double inequalities are valid:*

*(i)* *The double inequality*

$$
\begin{aligned}
\cos x+\frac{4}{15}(1-\cos x)^2-\frac{1}{945}x^6 &< \left(\frac{\sin x}{x}\right)^3\\
&< \cos x+\frac{4}{15}(1-\cos x)^2-\frac{256}{\pi^6}\left(\frac{1}{15}-\frac{2}{\pi^3}\right)x^6,\ \ x\in\left(0,\frac{\pi}{2}\right)
\end{aligned}
\quad (14)
$$

*holds with the best possible constants* $-\frac{1}{945}$ *and* $-\frac{256}{\pi^6}\left(\frac{1}{15}-\frac{2}{\pi^3}\right)$.

(ii) *The double inequality*

$$\cos x + \frac{4}{15}(1-\cos x)^2 - \frac{1}{945}x^6 < \left(\frac{\sin x}{x}\right)^3 < \cos x + \frac{4}{15}(1-\cos x)^2 - \frac{1}{15\pi^6}x^6, \quad x \in (0,\pi) \tag{15}$$

*holds with the best possible constants* $-\frac{1}{945}$ *and* $-\frac{1}{15\pi^6}$.

**Proof.** Applying Theorem 3, we obtain

$$g(0^+) < g(x) < g(\pi/2^-),$$

and

$$g(0^+) < g(x) < g(\pi^-).$$

The double inequalities (14) and (15) follow due to the limits

$$\lim_{x\to 0} g(x) = -\frac{1}{945}, \quad \lim_{x\to \pi/2^-} g(x) = -\frac{64}{\pi^6}\left(\frac{4}{15} - \frac{8}{\pi^3}\right), \quad \text{and} \quad \lim_{x\to\pi} g(x) = -\frac{1}{15\pi^6}.$$

□

**Remark 3.** *If we suppose*

$$L_2(x) = \cos x + \frac{4}{15}(1-\cos x)^2 - \frac{1}{945}x^6$$

*and*

$$U_2(x) = \cos x + \frac{4}{15}(1-\cos x)^2 - \frac{1}{15\pi^6}x^6,$$

*then it is obvious that* $U_2(x) < U_1(x)$ *for all* $x \in (0,\pi)$. *Based on Figure 2 and some numerical calculations, it is found that* $L_1(x) < L_2(x)$ *for* $x \in (0,\lambda^*)$ *where* $\lambda^* \approx 1.5969$.

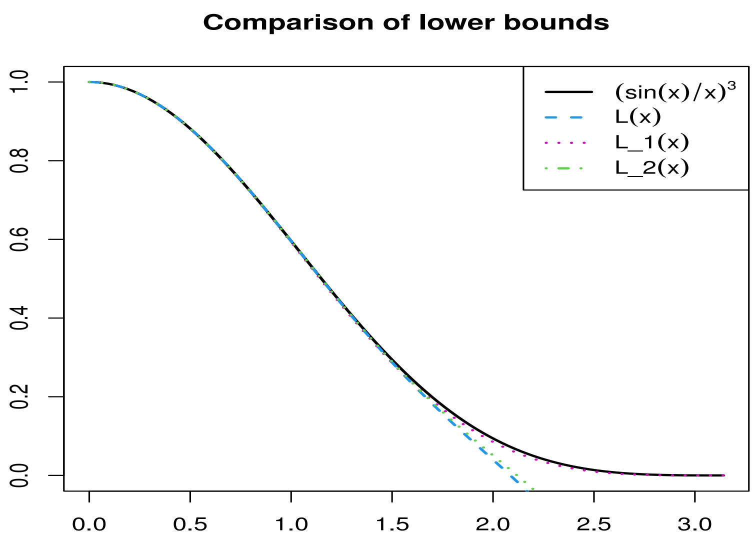
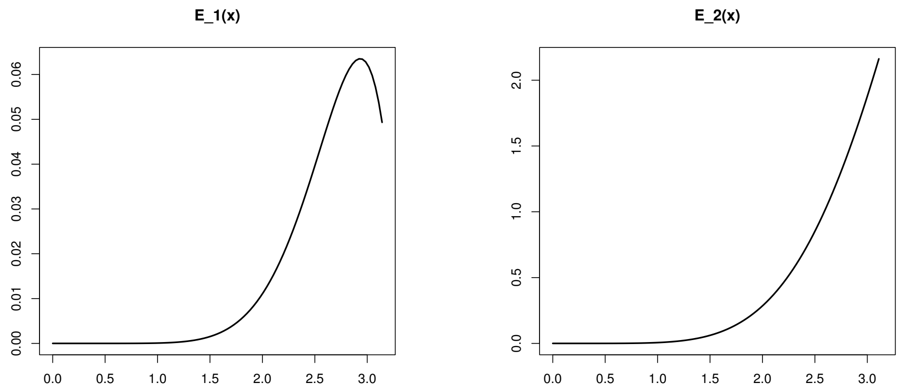



**Figure 2.** Graphs of the lower bounds of $\left(\frac{\sin x}{x}\right)^3$ in (2), (13) and (15) for $x \in (0,\pi)$.

*The curves* $E_1(x) = L_2(x) - L(x)$, *and* $E_2(x) = U(x) - U_2(x)$ *depicted in the following Figure 3 show that the double inequality (2) is completely refined to (15).*




(**a**) Graph of the function $E_1(x)$ (**b**) Graph of the function $E_2(x)$

**Figure 3.** Graphs of the functions $E_1(x)$ and $E_2(x)$ for $x \in (0, \pi)$.

**Lemma 7.** *Let*

$$\begin{aligned} T(k) = &(2^{2k-1} - 1) \cdot [16k^3(2^{2k+2} - 1) - (2176k + 1980) \cdot 2^{2k} + 544k + 720] \\ &- (2^{2k+1} - 1)(2k - 1)\pi^2 \cdot (2^{2k} + 90k - 136). \end{aligned}$$

*Then,* $T(k) > 0$ *for* $k = 7, 8, 9, \cdots$ *.*

**Proof.** We rewrite $T(k)$ as

$$\begin{aligned} T(k) = &(2^{2k+1} - 4) \cdot [4k^3(2^{2k+2} - 1) - (544k + 495) \cdot 2^{2k} + 136 + 180] \\ &- (2^{2k+1} - 1)(2k - 1)\pi^2 \cdot (2^{2k} + 90k - 136). \end{aligned}$$

Clearly,

$$2^{2k} + 2^{2k} > 2^{2k} + (90k - 132), \text{ for } k = 7, 8, 9, \cdots .$$

Therefore

$$(2^{2k+1} - 4) > (2^{2k} + 90k - 136), \; k = 7, 8, 9, \cdots . \tag{16}$$

Similarly,

$$16k^3 + 2\pi^2 > (544 + 4\pi^2)k + 495, \text{ for } k = 7, 8, 9, \cdots$$

implies

$$(16k^3 + 2\pi^2) \cdot 2^{2k} + (136 + 2\pi^2)k + (180 - \pi^2) > [(544 + 4\pi^2)k + 495] \cdot 2^{2k}.$$

Then, we have

$$4k^3(2^{2k+2} - 1) - (544k + 495) \cdot 2^{2k} + 136k + 180 > (2^{2k+1} - 1)(2k - 1)\pi^2 \tag{17}$$

for $k = 7, 8, 9, \cdots$ . From (16) and (17), we obtain $T(k) > 0$ for $k = 7, 8, 9, \cdots$ . □

It is still possible to refine the inequalities of Theorem 3. We present the following more refined results.

**Theorem 5.** *The function*

$$h(x) = \frac{\left[\left(\frac{\sin x}{x}\right)^3 - \cos x - \frac{4}{15}(1-\cos x)^2\right]}{x^2(1-\cos x)^2}$$

*is strictly increasing on* $(0, \pi)$.

**Proof.** By the differentiation of $h(x)$, we have

$$\begin{aligned}
[x(1-\cos x)]^4 \cdot h'(x) = {} & x(1-\cos x)\left(\frac{3\sin^2 x\cos x}{x^3} - \frac{3\sin^3 x}{x^4} + \frac{7}{15}\sin x + \frac{8}{15}\sin x\cos x\right) \\
& - (2 - 2\cos x + 2x\sin x)\left(\frac{\sin^3 x}{x^3} - \frac{7}{15}\cos x - \frac{4}{15} - \frac{4}{15}\cos^2 x\right) \\
= {} & \frac{3\sin^2 x\cos x}{x^2} - 5\frac{\sin^3 x}{x^3} + x\sin x + x\sin x\cos x - 3\frac{\sin^2 x}{x^2} + \frac{\sin^4 x}{x^2} \\
& + 5\frac{\sin^3 x\cos x}{x^3} - \frac{2}{15}\cos x + \frac{2}{15} + \frac{6}{15}\sin^2 x + \frac{8}{15}\sin^2 x\cos x \\
:= {} & \sin^4 x \cdot \lambda(x),
\end{aligned}$$

where

$$\begin{aligned}
\lambda(x) = {} & \frac{3}{x^2}\frac{\cos x}{\sin^2 x} - \frac{5}{x^3}\frac{1}{\sin x} + x\frac{1}{\sin^3 x} + x\frac{\cos x}{\sin^3 x} - \frac{3}{x^2}\frac{1}{\sin^2 x} + \frac{1}{x^2} \\
& + \frac{5}{x^3}\cot x - \frac{2}{15}\frac{\cos x}{\sin^4 x} + \frac{2}{15}\frac{1}{\sin^4 x} + \frac{6}{15}\frac{1}{\sin^2 x} + \frac{8}{15}\frac{\cos x}{\sin^2 x}.
\end{aligned}$$

We plan to use the series expansions given in Lemma 1 to obtain

$$\begin{aligned}
\lambda(x) = {} & \frac{3}{x^4} - \sum_{k=1}^{\infty}\frac{3(2^{2k}-2)(2k-1)}{(2k)!}|B_{2k}|x^{2k-4} - \frac{5}{x^4} - \sum_{k=1}^{\infty}\frac{5(2^{2k}-2)}{(2k)!}|B_{2k}|x^{2k-4} + \frac{1}{x^2} \\
& + \sum_{k=2}^{\infty}\frac{(2^{2k}-2)(2k-1)(2k-2)}{2\cdot(2k)!}|B_{2k}|x^{2k-2} + \frac{1}{2} + \sum_{k=1}^{\infty}\frac{(2^{2k}-2)}{2\cdot(2k)!}|B_{2k}|x^{2k} + \frac{1}{x^2} \\
& - \sum_{k=2}^{\infty}\frac{2^{2k}(2k-1)(k-1)}{(2k)!}|B_{2k}|x^{2k-2} - \frac{3}{x^2} - \sum_{k=1}^{\infty}\frac{3\cdot 2^{2k}(2k-1)}{(2k)!}|B_{2k}|x^{2k-4} + \frac{1}{x^2} + \frac{5}{x^4} \\
& - \sum_{k=1}^{\infty}\frac{5\cdot 2^{2k}}{(2k)!}|B_{2k}|x^{2k-4} + \frac{2}{45x^3} + \frac{1}{45x} + \sum_{k=1}^{\infty}\frac{(2^{2k}-2)(2k-1)}{45\cdot(2k)!}|B_{2k}|x^{2k-2} \\
& + \sum_{k=2}^{\infty}\frac{(2^{2k}-2)(2k-1)(2k-2)(2k-3)}{45\cdot(2k)!}|B_{2k}|x^{2k-4} + \frac{2}{15x^4} \\
& + \sum_{k=1}^{\infty}\frac{2^{2k}(2k-1)(2k-2)(2k-3)}{45\cdot(2k)!}|B_{2k}|x^{2k-4} + \frac{4}{45x^2} + \sum_{k=1}^{\infty}\frac{4}{45}\frac{2^{2k}(2k-1)}{(2k)!}|B_{2k}|x^{2k-2} \\
& + \frac{6}{15x^2} + \sum_{k=1}^{\infty}\frac{6}{15}\frac{2^{2k}(2k-1)}{(2k)!}|B_{2k}|x^{2k-2} + \frac{8}{15x^2} - \sum_{k=1}^{\infty}\frac{8}{15}\frac{(2^{2k}-2)(2k-1)}{(2k)!}|B_{2k}|x^{2k-2}.
\end{aligned}$$

After simplifying, $\lambda(x)$ can be written as

$$\begin{aligned}
\lambda(x) &= \frac{1}{2} + \frac{1}{45x} + \frac{181}{45x^2} + \frac{2}{45x^3} + \frac{2}{15x^4} + \sum_{k=1}^{\infty} \frac{(2^{2k}-2)}{(2k)!}|B_{2k}|x^{2k} \\
&+ \sum_{k=1}^{\infty} \frac{[4\cdot 2^{2k}+2^{2k}-2](2k-1)}{45\cdot(2k)!}|B_{2k}|x^{2k-2} + \sum_{k=2}^{\infty} \frac{[2^{2k}-2-2^{2k}](2k-1)(k-1)}{(2k)!}|B_{2k}|x^{2k-2} \\
&+ \sum_{k=1}^{\infty} \frac{\left[\frac{(2k-2)(2k-3)}{45}-3\right]\cdot 2^{2k}(2k-1)}{(2k)!}|B_{2k}|x^{2k-4} - \sum_{k=1}^{\infty} \frac{[3(2k-1)+5](2^{2k}-2)}{(2k)!}|B_{2k}|x^{2k-4} \\
&+ \sum_{k=2}^{\infty} \frac{(2^{2k}-2)(2k-1)(2k-2)(2k-3)}{45\cdot(2k)!}|B_{2k}|x^{2k-4} + \sum_{k=1}^{\infty}[3\cdot 2^{2k}-4(2^{2k}-2)]\frac{2(2k-1)}{15\cdot(2k)!}|B_{2k}|x^{2k-2}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\lambda(x) = &\left(\frac{1}{2} + \frac{1}{45x} + \frac{181}{45x^2} + \frac{2}{45x^3} + \frac{2}{15x^4}\right) + \sum_{k=2}^{\infty} \frac{(2^{2k-2}-2)}{(2k-2)!}|B_{2k-2}|x^{2k-2} \\
&+ \sum_{k=1}^{\infty} \frac{(5\cdot 2^{2k}-2)(2k-1)}{45\cdot(2k)!}|B_{2k}|x^{2k-2} - \sum_{k=2}^{\infty} \frac{2(2k-1)(k-1)}{(2k)!}|B_{2k}|x^{2k-2} \\
&+ \sum_{k=0}^{\infty} \frac{\left[\frac{2k(2k-1)}{45}-3\right]\cdot 2^{2k+2}(2k+1)}{(2k+2)!}|B_{2k+2}|x^{2k-2} - \sum_{k=0}^{\infty} \frac{(6k+8)(2^{2k+2}-2)}{(2k+2)!}|B_{2k+2}|x^{2k-2} \\
&+ \sum_{k=1}^{\infty} \frac{(2^{2k+2}-2)(2k+1)(2k)(2k-1)}{45\cdot(2k+2)!}|B_{2k+2}|x^{2k-2} - \sum_{k=1}^{\infty} \frac{(2^{2k}-8)\cdot 2(2k-1)}{15\cdot(2k)!}|B_{2k}|x^{2k-2},
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
\lambda(x) = &\left(\frac{1}{2} + \frac{1}{45x} + \frac{181}{45x^2} + \frac{2}{45x^3} + \frac{2}{15x^4}\right) + \left(1 - \frac{1}{x^2} - \frac{7}{3x^2}\right) + \sum_{k=1}^{\infty} \frac{(2^{2k-2}-2)}{(2k-2)!}|B_{2k-2}|x^{2k-2} \\
&+ \sum_{k=1}^{\infty} \frac{\left[\frac{(5\cdot 2^{2k}-2)}{45}-2(k-1)\right]}{(2k)!}(2k-1)|B_{2k}|x^{2k-2} \\
&+ \sum_{k=1}^{\infty} \frac{\left[\frac{2k(2k-1)}{45}-3\right]\cdot 2^{2k+2}(2k+1)}{(2k+2)!}|B_{2k+2}|x^{2k-2} - \sum_{k=1}^{\infty} \frac{(6k+8)(2^{2k+2}-2)}{(2k+2)!}|B_{2k+2}|x^{2k-2} \\
&+ \sum_{k=1}^{\infty} \frac{(2^{2k+2}-2)(2k+1)(2k)(2k-1)}{45\cdot(2k+2)!}|B_{2k+2}|x^{2k-2} - \sum_{k=1}^{\infty} \frac{2(2^{2k}-8)(2k-1)}{15\cdot(2k)!}|B_{2k}|x^{2k-2},
\end{aligned}$$

that is,

$$\lambda(x) = \left(\frac{3}{2} + \frac{1}{45x} + \frac{31}{45x^2} + \frac{2}{45x^3} + \frac{2}{15x^4}\right) + \sum_{k=1}^{\infty} \frac{c_k}{(2k-2)!}x^{2k-2}.$$

Here,

$$\begin{aligned}
c_k = &(2^{2k-2}-2)|B_{2k-2}| + \frac{[5\cdot 2^{2k}-2-90(k-1)]}{90k}|B_{2k}| + \frac{[2k(2k-1)-135]\cdot 2^{2k+2}}{45(2k+2)(2k)(2k-1)}|B_{2k+2}| \\
&- \frac{(6k+8)(2^{2k+2}-2)}{(2k+2)(2k+1)(2k)(2k-1)}|B_{2k+2}| + \frac{(2^{2k+2}-2)}{45(2k+2)}|B_{2k+2}| - \frac{(2^{2k}-8)}{15k}|B_{2k}|.
\end{aligned}$$

We write $c_k$ conveniently as

$$\begin{aligned} c_k &= (2^{2k-2}-2)|B_{2k-2}| + \frac{(5\cdot 2^{2k}-90k+88)}{90k}|B_{2k}| + \frac{(4k^2-2k-135)\cdot 2^{2k}}{45k(k+1)(2k-1)}|B_{2k+2}| \\ &\quad + \frac{(2^{2k+2}-2)(8k^3-272k-360)}{45(2k+2)(2k+1)(2k)(2k-1)}|B_{2k+2}| - \frac{(2^{2k}-8)}{15k}|B_{2k}| \\ &= (2^{2k-2}-2)|B_{2k-2}| + \frac{[16k^3(2^{2k+2}-1)-(2176k+1980)\cdot 2^{2k}+544k+720]}{45(2k+2)(2k+1)(2k)(2k-1)}|B_{2k+2}| \\ &\quad - \frac{(2^{2k}+90k-136)}{90k}|B_{2k}|. \end{aligned}$$

From this, we obtain $c_1 = -\frac{85}{54}$. So,

$$\lambda(x) = \left(-\frac{2}{27}+\frac{1}{45x}+\frac{31}{45x^2}+\frac{2}{45x^3}+\frac{2}{15x^4}\right) + \sum_{k=2}^{\infty}\frac{c_k}{(2k-2)!}x^{2k-2}$$

where the function in the parentheses is strictly decreasing on $(0,\pi)$. Hence,

$$\begin{aligned} \lambda(x) &> \left(-\frac{2}{27}+\frac{1}{45\pi}+\frac{31}{45\pi^2}+\frac{2}{45\pi^3}+\frac{2}{15\pi^4}\right) + \sum_{k=2}^{\infty}\frac{c_k}{(2k-2)!}x^{2k-2} \\ &= \left[\frac{(\pi^3+31\pi^2+2\pi+6)}{45\pi^4}-\frac{2}{27}\right] + \frac{37703}{38178\times(2!)}x^2 + \frac{449}{1890\times(4!)}x^4 + \frac{79073}{135135\times(6!)}x^6 \\ &\quad + \frac{56513281}{8918910\times(8!)}x^8 + \frac{13078211909763}{186943326570\times(10!)}x^{10} + \sum_{k=7}^{\infty}\frac{c_k}{(2k-2)!}x^{2k-2}. \end{aligned}$$

The values of $c_k$'s in the above expression for $k=2,3,\cdots,6$ are calculated using the Bernoulli numbers ([25], Chapter 23). For proving $\lambda(x)>0$ on $(0,\pi)$, it suffices to prove $c_k>0$ for $k=7,8,9,\cdots$. Now, we rewrite $c_k$ as follows:

$$\begin{aligned} c_k &= (2^{2k-2}-2)|B_{2k-2}| + |B_{2k}|\times \\ &\left\{\frac{[16k^3(2^{2k+2}-1)-(2176k+1980)\cdot 2^{2k}+544k+720]}{45(2k+2)(2k+1)(2k)(2k-1)}\frac{|B_{2k+2}|}{|B_{2k}|} - \frac{(2^{2k}+90k-136)}{90k}\right\}. \end{aligned}$$

Since

$$16k^3(2^{2k+2}-1)-(2176k+1980)\cdot 2^{2k}+544k+720>0 \ \text{ for } \ k=7,8,9,\cdots,$$

by using Qi's inequality (5) and applying Lemma 7. we have

$$\begin{aligned} c_k &> (2^{2k-2}-2)|B_{2k-2}| + |B_{2k}|\times \\ &\left\{\frac{[16k^3(2^{2k+2}-1)-(2176k+1980)\cdot 2^{2k}+544k+720](2^{2k-1}-1)}{45(2k)(2k-1)(2^{2k+1}-1)\pi^2} - \frac{(2^{2k}+90k-136)}{90k}\right\} \\ &= (2^{2k-2}-2)|B_{2k-2}| + \frac{|B_{2k}|\cdot T(k)}{90k(2k-1)(2^{2k+1}-1)\pi^2} > 0. \end{aligned}$$

Thus, $\lambda(x)>0$, implying that $h'(x)>0,\ \ x\in(0,\pi)$. Therefore, $h(x)$ is strictly increasing on $(0,\pi)$. The proof is completed. □

**Theorem 6.** *The following double inequalities are valid:*

*(i)* *The double inequality*

$$\cos x + \left[\frac{4}{15} - \frac{4}{945}x^2\right](1-\cos x)^2 < \left(\frac{\sin x}{x}\right)^3 < \cos x + \left[\frac{4}{15} - \frac{4}{\pi^2}\left(\frac{4}{15} - \frac{8}{\pi^3}\right)x^2\right](1-\cos x)^2, \quad x \in \left(0, \frac{\pi}{2}\right) \tag{18}$$

*holds with the best possible constants* $-\frac{4}{945}$ *and* $-\frac{4}{\pi^2}\left(\frac{4}{15} - \frac{8}{\pi^3}\right)$.

*(ii)* *The double inequality*

$$\cos x + \left(\frac{4}{15} - \frac{4}{945}x^2\right)(1-\cos x)^2 < \left(\frac{\sin x}{x}\right)^3 < \cos x + \left(\frac{4}{15} - \frac{1}{60\pi^2}x^2\right)(1-\cos x)^2, \quad x \in (0, \pi) \tag{19}$$

*holds with the best possible constants* $-\frac{4}{945}$ *and* $-\frac{1}{60\pi^2}$.

**Proof.** Applying Theorem 5, we obtain

$$h(0^+) < h(x) < h(\pi/2^-),$$

and

$$h(0^+) < h(x) < h(\pi^-).$$

Then, using the limits

$$\lim_{x\to 0^+} h(x) = -\frac{4}{945}, \quad \lim_{x\to \pi/2^-} h(x) = -\frac{4}{\pi^2}\left(\frac{4}{15} - \frac{8}{\pi^3}\right), \quad \text{and} \quad \lim_{x\to \pi^-} h(x) = -\frac{1}{60\pi^2},$$

we can prove our conclusion. □

**Remark 4.** *It is worth noting that the double inequality (21) is a refinement of (15). Because*

$$\frac{4}{15}(1-\cos x)^2 - \frac{1}{945}x^6 < \left(\frac{4}{15} - \frac{4}{945}x^2\right)(1-\cos x)^2 \iff 1 - \frac{x^2}{2} < \cos x$$

*and*

$$\left(\frac{4}{15} - \frac{1}{60\pi^2}x^2\right)(1-\cos x)^2 < \frac{4}{15}(1-\cos x)^2 - \frac{1}{15\pi^6}x^6 \iff \cos x < 1 - \frac{2x^2}{\pi^2}$$

*where the right-side inequalities are true due to the double inequality* $1 - \frac{x^2}{2} < \cos x < 1 - \frac{4x^2}{\pi^2}$ *as proved in [28].*

**Remark 5.** *In our new generalized Mitrinović–Adamović-type inequalities, we have compared our bounds of* $\left(\frac{\sin x}{x}\right)^3$ *in a larger interval* $(0, \pi)$ *with the old ones, i.e., with the corresponding bounds of (2). A similar numerical and/or graphical comparison can be made to show that the double inequalities (14) and (20) are superior to (3). In conclusion, we obtained stronger and superior bounds for the function* $\left(\frac{\sin x}{x}\right)^3$ *than those in (2)–(3). Moreover, all the main results of the paper hold in* $(-\pi, 0)$ *because of the the symmetry of curves involved. Thus, our bounds provide the better alternatives.*

**Remark 6.** *Our results can be used to bound the so-called sinc function, i.e., $\frac{\sin x}{x}$, which is extensively used in mathematics, physics and engineering. For instance, the double inequality (13) can be written as*

$$\left[\cos x + \frac{1}{4}(1-\cos x)^2\right]^{1/3} < \frac{\sin x}{x} < \left[\cos x + \frac{4}{15}(1-\cos x)^2\right]^{1/3}, \quad x \in (0,\pi).$$

**Remark 7.** *An interesting application of our new generalized Mitrinović–Adamović-type inequalities is to find the value $\frac{3\pi}{8}$ of the integral $\int_0^\infty \left(\frac{\sin x}{x}\right)^3 dx$. It is known that there are some complex standard methods; however, it is difficult to evaluate $\int_0^p \left(\frac{\sin x}{x}\right)^3 dx$, for any $p > 0$. In such a case, we need to rely on an approximate value of the integral. Here, we can better approximate the integral $\int_0^p \left(\frac{\sin x}{x}\right)^3 dx$, $0 < p \le \pi$ by using one of our main results. In particular, we approximate $\int_0^\pi \left(\frac{\sin x}{x}\right)^3 dx$ by selecting the inequality (21) as the best candidate whose bounds are tractable. Thus, integrating (21) from 0 to $\pi$, we have*

$$I_1 < \int_0^\pi \left(\frac{\sin x}{x}\right)^3 dx < I_2,$$

*where*

$$I_1 = \int_0^\pi \cos dx + \int_0^\pi \left(\frac{4}{15} - \frac{4}{945}x^2\right)(1-\cos x)^2 dx,$$

*and*

$$I_2 = \int_0^\pi \cos dx + \int_0^\pi \left(\frac{4}{15} - \frac{1}{60\pi^2}x^2\right)(1-\cos x)^2 dx.$$

*After expanding $(1-\cos x)^2$, using simple formulae of integration and integrating by parts, we obtain*

$$I_1 = \frac{(361-2\pi^2)\pi}{945} \quad \text{and} \quad I_2 = \frac{47\pi}{120} - \frac{17}{240\pi}.$$

*Then,*

$$\int_0^\pi \left(\frac{\sin x}{x}\right)^3 dx \approx \frac{I_1+I_2}{2} = \frac{1}{2}\left[\frac{(361-2\pi^2)\pi}{945} + \frac{47\pi}{120} - \frac{17}{240\pi}\right] \approx 1.1712$$

*which is very close to the exact value $\approx 1.1896$.*

## 4. Conclusions

In this paper, we established various new generalized Mitrinović–Adamović-type inequalities as follows:

- **(See Theorem 2):**
  - (i) The double inequality

$$\cos x + \frac{8}{\pi^3}(1-\cos x)^2 < \left(\frac{\sin x}{x}\right)^3 < \cos x + \frac{4}{15}(1-\cos x)^2, \quad x \in \left(0, \frac{\pi}{2}\right)$$

    holds with the best possible constants $\frac{8}{\pi^3}$ and $\frac{4}{15}$.
  - (ii) The double inequality

$$\cos x + \frac{1}{4}(1-\cos x)^2 < \left(\frac{\sin x}{x}\right)^3 < \cos x + \frac{4}{15}(1-\cos x)^2, \quad x \in (0,\pi)$$

    holds with the best possible constants $\frac{1}{4}$ and $\frac{4}{15}$.
- **(See Theorem 4):**

(i) The double inequality

$$\cos x + \frac{4}{15}(1-\cos x)^2 - \frac{1}{945}x^6 < \left(\frac{\sin x}{x}\right)^3 < \cos x + \frac{4}{15}(1-\cos x)^2 - \frac{256}{\pi^6}\left(\frac{1}{15} - \frac{2}{\pi^3}\right)x^6, \ \ x \in \left(0, \frac{\pi}{2}\right)$$

holds with the best possible constants $-\frac{1}{945}$ and $-\frac{256}{\pi^6}\left(\frac{1}{15} - \frac{2}{\pi^3}\right)$.

(ii) The double inequality

$$\cos x + \frac{4}{15}(1-\cos x)^2 - \frac{1}{945}x^6 < \left(\frac{\sin x}{x}\right)^3 < \cos x + \frac{4}{15}(1-\cos x)^2 - \frac{1}{15\pi^6}x^6, \ \ x \in (0, \pi)$$

holds with the best possible constants $-\frac{1}{945}$ and $-\frac{1}{15\pi^6}$.

- **(See Theorem 6):**

(i) The double inequality

$$\cos x + \left[\frac{4}{15} - \frac{4}{945}x^2\right](1-\cos x)^2 < \left(\frac{\sin x}{x}\right)^3 < \cos x + \left[\frac{4}{15} - \frac{4}{\pi^2}\left(\frac{4}{15} - \frac{8}{\pi^3}\right)x^2\right](1-\cos x)^2, \ \ x \in \left(0, \frac{\pi}{2}\right) \quad (20)$$

holds with the best possible constants $-\frac{4}{945}$ and $-\frac{4}{\pi^2}\left(\frac{4}{15} - \frac{8}{\pi^3}\right)$.

(ii) The double inequality

$$\cos x + \left(\frac{4}{15} - \frac{4}{945}x^2\right)(1-\cos x)^2 < \left(\frac{\sin x}{x}\right)^3 < \cos x + \left(\frac{4}{15} - \frac{1}{60\pi^2}x^2\right)(1-\cos x)^2, \ \ x \in (0, \pi) \quad (21)$$

holds with the best possible constants $-\frac{4}{945}$ and $-\frac{1}{60\pi^2}$.

In these new generalized Mitrinović–Adamović-type inequalities, we have compared our bounds of $\left(\frac{\sin x}{x}\right)^3$ in a larger interval $(0, \pi)$ with the old ones. We believe that our results will assist us in obtaining novel expression results related to other generalized Mitrinović–Adamović-type inequalities in future studies.

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