mathematics

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Harnack Inequality for Self-Repelling Diffusions Driven by Subordinated Brownian Motion

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Abstract: In this paper, we consider a self-repelling diffusion driven by the Lévy process. By using the coupling argument, we establish the corresponding Bismut formula and Harnack inequality.

Keywords: Harnack inequality; self-repelling diffusions; Lévy process; Bismut formula

MSC: 60E15; 60G18; 60H35

1. Introduction

In 1992, Durrett and Rogers [1] introduced a model for the shape of a growing polymer (Brownian polymer):

$$X_t = X_0 + B_t + \int_0^t \int_0^s f(X_s - X_u) duds, \tag{1}$$

where B is a d-dimensional standard Brownian motion and f is Lipschitz continuous. X_t corresponds to the location of the end of the polymer at time t. In general, the Equation (1) defines a self-interacting diffusion without any assumptions on f. We will call it self-repelling (resp. self-attracting) if, for all $x \in \mathbb{R}^d, x \cdot f(x) \geq 0$ (resp. ≤ 0), in other words, if it is more likely to stay away from (resp. come back to) the places it has already visited before. Numerous disciplines, including biophysics and financial mathematics, have made extensive use of the concept. For instance, the motion and interactions of colloid particles in a solution are similar to those of Brownian polymers in the field of colloid chemistry. Similar to the influence of the d-dimensional standard Brownian motion B_t in the equation, colloid particles are susceptible to the Brownian motion of solvent molecules. The self-interaction term $\int_0^t \int_0^s f(X_s - X_u)duds$ can be compared to the attractive or repulsive forces that exist among the particles. Determining the stability, aggregation behavior, and phase separation of colloid systems, among other phenomena, requires an understanding of these interactions.

In 2002, Benaïm et al. [2] considered a self-interacting diffusion with dependence on the (convolved) empirical measure as follows:

$$dX_t = \sqrt{2}dB_t - \left(\frac{1}{t}\int_0^t \nabla W(X_t - X_s)ds\right)dt,$$

where W is an interaction potential function. The fact that the drift term is divided by t distinguishes these diffusions from Brownian polymers. More works can be found in Benaïm et al. [3], He and Guo [4], Pan and Jiang [5], and the references therein.

Academic Editors: Francisco Chiclana and Paolo Mercorelli

Received: 18 February 2025
Revised: 2 April 2025
Accepted: 6 April 2025
Published: 11 April 2025

Citation: Sun, Y.; Yan, L. Harnack Inequality for Self-Repelling Diffusions Driven by Subordinated Brownian Motion. *Mathematics* **2025**, *13*, 1262. https://doi.org/10.3390/math13081262

In 2015, Benaïm et al. [3] considered the equation

$$X_t = x + B_t + \int_0^t b(X_s)ds - \int_0^t \int_0^s f(X_s - X_u)duds, \tag{2}$$

where $x \in \mathbb{R}$, $B = \{B_t, t \geq 0\}$ is a Brownian motion on $\mathbb{R}^1$, and f is a 2π-periodic function with sufficient regularity. For some suitable functions, they showed the existence and uniqueness of a solution with the Markov property and that it has the Feller property and an explicit form of an invariant probability. In this short note, we establish a Bismut-type formula and a Harnack inequality for self-interaction stochastic differential equations:

$$X_t = x + W_{S_t} + \int_0^t b(X_s)ds - \int_0^t \int_0^s f(X_s - X_u)duds + vt, \tag{3}$$

where $x, v \in \mathbb{R}^d$, $W = \{W_t, t \geq 0\}$ is a Brownian motion on $\mathbb{R}^d$ and S_t is the subordinator induced by a Bernstein function B of independent W, and b and f are two measurable functions. In order to obtain our results, we need the following condition:

There exist positive constants K_i, $i = 1, 2$, such that

$$\|\nabla b(\cdot)\|_\infty \leq K_1, \quad \|\nabla f(\cdot)\|_\infty \leq K_2,$$

where $\nabla b(x) = (\partial_{x_1} b(x), \cdots, \partial_{x_d} b(x))$, $\nabla f(x) = (\partial_{x_1} f(x), \cdots, \partial_{x_d} f(x))$ and $\|\cdot\|_\infty$ denotes the uniform norm with respect to x.

It is important to note that the solution of (3) is not a Markov process in general because of the explicit dependence of its dynamics on the entire historical path of X, rather than solely on its current state. On the other hand, as is well known, the functional inequalities and derivative formulas for Markov-type stochastic differential equations have been widely studied (see, for instance, Forte [6], Vespri et al. [7], Negro [8]), while the research on functional inequalities and derivative formulas related to non-Markov-type stochastic differential equations is still very scarce. Thus, the Bismut-type formula and the Harnack inequality for such equations as (3) appear to be worth studying. For simplicity, we will use C to represent a positive constant that depends only on the subscripts, and its value may vary depending on how it is presented.

Define the class $\{P_t, t \geq 0\}$ of operators by

$$P_t g(x) := \mathbb{E} g(X_t(x)), \quad t \geq 0, g \in C_b^1(\mathbb{R}^d), \tag{4}$$

where $X_t(x)$ is the solution of (3) with the initial value $x \in \mathbb{R}^d$, and $C_b^1(\mathbb{R}^d)$ is the space of all bounded functions on $\mathbb{R}^d$ with one-order continuous derivatives. At the same time, we can define

$$\nabla_h g(x) = \lim_{\varepsilon \to 0} \frac{g(x + \varepsilon h) - g(x)}{\varepsilon}, \quad x, h \in \mathbb{R}^d, g \in C_b^1(\mathbb{R}^d).$$

The main purpose of this note is to expound and prove the following theorems.

Theorem 1. *Under the condition* (C), *for any function* $g \in C_b^1(\mathbb{R}^d)$, $T > 0$ *and* $x, h \in \mathbb{R}^d$, *we have*

$$\nabla_h P_T g(x) = \mathbb{E}[g(X_T) N_T], \tag{5}$$

where

$$N_T = \int_0^T \left(\frac{S_T - s}{S_T} \nabla_h b(X_s) + \frac{h}{S_T} - \int_0^s \frac{s-u}{S_T} \nabla_h f(X_s - X_u) du \right) dW_{S_s}.$$

Theorem 2. *Under condition (C), for any non-negative $g \in C_b^1\left(\mathbb{R}^d\right)$, $p > 1$, $T > 0$ and $x, h \in \mathbb{R}^d$, the operator P_T satisfies that*

$$(P_T g(x))^p \leq P_T g^p(h) \exp\left[\frac{p}{p-1} C_{T,K_1,K_2} \|h - x\|^2\right].$$

2. Preliminaries

Consider the equation

$$X_t^\ell = x + W_{\ell_t} + \int_0^t b(X_s^\ell) ds - \int_0^t \int_0^s f(X_s^\ell - X_u^\ell) du ds + vt, \quad t \geq 0,$$

where $W = \{W_t, t \geq 0\}$ is a standard Brownian motion on $\mathbb{R}^d$ and ℓ is an absolutely continuous and strictly increasing function on $[0, +\infty)$ with $\ell_0 = 0$. Clearly, under the above assumption, this equation has a unique solution X_x^ℓ and the following Bismut formula holds. For more details, see Bao [9], Bismut [10], Fan [11], Guillin and Wang [12], and the references therein.

Lemma 1. *Under the condition (C), we have*

$$\nabla_h P_T^\ell g(x) = \mathbb{E}\left[g(X_T^\ell) L_T\right] \tag{6}$$

for any function $g \in C_b^1(\mathbb{R}^d)$, $T > 0$ and $x, h \in \mathbb{R}^d$, where

$$L_T = \int_0^T \left(\frac{\ell_T - s}{\ell_T} \nabla_h b(X_s^\ell) + \frac{h}{\ell_T} - \int_0^s \frac{s-u}{\ell_T} \nabla_h f(X_s^\ell - X_u^\ell) du\right) dW_{\ell_s}.$$

Proof. For $\varepsilon > 0$, we let Y^ℓ solve the equation

$$dY_s^\ell = dW_{\ell_s} + b(X_s^\ell) ds - \frac{\varepsilon h}{\ell_T} ds - \int_0^s f(X_s^\ell - X_u^\ell) du + v$$

with $Y_0^\ell = x + \varepsilon h$, $x, h \in \mathbb{R}^d$. Then $Y_s^\ell = X_s^\ell + \frac{\varepsilon(\ell_T - s)}{\ell_T} h$, $s \in [0, \ell_T]$. Denote

$$u_s = b(X_s^\ell) - b(Y_s^\ell) - \frac{\varepsilon h}{\ell_T} + \int_0^s \left[f(Y_s^\ell - Y_u^\ell) - f(X_s^\ell - X_u^\ell)\right] du$$

with $s \in [0, \ell_T]$. It follows from the Girsanov theorem that $\widetilde{W}_s = W_{\ell_t} - \int_0^t u_s ds, s \in [0, \ell_T]$ is also a Brownian motion under the probability $\mathrm{d}\mathbb{Q} := R_\varepsilon(\ell_T)\mathrm{d}\mathbb{P}$, where

$$R_\varepsilon(\ell_T) = \exp\left[-\int_0^{\ell_T} \langle u_s, dW_s \rangle - \frac{1}{2} \int_0^{\ell_T} |u_s|^2 ds\right].$$

Thus, under $\mathrm{d}\mathbb{Q} := R_\varepsilon \mathrm{d}\mathbb{P}$, the distribution of Y_y^ℓ coincides with that of $X_{x+\varepsilon h}^\ell$ under $\mathbb{P}$ by reformulating the equation for Y_s^ℓ as follows:

$$\begin{aligned} dY_s^\ell =& dW_{\ell_s} + b(X_s^\ell) ds - \frac{\varepsilon h}{\ell_T} ds - \int_0^s f(X_s^\ell - X_u^\ell) du + v \\ =& d\widetilde{W}_{\ell_s} + b(Y_s^\ell) ds - \int_0^s f(Y_s^\ell - Y_u^\ell) du + v, \end{aligned}$$

which implies that $\mathbb{E}\Big[g(Y_t^\ell)\Big] = \mathbb{E}\Big[R_\varepsilon g(X_t^\ell)\Big]$. Consider now the local martingale

$$\begin{aligned} M_T &= -\int_0^T u_s dW_{\ell_s} \\ &= \int_0^T \left(\frac{(\ell_T - s)\varepsilon}{\ell_T} \nabla_h b(X_s^\ell) + \frac{\varepsilon h}{\ell_T} - \int_0^s \frac{(s-u)\varepsilon}{\ell_T} \nabla_h f(X_s^\ell - X_u^\ell) du \right) dW_{\ell_s} := \varepsilon L_T. \end{aligned} \tag{7}$$

According to the condition (C), we obtain that

$$\langle M \rangle_T = \int_0^T \left| \frac{(\ell_T - s)\varepsilon}{\ell_T} \nabla_h b(X_s^\ell) + \frac{\varepsilon h}{\ell_T} - \int_0^s \frac{(s-u)\varepsilon}{\ell_T} \nabla_h f(X_s^\ell - X_u^\ell) du \right|^2 d\ell_s \le C_{K_1,K_2,T} \varepsilon^2$$

and

$$\begin{aligned} \nabla_h P_T^\ell g(x) &= \lim_{\varepsilon \to 0} \frac{P_T^\ell g(x + \varepsilon h) - P_T^\ell g(x)}{\varepsilon} = \lim_{\varepsilon \to 0} \mathbb{E}\left[g(X_T^\ell) \frac{R_\varepsilon - 1}{\varepsilon} \right] \\ &= \lim_{\varepsilon \to 0} \mathbb{E}\left[g(X_T^\ell) \frac{M_T}{\varepsilon} \right] = \mathbb{E}\Big[g(X_T^\ell) L_T \Big], \end{aligned}$$

and the theorem follows. □

Lemma 2. *Under the condition (C), the operator P_T^ℓ satisfies that*

$$\Big(P_T^\ell g(x)\Big)^p \le P_T^\ell g^p(h) \exp\left[\frac{p}{p-1} C_{T,K_1,K_2} \|h - x\|^2 \right]$$

for all $p > 1$, $T > 0$, $x, h \in \mathbb{R}^d$ and all non-negative $g \in C_b^1(\mathbb{R}^d)$.

Proof. By Young's inequality in Arnaudon et al. [13], we have, for all $\rho > 0$,

$$\Big| \nabla_h P_T^\ell g(x) \Big| \le \rho \Big[P_T^\ell (g \log g)(x) - P_T^\ell g(x) \Big(\log P_T^\ell g \Big)(x) \Big] + P_T^\ell g(x) \Big[\rho \log \mathbb{E} e^{\frac{|L_T|}{\rho}} \Big]. \tag{8}$$

Define $\beta(s) = 1 + s(p-1)$, $\delta(s) = x + s(h - x)$, $s \in [0,1]$ and $\rho = \frac{p-1}{\beta(s)}$. Then, we have

$$\begin{aligned} \frac{d}{ds} \log\Big(P_T^\ell g^{\beta(s)} \Big)^{\frac{p}{\beta(s)}} (\delta(s)) &\ge \frac{p}{\beta(s) P_T^\ell g^{\beta(s)}(\delta(s))} \left[\frac{p-1}{\beta(s)} \Big(P_T^\ell \Big(g^{\beta(s)} \log g^{\beta(s)} \Big)(\delta(s)) \right. \\ &\quad \left. - \Big(P_T^\ell g^{\beta(s)}(\delta(s)) \Big) \log P_T^\ell g^{\beta(s)}(\delta(s)) \Big) - \Big| \nabla_{h-x} P_T^\ell g^{\beta(s)}(\delta(s)) \Big| \right] \\ &\ge -\frac{p(p-1)}{\beta^2(s)} \log\left(\mathbb{E} e^{\frac{2\langle L \rangle_T}{\rho^2}} \right)^{\frac{1}{2}} \ge -\frac{p C_{K_1,K_2,T}}{p-1} \|h - x\|^2. \end{aligned}$$

It follows from (7) that

$$\begin{aligned} \langle L \rangle_T &= \int_0^T \frac{1}{\varepsilon^2} \left(b(Y_s^\ell) - b(X_s^\ell) + \frac{\varepsilon h}{\ell_T} - \int_0^s f(Y_s^\ell - Y_u^\ell) du + \int_0^s f(X_s^\ell - X_u^\ell) du \right)^2 d\ell_s \\ &\le \int_0^T \frac{1}{\varepsilon^2} \left(\|\nabla b\|_\infty \frac{\varepsilon(\ell_T - s)h}{\ell_T} + \frac{\varepsilon h}{\ell_T} + \int_0^s \|\nabla f\|_\infty \left| \frac{\varepsilon(\ell_T - u)h}{\ell_T} - \frac{\varepsilon(\ell_T - s)h}{\ell_T} \right| du \right)^2 d\ell_s \\ &\le \int_0^T \left(K_1 \frac{\ell_T - s}{\ell_T} + \frac{1}{\ell_T} + \int_0^s K_2 \frac{s-u}{\ell_T} du \right)^2 \|h\|^2 d\ell_s \le C_{K_1,K_2,T} \|h\|^2. \end{aligned}$$

This completes the proof. □

For the subordinator process $S = \{S_t, t \geq 0\}$, by Wang [14] and Zhang [15], we define the its regularization as follows:

$$S_t^\varepsilon := \frac{1}{\varepsilon}\int_t^{t+\varepsilon} S_s ds, \quad t \geq 0, \varepsilon > 0. \tag{9}$$

Then, S^ε is strictly increasing, absolutely continuous and $S^\varepsilon \to S$ a.s. as $\varepsilon \to 0$. For each $\varepsilon > 0$, we consider the following approximation equation of (3):

$$X_t^\varepsilon = x + W_{S_t^\varepsilon} + \int_0^t b(X_s^\varepsilon)ds - \int_0^t \int_0^s f(X_s^\varepsilon - X_u^\varepsilon)duds + vt, \quad t \geq 0. \tag{10}$$

In order to prove Theorem 1, we need some necessary lemmas.

Lemma 3. *For any $p \geq 1$, $T \geq 0$ and $x \in \mathbb{R}^d$, we have*

$$\lim_{\varepsilon \to 0} \sup_{0 \leq t \leq T} \mathbb{E}|X_t^\varepsilon(x) - X_t(x)|^p = 0.$$

Proof. Using (10), we have

$$\begin{aligned}
\sup_{0\leq t\leq T} \mathbb{E}|X_t^\varepsilon - X_t| &\leq \sup_{0\leq t\leq T} \mathbb{E}\Big|W_{S_t^\varepsilon} - W_{S_t}\Big| + \|\nabla b\|_\infty \int_0^t \sup_{0\leq t\leq T} \mathbb{E}|X_s^\varepsilon - X_s|ds \\
&\quad + \|\nabla f\|_\infty \int_0^t \int_0^s \sup_{0\leq t\leq T} \mathbb{E}|(X_s^\varepsilon - X_u^\varepsilon) - (X_s - X_u)|duds \\
&\leq \sup_{0\leq t\leq T} \mathbb{E}\Big|W_{S_t^\varepsilon} - W_{S_t}\Big| + K_1 \int_0^t \sup_{0\leq t\leq T} \mathbb{E}|X_s^\varepsilon - X_s|ds \\
&\quad + K_2 \int_0^t \int_0^s \sup_{0\leq t\leq T} \mathbb{E}|X_s^\varepsilon - X_s| + \sup_{0\leq t\leq T} \mathbb{E}|X_u^\varepsilon - X_u|duds \\
&\leq \sup_{0\leq t\leq T} \mathbb{E}\Big|W_{S_t^\varepsilon} - W_{S_t}\Big| + C_{K_1,K_2,T} \int_0^t \sup_{0\leq t\leq T} \mathbb{E}|X_s^\varepsilon - X_s|ds \\
&\quad + C_{K_2,T} \int_0^t \int_0^s \sup_{0\leq t\leq T} \mathbb{E}|X_u^\varepsilon - X_u|duds.
\end{aligned}$$

According to the Gronwall's inequality (given in Dragomir [16]), the lemma follows since $S_t^\varepsilon \downarrow S_t$, a.s. as $\varepsilon \downarrow 0$. □

Lemma 4 (Zhang [15]). *Assume that ξ_t is a bounded, continuous and $\left(\mathcal{F}_{\ell_t}^{\mathbb{W}}\right)$-adapted $\mathbb{R}^d$-valued process; we then have*

$$\lim_{\varepsilon \to 0} \mathbb{E}\left|\int_0^t \xi_s dW_{S_t^\varepsilon} - \int_0^t \xi_s dW_{S_t}\right|^p = 0$$

for each $p, t > 0$, where S_t^ε is defined by (9).

Lemma 5. *For $t > 0$ and $h \in \mathbb{R}^d$, we have*

$$\lim_{\varepsilon \to 0} \mathbb{E}\left(\int_0^t \nabla_h b(X_s^\varepsilon)dW_{S_s^\varepsilon}\right) = \mathbb{E}\left(\int_0^t \nabla_h b(X_s)dW_{S_s}\right) \tag{11}$$

and

$$\lim_{\varepsilon \to 0} \mathbb{E}\left(\int_0^t \int_0^s \nabla_h f(X_s^\varepsilon - X_u^\varepsilon)dudW_{S_s^\varepsilon}\right) = \mathbb{E}\left(\int_0^t \int_0^s \nabla_h f(X_s - X_u)dudW_{S_s}\right). \tag{12}$$

Proof. The lemma follows from Lemmas 3 and 4. □

3. Proofs

In this section, we give proofs of Theorems 1 and 2. Consider the operators

$$P_t^\varepsilon f(x) = \mathbb{E} f(X_t^\varepsilon(x)), \quad t \geq 0, f \in C_b^1(\mathbb{R}^d),$$

where $X^\varepsilon(x)$ is the solution of (10) with the initial value $x \in \mathbb{R}^d$. Then, Lemma 3 implies that

$$\lim_{\varepsilon \to 0} P_T^\varepsilon g(x) = P_T g(x), \quad g \in C_b^1(\mathbb{R}^d). \tag{13}$$

According to Lemma 1, we have

$$\nabla_h P_T^\ell g(x) = \mathbb{E}\Big[g(X_T^\ell) L_T\Big]$$

This holds true for both ℓ, where ℓ is an absolutely continuous and strictly increasing function on $[0, +\infty)$ with $\ell_0 = 0$, In particular, we take $\ell = S^\varepsilon$,

$$\begin{aligned} \nabla_h P_T^\varepsilon g(x) &= \mathbb{E}\Big[g(X_T^{S^\varepsilon}) L_T\Big] = \mathbb{E}[g(X_T^\varepsilon) L_T] \\ &= \mathbb{E}\left[g(X_T^\varepsilon) \int_0^T \left(\frac{S_T^\varepsilon - s}{S_T^\varepsilon} \nabla_h b(X_s^\varepsilon) + \frac{h}{S_T^\varepsilon} - \int_0^s \frac{s-u}{S_T^\varepsilon} \nabla_h f(X_s^\varepsilon - X_u^\varepsilon) du\right) dW_{S^\varepsilon}\right] \end{aligned} \tag{14}$$

for any function $g \in C_b^1(\mathbb{R}^d)$ and $h \in \mathbb{R}^d$ under the condition (C). Assuming $\varepsilon \downarrow 0$ and using Lemmas 3–5, and (14), we obtain that

$$\nabla_h P_T g(x) = \mathbb{E}[g(X_T) N_T]$$

for any function $g \in C_b^1(\mathbb{R}^d)$ and $h \in \mathbb{R}^d$, where

$$N_T = \int_0^T \left(\frac{S_T - s}{S_T} \nabla_h b(X_s) + \frac{h}{S_T} - \int_0^s \frac{s-u}{S_T} \nabla_h f(X_s - X_u) du\right) dW_{S_s}.$$

This completes the proof of Theorem 1.

Now, we check Theorem 2. Taking $\ell = S^\varepsilon$ and applying the same technique, we obtain

$$\begin{aligned} P_T^\varepsilon g(x) &= \mathbb{E}\Big[P_T^{S^\varepsilon} g(x)\Big] \\ &\leq \mathbb{E}\left[\left(P_T^{S^\varepsilon} g^p(x)\right)^{\frac{1}{p}} \left(\exp\left[\frac{p}{p-1} C_{T,K_1,K_2} \|h - x\|^2\right]\right)^{\frac{p-1}{p}}\right] \\ &\leq (P_T^\varepsilon g^p(x))^{\frac{1}{p}} \left(\exp\left[\frac{p}{p-1} C_{T,K_1,K_2} \|h - x\|^2\right]\right)^{\frac{p-1}{p}} \end{aligned}$$

for all $p > 1$, $T > 0$ and $x, h \in \mathbb{R}^d$, and all non-negative $g \in C_b^1(\mathbb{R}^d)$. Combining Lemmas 3–5, (13), and letting $\varepsilon \downarrow 0$ to lead that

$$P_T g(x) \leq (P_T g^p(h))^{\frac{1}{p}} \left(\exp\left[\frac{p}{p-1} C_{T,K_1,K_2} \|h - x\|^2\right]\right)^{\frac{p-1}{p}}$$

for all $x, h \in \mathbb{R}^d$ and all non-negative $g \in C_b^1(\mathbb{R}^d)$.

It remains to apply the approximation argument in Zhang [15] to finish the proof. Let

$$\tilde{b}(x) := b(x) - Kx, \quad \tilde{f}(x) := f(x) - Kx, \quad x \in \mathbb{R}^d.$$

According to Da Prato et al. [17], because for any $n \in \mathbb{N}$, the mapping $id - n^{-1}\tilde{b} : \mathbb{R}^d \to \mathbb{R}^d$ and $id - n^{-1}\tilde{f} : \mathbb{R}^d \to \mathbb{R}^d$ are injective under our condition, the maps

$$b^{(n)} := n\Big[(id - n^{-1}\tilde{b})^{-1}(x) - x\Big] + Kx, \quad n \in \mathbb{N}, x \in \mathbb{R}^d$$

and

$$f^{(n)} := n\Big[(id - n^{-1}\tilde{f})^{-1}(x) - x\Big] + Kx, \quad n \in \mathbb{N}, x \in \mathbb{R}^d$$

are globally Lipschitz continuous. Denote $(X_t^{(n)}(x))_{t\geq 0}$, solve the equation (3) with b, f replaced by $b^{(n)}$, $f^{(n)}$, respectively, and $P_t^{(n)}$ is the associated operator. Based on the previous proofs, the statements of Theorems 1 and 2 hold for $P_T^{(n)}$ in place of P_T.

On the other hand, we combine the argument in Wang [14] that

$$\lim_{n\to\infty} X_T^{(n)}(x) = X_T(x), a.s.,$$

for all $x \in \mathbb{R}^d$. And

$$\lim_{n\to\infty} P_T^{(n)} g = P_T g, \quad g \in C_b^1(\mathbb{R}^d).$$

Therefore, the claim follows if we let $n \to \infty$.

Author Contributions: Conceptualization, Y.S. and L.Y.; Methodology, Y.S. and L.Y.; Formal analysis, Y.S. and L.Y.; Investigation, Y.S. and L.Y.; Writing—original draft, Y.S. and L.Y.; writing—review and editing, Y.S. and L.Y. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by NSFC (11971101, 12171081) and the NSF of Shanghai (No. 24ZR1402900).

Data Availability Statement: Data are contained within the article.

Acknowledgments: We thank the anonymous referees for the careful reading of the manuscript and their suggestions, which helped to improve this paper.

Conflicts of Interest: The authors declare no conflicts of interest.

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