

Article

Nonparametric Estimation of Dynamic Value-at-Risk: Multifunctional GARCH Model Case

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Abstract: Value-at-Risk (VaR) estimation using the GARCH model is an important topic in financial data analysis. It allows for an increase in the accuracy of risk assessment by controlling time-varying volatility. In this paper, we enhance this feature by exploring the functional path of the financial data. More precisely, we study the nonparametric estimation of the multi-functional VaR function using the local linear method, construct an estimator, and establish its stochastic consistency. The derived asymptotic result provides a rigorous mathematical foundation that permits boosting the use of the VaR function in financial data analysis. Furthermore, an empirical analysis is performed in order to examine the efficiency of the proposed algorithm. Additionally, a real data application is created to highlight the multi-functionality of the VaR estimation for multi-asset risk management.

Keywords: Asymmetric least squares; value at risk; coherent risk measures; expected shortfall; expectile regression; prediction markets; decision making; big-data analysis

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1. Introduction

With rapid technological developments enabling the digital control of financial activities, real-time analysis of financial risks is becoming essential. However, it is well known that standard instruments obtained from multivariate parametric statistics cannot be used to solve this problem. Instead, in this paper, we propose using multi-functional data to develop a new data-driven approach that is capable of managing multi-financial risks in real time. Specifically, we assess market risk through the estimation of the VaR-function using multi-functional co-variate functions.

It is well-documented that financial volatility is an important factor in investment decisions. It permits us to offer a dynamic and robust measure of financial risk. This subject has been widely considered in the statistical literature. The first study was introduced by [1]. He considered the autoregressive conditional heteroscedasticity (ARCH) model. Such a concept was generalized by [2] to the (GARCH) model. The functional version of the GARCH model was introduced by [3]. He stated the necessary conditions for the existence of a strictly stationary solution of the process and established the consistency of the estimators. We return to [4] for an illustrative study showing the potentiality of the functional GARCG to fit the high-frequency data. An alternative estimation-based quasi-likelihood approach was introduced by [5]. They proved the asymptotic normality of the constructed estimator. We return to [6] or [7] for more relevant studies on the topic. As a

counterpart, VaR estimation is often constructed from the quantile function, which has been widely studied in mathematical statistics. In particular, estimation using quantile regression is a dominant approach because it makes it possible to fit the dynamics of quantiles. The first significant result on this subject was developed in [8]. In the latter reference, the problem of estimating the VaR was carried out under the conditional autoregressive structure. In addition, certain advances in risk management using the Conditional Autoregressive Value-at-Risk (CAVaR) models were obtained by numerous authors in the past (see, for example, [9] or [10]). Other alternative methods for estimating the CAVaR models are proposed in the literature. We cite, for example, the GARCH approach used in [11], the conditional Extreme Value (EV) method considered in [12], or the conditional Copula algorithm in [13]. For a comparative study between the three approaches using different stock market indices, we cite [14]. The authors of the last reference propose combining the GARCH, Copula, and EV functions to estimate the VaR of a given portfolio. Recently, the authors of [15] estimated the VaR under GARCH asymmetry models using the GARCH-EVT approach. This approach was applied to the financial time series on Kenya's currency exchange rate. In functional data analysis, the estimation of quantile regression was introduced by the authors of [16]. They used the B-spline smoothing approach to study a linear model of regression on quantiles when the explanatory variable is Hilbertian. They established the L_2 -convergence rate of the estimate. In the nonparametric context, the authors of [17] established the almost complete convergence of a kernel estimator of the quantile regression when the observations are i.i.d. [18] established the L_p convergence of the same estimator but in the dependent case. More recent investigations into quantile regression can be found in [19–22] and references therein. Despite the importance of multivariate functional data, the literature on the subject is limited. All the existing studies treat the unconditional quantile (see, [23,24] or [25]). Thus, the estimation of the conditional VaR-function using the local linear method is a challenging issue in functional statistics (see [26,27] for more information on the subject).

Value-at-Risk (VaR) in functional statistics (FVAR) allows us to tackle a more complex structure. Specifically, it permits us to analyze risk when the data comes in the form of entire curves. In financial markets, where prices fluctuate in milliseconds and risk evolves continuously, traditional Value-at-Risk (VaR) models often fail to follow the dynamic nature of modern trading. FVaR is more adequate for this situation. The literature on this topic is still limited. The first result on this model was developed by the authors of [28]; they proved the almost complete consistency of the local constant estimation of the FVAR model. The authors of [29] employed functional data modeling in financial risk management and applied their models to China's stock market. The authors of [30] estimated the conditional VaR as a functional linear quantile regression model and used their linear estimator for FVAR to analyze the intra-day return curves. The authors of [31] considered the nonparametric estimation of the functional expected shortfall regression as an alternative risk tool to the FVAR function. We refer to [32–34] for more alternative nonparametric functional models to FVAR risk models.

Estimation using multivariate functional data is a challenging issue in functional statistics. Clearly, the multivariate functional data analysis allows us to explore the complex relationships between multiple functional variables that the univariate methods fail to model. The multivariate functional data involves the usual structure in applied areas, namely, in finance, where there are potential interactions between different trajectories of the portfolios. The second novelty of this contribution is to analyze financial risk using the nonparametric functional GARCH model. This consideration is motivated by the popularity of the GARCH model as a principal tool to forecast the variability of asset returns over time. It models the time-varying volatility, which is crucial in risk management.

The combination of both of these structures is pivotal for financial risk. It permits us to enhance the accuracy of the prediction of fluctuations in financial dynamics by taking into account the multiple interactions of related financial movements. In the mathematical component of the paper, we construct an estimator of the dynamic VaR function using local linear estimation. Furthermore, we demonstrate the Borel convergence of the estimator within a multifunctional GARCH framework. It is widely recognized that local linear estimation provides a substantial benefit compared to local constant estimation, as it helps in minimizing the bias term. Thus, the local linear estimation method offers a more precise forecasting tool for managing financial risk. To measure the accuracy of the new estimator, we detail its convergence rate and describe it under broad assumptions that encompass the essential elements of the contributions, such as the functionality and correlation of the data, the nonparametric nature of the model, and the various parameters involved in the estimator. To illustrate real-world applicability, we examine the feasibility of our algorithm using simulated and real financial market data. The computational analysis of this contribution reveals the strong ability of the estimator to detect risk fluctuations.

This paper is organized as follows. In the next section, we define our statistical framework, and then we discuss the principal motivations to pursue this subject. We state the main results in Section 3. An empirical analysis is presented in Section 4. It contains illustrative examples covering different features of multivariate functional generalized autoregressive conditional heteroscedasticity (MFGARCH). A real data application is given in Section 5. In Section 6, we give a general conclusion and list some research perspectives allowing for further investigations in this area of statistics. The last section is an appendix (Appendix A) devoted to the demonstration of the intermediate results.

2. Statistical Framework Models and Estimators

In this paper, we introduce an innovative functional approach to analyze risk in multi-asset financial risk. Therefore, we consider, $(P_i(\cdot))_{i=1,\dots,n}$ as price trajectories of n -sample portfolios of m financial assets. Assume that $(P_i(\cdot))_{i=1,\dots,n}$ are multivariate functional GARCH processes (MFGARCH) defined by

$$P_i(\cdot) = \begin{pmatrix} P_i^1(\cdot) \\ P_i^2(\cdot) \\ \vdots \\ P_i^m(\cdot) \end{pmatrix}, \quad i = 1, \dots, n.$$

Such a functional modelization allows us to more accurately model complex market dynamics. Specifically, we focus on the functional version of the dynamic conditional correlation GARCH, introduced by [35], for which

$$P_i = \mathbf{H}_i^{1/2} \odot \Xi_i \quad i = 1, \dots, n, \quad (1)$$

where $\odot$ is the pointwise product, Ξ_i is a vector of independent functional random variables, and $\mathbf{H}_i$ is the conditional covariance matrix, which has the following volatility decomposition,

$$\mathbf{H}_i = \mathbf{D}_i \mathbf{R}_i \mathbf{D}_i \quad i = 1, \dots, n,$$

where $\mathbf{R}_i$ is the time-varying correlation matrix, and $\mathbf{D}_i$ is a functional diagonal matrix defined by

$$\mathbf{D}_i = \begin{pmatrix} h_i^1 & \dots & \dots & 0 \\ 0 & h_i^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & h_i^m \end{pmatrix}.$$

The diagonal is

$$\begin{pmatrix} h_i^1 \\ h_i^2 \\ \vdots \\ h_i^m \end{pmatrix} = \begin{pmatrix} w^1 \\ w^2 \\ \vdots \\ w^m \end{pmatrix} + \sum_{k=1}^p \alpha_k \begin{pmatrix} P_{i-k}^{1^2} \\ P_{i-k}^{2^2} \\ \vdots \\ P_{i-k}^{m^2} \end{pmatrix} + \sum_{j=1}^q \beta_j \begin{pmatrix} h_{i-j}^1 \\ h_{i-j}^2 \\ \vdots \\ h_{i-j}^m \end{pmatrix},$$

where $(\alpha_k)_k$ and $(\beta_j)_j$ are linear operators. This formulation generalizes the univariate functional GARCH process (introduced by [36]) to the multivariate setting. It permits us to highlight the time-varying nature of covariances and correlations of multiple assets in a continuous time. This situation is more realistic than the standard case. Indeed, in financial markets, the connections between assets are always in continuous change. This feature is not covered by the traditional method. Instead, the functional approach completely overlooks the market dynamics. Similar to [36], the strict stationarity and the ergodicity of the MFGARCH is related to the Lyapunov exponent of the sequence of matrix of operators by

$$\Psi_k^{(p,q)} = \begin{pmatrix} \otimes_k \alpha_1 & \dots & \otimes_k \alpha_p & \otimes_k \beta_1 & \dots & \otimes_k \beta_{q-1} & \otimes_k \beta_q \\ I & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & I & 0 & \dots & 0 & 0 \\ \alpha_1 & \dots & \alpha_p & \beta_1 & \dots & \beta_{q-1} & \beta_q \\ 0 & \dots & 0 & I & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & I & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & I & 0 \end{pmatrix},$$

where $\otimes_k$ are operators mapping $\otimes_k(\mathcal{X}) := \mathcal{X} \odot \Xi_k^2$. Typically, we assume the following:

PO1: The top Lyapunov exponent of the sequence $\Psi_k^{(p,q)}$ is such that

$$\gamma^{(p,q)} = \lim_{k \rightarrow \infty} \mathbb{E} \log \|\Psi_k^{(p,q)} \Psi_{k-1}^{(p,q)} \dots \Psi_1^{(p,q)}\| < 0,$$

where $\|\cdot\|$ is the norm over the functional matrix of order $p+q$. We refer to [37] for more discussions on this assumption and their variants. Recall that the financial risk is evaluated by forecasting the fluctuation of some future characteristics of the process $P(\cdot)$. Often, the claimed characteristics are fixed real values, such as the close values, the maximum values, the values at given time t_0 , or the variation at a given frame. We model this issue by representing the future characteristics as a response variable $(R_i)_{i=1,\dots,n}$ defined by

$$R_i = Fu(P_i), \quad i = 1, \dots, n. \quad (2)$$

Finally, the assessment of the financial risk is performed using the functional observations $O_i = (P_i, R_i)_{i=1,\dots,n}$ sampled from the MFGARCH process. For the sake of generality, we

suppose that $O_i = (P_i, R_i)_{i=1, \dots, n}$ are valued in $\mathcal{F} \times \mathbb{R}$, where $\mathcal{F} = \underbrace{\mathcal{F}^1 \times \mathcal{F}^2 \dots \times \mathcal{F}^m}_{m\text{-times}}$ is a product of the m -semi-metric space $\mathcal{F}^j$. The latter is equipped with a semi-metric d_j .

Furthermore, for all $\mathbf{P} = \begin{pmatrix} \mathbf{P}^1 \\ \mathbf{P}^2 \\ \vdots \\ \mathbf{P}^m \end{pmatrix} \in \mathcal{F}$, using $F^{\mathbf{P}}$ (resp. $f^{\mathbf{P}}$), we denote the conditional distribution (resp. conditional density) function of R_i given by $P_i = \mathbf{P}$ and by $\mathcal{N}_{\mathbf{P}}$ of a given neighborhood of $\mathbf{P}$. The neighborhood $\mathcal{N}_{\mathbf{P}}$ is defined according to the metric on $\mathcal{F}$ defined by $d(\cdot, \cdot) = \sum_{j=1}^m d_j(\cdot, \cdot)$. Theoretically, the functional VaR of order α given $P_i = \mathbf{P}$ is

$$VaR_{\alpha}(\mathbf{P}) = \arg \min_{t \in \mathbb{R}} \mathbb{E} \left[(R_i - t)(\alpha - \mathbb{1}_{[R_i < t]}) \mid P_i = \mathbf{P} \right], \quad \text{for } \alpha \in]0, 1[, \quad (3)$$

where $\mathbb{1}_A$ is the indicator function on the set A . As the local linear method is the more general and appropriate approach, we focus on its functional version that permits us to improve the nonparametric estimation of $VaR_{\alpha}(\mathbf{P})$. To accomplish that, we suppose

$$\text{for all } \mathcal{P} \in \mathcal{N}_{\mathbf{P}}, \quad VaR_{\alpha}(\mathcal{P}) = \xi + \zeta d(\mathcal{P}, \mathbf{P}),$$

where ξ, ζ are estimated by $\hat{\xi}$ and $\hat{\zeta}$ as the solutions of

$$\min_{\xi, \zeta \in \mathbb{R}^2} \frac{\sum_{i=1}^n \prod_{j=1}^m \mathbf{L}(b_n^{-1} d_j(\mathbf{P}^j, P_i^j)) (R_i - \xi - \zeta d(P_i, \mathbf{P})) (\alpha - \mathbb{1}_{[R_i - \xi - \zeta d(P_i, \mathbf{P}) < 0]})}{\sum_{i=1}^n \prod_{j=1}^m \mathbf{L}(b_n^{-1} d_j(\mathbf{P}^j, P_i^j))}, \quad (4)$$

where $\mathbf{L}$ is a kernel function, and b_n is a sequence of real positive numbers. Observe that if $d(\mathbf{P}, \mathbf{P}) = 0$, we have

$$VaR_{\alpha}(\mathbf{P}) = \xi \quad \text{and} \quad \widehat{VaR_{\alpha}}(\mathbf{P}) = \hat{\xi}.$$

We point out that if $b = 0$, we obtain, from (4), the Nadaraya–Watson estimator. In this work, we aim to establish the convergence rate of the estimator $\widehat{VaR_{\alpha}}(\mathbf{P})$ to $VaR_{\alpha}(\mathbf{P})$. Such a result constitutes a good mathematical support for the practical use of $\widehat{VaR_{\alpha}}(\mathbf{P})$ as a financial risk tool.

3. Notations, Assumptions and Comments

Now, to derive the claimed asymptotic result, we first denote, using C and C' , some strictly positive constants and, for $b > 0$, we define the ball

$$B(\mathbf{P}, b) := \left\{ \mathbf{P}' \in \mathcal{F} / d(\mathbf{P}', \mathbf{P}) = \sum_{j=1}^m d_j(\mathbf{P}^j, P_i^j) < b \right\}.$$

Moreover, for $i = 1, \dots, n$, we use $\mathfrak{F}_i$ to represent the σ -algebra generated by $((P_1, R_1), \dots, (P_i, R_i))$ and $\mathfrak{G}_i$ for the σ -algebra generated by $((P_1, R_1), \dots, (P_i, R_i), P_{i+1})$. We consider the following assumptions:

PO2: The processes $(P_i, R_i)_{k \in \mathbb{N}}$ satisfies

$$\begin{cases} \text{(i) The function } \chi(m, \mathbf{P}, b) := \mathbb{P}(P \in B(\mathbf{P}, b)) > 0, \quad \forall b > 0. \\ \text{(ii) For all } i = 1, \dots, n \text{ there exist a deterministic function } \chi_i(m, \mathbf{P}, \cdot) \text{ such that} \\ \quad 0 < \mathbb{P}(P_i \in B(\mathbf{P}, b) | \mathfrak{F}_{i-1}) \leq \chi_i(m, \mathbf{P}, b), \quad \forall b > 0. \\ \text{(iii) For all } b > 0, \quad \frac{1}{n \chi(m, \mathbf{P}, b)} \sum_{i=1}^n \mathbb{P}(P_i \in B(\mathbf{P}, b) | \mathfrak{F}_{i-1}) \rightarrow 1 \quad a.co. \end{cases}$$

PO3: There exists $\delta' > 0$ such that, for all $(t_1, t_2) \in [VaR_\alpha(\mathbf{P}) - \delta', VaR_\alpha(\mathbf{P})\delta']^2$ and $\forall (\mathbf{P}_1, \mathbf{P}_2) \in \mathcal{N}_{\mathbf{P}} \times \mathcal{N}_{\mathbf{P}}$, we have

$$|F^{\mathbf{P}_1}(t_1) - F^{\mathbf{P}_2}(t_2)| \leq C_1 \left(d^{k_1}(\mathbf{P}_1, \mathbf{P}_2) + |t_1 - t_2|^{k_2} \right), \text{ for } C_1, k_1, k_2 > 0.$$

PO4: The kernel $\mathbf{L}$ is a positive and differentiable function, which is supported within $(0, 1)$ such that

$$n^{-1} \begin{pmatrix} \mathbf{L}(1) - \int_0^1 \mathbf{L}'(t) \varphi(m, \mathbf{P}, t) dt & \mathbf{L}(1) - \int_0^1 (t\mathbf{L}(t))' \varphi(m, \mathbf{P}, t) dt \\ \mathbf{L}(1) - \int_0^1 (t\mathbf{L}(t))' \varphi(m, \mathbf{P}, t) dt & \mathbf{L}(1) - \int_0^1 (t^2\mathbf{L}(t))' \varphi(m, \mathbf{P}, t) dt \end{pmatrix}$$

is a positive definite matrix.

PO5:

$$\lim_{n \rightarrow \infty} b_n = 0 \text{ and } \lim_{n \rightarrow \infty} \frac{\varphi(m, \mathbf{P}, b_n) \log n}{n^2 \chi^2(m, \mathbf{P}, b_n)} = 0,$$

$$\text{where } \varphi(m, \mathbf{P}, b_n) = \sum_{i=1}^n \chi_i(m, \mathbf{P}, b_n).$$

Comments on the Hypotheses

All the considered assumptions are fairly mild in the nonparametric functional statistic. They permit us to involve a larger class of financial times series data. In fact, the MFGARCH with the DCC-mode is crucial in finance. It allows us to analyze the time-varying correlations between multiple financial assets. The exploration of the functional nature of this kind of process is very interesting for portfolio optimization or risk management. The functional structure is characterized by the assumption **PO2** that highlights the ergodicity property of the functional data. Clearly, this assumption is less restrictive to the assumption introduced by [38]. Indeed, unlike [38], in **PO2**, the decomposition of the concentration functions is not required. Thus, it is not necessary to use the Onsager–Machlup function (see [39]). The conditions **PO3–PO5** are also standard, and they are assumed by several works in this area of functional time series data (see [38]).

4. Results

Our primary result is given in the following theorem. It deals with the Borel almost completely (a.co.) (see [40]).

Theorem 1. Assume that **PO1–PO4** are satisfied, and if $f^{\mathbf{P}}(VaR_\alpha(\mathbf{P})) > 0$, then we have

$$\widehat{VaR}_\alpha(\mathbf{P}) - VaR_\alpha(\mathbf{P}) = O\left(b_n^{\min(k_1, k_2)}\right) + O\left(\sqrt{\frac{\varphi(m, \mathbf{P}, b_n) \log n}{n^2 \chi^2(m, \mathbf{P}, b_n)}}\right) \quad a.co.$$

Comments on the results: Clearly, this theorem gives a more robust version of stochastic convergence (BC-mode) that surpasses the usual almost-sure or probability consistencies. Although milder assumptions are adequate for these, more typical forms of consistency would support the practical application of our estimator; we emphasized this mathematical broadness. The BC-mode is not only encompasses these easy forms but also highlights a crucial trade-off between theoretical assurances and imposed assumptions. Therefore, our contribution lies in this dual guarantee: strong theoretically results that simultaneously advance practical applications in this field of financial risk management.

Proof of the Main Result

The proof is based on the application of the second part of Lemma 1 in [41]. Indeed, we apply the cited Lemma on the triplet $(Y_n, \mathbf{B}_n, v_n)$ defined by

- The vectorial sequence of functions

$$Y_n(v) = \frac{1}{n\chi(m, \mathbf{P}, b)} \sum_{i=1}^n \zeta_\alpha(v) \begin{pmatrix} 1 \\ b_n^{-1} \mathbf{d}_i \end{pmatrix} \mathbf{L}_i \quad \text{for } v = \begin{pmatrix} \kappa \\ \varkappa \end{pmatrix},$$

where

$$\zeta_\alpha(v) = \alpha - \mathbf{1}_{P_i \leq (\kappa + \xi) + (b_n^{-1} \varkappa + \varsigma) \mathbf{d}_i} \quad \text{and} \quad \mathbf{d}_i = d(P_i, \mathbf{P}).$$

- The vectorial sequence

$$\mathbf{B}_n = Y_n(v_0) \quad \text{with} \quad v_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

- The sequence $v_n = \begin{pmatrix} \hat{\xi} - \xi \\ b_n(\hat{\xi} - \varsigma) \end{pmatrix}$.

Since the function $\mathbf{1}_{[P_i \leq \cdot]}$ is a monotone and increasing function, then the first condition of Lemma 1 in [41] is satisfied. So, Theorem 1 is a consequence of the following lemmas.

Lemma 1. Under assumptions **PO1–PO5**, we have

$$\|\mathbf{B}_n\| = O(b_n^{\min(k_1, k_2)}) + O_{a.co.} \left(\sqrt{\frac{\varphi(m, \mathbf{P}, b_n) \log n}{n^2 \chi^2(m, \mathbf{P}, b_n)}} \right).$$

Lemma 2. Under assumptions **PO1–PO5**, and if $f^{\mathbf{P}}(\text{VaR}_\alpha(\mathbf{P})) > 0$, then, we have

$$\sup_{\|v\| \leq M} \|Y_n(v) + \lambda_0 Dv - \mathbf{B}_n\| = o_{a.co.}(1)$$

with

$$D = \begin{pmatrix} \mathbf{L}(1) - \int_{-1}^1 \mathbf{L}'(t) \chi(m, \mathbf{P}, t) dt & \mathbf{L}(1) - \int_{-1}^1 (t \mathbf{L}(t))' \chi(m, \mathbf{P}, t) dt \\ \mathbf{L}(1) - \int_{-1}^1 (t \mathbf{L}(t))' \chi(m, \mathbf{P}, t) dt & \mathbf{L}(1) - \int_{-1}^1 (t^2 \mathbf{L}(t))' \chi(m, \mathbf{P}, t) dt \end{pmatrix}$$

and

$$\lambda_0 = f^{\mathbf{P}}(\text{VaR}_\alpha(\mathbf{P})).$$

5. Empirical Data Analysis

The DCC model is a popular version of Multivariate GARCH models. It was popularized in financial risk assessment through its time-varying correlations, which reflect the usual situations in practice. Thus, the generalization of DCC in MFGARCH modeling is very interesting in practice. It permits us to enhance and improve the accuracy of risk detection by exploring the dynamic nature of financial markets and the time-varying volatility. Moreover, in this context of multi-functional data analysis, the VaR-function has a crucial role. It is used to quantify the extreme loss or the extreme gain of multiple financial assets and highlight the complex dependencies between different factors in financial time series data. The main aim of this section is to examine the computability of the

estimator $\widehat{VaR}_\alpha(\mathbf{P})$ using an artificial data drawn from the DCC-MFGARCH process. In fact, the computability of the $\widehat{VaR}_\alpha(\mathbf{P})$ is related to the determination of the different parameters used in the estimator. In this context of local linear smoothing, the choice of the smoothing parameter is a critical issue. Furthermore, in quantile regression, there exist many rules to select this parameter. A popular one is the cross-validation approach used by [40]. In this empirical analysis, we compare the cited rule to two other algorithms. The first one is the cross-validation method used by [42], for which the bandwidth is selected as

$$b_I^{opt} = \arg \min_{b \in \mathbf{H}_n} \frac{1}{n} \sum_{i=1}^n \left(VaR_\alpha^L(P_i) - \widehat{VaR}_\alpha^{-i}(P_i) \right)^2,$$

where $VaR_\alpha^L(P_i)$ denotes the local empirical VaR_α , computed from a neighbourhood of closest functionals to P_i (we used different neighbour sizes (4, 6, 10, 14, and 20)). Also, $\widehat{VaR}_\alpha^{-i}(P_i)$ is the conditional VaR estimator at P_i . The second rule (approach) is proposed by [43]. Its optimal bandwidth is given by

$$b_Y^{opt} = \arg \min_{b \in \mathbf{H}_n} \frac{1}{n} \sum_{i=1}^n \left((R_i - \widehat{VaR}_\alpha^{-i}(P_i))(\alpha - \mathbf{1}_{[R_i < \widehat{VaR}_\alpha^{-i}(P_i)]}) \right)^2. \quad (5)$$

Finally, we recall that the optimal bandwidth according to the cross-validation of [40] is

$$b_{FV}^{opt} = \arg \min_{b \in \mathbf{H}_n} \frac{1}{n} \sum_{i=1}^n \left(R_i - \widehat{VaR}_{0.5}^{-i}(P_i) \right)^2.$$

The three rules are optimized over $\mathbf{H}_n$, a subset of positive real numbers $b(k)$, for which the ball centered at P_i with radius $b(k)$ contains exactly k observation neighbors of P_i . The comparative study will be performed through a multi-variate functional GARCH process generated using the algorithm (1). We put $p = q = 1$, and $m = 2$ and the independent functional random variables are generated using the *mfdata* R-code (Version 4.2.2) in the R-package *fda.usc*. The operator α_1 and β_1 are defined by

$$\alpha_1(X_1, X_2) = \int_0^1 \alpha^1(t, s) X_1(s) ds + \int_0^1 \alpha^2(t, s) X_2(s) ds,$$

where $\alpha^1(t, s) = 0.12((s - 0.5)^2 + (t - 0.5)^2)$, $\alpha^2(t, s) = 0.12((1 - t)^2 s^2)$ and

$$\beta_1(X_1, X_2) = \int_0^1 \beta^1(t, s) X_1(s) ds + \int_0^1 \beta^2(t, s) X_2(s) ds,$$

with $\beta^1(t, s) = 0.12((1 - s)^2 t^2)$, $\beta^2(t, s) = 0.12((1 - t)^4 s^4)$. These operators are considered in order to incorporate the theoretical assumption, namely, the postulate **PO1**. The generated MFGARCH represents the day-dynamic of portfolios containing two financial assets. The price trajectories of the two assets are plotted in Figure 1.

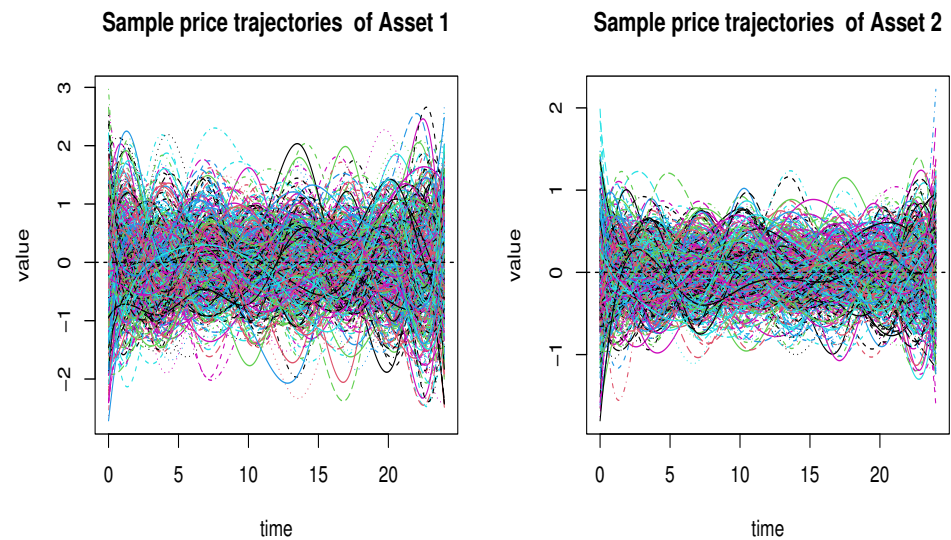


Figure 1. The MFGARCH covariate.

The curve of the dynamic correlations between the two components is displayed in Figure 2.

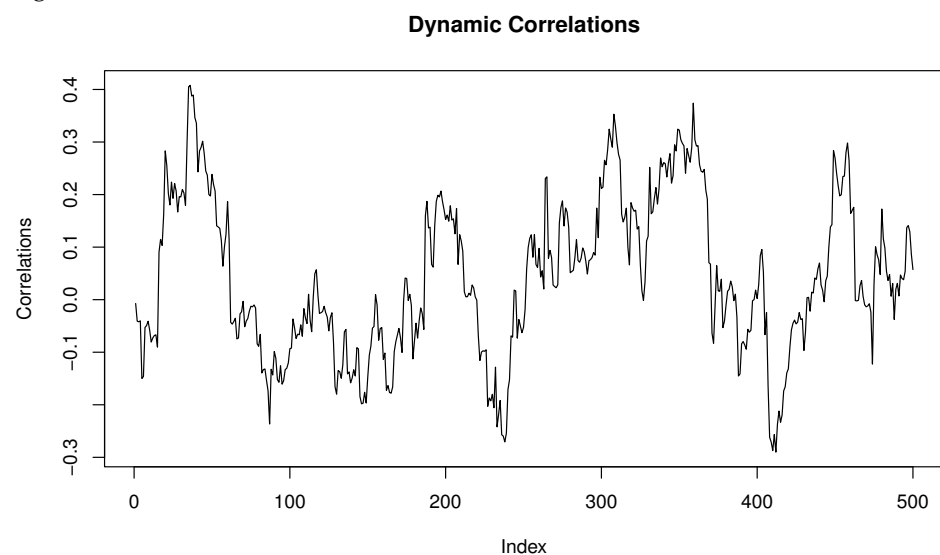


Figure 2. The curve of the dynamic correlations.

Now, the output variable is R_i , which represents the mean of the daily variations of the two assets. We examine the reliability and the behavior of $\widehat{VaR}_\alpha$ by using the quadratic kernel $L(\cdot)$ on $(0, 1)$ and the metric associated to the spline basis functions. Next, the three selectors of the smoothing parameters are optimized over selected values of $k \in \{5, 15, 25, \dots, 0.5n\}$. Finally, the efficiency of the VaR-estimation is checked using the backtesting measure,

$$EQ(\alpha) = \frac{1}{n} \sum_{k=1}^n \left(R_i - \widehat{VaR}_\alpha(P_i) \right) \left(\alpha - \mathbb{1}_{[R_i < \widehat{VaR}_\alpha(P_i)]} \right).$$

The obtained results are summarized in Table 1. It gives EQ-values for the three selectors based on the three values of $\alpha = 0.1, 0.05, 0.01$ and various sample sizes $n = 50, 150, 250$.

Table 1. The EQ(p) for different selectors.

Selectors	N	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.01$
b_I^{opt}	50	0.83	0.87	0.75
	150	0.42	0.39	0.32
	250	0.24	0.22	0.23
b_Y^{opt}	50	0.62	0.73	0.58
	150	0.29	0.26	0.24
	250	0.09	0.13	0.12
b_{FV}^{opt}	50	0.66	0.78	0.69
	150	0.38	0.31	0.33
	250	0.29	0.25	0.18

We observe that the estimator $\widehat{VaR}_\alpha$ has a good behavior compared to the three proposed selectors. However, it is clear that the rule (5) is more adequate than the other rules. So, despite the complexity of the considered structure, it appears that the computation of $\widehat{VaR}_\alpha$ is not very difficult. Moreover, its behaviour is less affected by the order of the threshold α , and the performance of the estimator is improved with respect to the sample size n .

6. Real Data Application

In this section, we highlight the importance of the nonparametric estimation of the VaR function in multi-functional time series data. In particular, we emphasize the local linear estimation of the VaR function by comparing it to the constant local estimator defined by

$$\widetilde{VaR}_\alpha(\mathbf{P}) = \arg \min_{\xi \in \mathbb{R}} \frac{\sum_{i=1}^n \prod_{j=1}^m \mathbf{L}(b_n^{-1} d_j(\mathbf{P}^j, P_k^j))(R_k - \xi)(\alpha - \mathbb{1}_{[R_k - \xi \leq 0]})}{\sum_{k=1}^n \prod_{j=1}^m \mathbf{L}(b_n^{-1} d_j(\mathbf{P}^j, P_k^j))}.$$

We point out that the local constant is a particular case of the local linear approach. Moreover, it is well documented that the local linear approach reduces the bias term in the local constant one. Thus, our principal purpose in this section is to highlight the superiority of the local linear approach over the local constant approach in financial risk management. Recall that the main feature of the present contribution is the multifunctionality of the data, allowing us to analyze the risk of multiple assets. Thus, we consider the daily price of a portfolio containing high values of three cryptocurrencies: Biconomy, Bitcoin, and BinaryX. The data are available on the website <https://stooq.com/db/h/> (accessed on 8 March 2025). We consider the data of weeks (from 18/11/2024 to 08/01/2025) with a frame of 5 mins. In order to examine the main characteristics of financial time series data, including volatility, skewness, kurtosis, and/or normality, we display the initial time series $Z(t)$ in Figures 3–5.

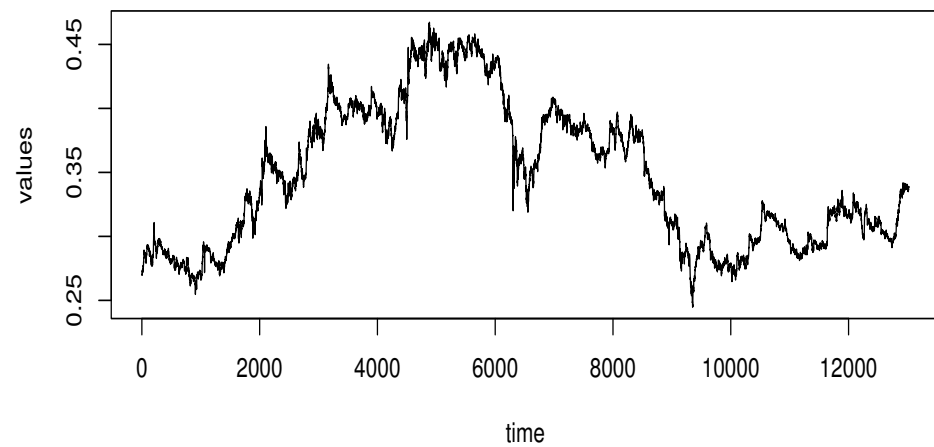


Figure 3. The function $R(t)$ of Biconomy.

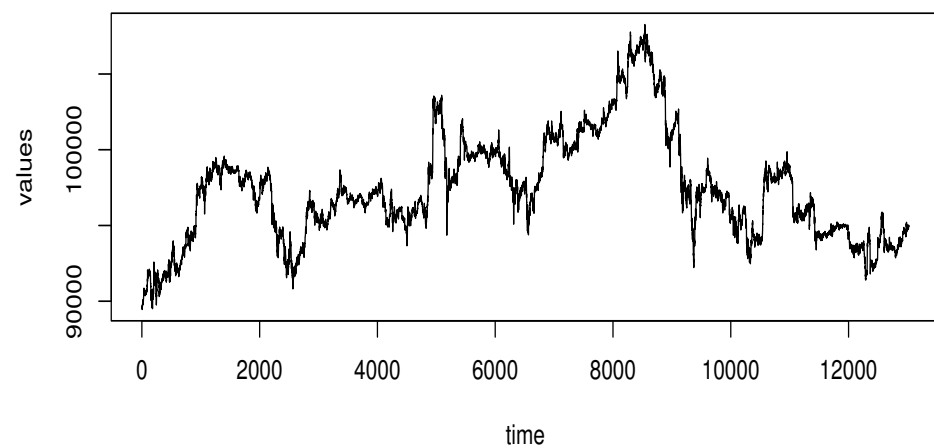


Figure 4. The function $R(t)$ of Bitcoin.

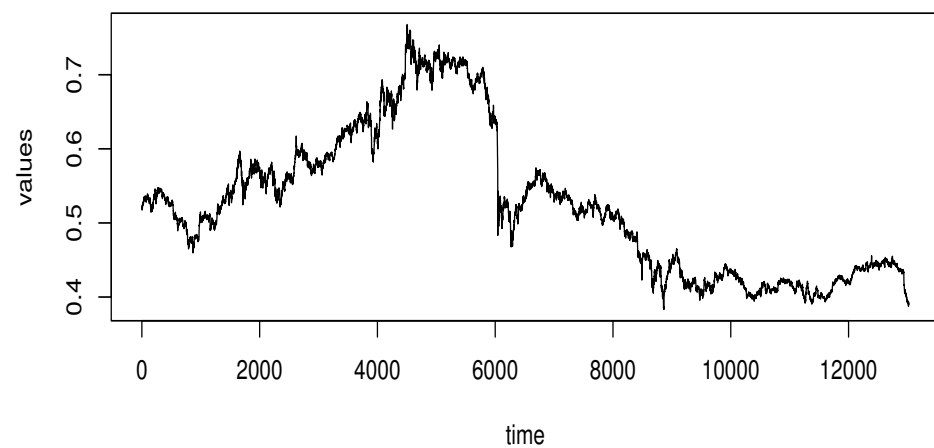


Figure 5. The function $R(t)$ BinaryX.

Biconomy is an important cryptocurrency market. The popularity of this cryptocurrency is justified by its simplicity as a decentralized application. It is based on simplifying Web3 interactions and cross-chain transfers. Biconomy is a creative solution and is playing a big role in making cryptocurrencies more accessible for regular users.

Bitcoin is one of the leading cryptocurrency markets. It is characterized by its rapid expansion and high volatility. As the first and most accessible digital currency, Bitcoin's

price movements often shape the overall market sentiment and serve as a crucial benchmark for assessing risks in crypto investments.

BinaryX (BNX) is a cryptocurrency platform known for its unpredictable price that increases and decreases, which defies general market trends. By including these three cryptocurrencies in our analysis, we specifically test our model's ability to handle these sudden and isolated price explosions that can make or break a portfolio's risk profile overnight. On the other hand, this type of high-frequency data requires special treatment because classical methods of multivariate statistics are unreliable in this situation. Usually, we are interested in the log-return, defined by

$$P(t) := 100 (\log(Z(t)) - \log(Z(t-1))),$$

and we predict the extreme return at a time T given the process $P(t)$ for $t \in [0, T]$.

To compare the effectiveness of the two risk measures $\widehat{\text{VaR}}_p(\mathbf{P})$ and $\widetilde{\text{VaR}}_p(\mathbf{P})$, we divide the studied financial time series data into two parts: the learning sample (400 of observations) and the testing sample (100 of observations). The learning sample is used to select the main parameters of the estimator, whereas the testing sample is used to assess the behavior of the financial risks $\widehat{\text{VaR}}_p(\mathbf{P})$ and $\widetilde{\text{VaR}}_p(\mathbf{P})$. So, for this comparison study, we keep the same strategies for selecting the principal parameters, such as b_n and kernel. In particular, we select the smoothing parameter using the following cross-validation rule:

$$b_Y^{opt} = \arg \min_{b \in \mathbf{H}_n} \frac{1}{n} \sum_{i=1}^n \left((R_i - \overline{\text{VaR}}_\alpha^{-i}(P_i)) (\alpha - \mathbf{1}_{[R_i < \overline{\text{VaR}}_\alpha^{-i}(P_i)])} \right),$$

where $\overline{\text{VaR}}_\alpha^{-i}(P_i)$ are the conditional VaR estimators ($\widehat{\text{VaR}}_p(\mathbf{P})$ and $\widetilde{\text{VaR}}_p(\mathbf{P})$) at P_i without this observation. Similar to the simulated data case, the subset of the positive real numbers $b(k)$ for which the ball centered at P_i with radius $b(k)$ contains exactly k observation neighbors of P_i . The integer k is varied in $k \in \{5, 15, 25, \dots, 0.5n\}$. Next, the metric d_i is the PCA-metric (see [44]) associated with the eigenvectors of the three largest eigenvalues. Finally, we point out that we used the quadratic kernel on (0,1). To examine the efficiency of both approaches, we repeat the splitting process between learning and testing samples (60 times) and plot the box plot of the backtesting measure,

$$\text{EQ}(\alpha) = \frac{1}{100} \sum_{k \in \text{testing sample}} \left(R_k - \widehat{\text{VaR}}_\alpha(P_k) \right) \left(\alpha - \mathbf{1}_{[R_k < \widehat{\text{VaR}}_\alpha(P_k)]} \right).$$

In fact, the use of backtesting measure is common in financial risk analysis. It permits us to evaluate VaR's conditional coverage via violation ratios—the proportion of observed returns breaching estimated quantiles. From the technical point of view, this backtesting measure helps traders validate whether their risk models can be trusted when markets turn volatile. Figures 6 and 7 display the obtained boxplots for two values of $\alpha = 0.1, 0.05$.

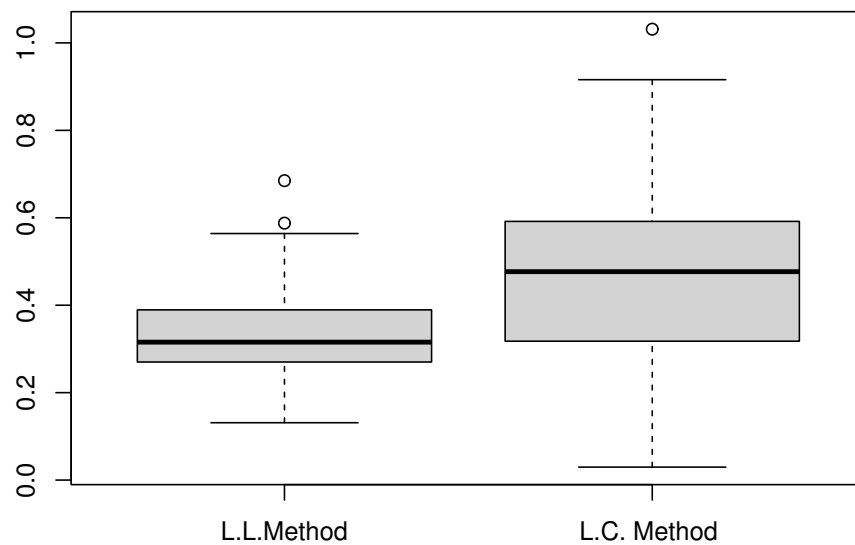


Figure 6. Comparison of EQ(0.1).

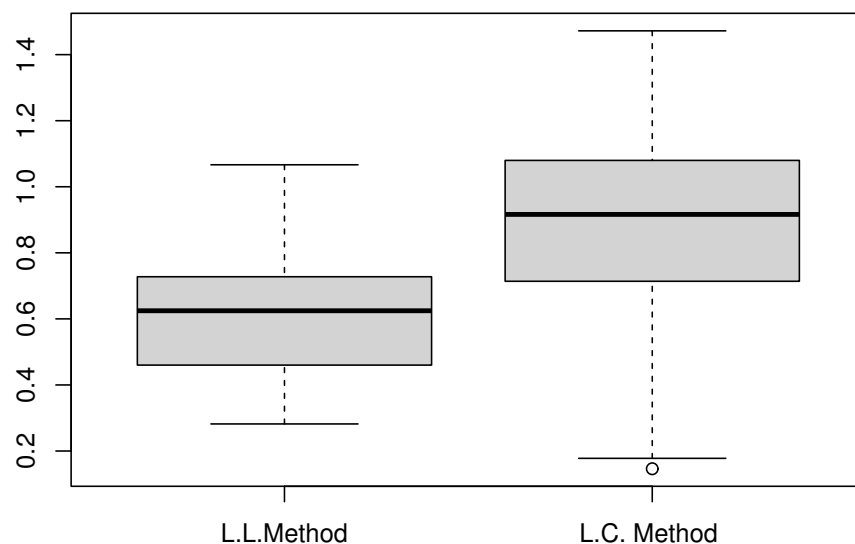


Figure 7. Comparison of EQ(0.05).

As expected, the local linear $\widehat{VaR}_\alpha$ is more appropriate than the local constant $\widehat{VaR}_\alpha$. This statement is confirmed by the interquartile values. Typically, the IQR of the $\widehat{VaR}_\alpha$ (L.L. Method) is 0.19 versus 0.25 for $\widehat{VaR}_\alpha$ (L.C. Method). Our findings validate the theoretical results that state that the local linear estimation outperforms local constant methods by significantly reducing bias. This gain is more important in functional statistics because the local linearity constitutes a structural aspect relating the functionality of the model to the functionality of the data. It highlights the importance of the functional structure on the performance of the estimator.

7. Conclusions

The digitalization of financial transactions make real-time financial risk management an inevitable issue for both investors and financial institutions. The real time analysis of financial risk is the principal motivation of this contribution. In pursuit of this aim, we

introduced a statistical algorithm combining the VaR function and the multi-GARCH with high-frequency data. Moreover, to overcome the limitation of the parametric approach, we employed the nonparametric local linear method. Unlike standard parametric approaches, which rely on restrictive linearity assumptions for modeling dynamics, our proposed method focuses on the general case without such constraints. From a theoretical point of view, we constructed an estimator of the local linear estimator of the VaR function and established its Borel convergence rate. The computational part demonstrates the easy implementation of the local linear VaR and proves that this estimator outperforms traditional kernel estimator in real-world financial applications. Through the conducted empirical analysis, we confirm two principal advantages of the proposed approach, which include its superior risk forecasting accuracy and computational efficiency. These advantages make it practical for making daily trading decisions regarding multiple assets. However, implementing this approach requires careful tuning of critical parameters, particularly the bandwidth b_n and the distance metric d_i , where the current methods still rely heavily on expert judgement rather than automatic selection. The parameter selection challenge currently limits the model's implementation and highlights a crucial need for future work to develop data-driven methods to automate these choices. In addition, our contribution opens multiple promising research directions, including three particularly exciting directions: (1) adapting the framework for complex data structures (spatial markets or incomplete datasets); (2) extending it to alternative risk measures, like Expected Shortfall, that better capture tail risk; and (3) enhancing the multifunctional model's predictive power by considering other functional time series cases. For risk managers, these advancements could eventually provide more risk models capable of incorporating the functional nature of the financial data.

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Appendix A

In order to simplify the demonstration of the preliminary results, we put

$$\mathbf{L}_i = \prod_{j=1}^m \mathbf{L}(b_n^{-1} \mathbf{z}(\mathbf{P}^i, P_k^i)), \quad i = 1, \dots, n,$$

$$\overline{\mathbf{Y}}_n(v) = \frac{1}{n\chi(m, \mathbf{P}, b)} \sum_{i=1}^n \mathbb{E} \left[\zeta_\alpha(v) \begin{pmatrix} 1 \\ b_n^{-1} \mathbf{d}_i \end{pmatrix} \mathbf{L}_i | \mathfrak{F}_{i-1} \right] \quad \text{and} \quad \overline{\mathbf{B}}_n = \overline{\mathbf{Y}}_n(v_0).$$

Proof of Lemma 1. First, set

$$\mathcal{Y}_i^1 = \left(\alpha - \mathbf{1}_{[Y_i \leq \xi + \varsigma \mathbf{d}_i]} \right) \mathbf{L}_i - \mathbb{E} \left[\left(\alpha - \mathbf{1}_{[Y_i \leq \xi + \varsigma \mathbf{d}_i]} \right) \mathbf{L}_i | \mathfrak{F}_{i-1} \right]$$

and

$$\mathcal{Y}_i^2 = \left(\alpha - \mathbf{1}_{[Y_i \leq \xi + \varsigma \mathbf{d}_i]} \right) \mathbf{d}_i \mathbf{L}_i - \mathbb{E} \left[\left(\alpha - \mathbf{1}_{[Y_i \leq \xi + \varsigma \mathbf{d}_i]} \right) \mathbf{d}_i \mathbf{L}_i | \mathfrak{F}_{i-1} \right].$$

Then,

$$\mathbf{B}_n - \overline{\mathbf{B}_n} = \begin{pmatrix} \mathbf{B}_n^1 \\ \mathbf{B}_n^2 \end{pmatrix},$$

where

$$\begin{cases} \mathbf{B}_n^1 = \frac{1}{n\chi(m, \mathbf{P}, b)} \sum_{i=1}^n \mathcal{Y}_i^1 \\ \mathbf{B}_n^2 = \frac{1}{nb_n\chi(m, \mathbf{P}, b)} \sum_{i=1}^n \mathcal{Y}_i^2. \end{cases}$$

Observe, for $p \geq 2$, that

$$\begin{aligned} \mathbb{E} \left[\mathcal{Y}_i^{1p} | \mathfrak{F}_{i-1} \right] &\leq C \mathbb{E} \left[\mathcal{Y}_i^{1^2} | \mathfrak{F}_{i-1} \right] \\ &\leq C \mathbb{E} \left[\mathbf{L}_i^2(x) | \mathfrak{F}_{i-1} \right] \\ &< C \mathbb{P}(P_i \in B(x, a_i) | \mathfrak{F}_{i-1}) \\ &\leq C \chi_i(m, \mathbf{P}, b_n). \end{aligned}$$

Using the same ideas,

$$\begin{aligned} \mathbb{E} \left[\mathcal{Y}_i^{2p} | \mathfrak{F}_{i-1} \right] &\leq C \mathbb{E} \left[\mathcal{Y}_i^{2^2} | \mathfrak{F}_{i-1} \right] \\ &\leq C b_n \mathbb{E} \left[\mathbf{L}_i^2(x) | \mathfrak{F}_{i-1} \right] \\ &< C b_n \mathbb{P}(P_i \in B(x, a_i) | \mathfrak{F}_{i-1}) \\ &\leq C b_n \chi_i(m, \mathbf{P}, b_n). \end{aligned}$$

From the exponential inequality of martingale difference to obtain for all $\varepsilon > 0$, we have

$$\begin{aligned} \mathbb{P} \left\{ \left| \mathbf{B}_n^1 \right| > \varepsilon \right\} &= \mathbb{P} \left\{ \left| \frac{1}{n\chi(m, \mathbf{P}, b)} \sum_{i=1}^n \mathcal{Y}_i^1 \right| > \varepsilon \right\} \\ &\leq 2 \exp \left\{ - \frac{\varepsilon^2 n^2 \chi^2(m, \mathbf{P}, b)}{2(\varphi(m, \mathbf{P}, b_n) + C \varepsilon n \chi(m, \mathbf{P}, b))} \right\} \\ &= 2 \exp \left\{ \frac{-\varepsilon^2 n^2 \chi^2(m, \mathbf{P}, b)}{2\varphi(m, \mathbf{P}, b_n)} \left(\frac{1}{1 + \frac{C \varepsilon n \chi(m, \mathbf{P}, b)}{\varphi(m, \mathbf{P}, b_n)}} \right) \right\}. \end{aligned}$$

Taking $\varepsilon = \varepsilon_0 \frac{\sqrt{\varphi(m, \mathbf{P}, b_n) \log n}}{n\chi(m, \mathbf{P}, b)}$, then,

$$\mathbb{P} \left\{ \left| \mathbf{B}_n^1 \right| > \varepsilon_0 \frac{\sqrt{\varphi(m, \mathbf{P}, b_n) \log n}}{n\chi(m, \mathbf{P}, b)} \right\} \leq 2 \exp \left\{ \frac{-\varepsilon_0^2 \log n}{2} \left(\frac{1}{1 + \frac{C \varepsilon_0 \sqrt{\log n}}{\sqrt{\varphi(m, \mathbf{P}, b_n)}}} \right) \right\}.$$

Under (PO2), we have

$$\mathbb{P} \left\{ \left| \mathbf{B}_n^1 \right| > \varepsilon_0 \frac{\sqrt{\varphi(m, \mathbf{P}, b_n) \log n}}{n\chi(m, \mathbf{P}, b)} \right\} \leq 2 \exp \left\{ \frac{-\varepsilon_0^2 \log n}{2C} \right\} \leq 2n^{-C'\varepsilon_0^2}.$$

and similarly,

$$\mathbb{P}\left\{\left|\mathbf{B}_n^2\right|>\epsilon_0\frac{\sqrt{\varphi(m,\mathbf{P},b_n)\log n}}{n\chi(m,\mathbf{P},b)}\right\}\leq 2\exp\left\{\frac{-\epsilon_0^2\log n}{2C}\right\}\leq 2n^{-C'\epsilon_0^2}.$$

Therefore,

$$\mathbf{B}_n^1=O_{a.co.}O\left(\sqrt{\frac{\varphi(m,\mathbf{P},b_n)\log n}{n^2\chi^2(m,\mathbf{P},b_n)}}\right)$$

and

$$\mathbf{B}_n^2=O_{a.co.}O\left(\sqrt{\frac{\varphi(m,\mathbf{P},b_n)\log n}{n^2\chi^2(m,\mathbf{P},b_n)}}\right).$$

Next, the first component of $\overline{\mathbf{B}}_n$ is

$$\begin{aligned}\overline{\mathbf{B}}_n &= \frac{1}{n\chi(m,\mathbf{P},b)}\sum_{i=1}^n\mathbb{E}\left[\left(\alpha-\mathbf{1}_{[P_i\leq\zeta+\varsigma\mathbf{d}_i]}\right)\mathbf{L}_i|\mathfrak{F}_{i-1}\right] \\ &= \frac{1}{n\chi(m,\mathbf{P},b)}\sum_{i=1}^n\mathbb{E}\left[\left(\alpha-\mathbf{1}_{[R_1\leq\zeta+\varsigma\mathbf{d}_1]}\right)\mathbf{L}_i|\mathfrak{F}_{i-1}\right] \\ &\leq \frac{1}{n\chi(m,\mathbf{P},b)}\sum_{i=1}^n\mathbb{E}\left[\left|F^{\mathbf{P}}VaR_{\alpha}(\mathbf{P})-F^{P_i}(\zeta+\varsigma\mathbf{d}_1)\mathbf{L}_i\right||\mathfrak{F}_{i-1}\right] \\ &= O(b_n^{\min(k_1,k_2)}).\end{aligned}$$

Similarly, for the second component of $\overline{\mathbf{B}}_n$, we have

$$\frac{1}{n\chi(m,\mathbf{P},b)}\sum_{i=1}^n\mathbb{E}\left[\left(\alpha-\mathbf{1}_{[P_i\leq\zeta+\varsigma\mathbf{d}_i]}\right)\mathbf{d}_i\mathbf{L}_i|\mathfrak{F}_{i-1}\right]=O(b_n^{\min(k_1,k_2)}).$$

We conclude that

$$\|\mathbf{B}_n\|=O(b_n^{\min(k_1,k_2)})+O_{a.co.}\left(\sqrt{\frac{\varphi(m,\mathbf{P},b_n)\log n}{n^2\chi^2(m,\mathbf{P},b_n)}}\right).$$

□

Proof of Lemma 2. We show that

$$\sup_{\|\delta\|\leq M}\|\mathbf{Y}_n(v)-\mathbf{B}_n-\overline{\mathbf{Y}}_n(v)+\overline{\mathbf{B}}_n\|=O_{a.co.}\left(\sqrt{\frac{\varphi(m,\mathbf{P},b_n)\log n}{n^2\chi^2(m,\mathbf{P},b_n)}}\right)\quad (\text{A1})$$

and

$$\sup_{\|\delta\|\leq M}\|\overline{\mathbf{Y}}_n(v)-\overline{\mathbf{B}}_n+f^{\mathbf{P}}(VaR_{\alpha}(\mathbf{P}))Dv\|=O(b_n^{\min(k_1,k_2)}).\quad (\text{A2})$$

For (A1), we use the compactness of $B(0,M)$ in $\mathbb{R}^2$, which implies

$$B(0,M)\subset\bigcup_{j=1}^{d_n}B(v_j,l_n),\ v_j=\begin{pmatrix}c_j\\d_j\end{pmatrix}\text{ and }l_n=d_n^{-1}=1/\sqrt{n}.$$

Write, for $j(v) = \arg \min_j |v - v_j|$, that

$$\begin{aligned} & \sup_{\|v\| \leq M} \|Y_n(v) - \mathbf{B}_n - \overline{Y_n}(v) + \overline{\mathbf{B}_n}\| \\ & \leq \sup_{\|v\| \leq M} \|Y_n(v) - Y_n(v_j)\| \\ & \quad + \sup_{\|v\| \leq M} \|Y_n(v_j) - \mathbf{B}_n - \overline{Y_n}(v_j) + \overline{\mathbf{B}_n}\| \\ & \quad + \sup_{\|v\| \leq M} \|\overline{Y_n}(v_j) - \overline{\mathbf{B}_n}\|. \end{aligned}$$

Since $|\mathbf{1}_{[P < a]} - \mathbf{1}_{[P < b]}| \leq \mathbf{1}_{[|P-b| \leq |a-b|]}$, then

$$\sup_{\|v\| \leq M} \|Y_n(v) - Y_n(v_j)\| \leq \frac{1}{n\chi(m, \mathbf{P}, b)} \sum_i Z_i$$

where

$$Z_i = \sup_{\|v\| \leq M} \mathbf{1}_{[|P_i - (c_j + \xi) - (b_n^{-1} \mathbf{x}_j + d) \mathbf{d}_i| \leq Cl_n]} \left\| \begin{pmatrix} 1 \\ b_n^{-1} \mathbf{d}_i \end{pmatrix} \right\| \mathbf{L}_i.$$

Clearly,

$$\mathbb{E}[Z_i | \mathfrak{F}_{i-1}] = O(l_n \chi_i(m, \mathbf{P}, b)) \quad \text{and} \quad \mathbb{E}[Z_i^p | \mathfrak{F}_{i-1}] = O(l_n \chi_i(m, \mathbf{P}, b)).$$

Use the fact that

$$l_n = o\left(\sqrt{\frac{\varphi(m, \mathbf{P}, b_n) \log n}{n^2 \chi^2(m, \mathbf{P}, b_n)}}\right) \quad (\text{A3})$$

to get

$$\sup_{\|v\| \leq M} \|Y_n(v) - Y_n(v_j)\| = O_{a.co.} \left(\sqrt{\frac{\varphi(m, \mathbf{P}, b_n) \log n}{n^2 \chi^2(m, \mathbf{P}, b_n)}} \right).$$

Furthermore, from (A3), we obtain from

$$\sup_{\|v\| \leq M} \|\overline{Y_n}(v_j) - \overline{\mathbf{B}_n}\| = O_{a.co.} \left(\sqrt{\frac{\varphi(m, \mathbf{P}, b_n) \log n}{n^2 \chi^2(m, \mathbf{P}, b_n)}} \right),$$

the quantity

$$\sup_{\|v\| \leq M} \|Y_n(v_j) - \mathbf{B}_n - \overline{Y_n}(v_j) + \overline{\mathbf{B}_n}\|.$$

We have

$$Y_n(v_j) - \mathbf{B}_n - \overline{Y_n}(v_j) + \overline{\mathbf{B}_n} = \begin{pmatrix} \mathbf{W}_n^1(v_j) \\ \mathbf{W}_n^2(v_j) \end{pmatrix},$$

where

$$\begin{cases} \mathbf{W}_n^1(v_j) = \frac{1}{n\chi(m, \mathbf{P}, b)} \sum_{i=1}^n \mathcal{X}_i^1 \\ \mathbf{W}_n^2(v_j) = \frac{1}{nb_n\chi(m, \mathbf{P}, b)} \sum_{i=1}^n \mathcal{X}_i^2 \end{cases}$$

with

$$\mathcal{X}_i^1 = (\zeta_\alpha(v_j) - \zeta_\alpha(v_0)) \mathbf{L}_i - \mathbb{E}[(\zeta_\alpha(v_j) - \zeta_\alpha(v_0)) \mathbf{L}_i | \mathfrak{F}_{i-1}]$$

and

$$\mathcal{X}_i^2 = (\zeta_\alpha(v_j) - \zeta_\alpha(v_0)) \mathbf{d}_i \mathbf{L}_i - \mathbb{E}[(\zeta_\alpha(v_j) - \zeta_\alpha(v_0)) \mathbf{d}_i \mathbf{L}_i | \mathfrak{F}_{i-1}].$$

For $p \geq 2$, we have

$$\mathbb{E}[\mathcal{X}_i^{1p} | \mathfrak{F}_{i-1}] \leq C\chi_i(m, \mathbf{P}, b_n) \quad \text{and} \quad \mathbb{E}[\mathcal{X}_i^{2p} | \mathfrak{F}_{i-1}] \leq Cb_n\chi_i(m, \mathbf{P}, b_n).$$

The latter proves

$$\begin{aligned} \sum_n \mathbb{P} \left(\sup_{\|v\| \leq M} \|\mathbf{W}_n^1(v)\| \geq \eta \left(\sqrt{\frac{\varphi(m, \mathbf{P}, b_n) \log n}{n^2 \chi^2(m, \mathbf{P}, b_n)}} \right) \right) \\ \leq \sum_n d_n \max_j \mathbb{P} \left(\|\mathbf{W}_n^1(v_j)\| \geq \eta \left(\sqrt{\frac{\varphi(m, \mathbf{P}, b_n) \log n}{n^2 \chi^2(m, \mathbf{P}, b_n)}} \right) \right) < \infty \end{aligned}$$

and

$$\begin{aligned} \sum_n \mathbb{P} \left(\sup_{\|v\| \leq M} \|\mathbf{W}_n^2(v)\| \geq \eta \left(\sqrt{\frac{\varphi(m, \mathbf{P}, b_n) \log n}{n^2 \chi^2(m, \mathbf{P}, b_n)}} \right) \right) \\ \leq \sum_n d_n \max_j \mathbb{P} \left(\|\mathbf{W}_n^2(v_j)\| \geq \eta \left(\sqrt{\frac{\varphi(m, \mathbf{P}, b_n) \log n}{n^2 \chi^2(m, \mathbf{P}, b_n)}} \right) \right) < \infty \end{aligned}$$

which leads to the conclusion (A1).

Concerning the term (A2), we use the same arguments for treating $\overline{\mathbf{B}}_n$. In fact, we write

$$\overline{\mathbf{Y}}_n(v) - \overline{\mathbf{B}}_n = \begin{pmatrix} \overline{\mathbf{W}}_n^1(v) \\ \overline{\mathbf{W}}_n^2(v) \end{pmatrix}$$

where

$$\begin{aligned} \overline{\mathbf{W}}_n^1(v) &= \frac{1}{n\chi(m, \mathbf{P}, b)} \sum_{i=1}^n \mathbb{E}[(\zeta_\alpha(v) - \zeta_\alpha(v_0)) \mathbf{L}_i | \mathfrak{F}_{i-1}], \\ \overline{\mathbf{W}}_n^2(v) &= \frac{1}{nb_n\chi(m, \mathbf{P}, b)} \sum_{i=1}^n \mathbb{E}[(\zeta_\alpha(v) - \zeta_\alpha(v_0)) \mathbf{d}_i \mathbf{L}_i | \mathfrak{F}_{i-1}] \end{aligned}$$

and use

$$\begin{aligned} \overline{\mathbf{W}}_n^1(v) &= -\frac{1}{n\chi(m, \mathbf{P}, b)} \sum_{i=1}^n \mathbb{E} \left[\left(\mathbf{1}_{[P_1 \leq (\kappa + \xi) + (b_n^{-1}\kappa + \xi)\mathbf{d}_1]} - \mathbf{1}_{[P_1 \leq \xi + \xi\mathbf{d}_1]} \right) \mathbf{L}_i | \mathfrak{F}_{i-1} \right] \\ &= -\frac{1}{n\chi(m, \mathbf{P}, b)} \sum_{i=1}^n \mathbb{E} \left[\left(F^{\mathbf{P}}((\kappa + \xi) + (b_n^{-1}\kappa + \xi)\mathbf{d}_1) - F^{\mathbf{P}}(\xi + \xi\mathbf{d}_1) \right) \mathbf{L}_i | \mathfrak{F}_{i-1} \right] \\ &\quad - \frac{1}{n\chi(m, \mathbf{P}, b)} \sum_{i=1}^n \mathbb{E} \left[\left(F^{\mathbf{P}}((\kappa + \xi) + (b_n^{-1}\kappa + \xi)\mathbf{d}_1) - F^{\mathbf{P}}(\xi + \xi\mathbf{d}_1) \right) \mathbf{L}_i \right] + O(h^{\mathbf{L}_1}) \\ &= -\frac{1}{n\chi(m, \mathbf{P}, b)} \sum_{i=1}^n \mathbb{E} \left[f^{\mathbf{P}}(\xi + \xi\mathbf{d}_1) \left(1, b_n^{-1}\mathbf{d}_1 \right) v \mathbf{L}_i | \mathfrak{F}_{i-1} \right] + O(h^{k_1}) + o(\|v\|) \\ &= -f^{\mathbf{P}}(VaR_\alpha(\mathbf{P})) \frac{1}{n\chi(m, \mathbf{P}, b)} \left(\sum_{i=1}^n \mathbb{E}[\mathbf{L}_i | \mathfrak{F}_{i-1}], b_n^{-1} \sum_{i=1}^n \mathbb{E}[\mathbf{d}_i \mathbf{L}_i | \mathfrak{F}_{i-1}] \right) v \\ &\quad + O(h^{\min(k_1, k_2)}) + o(\|v\|). \end{aligned}$$

Similarly,

$$\begin{aligned} \overline{\mathbf{W}}_n^2(v) &= -f^{\mathbf{P}}(VaR_\alpha(\mathbf{P})) \frac{1}{n\chi(m, \mathbf{P}, b)} \left(\sum_{i=1}^n \mathbb{E}[\mathbf{d}_i \mathbf{L}_i | \mathfrak{F}_{i-1}], b_n^{-1} \sum_{i=1}^n \mathbb{E}[\mathbf{d}_i^2 \mathbf{L}_i | \mathfrak{F}_{i-1}] \right) v \\ &\quad + O(h^{\min(k_1, k_2)}) + o(\|v\|). \end{aligned}$$

It follows that

$$\begin{aligned} \mathbb{E}[Y_n(v) - \mathbf{B}_n|\mathfrak{F}_{i-1}] &= -f^{\mathbf{P}}(VaR_\alpha(\mathbf{P})) \frac{1}{n\chi(m, \mathbf{P}, b)} \\ &\quad \left(\begin{array}{cc} \sum_{i=1}^n \mathbb{E}[\mathbf{L}_i|\mathfrak{F}_{i-1}] & \sum_{i=1}^n \mathbb{E}[\mathbf{L}_i b_n^{-1} \mathbf{d}_i|\mathfrak{F}_{i-1}] \\ \sum_{i=1}^n \mathbb{E}[\mathbf{L}_i b_n^{-1} \mathbf{d}_i|\mathfrak{F}_{i-1}] & \sum_{i=1}^n \mathbb{E}[h^{-2} \mathbf{d}_i^2 \mathbf{L}_i|\mathfrak{F}_{i-1}] \end{array} \right) v \\ &\quad + O(h^{\min(k_1, k_2)}) + o(\|v\|) \end{aligned}$$

It is proven in [45] that

$$h^{-a} \mathbb{E}[\mathbf{d}^{-a} \mathbf{L}_i^b] = \chi(m, \mathbf{P}, b) \left(K^b(1) - \int_{-1}^1 (u^a K^b(u))' \chi(m, \mathbf{P}, u) du \right) + o(\chi(m, \mathbf{P}, b)).$$

Hence, we can conclude that

$$\sup_{\|v\| \leq M} \|\overline{Y}_n(v) - \overline{\mathbf{B}}_n + f^{\mathbf{P}}(VaR_\alpha(\mathbf{P})) Dv + o(\|v\|)\| = O(b_n^{\min(k_1, k_2)})$$

which implies the result (A2). $\square$

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