

Article

The Studies of (Dual) Fusion Frames on Hilbert Space and Generalization of the Index Set

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Abstract

In this paper, we begin with the classical concept of tight frames in Hilbert spaces. First, we introduce the orthogonal projection P between H and $\theta(H)$ (the range of the frame transform θ associated with a traditional tight frame) and investigate the relationship between P and θ . We then explore fusion frames and extend the index set to an infinite set through a concrete example. Second, we examine the role of orthogonal projections in fusion frames with particular emphasis on robustness and redundancy illustrated by examples. Finally, we study dual fusion frames and establish several important results, especially concerning the relationship between the frame operators of two types of dual fusion frames.

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1. Introduction

Frame theory plays an important role in signal processing, image processing, data compression, sampling theory, and related fields. In 1952, Duffin and Schaeffer [1] introduced the concept of frames in Hilbert spaces in their study of non-harmonic Fourier series. However, this idea initially received little attention. It was not until 1986 that Daubechies, Grossman, and Meyer [2] carried out groundbreaking work in this area, after which frame theory began to be extensively studied.

Casazza and Kutyniok [3] later introduced fusion frames and investigated their properties, perturbations, and the approximation of inverse operators. In recent years, the theory of fusion frames has also undergone rapid development [4–7]. Fusion frames generalize traditional frames and are particularly well suited for processing block data or multi-channel signals, as they combine orthogonal projections of subspaces with weights.

More recently, considerable attention has been devoted to the study of dual frames. For example, J. Lopez, Han Deguang, Sun Wenchang, and others [8–16] have investigated the problem of selecting optimal dual frames from various perspectives. Their contributions have enabled frame theory to achieve significant progress in practical applications.

Orthogonal projection also plays a central role in frame theory. It is fundamental in contexts ranging from rigorous decomposition in Hilbert spaces to efficient approximation in machine learning, and from time-frequency localization in multiresolution analysis to geometric dimensionality reduction in optimization problems. Orthogonal projection provides an indispensable theoretical foundation for the modeling, analysis, and control

of complex systems. By balancing redundancy and efficiency through orthogonality constraints, it continues to play a key role in fields such as signal processing, communications, and control [17].

Throughout this paper, H is a Hilbert spaces, we write $W = \{(W_i, w_i)\}_{i \in I}$ as $W = \{(W_i, w_i P_{W_i})\}_{i \in I}$ in Section 3 for the convenience, and denote $\|f\|^2$ by $\langle f, f \rangle$. The symbol I_H denotes the identity operator on H . We now present the preliminary definitions and theorems used in this paper.

Definition 1 (see [18]). A sequence $\{f_i, i \in I\}$ for a Hilbert space H is called a frame if there exist constants $a, b > 0$ such that for any $f \in H$,

$$a\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq b\|f\|^2.$$

The constants a and b are called the lower and the upper frame bounds, respectively. A frame is called tight frame if $a = b$ and a normalized tight frame if $a = b = 1$.

Definition 2 (see [18]). Let $\{f_i, i \in I\}$ be a frame for a Hilbert space H , and let $\{e_i, i \in I\}$ be its orthonormal basis. The associated frame transform θ is defined by

$$\theta : H \rightarrow H \quad \text{with} \quad \theta(f) = \sum_{i \in I} \langle f, f_i \rangle e_i \quad \text{for any } f \in H.$$

Then θ is adjointable, its adjoint $\theta^* : H \rightarrow H$ is given by

$$\theta^*(f) = \sum_{i \in I} \langle f, e_i \rangle f_i,$$

where $\theta^*(e_i) = f_i$.

Moreover, we have

$$S(f) = \theta^* \theta(f) = \theta^* \left(\sum_{i \in I} \langle f, f_i \rangle e_i \right) = \sum_{i \in I} \langle f, f_i \rangle \theta^*(e_i) = \sum_{i \in I} \langle f, f_i \rangle f_i.$$

A direct calculation now yields

$$\langle S(f), f \rangle = \left\langle \sum_{i \in I} \langle f, f_i \rangle f_i, f \right\rangle = \sum_{i \in I} \langle f, f_i \rangle \langle f_i, f \rangle.$$

If $\{f_i, i \in I\}$ is a frame for a Hilbert space H , then the operator $S = \theta^* \theta$ is a positive, self-adjoint and invertible operator on H , called the frame operator.

It follows from Definition 2 that $\{f_i, i \in I\}$ is a -tight frame if and only if $S = \theta^* \theta = aI_H$ and a normalized tight frame if and only if $S = \theta^* \theta = I_H$.

Since $S(f) = \sum_{i \in I} \langle f, f_i \rangle f_i$, using $S^{-1}(f)$ to replace f , we obtain

$$f = SS^{-1}(f) = \sum_{i \in I} \langle S^{-1}(f), f_i \rangle f_i = \sum_{i \in I} \langle f, S^{-1}(f_i) \rangle f_i,$$

and

$$f = S^{-1}S(f) = S^{-1} \left(\sum_{i \in I} \langle f, f_i \rangle f_i \right) = \sum_{i \in I} \langle f, f_i \rangle S^{-1}(f_i).$$

Then,

$$f = \sum_{i \in I} \langle f, f_i \rangle S^{-1}(f_i) = \sum_{i \in I} \langle f, S^{-1}(f_i) \rangle f_i.$$

We now recall the definition of the dual frame.

Definition 3 (see [18]). Let $\{f_i, i \in I\}$ be a frame for a Hilbert space H with the frame operator S . Then $\{S^{-1}(f_i), i \in I\}$ is called the canonical dual frame of $\{f_i, i \in I\}$. If a frame $\{g_i, i \in I\}$ for H satisfies

$$\sum_{i \in I} \langle f, f_i \rangle g_i = \sum_{i \in I} \langle f, g_i \rangle f_i, \quad \text{for any } f \in H,$$

then $\{g_i, i \in I\}$ is called an alternate dual frame of $\{f_i, i \in I\}$.

Definition 4 (see [19]). Let I be a countable index set, let $\{W_i\}_{i \in I}$ be a family of closed subspaces of H , and let $\{w_i > 0\}_{i \in I}$ be a family of weights. Then $W = \{(W_i, w_i)\}_{i \in I}$ is a fusion frame if there exist constants $0 < A < B < +\infty$ so that

$$A \langle f, f \rangle \leq \sum_{i \in I} \langle w_i P_{W_i}(f), w_i P_{W_i}(f) \rangle \leq B \langle f, f \rangle \quad \text{for any } f \in H,$$

where $P_{W_i} : H \rightarrow W_i$ is the orthogonal projection. The constants A and B are called the lower and the upper fusion frame bounds, respectively.

The family $W = \{(W_i, w_i P_{W_i})\}_{i \in I}$ is called A -tight fusion frame if $A = B$ and a parseval fusion frame if $A = B = 1$. If only the right-hand inequality holds, it is called a Bessel fusion sequence with Bessel frame bound B . Family $W = \{(W_i, 1 P_{W_i})\}_{i \in I}$ is called 1-consistent Parseval fusion frame if

$$\sum_{i \in I} \langle P_{W_i}(f), P_{W_i}(f) \rangle = \langle f, f \rangle \quad \text{for any } f \in H,$$

or w -consistent Parseval fusion frame if

$$\sum_{i \in I} \langle w P_{W_i}(f), w P_{W_i}(f) \rangle = \langle f, f \rangle \quad \text{for any } f \in H.$$

Theorem 1 (see [20]). Let H be a Hilbert space, and let $\{P_i\}_{i \in I}$ be orthogonal projections. If $P_i(f) = f_i$ for any $f \in H$, $W_i = \overline{\text{span}\{f_i\}_{i \in I}}$. Then $\{f_i, i \in I\}$ is a frame for H if and only if $W = \{(W_i, w_i)\}_{i \in I}$ is fusion frame for H with the frame bounds unchanged and the weight set being $w_i = \|f_i\| > 0$.

Theorem 2 (see [20]). Let $\{W_i\}_{i \in I}$ be a sequence of closed subspaces of H , $\{w_i > 0\}_{i \in I}$ be the weight set, and let $\{\varphi_{ij}, j \in J\}$ be frames for W_i . Then $W = \{(W_i, w_i)\}_{i \in I}$ is a fusion frame for H if and only if $\{w_i \varphi_{ij}, i \in I, j \in J\}$ is a frame for H .

2. Results

2.1. Discovery of Fusion Frames on Hilbert Spaces and Generalization of the Index Set

Theorem 3. Let $\{f_i, i \in I\}$ be a-tight frame for H with the frame transform θ , and let $P : H \rightarrow \theta(H)$ be an orthogonal projection. Then $P(e_i) = \frac{1}{a} \theta(f_i)$ and $\theta \theta^* = aP$, where $\{e_i, i \in I\}$ is the orthonormal basis for H .

Proof. Since $\{f_i, i \in I\}$ is a-tight frame for H , $\theta^* \theta = aI_H$ and $P = I_H$ on $\theta(H)$, i.e., $P\theta(H) = \theta(H)$. Then for any $f \in H$,

$$\begin{aligned} \langle \theta(f), P(e_i) \rangle &= \langle P\theta(f), e_i \rangle = \langle \theta(f), e_i \rangle = \langle f, \theta^*(e_i) \rangle \\ &= \langle f, f_i \rangle = \frac{1}{a} \langle \theta(f), \theta(f_i) \rangle = \langle \theta(f), \frac{1}{a} \theta(f_i) \rangle, \end{aligned}$$

therefore, $P(e_i) = \frac{1}{a}\theta(f_i)$ or $\theta(f_i) = aP(e_i)$.

Since $f = \sum_{i \in I} \langle f, e_i \rangle e_i$, we obtain

$$\begin{aligned}\theta\theta^*(f) &= \theta\theta^*\left(\sum_{i \in I} \langle f, e_i \rangle e_i\right) = \theta\left(\sum_{i \in I} \langle f, e_i \rangle \theta^*(e_i)\right) = \theta\left(\sum_{i \in I} \langle f, e_i \rangle f_i\right) \\ &= \sum_{i \in I} \langle f, e_i \rangle \theta(f_i) = \sum_{i \in I} \langle f, e_i \rangle aP(e_i) \\ &= aP\left(\sum_{i \in I} \langle f, e_i \rangle e_i\right) = aP(f).\end{aligned}$$

By the arbitrariness of f , it follows that $\theta\theta^* = aP$.

In addition, from $P = \frac{1}{a}\theta\theta^*$ and $\theta^*\theta = aI$, we have

$$P^2 = \left(\frac{1}{a}\theta\theta^*\right)\left(\frac{1}{a}\theta\theta^*\right) = \frac{1}{a^2}\theta\theta^*\theta\theta^* = \frac{1}{a^2}\theta(aI)\theta^* = \frac{1}{a}\theta\theta^* = P.$$

Its self-adjointness is obvious. Thus, $P^2 = P = P^*$, which fully verifies that P acts as an orthogonal projection in the relevant context. $\square$

Let $\{f_{ij}, i \in I, j \in J\}$ be a_i -tight frame sequence for H with the frame transforms θ_i (where i is the number of frames and j is the dimension of the frames), and let $P_i : H \rightarrow \theta_i(H)$ be the orthogonal projections. Then $\theta_i\theta_i^* = a_iP_i$, and

$$\sum_{i \in I} (\sqrt{a_i})^2 P_i = \sum_{i \in I} a_i P_i = \sum_{i \in I} \theta_i \theta_i^*.$$

Therefore, $\{(\theta_i(H), \sqrt{a_i}P_{\theta_i(H)})\}_{i \in I}$ is a fusion frame if and only if there exist constants $0 < A < B < +\infty$ so that

$$AI_H \leq \sum_{i \in I} (\sqrt{a_i})^2 P_i = \sum_{i \in I} \theta_i \theta_i^* \leq BI_H.$$

Obviously, since $\sum_{i \in I} \theta_i \theta_i^* \leq \sum_{i \in I} \|\theta_i^*\|^2 I_H$, when I is a finite set, $\{(\theta_i(H), \sqrt{a_i}P_{\theta_i(H)})\}_{i \in I}$ must be a Bessel fusion sequence.

Our questions are, do such traditional frames and fusion frames exist? And, can I here be extended to an infinite set, i.e., $i \in \mathbb{N}^+$? Our answers are affirmative and we now give example below.

Example 1. Let $f_{ij} = \sqrt{a_i}e_j$ with $a_i > 0$ and $\sum_{i=1}^{+\infty} a_i = s$, Then $\{(\theta_i(H), \sqrt{a_i}P_{\theta_i(H)})\}_{i \in \mathbb{N}^+}$ is a s -tight fusion frame for H , where $\{e_j, j \in J\}$ is the standard orthonormal basis for H .

Proof. For any $f \in H$, we have

$$\sum_{j \in J} \langle f, f_{ij} \rangle \langle f_{ij}, f \rangle = \sum_{j \in J} \langle f, \sqrt{a_i}e_j \rangle \langle \sqrt{a_i}e_j, f \rangle = a_i \sum_{j \in J} \langle f, e_j \rangle \langle e_j, f \rangle = a_i \langle f, f \rangle,$$

so $\{f_{ij}, i \in \mathbb{N}^+, j \in J\}$ are a_i -tight frames for H . Since

$$\theta_i(f) = \sum_{j \in J} \langle f, \sqrt{a_i}e_j \rangle e_j = \sqrt{a_i} \sum_{j \in J} \langle f, e_j \rangle e_j = \sqrt{a_i}f,$$

i.e., $\theta_i = \sqrt{a_i}I_H$, $\theta_i^* = (\sqrt{a_i}I_H)^* = \sqrt{a_i}I_H$. Then

$$\theta_i \theta_i^* = (\sqrt{a_i}I_H)(\sqrt{a_i}I_H) = a_i I_H = a_i P_i,$$

which means that $P_i = I_H$ satisfying $P^2 = P = P^*$ and

$$\sum_{i=1}^{+\infty} (\sqrt{a_i})^2 P_i = \sum_{i=1}^{+\infty} a_i P_i = \sum_{i=1}^{+\infty} \theta_i \theta_i^* = \sum_{i=1}^{+\infty} a_i I_H = s I_H.$$

Therefore, $\{(\theta_i(H), \sqrt{a_i} P_{\theta_i(H)})\}_{i \in \mathbb{N}^+}$ is a s -tight fusion frame.

However, $\{(\theta_i(H), \sqrt{a_i} P_{\theta_i(H)})\}_{i \in \mathbb{N}^+}$ cannot be 1-uniform fusion frame or w -uniform fusion frame, because the constant series $\sum_{i=1}^{+\infty} w$ ($w > 0$) diverges forever unless I is a finite set. $\square$

We present it in the form of a certain theorem.

Theorem 4. Let H be a Hilbert space. Then there exists a sequence of a_i -tight frames $\{f_{ij}, i \in \mathbb{N}^+, j \in J\}$ for H such that $\{(\theta_i(H), \sqrt{a_i} P_{\theta_i(H)})\}_{i \in \mathbb{N}^+}$ is a tight fusion frame for H , where θ_i are the frame transforms of $\{f_{ij}, i \in I, j \in J\}$, $P_i : H \rightarrow \theta_i(H)$ are the orthogonal projections.

Remark 1. (1) Example 1 makes use of the convergence of positive-term series. For instance, if we take $a_i = \frac{1}{2^i}$, then $\sum_{i=1}^{+\infty} a_i = 1$, and $\{(\theta_i(H), \sqrt{a_i} P_{\theta_i(H)})\}_{i \in \mathbb{N}^+}$ forms a Parseval fusion frame, and in this case, $\sum_{i=1}^{+\infty} \theta_i \theta_i^* = I_H$.

In fact, for the positive term-series considered here, it is sufficient to establish convergence, there is no need to compute the exact sum. Moreover, many series can only be analyzed in terms of convergence, without an explicit evaluation of their sums. Even in such cases, $\{(\theta_i(H), \sqrt{a_i} P_{\theta_i(H)})\}_{i \in \mathbb{N}^+}$ still constitutes a (tight) fusion frame.

(2) The conclusion only holds for traditional tight frames, since the relationship between the frame transform and orthogonal projection exists exclusively in the tight-frame setting.

(3) Example 1 and Theorem 4 establish a close connection between traditional frames and fusion frames, and further realize the closed subspaces W_i of H by taking $W_i = P_{\theta_i(H)}$. Even more significantly, the index set I is generalized to infinite sets.

2.2. Orthogonal Projections in the Fusion Frames

The orthogonal projector P is a self-adjoint operator satisfying $P^2 = P$, i.e., $P^2 = P = P^*$. It projects element of H onto a closed subspace, and the projection is orthogonal to the complement of that subspace, that is, the inner product of the residual with the subspace is zero. Below, we present some simple examples in $L^2(\mathbb{R})$ and $L^2([a, b])$.

Example 2. In $L^2([a, b])$, let $M = \{c \mid c \in \mathbb{R}\}$. Then for any $f \in L^2[a, b]$ and for any $x \in [a, b]$,

$$P_M f = \frac{1}{b-a} \int_a^b f(x) dx.$$

Example 3. In $L^2[-\pi, \pi]$, let $M_n = \text{span}\{1, \cos x, \sin x, \dots, \cos nx, \sin nx\}$. Then

$$P_{M_n} f = \frac{a_0}{2} + \sum_{k=1}^n (a_k \cos kx + b_k \sin kx)$$

for any $f \in L^2[-\pi, \pi]$ and for any $x \in [-\pi, \pi]$, where the coefficients are the Fourier coefficients.

Example 4. In $L^2(\mathbb{R})$, let $I = [c, d]$ be a fixed interval, and let

$$M = \{g \mid g \in L^2(\mathbb{R}), g(x) = 0, \forall x \notin I\}.$$

Then

$$P_M f = \begin{cases} f(x) & x \in I \\ 0 & x \notin I \end{cases}.$$

A fusion frame is a subspace frame. Its central role lies in providing stable signal representation and reconstruction through subspace decomposition and weight balancing, while preserving the characteristics of each data source. Fusion frames thus serve as an important theoretical tool in multimodal and multi-sensor data processing. The following examples illustrate the role of orthogonal projection in the fusion frames and in the projection process as a whole.

Example 5. Let $H = \mathbb{R}^2$, $W_1 = \text{span}\{(1, 0)\}$, $W_2 = \text{span}\{(0, 1)\}$ (i.e., W_1 is the x -axis and W_2 is the y -axis), $v_1 = v_2 = 1$. Then for any $f = (a, b) \in \mathbb{R}^2$, $P_{W_1}(f) = (a, 0)$, $\|P_{W_1}(f)\|^2 = a^2$, $P_{W_2}(f) = (0, b)$, $\|P_{W_2}(f)\|^2 = b^2$. Thus, the weighted sum is

$$\sum_{i=1}^2 v_i^2 \|P_{W_i}(f)\|^2 = a^2 + b^2 = \|f\|^2.$$

Therefore, $\{(W_1, 1), (W_2, 1)\}$ is a Parseval fusion frame.

If W_2 fails, and only W_1 and v_1 are retained, then the weighted sum is a^2 , which cannot control $\|f\|^2$. However, if a third subspace $W_3 = \text{span}\{(a, b)\}$ is added (i.e., W_3 is the line $y = \frac{b}{a}x$ ($a \neq 0$)). Let $v_3 = 1$, then $P_{W_3}(f) = (a, b)$ and the weighted sum is

$$v_1^2 \|P_{W_1}(f)\|^2 + v_3^2 \|P_{W_3}(f)\|^2 = a^2 + (a^2 + b^2).$$

Since

$$a^2 + b^2 \leq a^2 + (a^2 + b^2) \leq 2(a^2 + b^2),$$

$$\|f\|^2 \leq v_1^2 \|P_{W_1}(f)\|^2 + v_3^2 \|P_{W_3}(f)\|^2 \leq 2\|f\|^2.$$

Therefore, $\{(W_1, 1), (W_3, 1)\}$ is a fusion frame, and the lower and upper fusion frame bounds are $A = 1$, $B = 2$, respectively. In other words, when W_2 fails, f can still be reconstructed through W_1 and W_3 , which demonstrates the robustness of the fusion frame.

In addition, we can also add two subspaces $W_4 = \text{span}\{(1, 1)\}$, $W_5 = \text{span}\{(-1, 1)\}$ (i.e., W_4 is the line $y = x$, W_5 is the line $y = -x$), $v_4 = v_5 = 1$. Then,

$$P_{W_4}(f) = \left(\frac{a+b}{2}, \frac{a+b}{2}\right), \quad P_{W_5}(f) = \left(\frac{a-b}{2}, \frac{b-a}{2}\right),$$

and the weighted sum is

$$v_1^2 \|P_{W_1}(f)\|^2 + v_4^2 \|P_{W_4}(f)\|^2 + v_5^2 \|P_{W_5}(f)\|^2 = a^2 + \frac{(a+b)^2}{2} + \frac{(a-b)^2}{2} = a^2 + (a^2 + b^2).$$

Thus,

$$\|f\|^2 \leq v_1^2 \|P_{W_1}(f)\|^2 + v_4^2 \|P_{W_4}(f)\|^2 + v_5^2 \|P_{W_5}(f)\|^2 \leq 2\|f\|^2.$$

Therefore, $\{(W_1, 1), (W_4, 1), (W_5, 1)\}$ is a fusion frame. In other words, f can still be reconstructed through W_1 , W_4 , and W_5 .

Orthogonal projection is also a fundamental tool for constructing tight frames. For instance, redundant frames can be transformed into Parseval frames by means of orthogonal projection. This property ensures lossless reconstruction in signal processing.

Example 6. Let $H = \mathbb{R}^2$, $W_1 = \text{span}\{(1,0)\}$, $W_2 = \text{span}\{(0,1)\}$, $W_3 = \text{span}\{(1,1)\}$, $v_1 = v_2 = v_3 = 1$. Then for any $f = (a,b)^T \in \mathbb{R}^2$, the weighted sum is

$$\|f\|^2 = a^2 + b^2 \leq \sum_{i=1}^3 v_i^2 \|P_{W_i}(f)\|^2 = a^2 + b^2 + \frac{(a+b)^2}{2} \leq 2(a^2 + b^2) = 2\|f\|^2.$$

Therefore, $\{(W_1, 1), (W_2, 1), (W_3, 1)\}$ is a fusion frame, but not a Parseval fusion frame.

Since W_3 can be linearly represented by W_1 and W_2 , it is therefore a redundant frame. Next, we transform it into a Parseval frame. we derive the analysis operator T of f . Since

$$\begin{aligned} Tf &= \sum_{i=1}^3 v_i^2 P_{W_i}(f) = (a,0)^T + (0,b)^T + \left(\frac{a+b}{2}, \frac{a+b}{2}\right)^T = \left(\frac{3a+b}{2}, \frac{a+3b}{2}\right)^T, \\ &= \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} f, \quad T = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}. \end{aligned}$$

Thus, T is a symmetric matrix, and

$$T^{-1} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix}.$$

Projecting $T^{-1}f = \left(\frac{3a-b}{4}, \frac{a-3b}{4}\right)^T$ onto the original subspaces yields

$$P_{W_1}(T^{-1}f) = \left(\frac{3a-b}{4}, 0\right), \quad P_{W_2}(T^{-1}f) = \left(0, \frac{a-3b}{4}\right), \quad P_{W_3}(T^{-1}f) = \left(\frac{a+b}{4}, \frac{a+b}{4}\right).$$

After adjusting the weights $v_1 = v_2 = 1, v_3 = 3$, the weighted sum is obtained as

$$\sum_{i=1}^3 v_i^2 \|P_{W_i}(T^{-1}f)\|^2 = a^2 + b^2 = \|f\|^2.$$

Thus, by projecting the subspaces of the redundant frame via T^{-1} and adjusting the weights, the redundant frame is converted into a Parseval fusion frame.

2.3. Research on Dual Fusion Frames on Hilbert Space

In this section, we begin with an example to explore how traditional dual frames in Hilbert spaces recover original signals in cases of data loss demonstrating the importance and research significance of dual frames.

Example 7. Let $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $x_2 = \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{5}}{2} \end{bmatrix}$, $x_3 = \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{5}}{2} \end{bmatrix}$. Then $\theta_x = \begin{bmatrix} 1 & 0 \\ -\frac{\sqrt{3}}{2} & \frac{\sqrt{5}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{\sqrt{5}}{2} \end{bmatrix}$,

$$S = \theta_x^* \theta_x = \theta_x^T \theta_x = \begin{bmatrix} 1 & -\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{5}}{2} & -\frac{\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{\sqrt{3}}{2} & \frac{\sqrt{5}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & 0 \\ 0 & \frac{5}{2} \end{bmatrix} = \frac{5}{2}I,$$

so $X = \{x_1, x_2, x_3\}$ is a tight frame for $\mathbb{R}^2$ with frame bound $\frac{5}{2}$, where θ_x is the frame transform of X and S is its frame operator.

Let the original signal vector be $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Then the frame coefficients are $c_1 = \langle x, x_1 \rangle = x^T x_1 = 1$, $c_2 = \langle x, x_2 \rangle = -\frac{\sqrt{3}+\sqrt{5}}{2}$, $c_3 = \langle x, x_3 \rangle = \frac{\sqrt{5}-\sqrt{3}}{2}$. Suppose that during transmission, the original signal loses c_3 , how can we recover the original signal?

We try to recover it using the alternate dual frame $Y = \{y_1, y_2, y_3\}$ of X .

Since c_3 is lost, let us set $y_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $y_1 = \begin{bmatrix} a \\ b \end{bmatrix}$, $y_2 = \begin{bmatrix} c \\ d \end{bmatrix}$. Then $\theta_Y = \begin{bmatrix} a & b \\ c & d \\ 0 & 0 \end{bmatrix}$. Since Y is the alternate dual frame of X , they are mutually alternate dual frames and $\theta_X^* \theta_Y = \theta_X^T \theta_Y = I$ or $\theta_Y^* \theta_X = \theta_Y^T \theta_X = I$. Then we have

$$\theta_X^T \theta_Y = \begin{bmatrix} 1 & -\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{5}}{2} & -\frac{\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

It is solved that $a = 1$, $b = \frac{\sqrt{3}}{\sqrt{5}}$, $c = 0$, $d = \frac{2}{\sqrt{5}}$. So $y_1 = \begin{bmatrix} 1 \\ \frac{\sqrt{3}}{\sqrt{5}} \end{bmatrix}$, $y_2 = \begin{bmatrix} 0 \\ \frac{2}{\sqrt{5}} \end{bmatrix}$, $y_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and

$$c_1 y_1 + c_2 y_2 = 1 \begin{bmatrix} 1 \\ \frac{\sqrt{3}}{\sqrt{5}} \end{bmatrix} + \left(-\frac{\sqrt{3}+\sqrt{5}}{2} \right) \begin{bmatrix} 0 \\ \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = x.$$

i.e., the alternate dual frame Y of X can recover the original signal.

The canonical dual frame $Z = \{z_1, z_2, z_3\}$ of X can also recover the original signal.

Since $S_X = \frac{5}{2}I$, $S_X^{-1} = \frac{2}{5}I$. Then we obtain

$$z_1 = S^{-1}(x_1) = \frac{2}{5}I \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} \\ 0 \end{bmatrix}, \quad z_2 = S^{-1}(x_2) = \frac{2}{5}I \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{3}}{5} \\ \frac{\sqrt{5}}{5} \end{bmatrix},$$

$$z_3 = S^{-1}(x_3) = \frac{2}{5}I \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{3}}{5} \\ -\frac{\sqrt{5}}{5} \end{bmatrix}.$$

Taking

$$c'_1 = \langle x, z_1 \rangle = x^T z_1 = \frac{2}{5}, \quad c'_2 = \langle x, z_2 \rangle = x^T z_2 = -\frac{\sqrt{3}+\sqrt{5}}{5},$$

$$c'_3 = \langle x, z_3 \rangle = x^T z_3 = \frac{\sqrt{5}-\sqrt{3}}{5},$$

then

$$c'_1 x_1 + c'_2 x_2 + c'_3 x_3 = \frac{2}{5} \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{\sqrt{3}+\sqrt{5}}{5} \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{5}}{2} \end{bmatrix} + \frac{\sqrt{5}-\sqrt{3}}{5} \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = x.$$

This means that the canonical dual frame Z of X can also recover the original signal.

Meanwhile, let $y_i = z_i + u_i$ ($i = 1, 2, 3$). Then

$$u_1 = \begin{bmatrix} \frac{3}{5} \\ \frac{\sqrt{3}}{\sqrt{5}} \end{bmatrix}, \quad u_2 = \begin{bmatrix} \frac{\sqrt{3}}{5} \\ \frac{1}{\sqrt{5}} \end{bmatrix}, \quad u_3 = \begin{bmatrix} \frac{\sqrt{3}}{5} \\ -\frac{1}{\sqrt{5}} \end{bmatrix},$$

and

$$c_1 u_1 + c_2 u_2 + c_3 u_3 = 1 \begin{bmatrix} \frac{3}{5} \\ \frac{\sqrt{3}}{\sqrt{5}} \end{bmatrix} + \left(-\frac{\sqrt{3} + \sqrt{5}}{2} \right) \begin{bmatrix} \frac{\sqrt{3}}{5} \\ \frac{1}{\sqrt{5}} \end{bmatrix} + \left(\frac{\sqrt{5} - \sqrt{3}}{2} \right) \begin{bmatrix} \frac{\sqrt{3}}{5} \\ -\frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We obtain $\sum_{i=1}^3 \langle x, x_i \rangle u_i = 0$, i.e., $\theta_x^* \theta_u = 0$, and of course $\theta_u^* \theta_x = 0$, where $U = \{u_1, u_2, u_3\}$.

It follows that the frames X and U are disjointed, meaning that any alternate dual frame of a frame can be expressed as the sum of its canonical dual frame and a frame can disjoint from itself, and the canonical dual frame is the “minimal” dual frame.

Next, we study dual fusion frames on Hilbert spaces.

Definition 5 (see [19]). Let $W = \{(W_i, w_i P_{W_i})\}_{i \in I}$ be a fusion frame for H , the analysis operator is defined by

$$T : H \rightarrow \left(\sum_{i \in I} \oplus W_i \right)_{\ell^2} \quad \text{with} \quad T(f) = \{w_i P_{W_i}(f)\}_{i \in I} \quad \text{for any } f \in H,$$

where $(\sum_{i \in I} \oplus W_i)_{\ell^2} = \{\{f_i\}_{i \in I} \mid f_i \in W_i \text{ and } \{\|f_i\|_{W_i}\}_{i \in I} \in \ell^2(I)\}$. It can easily be shown that the synthesis operator T^* , which is defined to be the adjoint operator, is given by

$$T^* : \left(\sum_{i \in I} \oplus W_i \right)_{\ell^2} \rightarrow H \quad \text{with} \quad T^*(\{f_i\}) = \sum_{i \in I} w_i P_{W_i}(f_i) \quad \text{for any } \{f_i\} \in \left(\sum_{i \in I} \oplus W_i \right)_{\ell^2}.$$

The fusion frame operator S for $W = \{(W_i, w_i P_{W_i})\}_{i \in I}$ is defined by

$$S : H \rightarrow H \quad \text{with} \quad S(f) = T^* T(f) = \sum_{i \in I} w_i^2 P_{W_i}(f).$$

If $W = \{(W_i, w_i P_{W_i})\}_{i \in I}$ be a fusion frame for H with fusion frame bound A and B , then the associated fusion frame operator S is a self-adjoint, positive, and invertible operator on H , and $AI_H \leq S = \sum_{i \in I} w_i^2 P_{W_i} \leq BI_H$.

It follows from Definition 5 that $W = \{(W_i, w_i P_{W_i})\}_{i \in I}$ is a B -tight fusion frame if and only if $S = BI_H$, and is a Parseval fusion frame if and only if $S = I_H$.

Obviously, if $W = \{(W_i, w_i P_{W_i})\}_{i \in I}$ is a fusion frame for H with fusion frame operator S , then $\{(W_i, w_i P_{W_i} S^{-1/2})\}_{i \in I}$ is a Parseval fusion frame for H . In fact,

$$\sum_{i \in I} w_i^2 (P_{W_i} S^{-1/2})(P_{W_i} S^{-1/2}) = S^{-1/2} \left(\sum_{i \in I} w_i^2 P_{W_i} \right) S^{-1/2} = S^{-1/2} S S^{-1/2} = I_H.$$

Theorem 5 (see [1]). Let T be a bounded linear operator on H , and let V be a closed subspace of H . Then $P_V T^* = P_V T^* P_{\overline{T(V)}}$. Moreover if T is a unitary operator, then $P_{\overline{T(V)}} T = T P_V$, where P is the orthogonal projection from H to V .

It follows from Theorem 5 that $(P_V T^*)^* = (P_V T^* P_{\overline{T(V)}})^*$, i.e., $T P_V = P_{\overline{T(V)}} T P_V$. So, if $W = \{(W_i, w_i P_{W_i})\}_{i \in I}$ is the fusion frame with fusion frame operator S , then

$$S^{-1} P_{W_i} = P_{S^{-1}(W_i)} S^{-1} P_{W_i}.$$

Therefore, for any $f \in H$, $S(f) = \sum_{i \in I} w_i^2 P_{W_i}(f)$, and

$$f = S^{-1} S(f) = \sum_{i \in I} w_i^2 S^{-1} P_{W_i}(f) = \sum_{i \in I} w_i^2 P_{S^{-1}(W_i)} S^{-1} P_{W_i}(f).$$

$S^{-1}(W) = \{(S^{-1}(W_i), w_i P_{S^{-1}(W_i)})\}_{i \in I}$ is called the canonical dual fusion frame of $W = \{(W_i, w_i P_{W_i})\}_{i \in I}$.

Corollary 1. Let $W = \{(W_i, w_i P_{W_i})\}_{i \in I}$ be the fusion frame for H with fusion frame operator S , and let $S^{-1}(W) = \{(S^{-1}(W_i), w_i P_{S^{-1}(W_i)})\}_{i \in I}$ be its the canonical dual fusion frame. Then $S_{S^{-1}(W)} = S^{-1}$.

Proof. For any $f \in H$,

$$\begin{aligned} S_{S^{-1}(W)}(f) &= \sum_{i \in I} w_i^2 P_{S^{-1}(W_i)} S^{-1} P_{W_i}(f) = S^{-1} \left(\sum_{i \in I} w_i^2 P_{W_i} \right) S^{-1}(f) \\ &= S^{-1} S S^{-1}(f) = S^{-1}(f). \end{aligned}$$

Therefore, the frame operator of the canonical dual fusion frame is the inverse of its own frame operator. It is obvious that if $W = \{(W_i, w_i P_{W_i})\}_{i \in I}$ is a parseval fusion frame, i.e., $S = I_H$, then its canonical dual fusion frame is itself. $\square$

Definition 6 (see [16]). Let $W = \{(W_i, w_i P_{W_i})\}_{i \in I}$ and $V = \{(V_i, v_i Q_{V_i})\}_{i \in I}$ be fusion frames for H . If for any $f \in H$,

$$f = \sum_{i \in I} w_i v_i Q_{V_i} S^{-1} P_{W_i}(f),$$

then $V = \{(V_i, v_i Q_{V_i})\}_{i \in I}$ is called an alternate dual fusion frame of $W = \{(W_i, w_i P_{W_i})\}_{i \in I}$, where S is the fusion frame operator of $W = \{(W_i, w_i P_{W_i})\}_{i \in I}$.

Remark 2. (1) The canonical dual fusion frame of $W = \{(W_i, w_i P_{W_i})\}_{i \in I}$ must be its alternate dual fusion frame. In particular, when $V_i = S^{-1}(W_i)$ and $v_i = w_i$, these two dual fusion frames are the same fusion frame.

(2) Traditional alternate dual frames are mutual alternate duals, while alternate dual fusion frames are not dual to each other.

In fact, if $V = \{(V_i, v_i Q_{V_i})\}_{i \in I}$ is the alternate dual fusion frame of $W = \{(W_i, w_i P_{W_i})\}_{i \in I}$, then $f = \sum_{i \in I} w_i v_i Q_{V_i} S_1^{-1} P_{W_i}(f)$ for any $f \in H$, where S_1 is the fusion frame operator of $W = \{(W_i, w_i P_{W_i})\}_{i \in I}$. If $W = \{(W_i, w_i P_{W_i})\}_{i \in I}$ is an alternate dual fusion frame of $V = \{(V_i, v_i Q_{V_i})\}_{i \in I}$, then $f = \sum_{i \in I} w_i v_i P_{W_i} S_2^{-1} Q_{V_i}(f)$ for any $f \in H$, where S_2 is the fusion frame operator of $V = \{(V_i, v_i Q_{V_i})\}_{i \in I}$.

In particular, if $S_1 = S_2 = S$, then the alternate dual fusion frames are mutual alternate duals.

The following are the necessary and sufficient conditions under which they are mutual alternate duals and their stability.

Theorem 6. Let $W = \{(W_i, w_i P_{W_i})\}_{i \in I}$ and $V = \{(V_i, v_i Q_{V_i})\}_{i \in I}$ be fusion frames for H , let T_1 and T_2 be their analysis operators respectively, let T_1^* and T_2^* be their synthesis operators, respectively, and let S_1 and S_2 be their frame operators respectively. Then

(1) $W = \{(W_i, w_i P_{W_i})\}_{i \in I}$ and $V = \{(V_i, v_i Q_{V_i})\}_{i \in I}$ are mutual alternate duals if and only if $T_2^* S_1^{-1} T_1 = T_1^* S_2^{-1} T_2 = I_H$.

(2) If $S_1 = S_2 = S$ and $V = \{(V_i, v_i Q_{V_i})\}_{i \in I}$ is alternate dual fusion frames of $W = \{(W_i, w_i P_{W_i})\}_{i \in I}$, then $W + V = \{(W_i + V_i, w_i S^{-1} P_{W_i} + v_i S^{-1} Q_{V_i})\}_{i \in I}$ is also a fusion frame for H .

Proof. (1) For any $f \in H$, we have

$$T_2^* S_1^{-1} T_1(f) = T_2^* S_1^{-1} \{(w_i P_{W_i}(f))\}_{i \in I} = \sum_{i \in I} v_i Q_{V_i} S_1^{-1} w_i P_{W_i}(f) = \sum_{i \in I} w_i v_i Q_{V_i} S_1^{-1} P_{W_i}(f),$$

$$T_1^* S_2^{-1} T_2(f) = T_1^* S_2^{-1} \{(v_i Q_{V_i}(f))\}_{i \in I} = \sum_{i \in I} w_i P_{W_i} S_2^{-1} v_i Q_{V_i}(f) = \sum_{i \in I} w_i v_i P_{W_i} S_2^{-1} Q_{V_i}(f).$$

Therefore, $W = \{(W_i, w_i P_{W_i})\}_{i \in I}$ and $V = \{(V_i, v_i Q_{V_i})\}_{i \in I}$ are mutual alternate duals if and only if

$$f = \sum_{i \in I} w_i v_i Q_{V_i} S_1^{-1} P_{W_i}(f) = \sum_{i \in I} w_i v_i P_{W_i} S_2^{-1} Q_{V_i}(f)$$

if and only if

$$f = T_2^* S_1^{-1} T_1(f) = T_1^* S_2^{-1} T_2(f),$$

by the arbitrariness of f , we obtain $T_2^* S_1^{-1} T_1 = T_1^* S_2^{-1} T_2 = I_H$.

(2) Since $S_1 = S_2 = S$, $W = \{(W_i, w_i P_{W_i})\}_{i \in I}$ and $V = \{(V_i, v_i Q_{V_i})\}_{i \in I}$ are mutual alternate duals. Combining with (1) of Theorem 6, for any $f \in H$, we have

$$T_2^* S^{-1} T_1(f) = \sum_{i \in I} w_i v_i Q_{V_i} S^{-1} P_{W_i}(f) = f, \quad T_1^* S^{-1} T_2(f) = \sum_{i \in I} w_i v_i P_{W_i} S^{-1} Q_{V_i}(f) = f.$$

And

$$T_1^* S^{-1} T_1(f) = \sum_{i \in I} w_i^2 P_{W_i} S^{-1} P_{W_i}(f), \quad T_2^* S^{-1} T_2(f) = \sum_{i \in I} v_i^2 Q_{V_i} S^{-1} Q_{V_i}(f).$$

Therefore,

$$\begin{aligned} & \sum_{i \in I} \langle (w_i S^{1/2} P_{W_i} + v_i S^{1/2} Q_{V_i})(f), (w_i S^{1/2} P_{W_i} + v_i S^{1/2} Q_{V_i})(f) \rangle \\ &= \sum_{i \in I} \langle w_i S^{1/2} P_{W_i}(f), w_i S^{1/2} P_{W_i}(f) \rangle + \sum_{i \in I} \langle w_i S^{1/2} P_{W_i}(f), v_i S^{1/2} Q_{V_i}(f) \rangle \\ & \quad + \sum_{i \in I} \langle v_i S^{1/2} Q_{V_i}(f), w_i S^{1/2} P_{W_i}(f) \rangle + \sum_{i \in I} \langle v_i S^{1/2} Q_{V_i}(f), v_i S^{1/2} Q_{V_i}(f) \rangle \\ &= \left\langle \sum_{i \in I} w_i^2 P_{W_i} S^{-1} P_{W_i}(f), f \right\rangle + \left\langle \sum_{i \in I} w_i v_i Q_{V_i} S^{-1} P_{W_i}(f), f \right\rangle \\ & \quad + \left\langle \sum_{i \in I} w_i v_i P_{W_i} S^{-1} Q_{V_i}(f), f \right\rangle + \left\langle \sum_{i \in I} v_i^2 Q_{V_i} S^{-1} Q_{V_i}(f), f \right\rangle \\ &= \langle T_1^* S^{-1} T_1(f), f \rangle + \langle T_2^* S^{-1} T_1(f), f \rangle + \langle T_1^* S^{-1} T_2(f), f \rangle + \langle T_2^* S^{-1} T_2(f), f \rangle \\ &= \langle T_1^* S^{-1} T_1(f), f \rangle + \langle f, f \rangle + \langle f, f \rangle + \langle T_2^* S^{-1} T_2(f), f \rangle \\ &= \langle (T_1^* S^{-1} T_1 + T_2^* S^{-1} T_2 + 2I_H)(f), f \rangle, \end{aligned}$$

that is to say,

$$\sum_{i \in I} (w_i S^{1/2} P_{W_i} + v_i S^{1/2} Q_{V_i})^* (w_i S^{1/2} P_{W_i} + v_i S^{1/2} Q_{V_i}) = T_1^* S^{-1} T_1 + T_2^* S^{-1} T_2 + 2I_H.$$

Moreover, since $W = \{(W_i, w_i P_{W_i})\}_{i \in I}$ and $V = \{(V_i, v_i Q_{V_i})\}_{i \in I}$ are fusion frames for H , there exist constants $0 < A < B < +\infty$ such that $AI_H \leq S_1 = S_2 = S \leq BI_H$. Therefore,

$$\frac{1}{B} I_H \leq S^{-1} \leq \frac{1}{A} I_H,$$

and

$$\frac{A}{B} I_H \leq \frac{1}{B} S = \frac{1}{B} T_1^* T_1 \leq T_1^* S^{-1} T_1 \leq \frac{1}{A} T_1^* T_1 = \frac{1}{A} S \leq \frac{B}{A} I_H.$$

Similarly, since

$$\frac{A}{B} I_H \leq T_2^* S^{-1} T_2 \leq \frac{B}{A} I_H,$$

we have

$$2\left(\frac{A}{B} + 1\right)I_H \leq T_1^*S^{-1}T_1 + T_2^*S^{-1}T_2 + 2I_H \leq 2\left(\frac{B}{A} + 1\right)I_H,$$

so $W + V = \{(W_i + V_i, w_iS^{-1}P_{W_i} + v_iS^{-1}Q_{V_i})\}_{i \in I}$ is also a fusion frame with frame operator $S_{W+V} = T_1^*S^{-1}T_1 + T_2^*S^{-1}T_2 + 2I_H$ and $2\left(\frac{A}{B} + 1\right)I_H \leq S_{W+V} \leq 2\left(\frac{B}{A} + 1\right)I_H$. $\square$

Theorem 7. Let $W = \{(W_i, w_iP_{W_i})\}_{i \in I}$ be a fusion frame for H with fusion frame operator S_1 , let $S_1^{-1}(W) = \{(S_1^{-1}(W_i), w_iP_{S_1^{-1}(W_i)})\}_{i \in I}$ and $V = \{(V_i, v_iQ_{V_i})\}_{i \in I}$ be its canonical dual fusion frame and alternate dual fusion frame respectively. Then $S_1^{-1} \leq \|S_1^{-1}\|^2 S_2$, where S_2 is the fusion frame operator of $V = \{(V_i, v_iQ_{V_i})\}_{i \in I}$.

Proof. Since for any $f \in H$,

$$\sum_{i \in I} w_i^2 P_{S_1^{-1}(W_i)} S_1^{-1} P_{W_i}(f) = f = \sum_{i \in I} w_i v_i Q_{V_i} S_1^{-1} P_{W_i}(f),$$

and

$$\sum_{i \in I} w_i^2 P_{S_1^{-1}(W_i)} S_1^{-1} P_{W_i}(f) = \sum_{i \in I} w_i^2 S_1^{-1} P_{W_i}(f),$$

we have

$$\sum_{i \in I} w_i^2 S_1^{-1} P_{W_i}(f) = f = \sum_{i \in I} w_i v_i Q_{V_i} S_1^{-1} P_{W_i}(f).$$

Replacing f with $S_1^{-1}(f)$,

$$\sum_{i \in I} w_i^2 S_1^{-1} P_{W_i}(S_1^{-1}(f)) = S_1^{-1}(f) = \sum_{i \in I} w_i v_i Q_{V_i} S_1^{-1} P_{W_i}(S_1^{-1}(f)).$$

Therefore,

$$\langle f, S_1^{-1}(f) \rangle = \left\langle f, \sum_{i \in I} w_i v_i Q_{V_i} S_1^{-1} P_{W_i}(S_1^{-1}(f)) \right\rangle = \left\langle f, \sum_{i \in I} w_i^2 S_1^{-1} P_{W_i}(S_1^{-1}(f)) \right\rangle,$$

and

$$\sum_{i \in I} \langle v_i S_1^{-1} Q_{V_i}(f), w_i P_{W_i} S_1^{-1}(f) \rangle = \sum_{i \in I} \langle w_i P_{W_i} S_1^{-1}(f), w_i P_{W_i} S_1^{-1}(f) \rangle,$$

that is to say,

$$\sum_{i \in I} \langle v_i S_1^{-1} Q_{V_i}(f) - w_i P_{W_i} S_1^{-1}(f), w_i P_{W_i} S_1^{-1}(f) \rangle = 0.$$

Moreover, since

$$\sum_{i \in I} \langle v_i S_1^{-1} Q_{V_i}(f) - w_i P_{W_i} S_1^{-1}(f), v_i S_1^{-1} Q_{V_i}(f) - w_i P_{W_i} S_1^{-1}(f) \rangle \geq 0,$$

we obtain

$$\begin{aligned}
& \sum_{i \in I} \langle v_i S_1^{-1} Q_{V_i}(f), v_i S_1^{-1} Q_{V_i}(f) \rangle \\
&= \sum_{i \in I} \langle v_i S_1^{-1} Q_{V_i}(f) - w_i P_{W_i} S_1^{-1}(f) + w_i P_{W_i} S_1^{-1}(f), v_i S_1^{-1} Q_{V_i}(f) - w_i P_{W_i} S_1^{-1}(f) + w_i P_{W_i} S_1^{-1}(f) \rangle \\
&= \sum_{i \in I} \langle v_i S_1^{-1} Q_{V_i}(f) - w_i P_{W_i} S_1^{-1}(f), v_i S_1^{-1} Q_{V_i}(f) - w_i P_{W_i} S_1^{-1}(f) \rangle \\
&\quad + \sum_{i \in I} \langle v_i S_1^{-1} Q_{V_i}(f) - w_i P_{W_i} S_1^{-1}(f), w_i P_{W_i} S_1^{-1}(f) \rangle \\
&\quad + \sum_{i \in I} \langle w_i P_{W_i} S_1^{-1}(f), v_i S_1^{-1} Q_{V_i}(f) - w_i P_{W_i} S_1^{-1}(f) \rangle + \sum_{i \in I} \langle w_i P_{W_i} S_1^{-1}(f), w_i P_{W_i} S_1^{-1}(f) \rangle \\
&\geq \sum_{i \in I} \langle w_i P_{W_i} S_1^{-1}(f), w_i P_{W_i} S_1^{-1}(f) \rangle.
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{i \in I} \langle w_i P_{W_i} S_1^{-1}(f), w_i P_{W_i} S_1^{-1}(f) \rangle &\leq \sum_{i \in I} \langle v_i S_1^{-1} Q_{V_i}(f), v_i S_1^{-1} Q_{V_i}(f) \rangle \\
&= \left\langle S_1^{-1} \left(\sum_{i \in I} v_i^2 Q_{V_i} \right) S_1^{-1}(f), f \right\rangle = \langle S_1^{-1} S_2 S_1^{-1}(f), f \rangle.
\end{aligned}$$

Furthermore, we have

$$\sum_{i \in I} w_i^2 \|P_{W_i} S_1^{-1}(f)\|^2 \leq \|S_1^{-1}\|^2 \langle S_2(f), f \rangle.$$

And

$$\sum_{i \in I} w_i^2 \|P_{W_i} S_1^{-1}(f)\|^2 = \langle S_1^{-1} \left(\sum_{i \in I} w_i^2 P_{W_i} \right) S_1^{-1}(f), f \rangle = \langle S_1^{-1} S_1 S_1^{-1}(f), f \rangle = \langle S_1^{-1}(f), f \rangle,$$

therefore,

$$\langle S_1^{-1}(f), f \rangle \leq \|S_1^{-1}\|^2 \langle S_2(f), f \rangle.$$

Since S_1^{-1} is the frame operator of the canonical dual fusion frame of $W = \{(W_i, w_i P_{W_i})\}_{i \in I}$, Theorem 7 reveals the relationship between the canonical dual fusion frame and the alternate dual fusion frame, especially the relationship between their respective fusion frame operators. $\square$

3. Conclusions

In this paper, we first uncover implicit fusion frames by studying the frame transform and orthogonal projection of traditional tight frames. Next, we present an example connecting traditional tight frames and fusion frames by combining the convergence of positive-term series in mathematical analysis with the orthonormal basis of a Hilbert space. In this context, the index set is extended to an infinite set and both types of frames are explicitly realized. Secondly, we investigated orthogonal projections in fusion frames, particularly their applications in transforming redundant fusion frames into Parseval fusion frames. Finally, we studied the dual fusion frames. Since alternate dual fusion frames are not dual to each other, we first established the necessary and sufficient conditions under which they are mutual alternate duals, and then examined their stability as well as the relationship between the two types of dual fusion frames.

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