

## Article

# On the Multiplication Operators from the Natural $\mu$ -Bloch-Type Space into Another Natural $\omega$ -Bloch-Type Space

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## Abstract

This paper investigates the boundedness of multiplication operators  $M_\psi$  between natural  $\mu$ -Bloch-type spaces  $B_{\mu,\text{nat}}(B_X)$  (or their little  $\mu$ -Bloch counterparts) and natural  $\omega$ -Bloch-type spaces  $B_{\omega,\text{nat}}(B_X)$  on the unit ball  $B_X$  of a complex Banach space  $X$ . We establish complete characterizations for the boundedness of  $M_\psi$  under varying conditions on the weight functions  $\mu$  and  $\omega$ , including specific cases such as logarithmic and power-weighted Bloch spaces. The results extend classical operator theory to infinite-dimensional settings, unifying prior work on finite-dimensional domains.

**Keywords:** the multiplication operator; Fréchet derivative; natural  $\mu$ -Bloch-type space; boundedness

**MSC:** 47B38; 47B01; 47B91; 46E50

## 1. Introduction and Preliminaries

Let  $X$  be a complex Banach space, which can either possess a finite or an infinite dimensional structure. Define  $\mathbb{B}_X := \{z \in X : \|z\| < 1\}$  as the unit ball of  $X$ . Let  $H(\mathbb{B}_X, \mathbb{C})$  represent the set of holomorphic mappings from  $\mathbb{B}_X$  into  $\mathbb{C}$ . Similarly, let  $H(\mathbb{B}_X, \mathbb{B}_X)$  denote the set of holomorphic mappings from  $\mathbb{B}_X$  into  $\mathbb{B}_X$ .

A positive continuous function  $\mu$ , defined on  $[0, 1)$  is called a normal, provided that there are positive constants  $s, t, 0 < s < t$  and  $t_0 \in [0, 1)$ , satisfying the condition

$$\frac{\mu(r)}{(1-r^2)^s} \text{ decreases for } t_0 \leq r < 1 \quad \text{and} \quad \lim_{r \rightarrow 1^-} \frac{\mu(r)}{(1-r^2)^s} = 0,$$

$$\frac{\mu(r)}{(1-r^2)^t} \text{ increases for } t_0 \leq r < 1 \quad \text{and} \quad \lim_{r \rightarrow 1^-} \frac{\mu(r)}{(1-r^2)^t} = \infty$$

(see, for example, [1]). From now on, if we state that a function  $\mu : \mathbb{B}_X \rightarrow (0, \infty)$  is normal, we will also presuppose that it is radial, that is to say  $\mu(z) = \mu(\|z\|)$ ,  $z \in \mathbb{B}_X$ .

Next, we will use the definition of Fréchet differentiable function and the chain rule of differentiation  $(f \circ g)'(x) = f'(g(x))g'(x)$  in [2]. It is widely recognized that constant functions  $f(x) := y_0$  are differentiable with  $f' = 0$ ; continuous linear maps are differentiable and  $T(x+h) = Tx + Th$ , so  $T'(x) = T$ .

We denote by  $Df(x) = f'(x)$  the Fréchet derivative of  $f$  at the point  $x$ .

Now, we shall introduce a new family of spaces, the natural Bloch-type space,  $\mathcal{B}_{\mu,\text{nat}}(\mathbb{B}_X)$ , which is defined as follows:



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**Definition 1.** For each  $f \in H(\mathbb{B}_X, \mathbb{C})$ , we define the natural  $\mu$ -Bloch semi-norm of  $f$  by

$$\|f\|_{\mu, \text{nat}} = \sup_{z \in \mathbb{B}_X} \{\mu(z) \|f'(z)\|\}.$$

The natural  $\mu$ -Bloch-type space  $\mathcal{B}_{\mu, \text{nat}}(\mathbb{B}_X)$  is given by

$$\mathcal{B}_{\mu, \text{nat}}(\mathbb{B}_X) = \{f \in H(\mathbb{B}_X, \mathbb{C}) : \|f\|_{\mu, \text{nat}} < \infty\}.$$

It is clear that  $\|\cdot\|_{\mu, \text{nat}}$  is a semi-norm for  $\mathcal{B}_{\mu, \text{nat}}(\mathbb{B}_X)$  and this space can be endowed with the norm  $\|f\|_{\mu} = |f(0)| + \|f\|_{\mu, \text{nat}}$ . Hence  $(\mathcal{B}_{\mu, \text{nat}}(\mathbb{B}_X), \|\cdot\|_{\mu})$  is a Banach space.

- (1) If  $X = \mathbb{C}$ ,  $\mathbb{B}_X = \mathbb{D}$  (the open unit disk of  $\mathbb{C}$ ) and  $\mu(z) = 1 - |z|^2$ , then  $\mathcal{B}_{\mu, \text{nat}}(\mathbb{B}_X)$  is the classical Bloch space  $\mathcal{B}(\mathbb{D})$  defined in [3,4].
- (2) If  $X = \mathbb{C}^n$ ,  $\mathbb{B}_X = \mathbb{B}_n$  (the open unit ball of  $\mathbb{C}^n$ ) and  $\mu(z) = 1 - |z|^2$ , then  $\mathcal{B}_{\mu, \text{nat}}(\mathbb{B}_X)$  is the natural Bloch-type space  $\mathcal{B}_{\text{nat}}(\mathbb{B}_n)$  (see, also, [5,6]).
- (3) When  $\mu(z) = 1 - \|z\|^2$ ,  $\mathcal{B}_{\mu, \text{nat}}(\mathbb{B}_X)$  is the natural Bloch space  $\mathcal{B}_{\text{nat}}(\mathbb{B}_X)$  defined in [7].
- (4) When  $\mu(z) = (1 - \|z\|^2)^{\alpha}$  ( $\alpha > 0$ ), write  $\mathcal{B}_{\mu, \text{nat}}(\mathbb{B}_X) = \mathcal{B}_{\alpha, \text{nat}}(\mathbb{B}_X)$ .
- (5) When  $\mu(z) = (1 - \|z\|^2) \ln \frac{e}{1 - \|z\|^2}$ , write  $\mathcal{B}_{\mu, \text{nat}}(\mathbb{B}_X) = \mathcal{B}_{\ln, \text{nat}}(\mathbb{B}_X)$ .  $\mathcal{B}_{\ln, \text{nat}}(\mathbb{D})$  is the logarithmic Bloch space, which emerged in characterizing the multipliers of the Bloch space  $\mathcal{B}(\mathbb{D})$  (see [6,8]).

It is easily proved that for  $0 < \alpha < 1 < \beta < \infty$ ,  $\mathcal{B}_{\alpha, \text{nat}}(\mathbb{B}_X) \subsetneq \mathcal{B}_{\ln, \text{nat}}(\mathbb{B}_X) \subsetneq \mathcal{B}_{\text{nat}}(\mathbb{B}_X) \subsetneq \mathcal{B}_{\beta, \text{nat}}(\mathbb{B}_X)$ . The Bloch space is a function space commonly used in mathematical physics and complex analysis. The subspace comprising functions with bounded differences (i.e., Lipschitz-type spaces) has been studied in the context of multiplication operators on trees. The Bloch space has been extensively examined on various homogeneous domains within  $\mathbb{C}^n$ , including the unit ball  $\mathbb{B}_n$ , the polydisc  $\mathbb{D}^n$ , and the open unit ball of a Hilbert space  $E$  (see, e.g., [6,9–12] and the references therein).

Let  $\psi \in H(\mathbb{B}_X, \mathbb{C})$ . The multiplication operator with symbol  $\psi$  is defined by

$$M_{\psi}f = \psi f, \quad f \in H(\mathbb{B}_X, \mathbb{C}).$$

The study of multiplication operators with symbols is fundamental to the exploration of Banach and Hilbert spaces of holomorphic functions. These operators hold a significant place in the theory of Hilbert space operators. A key application is the fact that every normal operator on a separable Hilbert space is unitarily equivalent to a multiplication operator. Furthermore, multiplication operators originate from spectral theory and continue to be actively researched in areas such as the theory of subnormal operators and the theory of Toeplitz operators. Recently, there has been a substantial increase in attention focused on the study of multiplication operators. Allen and Colonna in [13] examined isometries and spectra of multiplication operators on the Bloch space of the unit disk. Allen and Colonna in [14] investigated the multiplication operators on the Bloch space of bounded homogeneous domains. Douglas, Sun and Zheng in [15] studied multiplication operators on the Bergman space via analytic continuation. Guo and Huang in [16] studied multiplication operators on the Bergman space. De Jager and Labuschagne in [17] studied multiplication operators on non-commutative spaces. Reinwand and Kasprzak in [18] investigated the characteristics of multiplication operators operating on domains of real-valued functions with bounded variation on  $[0, 1]$ . Ghosh in [19] studied multiplication operator on the Bergman space by proper holomorphic mappings. Huang and Zheng in [20] studied multiplication operators on the Bergman space of bounded domains. Geng in [21] gave characterizations of the symbols  $\psi$  for which the multiplication operator  $M_{\psi}$  is isometries of BMOA. Han, Li and Wang in [22] studied multiplication operators on weighted Dirichlet

spaces. Liu and Yu in [23] studied products of composition, multiplication and radial derivative operators from logarithmic Bloch spaces to weighted-type spaces on the unit ball. Wang in [24] investigated the coefficient multipliers on  $H^2$  and  $\mathfrak{D}^2$  with Hyers–Ulam stability. Some research has also been dedicated to the situation when  $\mathbb{B}_X$  is the open unit ball of a Banach space  $X$  (see e.g., [25–31] and the references therein).

A primary purpose of this paper is to bring the current results on the multiplication operators from the natural  $\mu$ -Bloch-type spaces  $\mathcal{B}_{\mu,nat}(\mathbb{B}_X)$  (or the little  $\mu$ -Bloch-type spaces  $\mathcal{B}_{\mu,nat,0}(\mathbb{B}_X)$ ) into another natural  $\omega$ -Bloch-type spaces  $\mathcal{B}_{\omega,nat}(\mathbb{B}_X)$  (or another little  $\omega$ -Bloch-type spaces  $\mathcal{B}_{\omega,nat,0}(\mathbb{B}_X)$ ). Our hope is that this exposition will inspire more work in this area. We first present the following useful lemmas that will be utilized in the proofs of the main results.

**Lemma 1.** *Let  $\mu$  be a normal and  $f \in \mathcal{B}_{\mu,nat}(\mathbb{B}_X)$ . Then*

$$|f(z)| \leq \left(1 + \int_0^{\|z\|} \frac{1}{\mu(t)} dt\right) \|f\|_{\mu}, \text{ for } z \in \mathbb{B}_X.$$

**Proof.** If  $f \in \mathcal{B}_{\mu,nat}(\mathbb{B}_X)$ , then

$$\|f'(z)\| \leq \frac{\|f\|_{\mu,nat}}{\mu(z)} \text{ for } z \in \mathbb{B}_X,$$

we get

$$\begin{aligned} |f(z)| &\leq |f(0)| + \int_0^1 \|f'(tz)\| \|z\| dt \\ &\leq |f(0)| + \int_0^1 \frac{\|f\|_{\mu,nat} \|z\|}{\mu(tz)} dt \\ &= |f(0)| + \int_0^{\|z\|} \frac{1}{\mu(t)} dt \|f\|_{\mu,nat} \\ &\leq \left(1 + \int_0^{\|z\|} \frac{1}{\mu(t)} dt\right) \|f\|_{\mu}. \end{aligned}$$

□

**Remark 1.** *If  $f \in \mathcal{B}_{\mu,nat}(\mathbb{B}_X)$  and  $\int_0^1 \frac{1}{\mu(t)} dt < \infty$ , then  $\sup_{z \in \mathbb{B}_X} |f(z)| < \infty$ . Especially, if  $f \in \mathcal{B}_{\alpha,nat}(\mathbb{B}_X)$  and  $0 < \alpha < 1$ , then  $\int_0^1 \frac{1}{(1-t^2)^\alpha} dt < \infty$ , thus  $\sup_{z \in \mathbb{B}_X} |f(z)| < \infty$ . So, if  $0 < \alpha < 1$ , then  $\mathcal{B}_{\alpha,nat}(\mathbb{B}_X) \subset H^\infty(\mathbb{B}_X)$ , and the space comprises bounded holomorphic functions on  $\mathbb{B}_X$ .*

**Lemma 2.** *Let  $f \in \mathcal{B}_{nat}(\mathbb{B}_X)$ . There is a positive constant  $C$  such that*

$$|f(z)| \leq C \ln \frac{2}{1 - \|z\|^2} \|f\|_{nat-Bloch}, \text{ for } z \in \mathbb{B}_X.$$

**Proof.** If  $f \in \mathcal{B}_{nat}(\mathbb{B}_X)$ , taking  $\mu(z) = 1 - \|z\|^2$  in Lemma 1 we have

$$\begin{aligned} |f(z)| &\leq \left(1 + \int_0^{\|z\|} \frac{1}{1-t^2} dt\right) \|f\|_{nat} \\ &\leq \left(1 + \ln \frac{2}{1 - \|z\|^2}\right) \|f\|_{nat-Bloch} \\ &\leq C \ln \frac{2}{1 - \|z\|^2} \|f\|_{nat-Bloch} \end{aligned}$$

with  $C = \frac{1}{\ln 2} + 1$ .  $\square$

**Lemma 3.** Let  $\alpha > 1$  and  $f \in \mathcal{B}_{\alpha, \text{nat}}(\mathbb{B}_X)$ . There exists a positive constant  $C$  such that

$$|f(z)| \leq \frac{C}{(1 - \|z\|^2)^{\alpha-1}} \|f\|_{\alpha}, \text{ for } z \in \mathbb{B}_X.$$

**Proof.** If  $f \in \mathcal{B}_{\alpha, \text{nat}}(\mathbb{B}_X)$ , taking  $\mu(z) = (1 - \|z\|^2)^{\alpha}$  in Lemma 1 we have

$$\begin{aligned} |f(z)| &\leq \left(1 + \int_0^{\|z\|} \frac{1}{(1-t^2)^{\alpha}} dt\right) \|f\|_{\alpha, \text{nat}} \\ &\leq \left(1 + \int_0^{\|z\|} \frac{1}{(1-t)^{\alpha}} dt\right) \|f\|_{\alpha, \text{nat}} \\ &\leq \left(1 + \frac{1}{\alpha-1} \left(\frac{1}{(1-\|z\|)^{\alpha-1}} - 1\right)\right) \|f\|_{\alpha} \\ &\leq \frac{C}{(1 - \|z\|^2)^{\alpha-1}} \|f\|_{\alpha} \end{aligned}$$

with  $C = 1 + \frac{2^{\alpha-1}}{\alpha-1}$ .  $\square$

**Lemma 4.** Let  $f \in \mathcal{B}_{\ln, \text{nat}}(\mathbb{B}_X)$ . There exists a positive constant  $C$  such that

$$|f(z)| \leq C \ln \ln \frac{6}{1 - \|z\|^2} \|f\|_{\ln}, \text{ for } z \in \mathbb{B}_X.$$

**Proof.** If  $f \in \mathcal{B}_{\ln, \text{nat}}(\mathbb{B}_X)$ , taking  $\mu(z) = (1 - \|z\|^2) \ln \frac{e}{1-\|z\|}$  in Lemma 1 we have

$$\begin{aligned} |f(z)| &\leq \left(1 + \int_0^{\|z\|} \frac{1}{(1-t^2) \ln \frac{e}{1-t}} dt\right) \|f\|_{\ln} \\ &\leq \left(1 + \int_0^{\|z\|} \frac{1}{(1-t) \ln \frac{e}{1-t}} dt\right) \|f\|_{\ln} \\ &\leq \left(1 + \ln \ln \frac{e}{1-\|z\|}\right) \|f\|_{\ln} \\ &\leq C \ln \ln \frac{6}{1 - \|z\|^2} \|f\|_{\ln} \end{aligned}$$

with  $C = \frac{1}{\ln 6} + 1$ .  $\square$

The key findings of the paper are outlined in the subsequent sections.

## 2. The Boundedness of the Operator $M_{\psi} : \mathcal{B}_{\mu, \text{nat}}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega, \text{nat}}(\mathbb{B}_X)$

In this section, we examine the boundedness of the operator  $M_{\psi} : \mathcal{B}_{\mu, \text{nat}}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega, \text{nat}}(\mathbb{B}_X)$  and derive a necessary and sufficient condition, where  $\mu$  and  $\omega$  are normal. For  $x \in \mathbb{B}_X \setminus \{0\}$ , the set

$$T(x) = \{\ell_x \in X^* : \ell_x(x) = \|x\|, \|\ell_x\| = 1\}$$

of support functionals of  $x$  is nonempty by the Hahn–Banach theorem, where  $X^*$  is the dual space of  $X$ .

**Theorem 1.** (1) If  $\psi \in \mathcal{B}_{\omega, \text{nat}}(\mathbb{B}_X)$ ,

$$\sup_{z \in \mathbb{B}_X} \omega(z) \|\psi'(z)\| \int_0^{\|z\|} \frac{1}{\mu(t)} dt < \infty, \quad (1)$$

and

$$\sup_{z \in \mathbb{B}_X} \frac{\omega(z) |\psi(z)|}{\mu(z)} < \infty, \quad (2)$$

then the operator  $M_\psi : \mathcal{B}_{\mu, \text{nat}}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega, \text{nat}}(\mathbb{B}_X)$  is bounded.

(2) If  $0 < \alpha < 1$  and the operator  $M_\psi : \mathcal{B}_{\alpha, \text{nat}}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega, \text{nat}}(\mathbb{B}_X)$  is bounded, then  $\psi \in \mathcal{B}_{\omega, \text{nat}}(\mathbb{B}_X)$  and

$$\sup_{z \in \mathbb{B}_X} \frac{\omega(z) |\psi(z)|}{(1 - \|z\|^2)^\alpha} < \infty. \quad (3)$$

**Proof.** (1) Assume that  $\psi \in \mathcal{B}_{\omega, \text{nat}}(\mathbb{B}_X)$  and (1) and (2). Using the definition of the Fréchet derivative, it is easy to verify that for  $\psi, f \in H(\mathbb{B}_X, \mathbb{C})$ ,  $z \in \mathbb{B}_X$  and  $h$  in a neighborhood of 0,

$$\begin{aligned} & (\psi f)(z+h) - (\psi f)(z) \\ &= (\psi(z+h) - \psi(z))f(z+h) + (f(z+h) - f(z))\psi(z) \\ &= (\psi'(z)h + o(h))f(z+h) + (f'(z)h + o(h))\psi(z) \\ &= (\psi'(z)f(z+h) + f'(z)\psi(z))h + (f(z+h) - f(z))\psi(z) \\ &= (\psi'(z)f(z) + f'(z)\psi(z))h + \psi'(z)(f(z+h) - f(z))h \\ &\quad + (f(z+h) - f(z))\psi(z) \\ &= (\psi'(z)f(z) + f'(z)\psi(z))h + o(h), \end{aligned}$$

where  $\|o(h)\|/\|h\| \rightarrow 0$  as  $h \rightarrow 0$ , thus

$$(\psi f)'(z) = \psi(z)f'(z) + f(z)\psi'(z).$$

Using the triangle inequality, we get for  $z \in \mathbb{B}_X$ ,

$$\begin{aligned} & \omega(z) \|(\psi f)'(z)\| \\ & \leq \omega(z) \|\psi(z)f'(z)\| + \omega(z) |f(z)| \|\psi'(z)\| \\ & = \omega(z) |\psi(z)| \|f'(z)\| + \omega(z) |f(z)| \|\psi'(z)\|. \end{aligned} \quad (4)$$

Let  $f \in \mathcal{B}_{\mu, \text{nat}}(\mathbb{B}_X)$ . Then, by (4) and Lemma 1, we get

$$\begin{aligned} \|M_\psi f\|_{\omega, \text{nat}} &= \sup_{z \in \mathbb{B}_X} \omega(z) \|(\psi f)'(z)\| \\ &\leq \sup_{z \in \mathbb{B}_X} \omega(z) |\psi(z)| \frac{\|f\|_{\mu, \text{nat}}}{\mu(z)} + \sup_{z \in \mathbb{B}_X} \omega(z) \|\psi'(z)\| \left(1 + \int_0^{\|z\|} \frac{1}{\mu(t)} dt\right) \|f\|_\mu \\ &\leq \sup_{z \in \mathbb{B}_X} \frac{\omega(z) |\psi(z)|}{\mu(z)} \|f\|_\mu + \sup_{z \in \mathbb{B}_X} \omega(z) \|\psi'(z)\| \left(1 + \int_0^{\|z\|} \frac{1}{\mu(t)} dt\right) \|f\|_\mu. \end{aligned} \quad (5)$$

and

$$|M_\psi f(0)| = |\psi(0)f(0)| \leq |\psi(0)| \|f\|_\mu. \quad (6)$$

$\psi \in \mathcal{B}_{\omega, nat}(\mathbb{B}_X)$  together with (1)–(6), we can conclude the operator  $M_\psi : \mathcal{B}_{\mu, nat}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega, nat}(\mathbb{B}_X)$  is bounded.

(2) Assume that  $0 < \alpha < 1$  and the operator  $M_\psi : \mathcal{B}_{\alpha, nat}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega, nat}(\mathbb{B}_X)$  is bounded. Then, there exists a positive constant  $C$  such that

$$\|M_\psi f\|_\omega \leq C\|f\|_\alpha, \quad (7)$$

for all  $f \in \mathcal{B}_{\alpha, nat}(\mathbb{B}_X)$ . As usual, by taking the test function  $f(z) = 1$ , then  $f \in \mathcal{B}_{\alpha, nat}(\mathbb{B}_X)$  and  $\|f\|_\alpha = 1$ ; from this we obtain

$$\|M_\psi f\|_\omega = \|\psi\|_\omega \leq C\|f\|_\alpha = C < \infty,$$

that is,  $\psi \in \mathcal{B}_{\omega, nat}(\mathbb{B}_X)$ .

To demonstrate that (3) holds, fix  $a \in \mathbb{B}_X$ ; if  $a \in \mathbb{B}_X \setminus \{0\}$ , let  $\ell_a \in T(a)$  be fixed. We choose the test function  $e_a : \mathbb{B}_X \rightarrow \mathbb{C}$  given by

$$e_a(z) = \frac{1}{(1 - \|a\|\ell_a(z))^\alpha}, \quad z \in \mathbb{B}_X. \quad (8)$$

Furthermore, we explicitly have

$$e'_a(z) = \frac{\alpha\|a\|}{(1 - \|a\|\ell_a(z))^{\alpha+1}} \ell'_a(z),$$

and

$$e'_a(a) = \frac{\alpha\|a\|\ell'_a(a)}{(1 - \|a\|^2)^{\alpha+1}} = \frac{\alpha\|a\|\ell_a}{(1 - \|a\|^2)^{\alpha+1}}. \quad (9)$$

Thus,

$$\begin{aligned} \|e'_a(z)\| &\leq \frac{\alpha\|a\|}{|1 - \|a\|\ell_a(z)|^{\alpha+1}} \\ &\leq \frac{1}{(1 - \|a\|\|z\|)^{\alpha+1}} \leq \frac{1}{(1 - \|a\|)} \frac{1}{(1 - \|z\|)^\alpha}, \end{aligned}$$

which suggests that  $e_a \in \mathcal{B}_{\alpha, nat}(\mathbb{B}_X)$  with  $\|e_a\|_{\alpha, nat} \leq \frac{2^\alpha}{1 - \|a\|} \leq \frac{2^{\alpha+1}}{1 - \|a\|^2}$ . Hence, using the triangle inequality, (8) and (9) we would have that

$$\begin{aligned} \left(1 + \frac{2^{\alpha+1}}{1 - \|a\|^2}\right) \|M_\psi\| &\geq \|e_a\|_\alpha \|M_\psi\| \geq \\ &\geq \|M_\psi e_a\|_{\omega, nat} = \sup_{z \in \mathbb{B}_X} \omega(z) |(M_\psi e_a)'(z)| \\ &= \sup_{z \in \mathbb{B}_X} \omega(z) \|\psi(z) e'_a(z) + e_a(z) \psi'(z)\| \\ &\geq \omega(a) \|\psi(a) e'_a(a) + e_a(a) \psi'(a)\| \\ &\geq \omega(a) \|\psi(a)\| e'_a(a) - \omega(a) |e_a(a)| \|\psi'(a)\| \\ &= \frac{\alpha\omega(a) \|\psi(a)\| \|a\|}{(1 - \|a\|^2)^{\alpha+1}} - \frac{\omega(a) \|\psi'(a)\|}{(1 - \|a\|^2)^\alpha}. \end{aligned} \quad (10)$$

From (10) and  $0 < \alpha < 1$ , we easily get for  $r \in (0, 1)$ ,

$$\begin{aligned}
 & \sup_{r < \|z\| < 1} \frac{\omega(z)|\psi(z)|}{(1 - \|z\|^2)^\alpha} \\
 & \leq \frac{1}{r} \sup_{r < \|z\| < 1} \frac{\omega(z)\|z\||\psi(z)|}{(1 - \|z\|^2)^\alpha} \\
 & \leq \frac{1}{r\alpha} \sup_{r < \|z\| < 1} \left( (1 - \|z\|^2) \left( 1 + \frac{2^{\alpha+1}}{1 - \|z\|^2} \right) \|M_\psi\| + \sup_{r < \|z\| < 1} \frac{|\omega(z)||\psi'(z)|}{(1 - \|z\|^2)^{\alpha-1}} \right) \\
 & \leq \frac{1}{r\alpha} \left( (1 + 2^{\alpha+1}) \|M_\psi\| + \|\psi\|_\omega \right), \tag{11}
 \end{aligned}$$

and

$$\begin{aligned}
 & \sup_{z \in \mathbb{B}_X} \frac{\omega(z)|\psi(z)|}{(1 - \|z\|^2)^\alpha} \\
 & \leq \sup_{\|z\| \leq r} \frac{\omega(z)|\psi(z)|}{(1 - \|z\|^2)^\alpha} + \sup_{r < \|z\| < 1} \frac{\omega(z)\psi(z)}{(1 - \|z\|^2)^\alpha} \\
 & \leq \sup_{\|z\| \leq r} \frac{\omega(z)|\psi(z)|}{(1 - \|z\|^2)^\alpha} + \frac{1}{r\alpha} \left( (1 + 2^{\alpha+1}) \|M_\psi\| + \|\psi\|_\omega \right) \\
 & < \infty,
 \end{aligned}$$

condition (3) follows. The proof is successfully completed.  $\square$

**Corollary 1.** *If  $0 < \alpha < 1$ , then the operator*

*$M_\psi : \mathcal{B}_{\alpha, \text{nat}}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega, \text{nat}}(\mathbb{B}_X)$  is bounded if and only if  $\psi \in \mathcal{B}_{\omega, \text{nat}}(\mathbb{B}_X)$  and*

$$\sup_{z \in \mathbb{B}_X} \frac{\omega(z)|\psi(z)|}{(1 - \|z\|^2)^\alpha} < \infty.$$

**Proof.** It is easy.  $\square$

In the subsequent theorem, we delineate the boundedness properties of the operator  $M_\psi : \mathcal{B}_{\text{nat}}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega, \text{nat}}(\mathbb{B}_X)$ , where  $\omega$  is normal.

**Theorem 2.** *The operator  $M_\psi : \mathcal{B}_{\text{nat}}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega, \text{nat}}(\mathbb{B}_X)$  is bounded if and only if  $\psi \in \mathcal{B}_{\omega, \text{nat}}(\mathbb{B}_X)$ ,*

$$\sup_{z \in \mathbb{B}_X} \omega(z) \|\psi'(z)\| \ln \frac{2}{1 - \|z\|^2} < \infty, \tag{12}$$

and

$$\sup_{z \in \mathbb{B}_X} \frac{\omega(z)|\psi(z)|}{1 - \|z\|^2} < \infty. \tag{13}$$

**Proof.** Suppose that  $\psi \in \mathcal{B}_{\omega, \text{nat}}(\mathbb{B}_X)$  and (13) and (12). Let  $f \in \mathcal{B}_{\text{nat}}(\mathbb{B}_X)$ . Then by (4) and Lemma 2, we get

$$\begin{aligned}
 \|M_\psi f\|_{\omega, \text{nat}} &= \sup_{z \in \mathbb{B}_X} \omega(z) \|(\psi f)'(z)\| \\
 &\leq \sup_{z \in \mathbb{B}_X} \frac{\omega(z)|\psi(z)|}{1 - \|z\|^2} \|f\|_{\text{nat}} + C \sup_{z \in \mathbb{B}_X} \omega(z) \|\psi'(z)\| \ln \frac{2}{1 - \|z\|^2} \|f\|_{\text{nat-Bloch}}, \tag{14}
 \end{aligned}$$

and

$$|M_\psi f(0)| = |\psi(0)f(0)| \leq |\psi(0)|\|f\|_{\text{nat}}. \quad (15)$$

Applying conditions  $\psi \in \mathcal{B}_{\omega, \text{nat}}(\mathbb{B}_X)$  and (13)–(15), we can conclude the operator  $M_\psi : \mathcal{B}_{\text{nat}}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega, \text{nat}}(\mathbb{B}_X)$  is bounded.

Assume that the operator  $M_\psi : \mathcal{B}_{\text{nat}}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega, \text{nat}}(\mathbb{B}_X)$  is bounded. Then, by (7), we get  $\psi \in \mathcal{B}_{\omega, \text{nat}}(\mathbb{B}_X)$ .

(1) First we prove (13). Fix  $a \in \mathbb{B}_X$ ; if  $a \in \mathbb{B}_X \setminus \{0\}$ , let  $\ell_a \in T(a)$  be fixed. We choose the test function  $f_a : \mathbb{B}_X \rightarrow \mathbb{C}$  given by

$$f_a(z) = \frac{1}{1 - \|a\|\ell_a(z)}, \quad z \in \mathbb{B}_X. \quad (16)$$

Then

$$f'_a(z) = \frac{\|a\|}{(1 - \|a\|\ell_a(z))^2} \ell'_a(z),$$

and

$$f'_a(a) = \frac{\|a\|\ell'_a(a)}{(1 - \|a\|^2)^2} = \frac{\|a\|\ell_a}{(1 - \|a\|^2)^2}. \quad (17)$$

Thus,

$$\|f'_a(z)\| \leq \frac{1}{(1 - \|a\|)} \frac{1}{(1 - \|z\|)},$$

which suggests that  $f_a \in \mathcal{B}_{\text{nat}}(\mathbb{B}_X)$  with  $\|f_a\|_{\text{nat}} \leq \frac{2}{1 - \|a\|} \leq \frac{4}{1 - \|a\|^2}$ . Therefore, employing the triangle inequality, (16) and (17) we get

$$\begin{aligned} & \left(1 + \frac{4}{1 - \|a\|^2}\right) \|M_\psi\| \geq \|f_a\|_{\text{nat-Bloch}} \|M_\psi\| \\ & \geq \|M_\psi f_a\|_\omega \geq \|M_\psi f_a\|_{\omega, \text{nat}} \\ & = \sup_{z \in \mathbb{B}_X} \omega(z) \|(M_\psi f_a)'(z)\| \\ & \geq \frac{\omega(a) \|\psi(a)\| \|a\|}{(1 - \|a\|^2)^2} - \frac{\omega(a) \|\psi'(a)\|}{1 - \|a\|^2}. \end{aligned} \quad (18)$$

From (18), it follows that for  $r \in (0, 1)$

$$\begin{aligned} & \sup_{r < \|z\| < 1} \frac{\omega(z) \|\psi(z)\|}{1 - \|z\|^2} \\ & \leq \frac{1}{r} \sup_{r < \|z\| < 1} \frac{\omega(z) \|z\| \|\psi(z)\|}{1 - \|z\|^2} \\ & \leq \frac{1}{r} \left( (5 - r^2) \|M_\psi\| + \sup_{r < \|z\| < 1} |\omega(z)| \|\psi'(z)\| \right) \\ & \leq \frac{1}{r} \left( (5 - r^2) \|M_\psi\| + \|\psi\|_\omega \right). \end{aligned} \quad (19)$$

So

$$\begin{aligned}
 & \sup_{z \in \mathbb{B}_X} \frac{\omega(z)|\psi(z)|}{1 - \|z\|^2} \\
 & \leq \sup_{\|z\| \leq r} \frac{\omega(z)|\psi(z)|}{1 - \|z\|^2} + \sup_{r < \|z\| < 1} \frac{\omega(z)|\psi(z)|}{1 - \|z\|^2} \\
 & \leq \sup_{\|z\| \leq r} \frac{\omega(z)|\psi(z)|}{1 - \|z\|^2} + \frac{1}{r} \left( (5 - r^2) \|M_\psi\| + \|\psi\|_\omega \right) \\
 & < \infty,
 \end{aligned}$$

that is, inequality (13) holds.

(2) Next, we will proceed to prove (12). For the given  $a \in \mathbb{B}_X \setminus \{0\}$ , consider the function  $g_a$  given by

$$g_a(z) = 2 \ln \frac{2}{1 - \|a\| \ell_a(z)} - \left( \ln \frac{2}{1 - \|a\| \ell_a(z)} \right)^2 / \ln \frac{2}{1 - \|a\|^2}, \quad (20)$$

for  $z \in \mathbb{B}_X$ . Then

$$g'_a(z) = \frac{2\|a\|\ell_a}{1 - \|a\|\ell_a(z)} - \frac{2 \ln \frac{2}{1 - \|a\|\ell_a(z)}}{\ln \frac{2}{1 - \|a\|^2}} \frac{\|a\|\ell_a}{1 - \|a\|\ell_a(z)}, \quad (21)$$

for  $z \in \mathbb{B}_X$ , so that

$$|g_a(0)| \leq 3 \ln 2, \quad (22)$$

$$g'_a(a) = \theta \text{ (null operator) and } g_a(a) = \ln \frac{2}{1 - \|a\|^2}. \quad (23)$$

On the other hand, we obtain from the estimate in [32]

$$\|g'_a(z)\| \leq \frac{C}{1 - \|z\|^2}, \text{ for } z \in \mathbb{B}_X, \quad (24)$$

$g_a \in \mathcal{B}_{nat}(\mathbb{B}_X)$  for  $a \in \mathbb{B}_X \setminus \{0\}$  and  $\sup_{a \in \mathbb{B}_X \setminus \{0\}} \|g_a\|_{nat} \leq C$ . By (22)–(24) we obtain for  $\|a\| > r > 0$

$$\begin{aligned}
 & \omega(z) \|g_a(z) \psi'(z)\| \\
 & \leq \sup_{z \in \mathbb{B}_X} \omega(z) \|(M_\psi g_a)'(z)\| + \sup_{z \in \mathbb{B}_X} \omega(z) |\psi(z)| \|g'_a(z)\| \\
 & \leq \|M_\psi g_a\|_\omega + \sup_{z \in \mathbb{B}_X} \frac{\omega(z) |\psi(z)|}{1 - \|z\|^2} \\
 & \leq \|M_\psi\| \|g_a\|_{nat-Bloch} + \sup_{z \in \mathbb{B}_X} \frac{\omega(z) |\psi(z)|}{1 - \|z\|^2} \\
 & \leq \|M_\psi\| (C + 3 \ln 2) + \sup_{z \in \mathbb{B}_X} \frac{\omega(z) |\psi(z)|}{1 - \|z\|^2}.
 \end{aligned} \quad (25)$$

Taking  $z = a$  in (25), we get

$$\begin{aligned}
 & \omega(a) \|\psi'(a)\| \ln \frac{2}{1 - \|a\|^2} = \omega(a) \|g_a(a) \psi'(a)\| \\
 & \leq \|M_\psi\| (C + 3 \ln 2) + \sup_{z \in \mathbb{B}_X} \frac{\omega(z) |\psi(z)|}{1 - \|z\|^2}.
 \end{aligned}$$

From this and (13) we obtain

$$\sup_{\|z\|>r>0} \omega(z) \|\psi'(z)\| \ln \frac{2}{1-\|z\|^2} < \infty. \quad (26)$$

If  $\|z\| \leq r < 1$ , using  $\psi \in \mathcal{B}_{\omega, \text{nat}}(\mathbb{B}_X)$  we have

$$\begin{aligned} & \omega(z) \|\psi'(z)\| \ln \frac{2}{1-\|z\|^2} \\ & \leq \ln \frac{2}{1-r^2} \omega(z) \|\psi'(z)\| \leq \ln \frac{2}{1-r^2} \|\psi\|_{\omega} < \infty, \end{aligned}$$

which, in conjunction with (26), demonstrates that the condition in (12) is indeed necessary.  $\square$

In the subsequent theorem, we examine the boundedness of the operator  $M_{\psi} : \mathcal{B}_{\alpha, \text{nat}}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega, \text{nat}}(\mathbb{B}_X)$  ( $\alpha > 1$ ) and obtain a sufficient and necessary condition, where  $\omega$  is normal.

**Theorem 3.** Let  $\alpha > 1$ ; then, the operator  $M_{\psi} : \mathcal{B}_{\alpha, \text{nat}}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega, \text{nat}}(\mathbb{B}_X)$  is bounded if and only if

$$\sup_{z \in \mathbb{B}_X} \frac{\omega(z) |\psi(z)|}{(1-\|z\|^2)^{\alpha}} < \infty, \quad (27)$$

and

$$\sup_{z \in \mathbb{B}_X} \frac{\omega(z) \|\psi'(z)\|}{(1-\|z\|^2)^{\alpha-1}} < \infty. \quad (28)$$

**Proof.** Suppose that (27) and (28). Let  $f \in \mathcal{B}_{\alpha, \text{nat}}(\mathbb{B}_X)$ . Then by (4) and Lemma 3, we get

$$\begin{aligned} \|M_{\psi} f\|_{\omega, \text{nat}} & \leq \sup_{z \in \mathbb{B}_X} \omega(z) |\psi(z)| \|f'(z)\| + \sup_{z \in \mathbb{B}_X} \omega(z) |f(z)| \|\psi'(z)\| \\ & \leq \sup_{z \in \mathbb{B}_X} \frac{\omega(z) |\psi(z)|}{(1-\|z\|^2)^{\alpha}} \|f\|_{\alpha, \text{nat}} + C \sup_{z \in \mathbb{B}_X} \frac{\omega(z) \|\psi'(z)\|}{(1-\|z\|^2)^{\alpha-1}} \|f\|_{\alpha, \text{nat}}, \end{aligned} \quad (29)$$

and

$$|M_{\psi} f(0)| = |\psi(0) f(0)| \leq |\psi(0)| \|f\|_{\alpha}. \quad (30)$$

Using conditions (27)–(30), we can conclude the operator  $M_{\psi} : \mathcal{B}_{\alpha, \text{nat}}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega, \text{nat}}(\mathbb{B}_X)$  is bounded.

Assume that the operator  $M_{\psi} : \mathcal{B}_{\alpha, \text{nat}}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega, \text{nat}}(\mathbb{B}_X)$  is bounded. Then  $\psi \in \mathcal{B}_{\omega, \text{nat}}(\mathbb{B}_X)$ .

(1) First we prove (27). For  $a \in \mathbb{B}_X \setminus \{0\}$ , using  $l_a \in \mathcal{B}_{\alpha, \text{nat}}(\mathbb{B}_X)$  and  $\psi \in \mathcal{B}_{\omega, \text{nat}}(\mathbb{B}_X)$ , we get

$$\begin{aligned} \omega(z) \|l_a\| |\psi(z)| & = \omega(z) \|l'_a(z)\| |\psi(z)| \leq \sup_{z \in \mathbb{B}_X} \omega(z) \|\psi(z) l'_a(z)\| \\ & \leq \sup_{z \in \mathbb{B}_X} \omega(z) \|\psi(z) l'_a(z) + l_a(z) \psi'(z)\| + \sup_{z \in \mathbb{B}_X} \omega(z) \|\psi'(z)\| \|l_a(z)\| \\ & \leq \|M_{\psi}\| + \|\psi\|_{\omega}. \end{aligned}$$

From which we have

$$\begin{aligned}
& \sup_{z \in \mathbb{B}_X} \frac{\omega(z)|\psi(z)|}{(1 - \|z\|^2)^\alpha} \\
& \leq \sup_{\|z\| \leq r} \frac{\omega(z)|\psi(z)|}{(1 - \|z\|^2)^\alpha} + \sup_{r < \|z\| < 1} \frac{\omega(z)|\psi(z)|}{(1 - \|z\|^2)^\alpha} \\
& \leq \sup_{\|z\| \leq r} \frac{\omega(z)|\psi(z)|}{(1 - \|z\|^2)^\alpha} + \frac{1}{(1 - r^2)^\alpha} \sup_{r < \|z\| < 1} |\omega(z)| |\psi(z)| \\
& \leq \sup_{\|z\| \leq r} \frac{\omega(z)|\psi(z)|}{(1 - \|z\|^2)^\alpha} + \frac{1}{(1 - r^2)^\alpha} (\|M_\psi\| + \|\psi\|_\omega) \\
& < \infty,
\end{aligned}$$

that is, Equation (27) follows.

(2) Next, we will prove (28). For the given  $a \in \mathbb{B}_X \setminus \{0\}$ , consider the test function  $h_a$  given by

$$h_a(z) = \frac{1}{(1 - \|a\|\ell_a(z))^{\alpha-1}}, \text{ for } z \in \mathbb{B}_X. \quad (31)$$

Then

$$\begin{aligned}
h'_a(z) &= \frac{(1 - \alpha)\|a\|\ell'_a(z)}{(1 - \|a\|\ell_a(z))^\alpha} \\
&= \frac{(1 - \alpha)\|a\|\ell_a}{(1 - \|a\|\ell_a(z))^\alpha}, \text{ for } z \in \mathbb{B}_X,
\end{aligned} \quad (32)$$

so that

$$|h_a(0)| = 1, \quad (33)$$

$$h_a(a) = \frac{1}{(1 - \|a\|^2)^{\alpha-1}}, \quad (34)$$

$$\text{and } h'_a(a) = \frac{(1 - \alpha)\|a\|\ell_a}{(1 - \|a\|^2)^\alpha}. \quad (35)$$

On the flip side, we get for  $z \in \mathbb{B}_X$ ,

$$\|h'_a(z)\| \leq \frac{(\alpha - 1)\|a\|\|\ell_a\|}{(1 - \|\ell_a(z)\|)^\alpha} \leq \frac{(\alpha - 1)}{(1 - \|z\|)^\alpha}, \quad (36)$$

hence  $h_a \in \mathcal{B}_{\alpha, \text{nat}}(\mathbb{B}_X)$  for  $a \in \mathbb{B}_X \setminus \{0\}$  and  $\sup_{a \in \mathbb{B}_X \setminus \{0\}} \|h_a\|_{\alpha, \text{nat}} \leq 2^\alpha(\alpha - 1)$ . By (31)–(36) we obtain for all  $\|a\| > r > 0$  and  $z \in \mathbb{B}_X$

$$\begin{aligned}
& \omega(z)|h_a(z)|\|\psi'(z)\| \\
& \leq \sup_{z \in \mathbb{B}_X} \omega(z)\|(M_\psi h_z)'\| + \sup_{z \in \mathbb{B}_X} \omega(z)|\psi(z)| \frac{2^\alpha(\alpha - 1)}{(1 - \|z\|^2)^\alpha} \\
& = \|M_\psi h_a\|_{\omega, \text{nat}} + \sup_{z \in \mathbb{B}_X} \omega(z)|\psi(z)| \frac{2^\alpha(\alpha - 1)}{(1 - \|z\|^2)^\alpha} \\
& \leq \|M_\psi\| \|h_a\|_\alpha + \sup_{z \in \mathbb{B}_X} \omega(z)|\psi(z)| \frac{2^\alpha(\alpha - 1)}{(1 - \|z\|^2)^\alpha} \\
& = \|M_\psi\| (\|h_a\|_{\alpha, \text{nat}} + |h_a(0)|) + \sup_{z \in \mathbb{B}_X} \omega(z)|\psi(z)| \frac{2^\alpha(\alpha - 1)}{(1 - \|z\|^2)^\alpha} \\
& \leq \|M_\psi\| (2^\alpha(\alpha - 1) + 1) + 2^\alpha(\alpha - 1) \sup_{z \in \mathbb{B}_X} \frac{\omega(z)|\psi(z)|}{(1 - \|z\|^2)^\alpha}.
\end{aligned} \quad (37)$$

From (27), (37) we obtain

$$\begin{aligned} & \frac{\omega(a)\|\psi'(a)\|}{(1-\|a\|^2)^{\alpha-1}} \\ & \leq \|M_\psi\|(2^\alpha(\alpha-1)+1)+2^\alpha(\alpha-1)\sup_{z\in\mathbb{B}_X}\frac{\omega(z)|\psi(z)|}{(1-\|z\|^2)^\alpha}<\infty \end{aligned} \quad (38)$$

for all  $\|a\|>r>0$ . If  $\|a\|\leq r<1$ , using  $\psi\in\mathcal{B}_{\omega,nat}(\mathbb{B}_X)$  we have

$$\begin{aligned} & \frac{\omega(a)\|\psi'(a)\|}{(1-\|a\|^2)^{\alpha-1}} \\ & \leq \frac{1}{(1-r^2)^{\alpha-1}}\omega(a)\|\psi'(a)\|\leq\frac{1}{(1-r^2)^{\alpha-1}}\|\psi\|_\omega<\infty, \end{aligned}$$

which together with (38) implies that the condition in (28) is necessary.  $\square$

**Remark 2.** The method of proving (27) is also suitable for the proof of conditions (3) and (13).

**Example 1.** Case (1)  $0<\alpha<1$ . Let  $\omega(z)=1-\|z\|^2$  and  $a\in\mathbb{B}_X\setminus\{0\}$ ,  $\psi_{a,\beta}(z)=\frac{1}{(1-\|a\|\ell_a(z))^\beta}$  ( $\beta>0$ ). Then  $\psi_{a,\beta}\in\mathcal{B}_{nat}(\mathbb{B}_X)$  and

$$\begin{aligned} & \sup_{z\in\mathbb{B}_X}\frac{\omega(z)|\psi_{a,\beta}(z)|}{(1-\|z\|^2)^\alpha}=\sup_{z\in\mathbb{B}_X}\frac{1-\|z\|^2}{(1-\|z\|^2)^\alpha|1-\|a\|\ell_a(z)|^\beta} \\ & \leq\sup_{z\in\mathbb{B}_X}\frac{1-\|z\|^2}{(1-\|z\|^2)^\alpha(1-\|a\|)^\beta}\leq\frac{1}{(1-\|a\|)^\beta}. \end{aligned}$$

By Corollary 1, the operator  $M_{\psi_{a,\beta}}:\mathcal{B}_{\alpha,nat}(\mathbb{B}_X)\rightarrow\mathcal{B}_{nat}(\mathbb{B}_X)$  is bounded.

Case (2)  $\alpha=1$ . Let  $\mu(z)=\omega(z)=1-\|z\|^2$ . Let  $a\in\mathbb{B}_X\setminus\{0\}$ ,  $\psi_{a,1}(z)=\frac{1}{1-\|a\|\ell_a(z)}$ . Then  $\psi_{a,1}\in\mathcal{B}_{nat}(\mathbb{B}_X)$ ,

$$\begin{aligned} & \sup_{z\in\mathbb{B}_X}\frac{\omega(z)|\psi_{a,1}(z)|}{1-\|z\|^2}=\sup_{z\in\mathbb{B}_X}\frac{1-\|z\|^2}{|1-\|a\|\ell_a(z)|(1-\|z\|^2)} \\ & \leq\frac{1}{1-\|a\|} \end{aligned}$$

and

$$\begin{aligned} & \sup_{z\in\mathbb{B}_X}\omega(z)\|\psi'_{a,1}(z)\|\ln\frac{2}{1-\|z\|^2}=\sup_{z\in\mathbb{B}_X}\frac{(1-\|z\|^2)\|a\|\ell'_a(z)|}{|1-\|a\|\ell_a(z)|^2}\ln\frac{2}{1-\|z\|^2} \\ & \leq\sup_{z\in\mathbb{B}_X}\frac{1-\|z\|^2}{(1-\|a\|)^2}\ln\frac{2}{1-\|z\|^2}\leq\frac{C}{(1-\|a\|)^2}. \end{aligned}$$

By Theorem 5, the operator  $M_{\psi_{a,1}}:\mathcal{B}_{nat}(\mathbb{B}_X)\rightarrow\mathcal{B}_{nat}(\mathbb{B}_X)$  is bounded.

Case (3)  $\alpha>1$ . Let  $\omega(z)=(1-\|z\|^2)^\alpha$  and  $a\in\mathbb{B}_X\setminus\{0\}$ ,  $\psi_{a,1}(z)=\frac{1}{1-\|a\|\ell_a(z)}$ . Then  $\psi_{a,1}\in\mathcal{B}_{\alpha,nat}(\mathbb{B}_X)$ ,

$$\begin{aligned} & \sup_{z\in\mathbb{B}_X}\frac{\omega(z)|\psi_{a,1}(z)|}{(1-\|z\|^2)^\alpha}=\sup_{z\in\mathbb{B}_X}\frac{(1-\|z\|^2)^\alpha}{|1-\|a\|\ell_a(z)|(1-\|z\|^2)^\alpha} \\ & \leq\frac{1}{1-\|a\|} \end{aligned}$$

and

$$\begin{aligned} \sup_{z \in \mathbb{B}_X} \frac{\omega(z) \|\psi'_{a,1}(z)\|}{(1 - \|z\|^2)^{\alpha-1}} &= \sup_{z \in \mathbb{B}_X} \frac{(1 - \|z\|^2)^\alpha \|a\| |\ell'_a(z)|}{(|1 - \|a\| \ell_a(z)|)^2 (1 - \|z\|^2)^{\alpha-1}} \\ &\leq \frac{2}{1 - \|a\|}. \end{aligned}$$

By Theorem 6, the operator  $M_{\psi_{a,1}} : \mathcal{B}_{\alpha,nat}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\alpha,nat}(\mathbb{B}_X)$  is bounded.

Finally, since

$$\begin{aligned} \sup_{z \in \mathbb{B}_X} \frac{(1 - \|z\|^2) |\psi_{a,1}(z)|}{(1 - \|z\|^2)^\alpha} &= \sup_{z \in \mathbb{B}_X} \frac{1 - \|z\|^2}{|1 - \|a\| \ell_a(z)| (1 - \|z\|^2)^\alpha} \\ &= \sup_{z \in \mathbb{B}_X} \frac{1}{|1 - \|a\| \ell_a(z)| (1 - \|z\|^2)^{\alpha-1}} \\ &\geq \sup_{z \in \mathbb{B}_X} \frac{1}{2(1 - \|z\|^2)^{\alpha-1}}, \end{aligned}$$

so by Theorem 6, the operator  $M_{\psi_{a,1}} : \mathcal{B}_{\alpha,nat}(\mathbb{B}_X) \rightarrow \mathcal{B}_{nat}(\mathbb{B}_X)$  is not bounded.

### 3. The Boundedness of the Operator $M_\psi : \mathcal{B}_{\mu,nat,0}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega,nat,0}(\mathbb{B}_X)$

In this section, we will explore the generalization of the little Bloch space  $\mathcal{B}_0(\mathbb{D})$ . Thus, we first present the natural little  $\mu$ -Bloch-type space  $\mathcal{B}_{\mu,nat,0}(\mathbb{B}_X)$ . Then, we investigate the boundedness of the operator  $M_\psi : \mathcal{B}_{\mu,nat,0}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega,nat,0}(\mathbb{B}_X)$ .

**Definition 2.** The natural little  $\mu$ -Bloch-type space  $\mathcal{B}_{\mu,nat,0}(\mathbb{B}_X)$  is a subspace of  $\mathcal{B}_{\mu,nat}(\mathbb{B}_X)$  consisting of all  $f$  such that

$$\lim_{\|z\| \rightarrow 1} \mu(z) \|f'(z)\| = 0.$$

Next, we develop and validate several ancillary results.

**Lemma 5.** If  $f \in \mathcal{B}_{\mu,nat,0}(\mathbb{B}_X)$  and  $\int_0^1 \frac{1}{\mu(t)} dt = \infty$ , then

$$\lim_{\|z\| \rightarrow 1} \frac{|f(z)|}{\int_0^{\|z\|} \frac{1}{\mu(t)} dt} = 0.$$

**Proof.** If  $f \in \mathcal{B}_{\mu,nat,0}(\mathbb{B}_X)$  and  $\int_0^1 \frac{1}{\mu(t)} dt = \infty$ , then for every  $\epsilon > 0$ , there is a  $\delta > 0$  such that

$$\|f'(z)\| < \frac{\epsilon}{2\mu(z)} \quad (39)$$

for all  $z$  with  $\delta < \|z\| < 1$ . Since

$$\lim_{\|z\| \rightarrow 1} \int_0^{\|z\|} \frac{1}{\mu(t)} dt = \infty,$$

there is a  $\tau \in (\delta, 1)$  such that

$$\frac{|f(0)| + \int_0^\delta \frac{1}{\mu(t)} dt \|f\|_\mu}{\int_0^{\|z\|} \frac{1}{\mu(t)} dt} < \frac{\epsilon}{2}, \quad (40)$$

for all  $z$  with  $\tau < \|z\| < 1$ .

By (39) and (40) we have

$$\begin{aligned}
 |f(z)| &\leq |f(0)| + \int_0^1 \|f'(tz)\| \|z\| dt \\
 &= |f(0)| + \int_0^{\frac{\delta}{\|z\|}} \|f'(tz)\| \|z\| dt + \int_{\frac{\delta}{\|z\|}}^1 \|f'(tz)\| \|z\| dt \\
 &\leq |f(0)| + \|z\| \int_0^{\frac{\delta}{\|z\|}} \frac{\|f\|_\mu}{\mu(t\|z\|)} dt + \int_{\frac{\delta}{\|z\|}}^1 \frac{\epsilon \|z\|}{2\mu(t\|z\|)} dt \\
 &\leq |f(0)| + \int_0^\delta \frac{1}{\mu(t)} dt \|f\|_\mu + \int_\delta^{\|z\|} \frac{\epsilon}{2\mu(t)} dt \\
 &\leq \int_0^{\|z\|} \frac{1}{\mu(t)} dt \frac{\epsilon}{2} + \int_0^{\|z\|} \frac{1}{\mu(t)} dt \frac{\epsilon}{2} \\
 &= \int_0^{\|z\|} \frac{1}{\mu(t)} dt \epsilon,
 \end{aligned}$$

for all  $z$  with  $\tau < \|z\| < 1$ , that is

$$\lim_{\|z\| \rightarrow 1} \frac{|f(z)|}{\int_0^{\|z\|} \frac{1}{\mu(t)} dt} = 0.$$

□

**Proposition 1.** The natural little  $\mu$ -Bloch-type space  $\mathcal{B}_{\mu,nat,0}(\mathbb{B}_X)$  is a closed subspace of  $\mathcal{B}_{\mu,nat}(\mathbb{B}_X)$ .

**Proof.** The proof of this proposition is similar to the proof of [32] (Proposition 3.3). □

We now turn to describe the boundedness of  $M_\psi : \mathcal{B}_{\mu,nat,0}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega,nat,0}(\mathbb{B}_X)$ .

**Theorem 4.** Suppose  $\psi \in H(\mathbb{B}_X, \mathbb{C})$ .

(1) If  $\int_0^1 \frac{1}{\mu(t)} dt = \infty$ ,

$$M := \sup_{z \in \mathbb{B}_X} \omega(z) \|\psi'(z)\| \int_0^{\|z\|} \frac{1}{\mu(t)} dt < \infty \quad (41)$$

and

$$L := \sup_{z \in \mathbb{B}_X} \frac{\omega(z) |\psi(z)|}{\mu(z)} < \infty, \quad (42)$$

then the operator  $M_\psi : \mathcal{B}_{\mu,nat,0}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega,nat,0}(\mathbb{B}_X)$  is bounded.

(2) If  $\alpha > 0$  and the operator  $M_\psi : \mathcal{B}_{\alpha,nat,0}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega,nat,0}(\mathbb{B}_X)$  is bounded, then  $\psi \in \mathcal{B}_{\omega,nat,0}(\mathbb{B}_X)$  and

$$\sup_{z \in \mathbb{B}_X} \frac{\omega(z) |\psi(z)|}{(1 - \|z\|^2)^\alpha} < \infty. \quad (43)$$

**Proof.** (1) First assume that (41) and (42) holds. From Theorem 1, it follows that the operator  $M_\psi : \mathcal{B}_{\mu,nat}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega,nat}(\mathbb{B}_X)$  is bounded. Since  $\mathcal{B}_{\mu,nat,0}(\mathbb{B}_X) \subset \mathcal{B}_{\mu,nat}(\mathbb{B}_X)$ ,  $M_\psi : \mathcal{B}_{\mu,nat,0}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega,nat}(\mathbb{B}_X)$  is bounded. According to the Closed Graph Theorem, we only need to prove that  $M_\psi f \in \mathcal{B}_{\omega,nat,0}(\mathbb{B}_X)$  for all  $f \in \mathcal{B}_{\mu,0}(\mathbb{B}_X)$ . For an arbitrarily  $\epsilon > 0$ ,

if  $f \in \mathcal{B}_{\mu, \text{nat}, 0}(\mathbb{B}_X)$ , from  $\int_0^1 \frac{1}{\mu(t)} dt = \infty$  and Lemma 5, we have that there is a  $0 < \delta_1 < 1$  such that

$$\mu(z) \|f'(z)\| < \frac{\epsilon}{2L},$$

$$\left( \int_0^{\|z\|} \frac{1}{\mu(t)} dt \right)^{-1} |f(z)| < \frac{\epsilon}{2M},$$

for all  $z$  with  $\delta_1 < \|z\| < 1$ . Using (41) and (42), we get

$$\begin{aligned} & \omega(z) \|(M_\psi f)'(z)\| \\ & \leq \omega(z) |\psi(z)| \|f'(z)\| + \omega(z) |f(z)| \|\psi'(z)\| \\ & \leq \frac{\omega(z) |\psi(z)| \epsilon}{2L\mu(z)} + \omega(z) \|\psi'(z)\| \int_0^{\|z\|} \frac{1}{\mu(t)} dt \frac{\epsilon}{2M} \\ & < \epsilon \end{aligned} \quad (44)$$

for all  $z$  with  $\delta_1 < \|z\| < 1$ . From (44), we conclude that

$$\lim_{\|z\| \rightarrow 1} \omega(z) \|(M_\psi f)'(z)\| = 0.$$

Hence  $M_\psi f \in \mathcal{B}_{\omega, \text{nat}, 0}(\mathbb{B}_X)$  for all  $f \in \mathcal{B}_{\mu, \text{nat}, 0}(\mathbb{B}_X)$ . So,  $M_\psi : \mathcal{B}_{\mu, \text{nat}, 0}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega, \text{nat}, 0}(\mathbb{B}_X)$  is bounded.

(2) Suppose the operator  $M_\psi : \mathcal{B}_{\alpha, \text{nat}, 0}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega, \text{nat}, 0}(\mathbb{B}_X)$  is bounded. That means that  $M_\psi f \in \mathcal{B}_{\omega, \text{nat}, 0}(\mathbb{B}_X)$  for all  $f \in \mathcal{B}_{\alpha, \text{nat}, 0}(\mathbb{B}_X)$ . Choose  $f(z) = 1$ , then

$$\lim_{\|z\| \rightarrow 1} \omega(z) \|\psi'(z)\| = \lim_{\|z\| \rightarrow 1} \omega(z) \|(M_\psi f)'(z)\| = 0,$$

that is,  $\psi \in \mathcal{B}_{\omega, \text{nat}, 0}(\mathbb{B}_X)$ .

To prove (43) holds, fix  $a \in \mathbb{B}_X \setminus \{0\}$  and let  $\ell_a \in T(a)$  be fixed. We take the test function  $l_a : \mathbb{B}_X \rightarrow \mathbb{C}$ . Since

$$\|l'_a(z)\| \leq 1,$$

so  $l_a \in \mathcal{B}_{\alpha, \text{nat}, 0}(\mathbb{B}_X)$  with  $\|l_a\|_{\alpha, \text{nat}} \leq 1$ . Hence, using the proof of (27) in Theorem 3, condition (43) follows.  $\square$

In the following theorem, we consider the boundedness of the operator  $M_\psi : \mathcal{B}_{\alpha, \text{nat}, 0}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega, \text{nat}, 0}(\mathbb{B}_X)$ .

**Theorem 5.** *If  $0 < \alpha < 1$ , then the operator  $M_\psi : \mathcal{B}_{\alpha, \text{nat}, 0}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega, \text{nat}, 0}(\mathbb{B}_X)$  is bounded if and only if  $M_\psi : \mathcal{B}_{\alpha, \text{nat}, 0}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega, \text{nat}}(\mathbb{B}_X)$  is bounded,  $\psi \in \mathcal{B}_{\omega, \text{nat}, 0}(\mathbb{B}_X)$ ,*

$$L := \sup_{z \in \mathbb{B}_X} \frac{\omega(z) |\psi(z)|}{(1 - \|z\|^2)^\alpha} < \infty \quad (45)$$

and

$$\lim_{\|z\| \rightarrow 1} \omega(z) \|\psi'(z)\| \int_0^{\|z\|} \frac{1}{(1 - t^2)^\alpha} dt = 0. \quad (46)$$

**Proof.** If  $0 < \alpha < 1$  and the operator  $M_\psi : \mathcal{B}_{\alpha, \text{nat}, 0}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega, \text{nat}, 0}(\mathbb{B}_X)$  is bounded, then the operator  $M_\psi : \mathcal{B}_{\alpha, \text{nat}, 0}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega, \text{nat}}(\mathbb{B}_X)$  is bounded and  $\psi \in \mathcal{B}_{\omega, \text{nat}, 0}(\mathbb{B}_X)$ . Since  $l_a \in \mathcal{B}_{\alpha, \text{nat}, 0}(\mathbb{B}_X)$ , as in Theorem 4, (45) holds. Since

$$\int_0^1 \frac{1}{(1-t^2)^\alpha} dt < \infty,$$

(46) follows.

On the other hand, to prove the operator  $M_\psi : \mathcal{B}_{\alpha, \text{nat}, 0}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega, \text{nat}, 0}(\mathbb{B}_X)$  is bounded, we only need to prove that  $M_\psi f \in \mathcal{B}_{\omega, \text{nat}, 0}(\mathbb{B}_X)$ , for  $f \in \mathcal{B}_{\alpha, \text{nat}, 0}(\mathbb{B}_X)$ . For any  $\epsilon > 0$ , there is a  $0 < \delta_2 < 1$  such that

$$\begin{aligned} & \omega(z) \|(M_\psi f)'(z)\| \\ & \leq \omega(z) |\psi(z)| \|f'(z)\| + \omega(z) |f(z)| \|\psi'(z)\| \\ & \leq \frac{\omega(z) |\psi(z)| \epsilon}{2L(1-\|z\|^2)^\alpha} + M\omega(z) \|\psi'(z)\| \left( |f(0)| + \int_0^{\|z\|} \frac{1}{(1-t^2)^\alpha} dt \|f\|_{\alpha, \text{nat}} \right) \\ & < \epsilon + M\omega(z) \|\psi'(z)\| \left( |f(0)| + \int_0^{\|z\|} \frac{1}{(1-t^2)^\alpha} dt \|f\|_{\alpha, \text{nat}} \right), \end{aligned}$$

for  $\delta_2 < \|z\| < 1$ , condition  $\psi \in \mathcal{B}_{\omega, \text{nat}, 0}(\mathbb{B}_X)$  and (46) implies that  $M_\psi f \in \mathcal{B}_{\omega, \text{nat}, 0}(\mathbb{B}_X)$ . Thus the operator  $M_\psi : \mathcal{B}_{\alpha, \text{nat}, 0}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega, \text{nat}, 0}(\mathbb{B}_X)$  is bounded.  $\square$

In the following theorem, we characterize the boundedness of the operator  $M_\psi : \mathcal{B}_{\text{nat}, 0}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega, \text{nat}, 0}(\mathbb{B}_X)$ .

**Theorem 6.** The operator  $M_\psi : \mathcal{B}_{\text{nat}, 0}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega, \text{nat}, 0}(\mathbb{B}_X)$  is bounded if and only if the operator  $M_\psi : \mathcal{B}_{\text{nat}, 0}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega, \text{nat}}(\mathbb{B}_X)$  is bounded,  $\psi \in \mathcal{B}_{\omega, \text{nat}, 0}(\mathbb{B}_X)$ ,

$$K := \sup_{z \in \mathbb{B}_X} \omega(z) \|\psi'(z)\| \ln \frac{2}{1-\|z\|^2} < \infty \quad (47)$$

and

$$N := \sup_{z \in \mathbb{B}_X} \frac{\omega(z) |\psi(z)|}{1-\|z\|^2} < \infty. \quad (48)$$

**Proof.** If the operator  $M_\psi : \mathcal{B}_{\text{nat}, 0}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega, \text{nat}}(\mathbb{B}_X)$  is bounded,  $\psi \in \mathcal{B}_{\omega, \text{nat}, 0}(\mathbb{B}_X)$  and (47) and (48) holds, and we only need to prove that  $M_\psi f \in \mathcal{B}_{\omega, 0}(\mathbb{B}_X)$ , for  $f \in \mathcal{B}_{\text{nat}, 0}(\mathbb{B}_X)$ . If  $f \in \mathcal{B}_{\text{nat}, 0}(\mathbb{B}_X)$ , then for every  $\epsilon > 0$ , there exists a  $\delta > 0$  such that

$$\|f'(z)\| < \frac{\epsilon}{3(1-\|z\|^2)} \quad (49)$$

and

$$\frac{|f(0)| + \ln 2 \|f\|_{\text{nat}}}{\ln \frac{2}{1-\|z\|^2}} < \frac{\epsilon}{3}, \quad (50)$$

for all  $z$  with  $\delta < \|z\| < 1$ . Using the following limit again

$$\lim_{\|z\| \rightarrow 1} \frac{1}{\ln \frac{2}{1-\|z\|^2}} = 0,$$

there is a  $\tau \in (\delta, 1)$ , such that

$$\frac{1}{\ln \frac{2}{1-\|z\|^2}} < \frac{\epsilon}{3 \ln \frac{2}{1-\delta} \|f\|_{nat}}, \quad (51)$$

for all  $z$  with  $\tau < \|z\| < 1$ . By (49)–(51) we have

$$\begin{aligned} |f(z)| &\leq |f(0)| + \int_0^1 \|f'(tz)\| \|z\| dt \\ &= |f(0)| + \int_0^{\frac{\delta}{\|z\|}} \|f'(tz)\| \|z\| dt + \int_{\frac{\delta}{\|z\|}}^1 \|f'(tz)\| \|z\| dt \\ &\leq |f(0)| + \|z\| \int_0^{\frac{\delta}{\|z\|}} \frac{\|f\|_{nat}}{1-\|tz\|^2} dt + \int_{\frac{\delta}{\|z\|}}^1 \frac{\epsilon \|z\|}{3(1-t^2\|z\|^2)} dt \\ &\leq |f(0)| + \frac{1}{2} \ln \frac{1+\delta}{1-\delta} \|f\|_{nat} + \frac{1}{2} \ln \frac{1+\|z\|}{1-\|z\|} \frac{\epsilon}{3} \\ &\leq |f(0)| + \ln \frac{2}{1-\delta} \|f\|_{nat} + \ln \frac{2}{1-\|z\|^2} \frac{\epsilon}{3} \\ &< \ln \frac{2}{1-\|z\|^2} \frac{\epsilon}{3} + \ln \frac{2}{1-\|z\|^2} \frac{\epsilon}{3} + \ln \frac{2}{1-\|z\|^2} \frac{\epsilon}{3} \\ &= \ln \frac{2}{1-\|z\|^2} \epsilon, \end{aligned}$$

and

$$\begin{aligned} \omega(z) \|(M_\psi f)'(z)\| &\leq \omega(z) |\psi(z)| \|f'(z)\| + \omega(z) |f(z)| \|\psi'(z)\| \\ &\leq \frac{\omega(z) |\psi(z)| \epsilon}{2N(1-\|z\|^2)} + \omega(z) \|\psi'(z)\| \ln \frac{2}{1-\|z\|^2} \epsilon \\ &< \epsilon + K\epsilon, \end{aligned}$$

for all  $z$  with  $\tau < \|z\| < 1$ , so,  $M_\psi f \in \mathcal{B}_{\omega, nat, 0}(\mathbb{B}_X)$ . Thus the operator  $M_\psi : \mathcal{B}_{nat, 0}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega, nat, 0}(\mathbb{B}_X)$  is bounded.

If the operator  $M_\psi : \mathcal{B}_{nat, 0}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega, nat, 0}(\mathbb{B}_X)$  is bounded, then  $M_\psi : \mathcal{B}_{nat, 0}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega, nat}(\mathbb{B}_X)$  is bounded. Using the proof of Theorem 4,  $\psi \in \mathcal{B}_{\omega, nat, 0}(\mathbb{B}_X)$  and (48) holds.

To prove (47) holds, we take the test function  $g_a : \mathbb{B}_X \rightarrow \mathbb{C}$  given by (20) and get

$$\begin{aligned} \|g'_a(z)\| &= \left\| \frac{2\|a\|\ell_a}{1-\|a\|\ell_a(z)} - \frac{2 \ln \frac{2}{1-\|a\|\ell_a(z)}}{\ln \frac{2}{1-\|a\|^2}} \frac{\|a\|\ell_a}{1-\|a\|\ell_a(z)} \right\| \\ &\leq \frac{2\|\ell_a\|}{1-\|a\|} + \left| \frac{2 \ln \frac{2}{1-\|a\|\ell_a(z)}}{\ln \frac{2}{1-\|a\|^2}} \right| \frac{2\|\ell_a\|}{1-\|a\|} \\ &\leq \frac{2}{1-\|a\|} + \frac{4 \left( \ln \left| \frac{2}{1-\|a\|\ell_a(z)} \right| + \frac{\pi}{2} \right)}{(1-\|a\|) \ln \frac{2}{1-\|a\|^2}} \\ &\leq \frac{2}{1-\|a\|} + \frac{4 \left( \ln \frac{4}{1-\|a\|^2} + \frac{\pi}{2} \right)}{(1-\|a\|) \ln \frac{2}{1-\|a\|^2}} \\ &\leq \frac{2}{1-\|a\|} + \frac{4}{1-\|a\|} \left( 2 + \frac{\pi}{2 \ln 2} \right) \\ &\leq \frac{C}{1-\|a\|^2}, \text{ for } z \in \mathbb{B}_X, \end{aligned}$$

so  $g_a \in \mathcal{B}_{nat,0}(\mathbb{B}_X)$ . Using the proof of (12) in Theorem 2, condition (47) holds.  $\square$

We characterize the boundedness of the operator  $M_\psi : \mathcal{B}_{\alpha,nat,0}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega,nat,0}(\mathbb{B}_X)$  ( $\alpha > 1$ ).

**Theorem 7.** *If  $\alpha > 1$ , then the operator  $M_\psi : \mathcal{B}_{\alpha,nat,0}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega,nat,0}(\mathbb{B}_X)$  is bounded if and only if the operator  $M_\psi : \mathcal{B}_{\alpha,nat,0}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega,nat}(\mathbb{B}_X)$  is bounded,  $\psi \in \mathcal{B}_{\omega,nat,0}(\mathbb{B}_X)$ ,*

$$P := \sup_{z \in \mathbb{B}_X} \omega(z) \|\psi'(z)\| \frac{1}{(1 - \|z\|^2)^{\alpha-1}} < \infty \quad (52)$$

and

$$Q := \sup_{z \in \mathbb{B}_X} \frac{\omega(z) |\psi(z)|}{(1 - \|z\|^2)^\alpha} < \infty. \quad (53)$$

**Proof.** In terms of sufficiency, if the operator  $M_\psi : \mathcal{B}_{\alpha,nat,0}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega,nat}(\mathbb{B}_X)$  is bounded,  $\psi \in \mathcal{B}_{\omega,nat,0}(\mathbb{B}_X)$  and (52) and (53) holds; we only need to prove that  $M_\psi f \in \mathcal{B}_{\omega,nat,0}(\mathbb{B}_X)$ , for  $f \in \mathcal{B}_{\alpha,nat,0}(\mathbb{B}_X)$ . If  $f \in \mathcal{B}_{\alpha,nat,0}(\mathbb{B}_X)$ , then for every  $\epsilon > 0$ , there is a  $\delta > 0$  such that

$$\|f'(z)\| < \frac{\epsilon}{2Q(1 - \|z\|^2)^\alpha} \quad (54)$$

for all  $z$  with  $\delta < \|z\| < 1$ .

Using (54) we deduce

$$\begin{aligned} |f(z)| &\leq |f(0)| + \int_0^1 \|f'(tz)\| \|z\| dt \\ &\leq |f(0)| + \|z\| \int_0^{\frac{\delta}{\|z\|}} \frac{\|f\|_\alpha}{(1 - \|tz\|^2)^\alpha} dt + \int_{\frac{\delta}{\|z\|}}^1 \frac{\epsilon \|z\|}{2Q(1 - t^2 \|z\|^2)^\alpha} dt \\ &\leq |f(0)| + \int_0^\delta \frac{\|f\|_\alpha}{(1 - \|t\|)^\alpha} dt + \int_\delta^{\|z\|} \frac{\epsilon}{2Q(1 - t)^\alpha} dt \\ &\leq |f(0)| + \frac{\delta}{(1 - \delta)^\alpha} \|f\|_\alpha + \frac{\epsilon}{2Q(1 - \delta)^{\alpha-1}} + \frac{\epsilon}{2Q(\alpha - 1)(1 - \|z\|)^{\alpha-1}} \\ &\leq |f(0)| + \frac{\delta}{(1 - \delta)^\alpha} \|f\|_\alpha + \frac{1}{2Q(1 - \delta)^{\alpha-1}} + \frac{\epsilon}{2Q(\alpha - 1)(1 - \|z\|)^{\alpha-1}} \\ &= C_1 + \frac{\epsilon}{2Q(\alpha - 1)(1 - \|z\|^2)^{\alpha-1}}, \end{aligned}$$

with  $C_1 = |f(0)| + \frac{\delta}{(1 - \delta)^\alpha} \|f\|_\alpha + \frac{1}{2Q(1 - \delta)^{\alpha-1}}$  and

$$\begin{aligned} &\omega(z) \|(M_\psi f)'(z)\| \\ &\leq \omega(z) |\psi(z)| \|f'(z)\| + \omega(z) |f(z)| \|\psi'(z)\| \\ &\leq \frac{\omega(z) |\psi(z)| \epsilon}{2Q(1 - \|z\|^2)^\alpha} + \omega(z) \|\psi'(z)\| \left( C_1 + \frac{\epsilon}{2Q(\alpha - 1)(1 - \|z\|^2)^{\alpha-1}} \right) \\ &\leq \epsilon + \omega(z) \|\psi'(z)\| \left( C_1 + \frac{\epsilon}{2Q(\alpha - 1)(1 - \|z\|^2)^{\alpha-1}} \right) \\ &\leq \epsilon + C_1 \omega(z) \|\psi'(z)\| + \sup_{z \in \mathbb{B}_X} \omega(z) \|\psi'(z)\| \frac{1}{(1 - \|z\|^2)^{\alpha-1}} \frac{\epsilon}{2Q(\alpha - 1)} \\ &\leq \epsilon + \frac{P}{2Q(\alpha - 1)} \epsilon + C_1 \omega(z) \|\psi'(z)\|, \end{aligned}$$

for all  $z$  with  $\delta < \|z\| < 1$ . By the condition  $\psi \in \mathcal{B}_{\omega, \text{nat}, 0}(\mathbb{B}_X)$ , we have

$$\lim_{\|z\| \rightarrow 1} \omega(z) \|(M_\psi f)'(z)\| = 0,$$

so  $M_\psi f \in \mathcal{B}_{\omega, \text{nat}, 0}(\mathbb{B}_X)$ . Thus the operator  $M_\psi : \mathcal{B}_{\alpha, \text{nat}, 0}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega, \text{nat}, 0}(\mathbb{B}_X)$  is bounded.

In terms of necessity, if the operator  $M_\psi : \mathcal{B}_{\alpha, \text{nat}, 0}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega, \text{nat}, 0}(\mathbb{B}_X)$  is bounded, then  $M_\psi : \mathcal{B}_{\alpha, \text{nat}, 0}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\omega, \text{nat}}(\mathbb{B}_X)$  is bounded. By the proof of (43) in Theorem 4,  $\psi \in \mathcal{B}_{\omega, \text{nat}, 0}(\mathbb{B}_X)$  and (53) holds.

To prove that (52) holds, we take the test function  $h_a : \mathbb{B}_X \rightarrow \mathbb{C}$  given by (31) and have for  $z \in \mathbb{B}_X$ ,

$$\|h'_a(z)\| \leq \frac{(\alpha - 1)\|a\|}{|1 - \|a\|\ell_a(z)|^\alpha} \leq \frac{(\alpha - 1)\|a\|}{|1 - \|a\||^\alpha}, \quad (55)$$

It follows from (55) that

$$\lim_{\|z\| \rightarrow 1} (1 - \|z\|^2)^\alpha \|h'_a(z)\| = 0,$$

so  $h_a \in \mathcal{B}_{\alpha, \text{nat}, 0}(\mathbb{B}_X)$ . Using the proof of (28) in Theorem 3, condition (52) follows. This completes the proof of Theorem 7.  $\square$

Finally, we provide a sufficient condition of the boundedness of the operator  $M_\psi : \mathcal{B}_{\text{ln}, \text{nat}}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\text{nat}}(\mathbb{B}_X)$ .

**Theorem 8.** (1) If

$$\sup_{z \in \mathbb{B}_X} \frac{|\psi(z)|}{\ln \frac{e}{1 - \|z\|}} < \infty, \quad (56)$$

and

$$\sup_{z \in \mathbb{B}_X} (1 - \|z\|^2) \|\psi'(z)\| \ln \ln \frac{6}{1 - \|z\|^2} < \infty. \quad (57)$$

Then the operator  $M_\psi : \mathcal{B}_{\text{ln}, \text{nat}}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\text{nat}}(\mathbb{B}_X)$  is bounded.

(2) If the operator  $M_\psi : \mathcal{B}_{\text{ln}, \text{nat}}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\text{nat}}(\mathbb{B}_X)$  is bounded, then (56) holds.

**Proof.** (1) Suppose that (56) and (57). Let  $f \in \mathcal{B}_{\text{ln}, \text{nat}}(\mathbb{B}_X)$ . Then by (4) and Lemma 4, we get

$$\begin{aligned} \|M_\psi f\|_{\text{nat}} &= \sup_{z \in \mathbb{B}_X} (1 - \|z\|^2) \|(\psi f)'(z)\| \\ &\leq \sup_{z \in \mathbb{B}_X} \frac{|\psi(z)|}{\ln \frac{e}{1 - \|z\|}} \|f\|_{\text{ln}} + C \sup_{z \in \mathbb{B}_X} (1 - \|z\|^2) \|\psi'(z)\| \ln \ln \frac{6}{1 - \|z\|^2} \|f\|_{\text{ln}}, \end{aligned} \quad (58)$$

and

$$|M_\psi f(0)| = |\psi(0)f(0)| \leq |\psi(0)| \|f\|_{\text{ln}}. \quad (59)$$

Applying conditions (56)–(59), we can conclude the operator  $M_\psi : \mathcal{B}_{\text{ln}, \text{nat}}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\text{nat}}(\mathbb{B}_X)$  is bounded.

(2) For  $a \in \mathbb{B}_X \setminus \{0\}$ , using  $l_a \in \mathcal{B}_{\ln, \text{nat}}(\mathbb{B}_X)$  and  $\psi \in \mathcal{B}_{\text{nat}}(\mathbb{B}_X)$ , we get

$$\begin{aligned} (1 - \|z\|^2) \|l_a\| |\psi(z)| &= (1 - \|z\|^2) \|l'_a(z)\| |\psi(z)| \leq \sup_{z \in \mathbb{B}_X} (1 - \|z\|^2) \|\psi(z) l'_a(z)\| \\ &\leq \sup_{z \in \mathbb{B}_X} (1 - \|z\|^2) \|\psi(z) l'_a(z) + l_a(z) \psi'(z)\| + \sup_{z \in \mathbb{B}_X} (1 - \|z\|^2) \|\psi'(z)\| \|l_a(z)\| \\ &\leq \|M_\psi l_a\|_{\text{nat}} + \sup_{z \in \mathbb{B}_X} (1 - \|z\|^2) \|\psi'(z)\| \\ &\leq \|M_\psi\| + \|\psi\|_{\text{nat}}. \end{aligned}$$

From which we have

$$\begin{aligned} &\sup_{z \in \mathbb{B}_X} \frac{|\psi(z)|}{\ln \frac{e}{1-\|z\|}} \\ &\leq \sup_{\|z\| \leq r} \frac{|\psi(z)|}{\ln \frac{e}{1-\|z\|}} + \frac{1}{(1-r^2)} \sup_{r < \|z\| < 1} \frac{(1 - \|z\|^2) |\psi(z)|}{\ln \frac{e}{1-\|z\|}} \\ &\leq \sup_{\|z\| \leq r} \frac{|\psi(z)|}{\ln \frac{e}{1-\|z\|}} + \frac{1}{(1-r^2) \ln \frac{e}{1-r}} \sup_{r < \|z\| < 1} (1 - \|z\|^2) |\psi(z)| \\ &\leq \sup_{\|z\| \leq r} \frac{|\psi(z)|}{\ln \frac{e}{1-\|z\|}} + \frac{1}{(1-r^2)} (\|M_\psi\| + \|\psi\|_{\text{nat}}) \\ &< \infty, \end{aligned}$$

that is, (56) holds.  $\square$

**Remark 3.** At the moment, we are not sure if the boundedness of the operator  $M_\psi : \mathcal{B}_{\ln, \text{nat}}(\mathbb{B}_X) \rightarrow \mathcal{B}_{\text{nat}}(\mathbb{B}_X)$  implies that condition (57) holds. Hence, for interested readers, we leave this as an open problem.

## 4. Conclusions

There has been significant interest in the operators on subspaces of  $H(\mathbb{B}_X)$ . We provide the definition of the natural  $\mu$ -Bloch-type space  $\mathcal{B}_{\mu, \text{nat}}(\mathbb{B}_X)$ . Thus, our aspiration is that this exposition will spur on further research and activity in this field. In the current work, our aim is to investigate the boundedness of the multiplication operators from  $\mathcal{B}_{\alpha, \text{nat}}(\mathbb{B}_X)$  (or  $\mathcal{B}_{\alpha, \text{nat}, 0}(\mathbb{B}_X)$ ) into  $\mathcal{B}_{\omega, \text{nat}}(\mathbb{B}_X)$  (or  $\mathcal{B}_{\omega, \text{nat}, 0}(\mathbb{B}_X)$ ) on the unit ball  $\mathbb{B}_X$ . This serves as an excellent foundation for discussion and further inquiry. Naturally, collaborating with operators on the unit ball of Banach spaces  $X$  presents certain challenges, in contrast to working with multiplication operators on the subspace comprising all holomorphic functions within the open unit disc or the unit ball. This is mainly because the test function in the natural  $\alpha$ -Bloch space  $\mathcal{B}_{\alpha, \text{nat}}(\mathbb{B}_X)$  or  $\mathcal{B}_{\alpha, \text{nat}, 0}(\mathbb{B}_X)$  ( $\alpha > 0$ ) is not easy to obtain. The methods, ideas and tricks presented here, with some modifications, can be used in some other settings, which should lead to some further investigations in this direction. For future research, one possible direction would be to investigate the compactness of the multiplication operators from  $\mathcal{B}_{\alpha, \text{nat}}(\mathbb{B}_X)$  (or  $\mathcal{B}_{\alpha, \text{nat}, 0}(\mathbb{B}_X)$ ) into  $\mathcal{B}_{\omega, \text{nat}}(\mathbb{B}_X)$  (or  $\mathcal{B}_{\omega, \text{nat}, 0}(\mathbb{B}_X)$ ).

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