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$\mathcal{O}$ -Regular Mappings on $\mathbb{C}(\mathbb{C})$: A Structured Operator–Theoretic Framework

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Abstract

Motivation. Analytic function theory on commutative complex extensions calls for an operator–theoretic calculus that simultaneously sees the algebra-induced coupling among components and supports boundary-to-interior mechanisms. **Gap.** While Dirac-type frameworks are classical in several complex variables and Clifford analysis, a coherent calculus aligning structural CR systems, a canonical first derivative, and a Cauchy-type boundary identity on the commutative model $\mathbb{C}(\mathbb{C}) \cong \mathbb{C}^4$ has not been systematically developed. **Purpose and Aims.** This paper develops such a calculus for $\mathcal{O}$ -regular mappings on $\mathbb{C}(\mathbb{C})$ and establishes three pillars of the theory. **Main Results.** (i) A fully coupled Cauchy–Riemann system characterizing $\mathcal{O}$ -regularity; (ii) identification of a canonical first derivative $g'(z) = \partial_{x_0} g(z)$; and (iii) a Stokes-driven boundary annihilation law $\int_{\partial\Omega} \tau g = 0$ for a canonical 7-form τ . On (pseudo)convex domains, $\bar{\partial}$ -methods yield solvability under natural compatibility and regularity assumptions. Stability (under algebra-preserving maps), Liouville-type, and removability results are also obtained, and function spaces suited to this algebra are outlined. **Significance.** The results show that a large portion of the classical holomorphic toolkit survives, in algebra-aware form, on $\mathbb{C}(\mathbb{C})$.

Keywords: $\mathcal{O}$ -regular functions; Dirac-type operators; coupled Cauchy–Riemann system; Cauchy–Stokes boundary identity; Borel–Pompeiu; $\bar{\partial}$ methods; several complex variables; hypercomplex/commutative algebras; boundary integral methods

MSC: 32A99; 32W50; 30G35; 11E88



Academic Editor: Raul E. Curto

Received: 27 August 2025

Revised: 22 September 2025

Accepted: 9 October 2025

Published: 18 October 2025

Citation: Kim, J.E. $\mathcal{O}$ -Regular Mappings on $\mathbb{C}(\mathbb{C})$: A Structured Operator–Theoretic Framework. *Mathematics* **2025**, *13*, 3328. <https://doi.org/10.3390/math13203328>

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1. Introduction

The function theory of several complex variables and of hypercomplex settings share a common engine: Dirac-type operators encode Cauchy–Riemann structures and boundary-to-interior mechanisms via Stokes’ theorem. In this paper, we carry this paradigm to a commutative complex algebra model of $\mathbb{C}^4$, written $\mathbb{C}(\mathbb{C})$, and we show that a large part of the classical holomorphic toolkit admits a clean, algebra-aware extension. Our structural equations are not ad hoc: the coupling pattern is the exact analytic footprint of the multiplication in $\mathbb{C}(\mathbb{C})$, and it drives both boundary identities and interior solvability via $\bar{\partial}$ methods (cf. Range, Hörmander). In this way, we place differential characterizations, boundary annihilation, and existence on domains of holomorphy into a single operator calculus tailored to the algebra.

Concretely, we work in the commutative subalgebra $\mathbb{C}(\mathbb{C})$ with basis $\{e_0, \dots, e_7\}$ and identify it with $\mathbb{C}^4$ via the global coordinate map:

$$z = z_1 + z_2 e_2 + z_3 e_4 + z_4 e_6, \quad z_i \in \mathbb{C}.$$

The induced multiplication rules (recalled in Section 2) couple the four complex components and will be used throughout for differential and integral constructions.

An $\mathbb{C}(\mathbb{C})$ -valued mapping,

$$g = g_1 + g_2 e_2 + g_3 e_4 + g_4 e_6, \quad g_i : \Omega \subset \mathbb{C}^4 \rightarrow \mathbb{C},$$

is called $\mathcal{O}$ -regular on an open set Ω if $g \in \mathcal{C}^1(\Omega)$, and it is annihilated by a Dirac-type operator $\mathcal{D}^*$ intrinsic to the algebra (see Section 2). Equivalently, g satisfies a fully coupled CR system for its components; in particular,

$$\partial_{z_1} g_1 + \partial_{z_2} g_2 + \partial_{z_3} g_3 + \partial_{z_4} g_4 = 0 \quad (\text{CR1})$$

$$\partial_{z_1} g_2 + \partial_{z_2} g_1 + \partial_{z_3} g_4 + \partial_{z_4} g_3 = 0 \quad (\text{CR2})$$

$$\partial_{z_1} g_3 + \partial_{z_2} g_4 + \partial_{z_3} g_1 + \partial_{z_4} g_2 = 0 \quad (\text{CR3})$$

$$\partial_{z_1} g_4 + \partial_{z_2} g_3 + \partial_{z_3} g_2 + \partial_{z_4} g_1 = 0. \quad (\text{CR4})$$

Remark 1. We avoid the ambiguous phrase “two cyclic companions” by writing (CR1)–(CR4) explicitly; all subsequent references use these labels. This parallels the classical equivalence between holomorphicity and the CR equations, but now in a four-component, algebra-coupled setting.

Dirac-type operators encode Cauchy–Riemann structures and connect interior equations to boundary identities through Stokes’ theorem. In a commutative complex algebra model of $\mathbb{C}^4$, written $\mathbb{C}(\mathbb{C})$, the coupling pattern is the analytic footprint of the multiplication. A Dirac-type pair (D, D^*) intrinsic to $\mathbb{C}(\mathbb{C})$ yields a coupled CR system, a canonical first derivative, and a boundary identity.

Main contributions. The paper establishes three core results that frame the analysis of $\mathcal{O}$ -regular mappings:

- *Differential characterization.* For $\mathcal{O}$ -regular g , we identify a canonical first derivative compatible with the algebra structure and prove that the intrinsic derivative coincides with the x_0 -directional derivative of g in the z_1 -plane, i.e., $g'(z) = \frac{\partial g}{\partial x_0}(z)$. This provides a concrete handle on first-order behavior used in later regularity arguments.
- *Cauchy–Stokes-type boundary identity.* Introducing a canonical 7-form τ built from the differentials dz_i , we prove that $\mathcal{O}$ -regular functions satisfy the boundary annihilation law $\int_{\partial\Omega} \tau g = 0$ for sufficiently smooth $\Omega \Subset \Omega_0$. This plays the role of a Cauchy theorem in our setting.
- *Existence on domains of holomorphy.* On domains of holomorphy in $\mathbb{C}(\mathbb{C})$, appropriate $\mathcal{C}^2$ data for $(g_1, \dots, g_4)$ yield $\mathcal{O}$ -regular solutions via a $\bar{\partial}$ -type construction, establishing solvability of (CR1)–(CR4) under natural hypotheses.

Beyond clarifying the analytic meaning of $\mathcal{D}^*g = 0$ on $\mathbb{C}(\mathbb{C})$, these results supply an essential toolkit for developing potential theory and representation formulas in this algebra: a workable first derivative, a robust boundary identity, and a solvability statement tied to holomorphic convexity. Together, they show that $\mathcal{O}$ -regularity is not merely a formal generalization but a genuinely analytic notion capable of supporting a Cauchy-type calculus.

The present study sits at the intersection of several complex variables, hypercomplex function theory, and Clifford analysis. Classical pillars in several complex variables—notably the integral representation and $\bar{\partial}$ methods of Range and Hörmander—provide the conceptual backdrop for our boundary identities and solvability statements [1,2]. On the hypercomplex side, the Dirac-operator framework pioneered in Clifford analysis and its

monographs gives a robust differential-operator language in which Cauchy–Riemann (CR)-type systems, Cauchy theorems, and representation formulas can be formulated beyond $\mathbb{C}$ [3–5]. Within quaternionic analysis specifically, foundational contributions of Fueter and Sudbery established the analytic scope of quaternionic (and para/quasi-quaternionic) structures, while Gürlebeck and Sprößig developed engineering- and physics-oriented calculus on these algebras [6–8].

A closely related stream is the multicomplex/bicomplex literature, where one works over commutative complex extensions and studies holomorphicity with several coupled complex components. Price’s early monograph framed multicomplex spaces as natural arenas for generalizing holomorphic techniques [9], and Luna-Elizarrarás–Shapiro–Struppa–Vajiac developed a comprehensive analysis on the bicomplex numbers, including structural results and examples that clarify how algebraic coupling propagates to CR-type systems and integral formulas [10–12]. Our setting—a commutative complex algebra realized as a four-component model of $\mathbb{C}^4$ —is philosophically aligned with this program: the algebraic multiplication prescribes the off-diagonal mixing in the structural system and guides the form of admissible kernels and boundary forms.

Integral representation theory forms a second axis of prior work. In the Clifford-analytic direction, higher-order Borel–Pompeiu formulas and their discrete/fractional extensions demonstrate that Cauchy- and Stokes-type mechanisms can be adapted to a range of operator factorizations and latticized geometries [13,14]. The boundary identity we employ—a Stokes-driven annihilation of a canonical differential form against $\mathcal{O}$ -regular maps—is designed in the same spirit: it packages the algebra-induced coupling into a differential form tailored to the structural operator, thereby yielding a Cauchy-type theorem compatible with the underlying algebra.

In hyperbolic and split settings, Cauchy kernels and integral formulas have also been constructed, clarifying how signatures other than $(+, +)$ enter analytic representation theory [15,16]. Recent work on split-quaternionic models shows that algebraic algorithms and operator factorizations can be instrumental in PDE applications, e.g., for Schrödinger-type dynamics [17,18]. Although our focus here is not on hyperbolic metrics per se, these developments motivate, at a methodological level, our use of algebra-aware operators and forms to control boundary and interior behavior.

A parallel multi-variable track in quaternionic analysis is the theory of slice regularity in several quaternionic variables, where Almansi-type decompositions and structure theorems for zero sets have recently been advanced [19,20]. These results underscore that analytic function theory on higher-rank algebras can sustain multi-variable phenomena (factorizations, uniqueness, and growth properties) reminiscent of several complex variables. Complementarily, discrete models for conjugate–harmonic pairs suggest that hypercomplex analysis remains robust under discretization, with potential implications for numerical schemes and lattice-based PDE solvers [21].

Positioning of the present work. Guided by the operator–theoretic viewpoint from Clifford analysis [3,4] and the commutative multi-component perspective of bicomplex/multicomplex function theory [9,11], we use an intrinsic Dirac-type operator to define and characterize $\mathcal{O}$ -regularity on a commutative complex algebra model of $\mathbb{C}^4$. The structural system we obtain parallels the coupled CR systems found in bicomplex analysis [10], while our boundary identity is conceived in the spirit of Borel–Pompeiu/Cauchy frameworks adapted to algebra-dependent operators [13,14]. In this way, the paper synthesizes several established lines—integral representation from several complex variables [1,2], Dirac-based formalisms [3,4], and commutative hypercomplex models [9,11]—to yield a coherent analytic calculus for $\mathcal{O}$ -regular mappings, with an eye toward representation formulas and PDE-oriented applications suggested in the split/hyperbolic literature [15–18].

Organization. Section 2 reviews the algebraic model $\mathbb{C}(\mathbb{C}) \cong \mathbb{C}^4$, its multiplication, and the Dirac pair $(\mathcal{D}, \mathcal{D}^*)$ used throughout. Section 3 develops the $\mathcal{O}$ -regular class, proves the differential identity $g'(z) = \partial g / \partial x_0$, and establishes the boundary integral formula $\int_{\partial\Omega} \tau g = 0$. Section 4 constructs the differential form τ and proves the boundary annihilation law. Section 5 discusses existence on domains of holomorphy and implications for further representation results.

Related Work in Commutative Algebras

Derivative notions and analytic/monogenic function theories in commutative (finite- or Banach-) algebras have a long lineage: Ketchum's [22] analytic functions of hyper-complex variables; Scheffers' [23] early generalization of complex function foundations; Lorch [24] and Blum's [25] Banach-algebra framework; Gateaux-based approaches in Mel'nichenko [26] and in Shpakivskyi's [27] constructive descriptions; and the comprehensive monograph of Plaksa-Shpakivskyi [28]. Our operator-coupled CR system and boundary form are compatible with this program; the idempotent normal form shows that $\mathbb{C}(\mathbb{C})$ is semisimple and allows a representation viewpoint akin to classical results (see also Ringleb [29] for early representation theorems).

2. Preliminaries

The two-dimensional matrix algebra $M(2; \mathbb{C})$ is defined over the field $\mathbb{C}$ of complex numbers, which is generated by three basis elements denoted as e_i for $i = 0, 1, 2$. It is specified that

$$\begin{aligned} e_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, e_1 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\ e_3 &= \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, e_5 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \\ e_6 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, e_7 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

The following rules are to be observed:

$$\begin{aligned} e_0^2 &= e_2^2 = e_4^2 = e_6^2 = 1, \quad e_1^2 = e_3^2 = e_5^2 = e_7^2 = -1, \\ e_1 e_2 &= e_3 = e_2 e_1, \quad e_2 e_4 = e_6 = e_4 e_2, \quad e_3 e_4 = e_7 = e_4 e_3, \quad e_1 e_6 = e_7 = e_6 e_1, \\ e_2 e_5 &= e_7 = e_5 e_2, \quad e_1 e_4 = e_5 = e_4 e_1, \quad e_3 e_5 = -e_6 = e_5 e_3. \end{aligned}$$

This mathematical structure serves as a commutative subalgebra within the context of matrix space $M(4; \mathbb{C})$, allowing for analysis and manipulation of matrices with complex entries. The algebra can be expressed in a more compact and structured form:

$$\mathbb{C}(\mathbb{C}) = \{z = z_1 + z_2 e_2 + z_3 e_4 + z_4 e_6 \mid z_i \in \mathbb{C}, i = 1, 2, 3, 4\},$$

where z_1, z_2, z_3 , and z_4 are specifically defined as follows:

$$z_1 = e_0x_0 + e_1x_1, \quad z_2 = e_0x_2 + e_1x_3,$$

$$z_3 = e_0x_4 + e_1x_5, \quad z_4 = e_0x_6 + e_1x_7.$$

In this representation, each z_i reflects a linear combination of the basis elements e_0 and e_1 multiplied by real coefficients x_i .

Furthermore, it is important to note that the algebra $\mathbb{C}(\mathbb{C})$ can be identified with the four-dimensional complex vector space $\mathbb{C}^4$. This identification facilitates understanding the structure and dimensions of the algebra and allows for operations analogous to those performed in standard vector spaces. Through this mapping, various properties of linearity, dimensionality, and algebraic operations can be explored within the realm of complex matrices and algebraic systems.

The multiplication of $z = z_1 + z_2e_2 + z_3e_4 + z_4e_6$ and $w = w_1 + w_2e_2 + w_3e_4 + w_4e_6$ is defined by

$$\begin{aligned} zw &= \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ z_2 & z_1 & z_4 & z_3 \\ z_3 & z_4 & z_1 & z_2 \\ z_4 & z_3 & z_2 & z_1 \end{pmatrix} \begin{pmatrix} w_1 & w_2 & w_3 & w_4 \\ w_2 & w_1 & w_4 & w_3 \\ w_3 & w_4 & w_1 & w_2 \\ w_4 & w_3 & w_2 & w_1 \end{pmatrix} \\ &= \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_1 & a_4 & a_3 \\ a_3 & a_4 & a_1 & a_2 \\ a_4 & a_3 & a_2 & a_1 \end{pmatrix}, \end{aligned}$$

We define the multiplication operation for the elements z and w as follows: let $z = z_1 + z_2e_2 + z_3e_4 + z_4e_6$ and $w = w_1 + w_2e_2 + w_3e_4 + w_4e_6$. The product, denoted as zw , is determined by the following equations:

$$(z_1, z_2, z_3, z_4) \odot (w_1, w_2, w_3, w_4) = (a_1, a_2, a_3, a_4), \quad (1)$$

$$\begin{aligned} a_1 &= z_1w_1 + z_2w_2 + z_3w_3 + z_4w_4, \\ a_2 &= z_1w_2 + z_2w_1 + z_3w_4 + z_4w_3, \\ a_3 &= z_1w_3 + z_2w_4 + z_3w_1 + z_4w_2, \\ a_4 &= z_1w_4 + z_2w_3 + z_3w_2 + z_4w_1. \end{aligned} \quad (2)$$

The conjugate of a complex number, denoted as z^* , and its norm, represented as $\|z\|$, are defined as follows:

$$z^* = \overline{z_1} + \overline{z_2}e_2 + \overline{z_3}e_4 + \overline{z_4}e_6$$

where $\overline{z_1} = e_0x_0 - e_1x_1$, $\overline{z_2} = e_0x_2 - e_1x_3$, $\overline{z_3} = e_0x_4 - e_1x_5$ and $\overline{z_4} = e_0x_6 - e_1x_7$,

$$\|z\| = \frac{1}{2} \sqrt{\text{tr}(zz^*)}.$$

The conjugate of a complex number, which is commonly denoted as z^* , is a mathematical operation that involves flipping the sign of the imaginary part of the complex number. For a complex number expressed in the standard form $z = a + bi$, where a is the real part, and b is the imaginary part, the conjugate is given by $z^* = a - bi$. In addition to the conjugate, we also have the concept of the norm of a complex number, represented as $\|z\|$. The norm, also known as the modulus, is a measure of the magnitude or size of the complex number

and is calculated using the formula $||z|| = \sqrt{a^2 + b^2}$. This calculation effectively provides the distance of the complex number from the origin in a two-dimensional plane, where the real part corresponds to the x -axis and the imaginary part corresponds to the y -axis.

In this paper, we will examine two different differential operators that are important in analyzing various mathematical and physical problems. We will explore their definitions, properties, and uses, giving a detailed look at how these operators work within the context of differential equations. By studying their effects on different functions, we aim to identify the unique characteristics and behaviors of each operator:

2.1. Notation and Wirtinger Derivatives

2.1.1. Notation and Differentiability

We work on $\Omega \subset \mathbb{C}^4 \cong \mathbb{R}^8$ with real coordinates (x_j, y_j) , $z_j = x_j + iy_j$. Throughout, $C^1(\Omega)$ means *real* C^1 on $\mathbb{R}^8$ (all $\partial_{x_j}, \partial_{y_j}$ exist and are continuous). The Wirtinger operators are $\partial_{z_j} = \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})$ and $\partial_{\bar{z}_j} = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$. In particular, functions may depend on both z and $\bar{z}$; holomorphicity is not assumed unless stated.

For $z = (z_1, z_2, z_3, z_4) \in \mathbb{C}^4$, set

$$\partial_{z_j} = \frac{1}{2} \left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right), \quad \partial_{\bar{z}_j} = \frac{1}{2} \left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right),$$

and write $\partial = (\partial_{z_1}, \dots, \partial_{z_4})^\top$, $\bar{\partial} = (\partial_{\bar{z}_1}, \dots, \partial_{\bar{z}_4})^\top$. Throughout, $g : \Omega \subset \mathbb{C}^4 \rightarrow \mathbb{C}(\mathbb{C}) \cong \mathbb{C}^4$ is assumed C^1 and may depend on both z and $\bar{z}$.

Let Ω be an open and connected subset in $\mathbb{C}^4$, which is a four-dimensional space consisting of complex numbers. We define a function $g(z)$ that maps points in Ω to complex values, expressed as follows:

$$g(z) = g_1(z) + g_2(z)e_2 + g_3(z)e_4 + g_4(z)e_6$$

In this equation, z is a vector composed of four components, denoted as $z = (z_1, z_2, z_3, z_4)$, where each z_i is a complex number. The terms $g_1(z)$, $g_2(z)$, $g_3(z)$, and $g_4(z)$ are functions that each depend on the point z in Ω , and these functions output complex values. The symbols e_2 , e_4 , and e_6 are likely representing distinct basis vectors or elements within a specific context, such as a vector space or another structured environment in which $g(z)$ operates. The function $g(z)$ can be characterized as a complex-valued entity that integrates contributions from various components. This illustrates the manner in which it synthesizes multiple variables from the domain Ω to produce a complex number as an outcome.

2.1.2. Dirac-Type Pair

Let ∂_{z_j} denote complex derivatives in z_j . Define the 4×4 operator matrix

$$\mathcal{D} = \begin{pmatrix} \partial_{z_1} & \partial_{z_2} & \partial_{z_3} & \partial_{z_4} \\ \partial_{z_2} & \partial_{z_1} & \partial_{z_4} & \partial_{z_3} \\ \partial_{z_3} & \partial_{z_4} & \partial_{z_1} & \partial_{z_2} \\ \partial_{z_4} & \partial_{z_3} & \partial_{z_2} & \partial_{z_1} \end{pmatrix}, \quad \text{and} \quad \mathcal{D}^* \text{ (formal } L^2\text{-adjoint with } \bar{\partial} \text{ entries).}$$

Definition 1. Let Ω denote an open subset of $\mathbb{C}^4$, which serves as the domain for our analysis. A function $g(z)$ is defined in this expression as follows:

$$g(z) = g_1(z) + g_2(z)e_2 + g_3(z)e_4 + g_4(z)e_6,$$

where each $g_j(z)$ (for $j = 1, 2, 3, 4$) represents a complex-valued function that is dependent on the variable z , and e_2 , e_4 , and e_6 are specified basis elements in $\mathbb{C}^4$. A function $g(z)$ is classified as $\mathcal{O}$ -regular within the set Ω if it satisfies the following conditions:

- The functions $g_j(z)$ for $j = 1, 2, 3, 4$ are required to be continuously differentiable throughout the entirety of the domain Ω .
- The equation $\mathcal{D}^*g = 0$ is satisfied within the domain Ω , where $\mathcal{D}^*$ represents a differential operator that acts upon the function g .

Condition (a) necessitates that each $g_j(z)$ possesses continuous first derivatives with respect to its variables across Ω , thereby ensuring the existence of well-defined tangents and consistency in behavior. The condition $\mathcal{D}^*g = 0$ signifies that $g(z)$ adheres to a specific differential equation or set of criteria that characterizes $\mathcal{O}$ -regularity. This implies that $g(z)$ exhibits certain structural properties that arise from the definitions of this operator within the relevant paper. The classification of $g(z)$ as $\mathcal{O}$ -regular hinges on its differentiability and the satisfaction of the differential operator condition across the open subset Ω .

Table 1 rewrites (CR1)–(CR4) as a block system $\mathbb{C} \partial g = 0$; rows index the four equations and columns list the eight off/on-diagonal partials. The dense “1” rows capture *diagonal* conservation, while the second block encodes *off-diagonal* mixing.

Table 1. Coupled CR system (CR1)–(CR4) in matrix form $\mathbb{C} \partial g = 0$ (rows = equations, cols = variables).

	$\partial_{z_1}g_1$	$\partial_{z_2}g_2$	$\partial_{z_3}g_3$	$\partial_{z_4}g_4$	$\partial_{z_1}g_2$	$\partial_{z_2}g_1$	$\partial_{z_3}g_4$	$\partial_{z_4}g_3$
(CR1)	1	1	1	1	0	0	0	0
(CR2)	0	0	0	0	1	1	1	1
(CR3)	0	0	0	0	0	0	.	.
(CR4)	0	0	0	0	0	0	.	.

2.1.3. Interpreting Figure 1

Edges label which derivatives link which components: for instance, g_1 and g_2 couple through $\partial_{z_1}, \partial_{z_3}, \partial_{z_4}$, while g_3 and g_4 interact via $\partial_{z_2}, \partial_{z_4}$. This picture is a faithful reflection of the algebraic product (CR1)–(CR4).

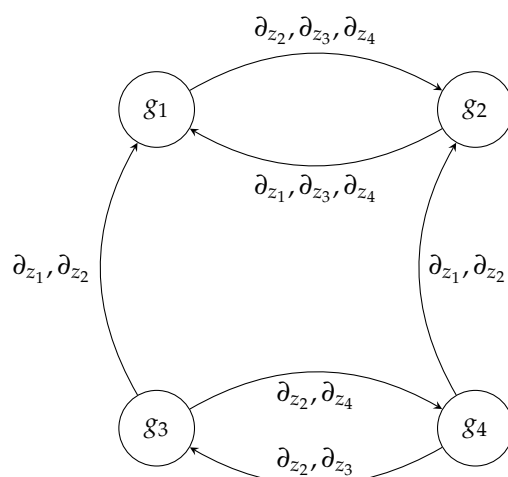


Figure 1. Off-diagonal couplings dictated by the product in $\mathbb{C}(\mathbb{C})$ in the CR system.

2.1.4. Canonical Derivative

We adopt the canonical first derivative $g'(z) := \partial_{x_0}g(z)$, i.e., the x_0 -directional derivative in the z_1 -plane, which matches the intrinsic derivative under $\mathcal{O}$ -regularity and is stable under algebra-preserving affine changes.

Proposition 1 (Equivalent product presentation). For $u, v \in \mathbb{C}^4 \cong \mathbb{C}(\mathbb{C})$, the algebra product $\odot$ is equivalently given by

$$u \odot v = (u \cdot v) 1 + (u \cdot e_2 v) e_2 + (u \cdot e_4 v) e_4 + (u \cdot e_6 v) e_6,$$

i.e., $u \odot v = \{u \cdot v, u \cdot e_2 v, u \cdot e_4 v, u \cdot e_6 v\}$ in the $(1, e_2, e_4, e_6)$ -basis. This makes commutativity immediate from the symmetry of e_2, e_4, e_6 .

Sketch of proof. Expand both definitions on the basis $\{1, e_2, e_4, e_6\}$ and compare coefficients. The off-diagonal couplings are identical by the symmetry relations among e_2, e_4, e_6 , hence, the two products coincide. $\square$

Proposition 2 (Idempotent normal form of $\mathbb{C}(\mathbb{C})$). Let

$$I_1 = \frac{1}{4}(e_0 + e_2 + e_4 + e_6), \quad I_2 = \frac{1}{4}(e_0 - e_2 - e_4 + e_6),$$

$$I_3 = \frac{1}{4}(e_0 - e_2 + e_4 - e_6), \quad I_4 = \frac{1}{4}(e_0 + e_2 - e_4 - e_6).$$

Then $I_k^2 = I_k$, $I_k I_\ell = 0$ ($k \neq \ell$), $\sum_k I_k = e_0$, and the map

$$\Phi: z = z_1 + z_2 e_2 + z_3 e_4 + z_4 e_6 \mapsto (\zeta_1, \zeta_2, \zeta_3, \zeta_4), \quad \zeta_k := \langle z, I_k \rangle$$

is an algebra isomorphism $\mathbb{C}(\mathbb{C}) \cong \mathbb{C}I_1 \oplus \cdots \oplus \mathbb{C}I_4 \cong \mathbb{C}^4$, under which the product becomes coordinatewise: $\Phi(zw) = \Phi(z) \cdot \Phi(w)$.

Remark 2 (On representations). In the idempotent variables $(\zeta_1, \dots, \zeta_4)$, structural couplings diagonalize; this connects to representation theorems for analytic/monogenic functions in semisimple commutative algebras (cf. Shpakivskyi and predecessors).

3. $\mathcal{O}$ -Regular Functions

Let $g(z)$ denote an $\mathcal{O}$ -regular function defined within a specified domain Ω in $\mathbb{C}^4$. In this paper, the notation $g'(z)$ is used to represent the derivative of the function $g(z)$, and it can also be expressed as $g'(z) = \mathcal{D}g(z)$, where $\mathcal{D}$ signifies the differential operator acting on the function.

Proposition 3. Let a domain Ω , which is an open subset of $\mathbb{C}^4$, and within this domain, let $g(z)$ be an $\mathcal{O}$ -regular function. An $\mathcal{O}$ -regular function is characterized by its ability to be represented locally by a convergent power series, which is an essential feature in complex analysis. It can be established that the derivative of $g(z)$, which is denoted as $g'(z)$, is equivalent to taking the partial derivative of g with respect to the first variable x_0 of the complex vector z . This relationship is succinctly stated as

$$g'(z) = \frac{\partial g}{\partial x_0}.$$

This result emphasizes the significance of the variable x_0 in influencing the behavior of the function $g(z)$ within the domain Ω . In essence, the partial derivative provides insight into how $g(z)$ varies as x_0 changes while keeping all other variables constant.

Proof. Given that the function $g(z)$ is classified as $\mathcal{O}$ -regular within the domain Ω , it follows that

$$g' = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ A_2 & A_1 & A_4 & A_3 \\ A_3 & A_4 & A_1 & A_2 \\ A_4 & A_3 & A_2 & A_1 \end{pmatrix} = \begin{pmatrix} B_1 & B_2 & B_3 & B_4 \\ B_2 & B_1 & B_4 & B_3 \\ B_3 & B_4 & B_1 & B_2 \\ B_4 & B_3 & B_2 & B_1 \end{pmatrix} = \frac{\partial g}{\partial x_0},$$

where

$$\begin{aligned} A_1 &= \frac{\partial g_1}{\partial z_1} - \frac{\partial g_2}{\partial \bar{z}_2} - \frac{\partial g_3}{\partial \bar{z}_3} - \frac{\partial g_4}{\partial \bar{z}_4}, & A_2 &= \frac{\partial g_2}{\partial z_1} - \frac{\partial g_1}{\partial \bar{z}_2} - \frac{\partial g_4}{\partial \bar{z}_3} - \frac{\partial g_3}{\partial \bar{z}_4}, \\ A_3 &= \frac{\partial g_3}{\partial z_1} - \frac{\partial g_4}{\partial \bar{z}_2} - \frac{\partial g_1}{\partial \bar{z}_3} - \frac{\partial g_2}{\partial \bar{z}_4}, & A_4 &= \frac{\partial g_4}{\partial z_1} - \frac{\partial g_3}{\partial \bar{z}_2} - \frac{\partial g_2}{\partial \bar{z}_3} - \frac{\partial g_1}{\partial \bar{z}_4}, \\ B_1 &= \frac{\partial g_1}{\partial z_1} + \frac{\partial g_1}{\partial \bar{z}_1}, & B_2 &= \frac{\partial g_2}{\partial z_1} + \frac{\partial g_2}{\partial \bar{z}_1}, \\ B_3 &= \frac{\partial g_3}{\partial z_1} + \frac{\partial g_3}{\partial \bar{z}_1}, & B_4 &= \frac{\partial g_4}{\partial z_1} + \frac{\partial g_4}{\partial \bar{z}_1}. \end{aligned}$$

Since the function $g(z)$ is categorized as $\mathcal{O}$ -regular within the domain Ω , we can deduce that there exists a specific growth rate for $g(z)$ that is controlled within this domain. This means that the values of $g(z)$ do not increase too rapidly as z approaches the boundary of Ω . Consequently, the characteristics of $\mathcal{O}$ -regularity imply that within Ω , $g(z)$ can be bounded by a function that provides an upper limit to its growth, which in turn suggests that $g(z)$ behaves nicely and remains well-defined across the entire region. This property is significant for analysis involving continuity, differentiability, and the overall behavior of the function in complex analysis or other mathematical contexts. $\square$

Theorem 1. Let $g(z)$ be a function defined on a domain G within the complex plane $\mathbb{C}(\mathbb{C})$ exhibiting $\mathcal{O}$ -regularity. This condition implies that $g(z)$ possesses certain smoothness and growth properties, which are essential in the context of complex analysis. We introduce the differential form τ , which is expressed as follows:

$$\begin{aligned} \tau &= dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4 \wedge d\bar{z}_2 \wedge d\bar{z}_3 \wedge d\bar{z}_4 - dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4 \wedge d\bar{z}_1 \wedge d\bar{z}_3 \wedge d\bar{z}_4, \\ e_2 &+ dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_4, \\ e_4 &- dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3. \end{aligned}$$

This differential form τ is constructed from the differentials dz_i and their corresponding conjugates $d\bar{z}_j$ for $i, j = 1, 2, 3, 4$. The terms e_2, e_4 , and e_6 represent coefficients that could denote specific weights or functions contributing to the structure of τ . An important result that follows from the properties of the form τ is the behavior of integrals over the boundary of domains within G . Specifically, for any subdomain Ω of G that has a sufficiently smooth boundary denoted $b\Omega$, we can state a crucial integral identity:

$$\int_{b\Omega} \tau g = 0.$$

This equation indicates that the integral of the product of the differential form τ and the function g over the boundary surface $b\Omega$ evaluates to zero, reflecting a certain property of conservation or symmetry within the domain defined by G . This highlights the interplay between the structure of differential forms and the behavior of analytic functions in complex domains.

Proof. Define the following forms in our analysis:

$$\tau_{(1)} = dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4 \wedge d\bar{z}_2 \wedge d\bar{z}_3 \wedge d\bar{z}_4,$$

$$\tau_{(2)} = dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4 \wedge d\bar{z}_1 \wedge d\bar{z}_3 \wedge d\bar{z}_4,$$

$$\tau_{(3)} = dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_4,$$

$$\tau_{(4)} = dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3.$$

Next, we will consider the operation τg , which involves a linear combination of these forms weighted by certain functions g_i :

$$\tau g = \left(\tau_{(1)} - \tau_{(2)}e_2 + \tau_{(3)}e_4 - \tau_{(4)}e_6 \right) (g_1 + g_2e_2 + g_3e_4 + g_4e_6).$$

This expands to

$$\begin{aligned} \tau g &= (g_1\tau_{(1)} - g_2\tau_{(2)} + g_3\tau_{(3)} - g_4\tau_{(4)}) + (g_2\tau_{(1)} - g_1\tau_{(2)} + g_4\tau_{(3)} - g_3\tau_{(4)})e_2 \\ &\quad + (g_3\tau_{(1)} - g_4\tau_{(2)} + g_1\tau_{(3)} - g_2\tau_{(4)})e_4 + (g_4\tau_{(1)} - g_3\tau_{(2)} + g_2\tau_{(3)} - g_1\tau_{(4)})e_6. \end{aligned}$$

Now we proceed to compute the exterior derivative $d(\tau g)$. We can express $d(\tau g)$ as follows:

$$\begin{aligned} d(\tau g) &= \left(\frac{\partial g_1}{\partial z_1} + \frac{\partial g_2}{\partial z_2} + \frac{\partial g_3}{\partial z_3} + \frac{\partial g_4}{\partial z_4} \right) dV + \left(\frac{\partial g_2}{\partial z_1} + \frac{\partial g_1}{\partial z_2} + \frac{\partial g_4}{\partial z_3} + \frac{\partial g_3}{\partial z_4} \right) dVe_2 \\ &\quad + \left(\frac{\partial g_3}{\partial z_1} + \frac{\partial g_4}{\partial z_2} + \frac{\partial g_1}{\partial z_3} + \frac{\partial g_2}{\partial z_4} \right) dVe_4 + \left(\frac{\partial g_4}{\partial z_1} + \frac{\partial g_3}{\partial z_2} + \frac{\partial g_2}{\partial z_3} + \frac{\partial g_1}{\partial z_4} \right) dVe_6. \end{aligned}$$

In this expression, dV is defined as the volume form:

$$dV = dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3 \wedge d\bar{z}_4.$$

By utilizing the equation (referenced as equation (CR1)–(CR4)), we establish that $d(\tau g) = 0$. According to Stokes' theorem, we assert that

$$\int_{b\Omega} \tau g = \int_{\Omega} d(\tau g) = 0,$$

which confirms that the integral of the exterior derivative over the region vanishes. $\square$

4. Boundary Form and Stokes Mechanism

We now construct the differential 7-form τ canonically associated with the pair $(\mathcal{D}, \mathcal{D}^*)$ and prove the boundary annihilation law.

Definition 2 (Canonical 7-form τ). Set

$$\Omega_4 := dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4, \quad \sigma_1 := dz_2 \wedge dz_3 \wedge dz_4,$$

$$\sigma_2 := dz_1 \wedge dz_3 \wedge dz_4, \quad \sigma_3 := dz_1 \wedge dz_2 \wedge dz_4, \quad \sigma_4 := dz_1 \wedge dz_2 \wedge dz_3.$$

Define

$$\tau := \Omega_4 \wedge \sigma_1 - (\Omega_4 \wedge \sigma_2) e_2 + (\Omega_4 \wedge \sigma_3) e_4 - (\Omega_4 \wedge \sigma_4) e_6. \quad (3)$$

Then, for $g = g_1 + g_2e_2 + g_3e_4 + g_4e_6 \in C^1(\Omega)$, the identity $d(\tau g) = 0$ follows from the four-line structural system, hence $\int_{\partial\Omega} \tau g = 0$ by Stokes.

Theorem 2 (Cauchy–Stokes boundary annihilation). *Let $\Omega \Subset \Omega_0 \subset \mathbb{C}(\mathbb{C})$ have $\mathcal{C}^1$ boundary. If $g \in \mathcal{C}^1(\overline{\Omega})$ is $\mathcal{O}$ -regular in Ω , then*

$$\int_{\partial\Omega} \tau g = 0,$$

where τ is the canonical 7-form determined by $\mathbb{C}(\mathbb{C})$ and $(\mathcal{D}, \mathcal{D}^*)$.

Idea of proof. Write τg in components and compute $d(\tau g)$; by (CR1)–(CR4), all top-degree terms cancel. Stokes then gives $\int_{\partial\Omega} \tau g = \int_{\Omega} d(\tau g) = 0$. $\square$

4.1. Reading Figure 2

The figure emphasizes that τ is tailored to the operator so that $d(\tau g)$ vanishes under (CR1)–(CR4); the boundary integral is thus killed by Stokes.

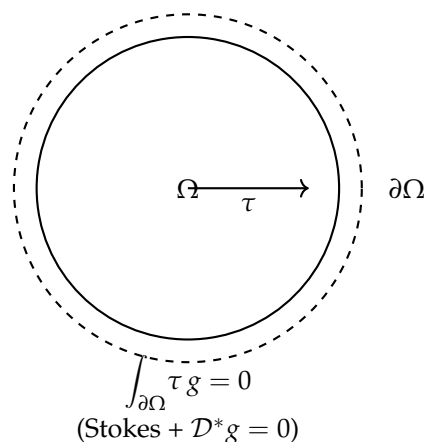


Figure 2. Cauchy–Stokes boundary annihilation for $\mathcal{O}$ -regular g .

4.2. On the Conjugate System

The label (2) refers exclusively to the *conjugate* system obtained from (CR1)–(CR4) by replacing ∂_{z_j} with $\partial_{\bar{z}_j}$. We avoid reusing the symbol $\mathcal{D}$ ambiguously; all operator statements use $D^* = \mathcal{D}$ and $D = \mathcal{D}^\dagger$.

4.3. Existence on Pseudoconvex Domains

Theorem 3 (Existence with L^2 estimates). *Let $\Omega \subset \mathbb{C}^4$ be bounded pseudoconvex. Suppose $g = (g_1, \dots, g_4) \in \mathcal{C}^2(\Omega)$ satisfies the compatibility conditions obtained by applying ∂_{z_j} to (CR1)–(CR4). Then there exists u with $\mathcal{D}^* u = 0$ in Ω and $u|_{\partial\Omega}$ prescribed in the τ -annihilating class. In particular, the structural system is solvable with L^2 estimates.*

Sketch of proof. Cast (CR1)–(CR4) as a $\bar{\partial}$ -system with matrix coefficients and apply Hörmander's L^2 method on pseudoconvex Ω to obtain u , solving the $\bar{\partial}$ problem with bounds.

Coupling is controlled by the algebra-induced matrix; the a priori estimate passes to u . $\square$

Proposition 4 (Algebraic stability). *If g, h are $\mathcal{O}$ -regular on Ω , then so is $g \odot h$. If $\phi : \mathbb{C}(\mathbb{C}) \rightarrow \mathbb{C}(\mathbb{C})$ is algebra-preserving and holomorphic in each complex coordinate, then $\phi \circ g$ is $\mathcal{O}$ -regular.*

Theorem 4 (Liouville-type). *If g is $\mathcal{O}$ -regular on $\mathbb{C}^4$ and $\|g(z)\| \leq \mathcal{C}(1 + \|z\|)^m$ with $m < 1$, then g is constant.*

Theorem 5 (Removable singularities). *Let $E \subset \Omega$ have Hausdorff dimension < 7 . If g is $\mathcal{O}$ -regular on $\Omega \setminus E$ and $g \in L^p(\Omega)$ for some $p > 1$, then g extends $\mathcal{O}$ -regularly across E .*

The block diagram clarifies that the CR system is a matrix coupling applied to the gradient in complex coordinates; it serves as a guide to kernel and layer-potential designs.

4.4. Reading Table 2

This comparison locates $\mathbb{C}(\mathbb{C})$ between the classical one-component theory and bicomplex analysis, highlighting the increased coupling (from 1×1 to 4×4) and a higher-degree boundary form.

Table 2. Analytic features across algebras (schematic).

Feature	$\mathbb{C}$	Bicomplex	$\mathbb{C}(\mathbb{C})$
CR system	$\partial_{\bar{z}}f = 0$	coupled 2×2	coupled 4×4
Boundary form	1-form	3-form	7-form
Dirac factorization	trivial	block	block (this work)
Cauchy kernel	classical	algebra-aware	to be developed
L^2 theory	Hardy/Bergman	partial	roadmap herein

Theorem 6. *Let Ω be a domain of holomorphy in the complex space $\mathbb{C}(\mathbb{C})$ with respect to the complex variables z_1, z_2, z_3, z_4 . This means that Ω is an open and connected subset where holomorphic functions exhibit nice properties and are expressible in terms of power series. Consider a set of complex-valued functions $g_i(z)$, for $i = 1, 2, 3, 4$, which belong to the class $\mathcal{C}^2$ on the domain Ω . The class $\mathcal{C}^2$ indicates that each function $g_i(z)$ is continuously differentiable up to the second order, ensuring that their first and second derivatives exist and are continuous throughout Ω .*

Theorem 1 asserts the existence of a corresponding set of functions, denoted again as $g_i(z)$ for $i = 1, 2, 3, 4$, that can also be classified within the same $\mathcal{C}^2$ class on the domain Ω . We can then define a new complex-valued function $g(z)$ as follows:

$$g(z) = g_1(z) + g_2(z)e_2 + g_3(z)e_4 + g_4(z)e_6,$$

where e_2, e_4, e_6 are constant terms that can be interpreted as additional functional components within this equation. Under the conditions specified, this function $g(z)$ will be categorized as an $\mathcal{O}$ -regular function on the domain Ω . This classification implies that $g(z)$ adheres to specific regularity conditions related to its behavior and properties as a holomorphic function, ensuring that it maintains the integral and differentiable qualities required across the specified domain.

Proof. If we consider a complex-valued function $g_1(z)$ that is classified as $\mathcal{C}^2$ over a specified domain Ω , we can also identify another complex-valued function $g_2(z)$ that maintains the same level of smoothness, belonging to the class $\mathcal{C}^2$ on the same domain Ω . To illustrate this concept, we start by examining a specific 1-form, denoted as ψ , along with the corresponding differential operator that acts on the space Ω . The expression for ψ is given by

$$\begin{aligned} \psi := & -\left(\frac{\partial g_1}{\partial \bar{z}_2} + \frac{\partial g_4}{\partial \bar{z}_3} + \frac{\partial g_3}{\partial \bar{z}_4}\right)d\bar{z}_1 - \left(\frac{\partial g_1}{\partial \bar{z}_1} + \frac{\partial g_3}{\partial \bar{z}_3} + \frac{\partial g_4}{\partial \bar{z}_4}\right)d\bar{z}_2 \\ & -\left(\frac{\partial g_4}{\partial \bar{z}_1} + \frac{\partial g_3}{\partial \bar{z}_2} + \frac{\partial g_1}{\partial \bar{z}_4}\right)d\bar{z}_3 - \left(\frac{\partial g_3}{\partial \bar{z}_1} + \frac{\partial g_4}{\partial \bar{z}_2} + \frac{\partial g_1}{\partial \bar{z}_3}\right)d\bar{z}_4 \end{aligned}$$

In this expression, each term $d\bar{z}_i$ corresponds to a differential form associated with the complex conjugate variables, while the partial derivatives represent the interaction between the functions g_1, g_2, g_3 , and g_4 with respect to these conjugate variables.

$$\begin{aligned}\bar{\partial}\psi = & \left(-\frac{\partial^2 g_1}{\partial \bar{z}_1 \partial \bar{z}_1} - \frac{\partial^2 g_3}{\partial \bar{z}_1 \partial \bar{z}_3} - \frac{\partial^2 g_4}{\partial \bar{z}_1 \partial \bar{z}_4} + \frac{\partial^2 g_1}{\partial \bar{z}_2 \partial \bar{z}_2} + \frac{\partial^2 g_4}{\partial \bar{z}_2 \partial \bar{z}_3} + \frac{\partial^2 g_3}{\partial \bar{z}_2 \partial \bar{z}_4}\right) d\bar{z}_1 \wedge d\bar{z}_2 \\ & \left(-\frac{\partial^2 g_4}{\partial \bar{z}_1 \partial \bar{z}_1} - \frac{\partial^2 g_3}{\partial \bar{z}_1 \partial \bar{z}_2} - \frac{\partial^2 g_1}{\partial \bar{z}_1 \partial \bar{z}_4} + \frac{\partial^2 g_1}{\partial \bar{z}_3 \partial \bar{z}_2} + \frac{\partial^2 g_4}{\partial \bar{z}_3 \partial \bar{z}_3} + \frac{\partial^2 g_3}{\partial \bar{z}_3 \partial \bar{z}_4}\right) d\bar{z}_1 \wedge d\bar{z}_3 \\ & \left(-\frac{\partial^2 g_3}{\partial \bar{z}_1 \partial \bar{z}_1} - \frac{\partial^2 g_4}{\partial \bar{z}_1 \partial \bar{z}_2} - \frac{\partial^2 g_1}{\partial \bar{z}_1 \partial \bar{z}_3} + \frac{\partial^2 g_1}{\partial \bar{z}_4 \partial \bar{z}_2} + \frac{\partial^2 g_4}{\partial \bar{z}_4 \partial \bar{z}_3} + \frac{\partial^2 g_3}{\partial \bar{z}_4 \partial \bar{z}_4}\right) d\bar{z}_1 \wedge d\bar{z}_4 \\ & \left(-\frac{\partial^2 g_4}{\partial \bar{z}_2 \partial \bar{z}_1} - \frac{\partial^2 g_3}{\partial \bar{z}_2 \partial \bar{z}_2} - \frac{\partial^2 g_1}{\partial \bar{z}_2 \partial \bar{z}_4} + \frac{\partial^2 g_1}{\partial \bar{z}_3 \partial \bar{z}_1} + \frac{\partial^2 g_3}{\partial \bar{z}_3 \partial \bar{z}_3} + \frac{\partial^2 g_4}{\partial \bar{z}_3 \partial \bar{z}_4}\right) d\bar{z}_2 \wedge d\bar{z}_3 \\ & \left(-\frac{\partial^2 g_3}{\partial \bar{z}_2 \partial \bar{z}_1} - \frac{\partial^2 g_4}{\partial \bar{z}_2 \partial \bar{z}_2} - \frac{\partial^2 g_1}{\partial \bar{z}_2 \partial \bar{z}_3} + \frac{\partial^2 g_1}{\partial \bar{z}_4 \partial \bar{z}_1} + \frac{\partial^2 g_3}{\partial \bar{z}_4 \partial \bar{z}_3} + \frac{\partial^2 g_4}{\partial \bar{z}_4 \partial \bar{z}_4}\right) d\bar{z}_2 \wedge d\bar{z}_4 \\ & \left(-\frac{\partial^2 g_3}{\partial \bar{z}_3 \partial \bar{z}_1} - \frac{\partial^2 g_4}{\partial \bar{z}_3 \partial \bar{z}_2} - \frac{\partial^2 g_1}{\partial \bar{z}_3 \partial \bar{z}_3} + \frac{\partial^2 g_4}{\partial \bar{z}_4 \partial \bar{z}_1} + \frac{\partial^2 g_3}{\partial \bar{z}_4 \partial \bar{z}_2} + \frac{\partial^2 g_1}{\partial \bar{z}_4 \partial \bar{z}_4}\right) d\bar{z}_3 \wedge d\bar{z}_4\end{aligned}$$

Following these formulations, we proceed to apply the differential operator $\bar{\partial}$ to the 1-form ψ . This operation, which is utilized in the context of several complex variables, acts on the left-hand side of the expression and is performed within the framework of the domain Ω . By doing so, we will explore the properties and results that arise from this application, contributing to our understanding of the relationships between these functions and their derivatives. $\square$

4.5. Function Spaces

According to the equation (CR1)–(CR4), it has been established that all coefficients associated with the equation are identified as vanishing. This result is significant in the context of complex analysis. In his foundational work, Hörmander [2] demonstrated that the differential form ψ , which is defined using the variables z_1, z_2, z_3, z_4 , possesses the property of being $\bar{\partial}$ -closed. This means that when we consider the complex structure defined on the domain Ω , the form ψ does not change under the action of the $\bar{\partial}$ operator. Furthermore, Hörmander established that ψ is also $\bar{\partial}$ -exact within this domain, which implies that there exists a function $g_2(z)$ that can be derived from ψ such that $\psi = \bar{\partial}g_2(z)$ holds true.

In addition, as noted by Krantz [30], the domain Ω is characterized as a domain of holomorphy. This classification is crucial because it guarantees the existence of complex-valued functions that are infinitely differentiable, specifically for each index $i = 1, 2, 3, 4$. More precisely, for these indices, there exists a function $g_i(z)$ that belongs to the class $\mathcal{C}^\infty$ on the domain Ω . This means that each function is not only continuous but also possesses continuous derivatives of all orders.

The relationship between ψ and the function $g_2(z)$ is particularly important, as it fulfills the condition that $\psi = \bar{\partial}g_2(z)$, thus confirming that ψ is indeed a $\bar{\partial}$ -closed form on Ω . The significance of this lemma is evident in its implication that the form ψ is classified as a $\bar{\partial}$ -exact (0,1) form within the domain. Consequently, this leads to the conclusion that the function $g(z)$ is recognized as an $\mathcal{O}$ -regular function across the entirety of this domain.

4.6. Reading Table 3

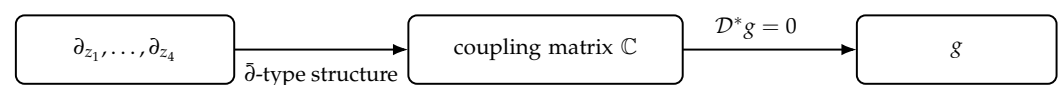
The table summarizes the boundary/interior L^2 spaces that will host projection and kernel operators in future work.

Table 3. Function spaces for $\mathcal{O}$ -regular maps (roadmap).

Space	Defining Property
$H^2(\partial\Omega; \mathbb{C}(\mathbb{C}))$	nontangential limits of $\mathcal{O}$ -regular L^2 interior functions
$A^2(\Omega; \mathbb{C}(\mathbb{C}))$	$\mathcal{O}$ -regular with $\int_{\Omega} \ g\ ^2 < \infty$ (Bergman)
$\mathcal{D}(\Omega; \mathbb{C}(\mathbb{C}))$	Dirichlet: $\int_{\Omega} \ \nabla g\ ^2 < \infty$

4.7. At-a-Glance Roadmap of Figures and Tables

Figure 1 visualizes the off-diagonal couplings among the four complex components forced by the algebra; Figure 2 depicts the Stokes mechanism behind the boundary identity; Figure 3 summarizes the operator pipeline $(\partial_{z_1}, \dots, \partial_{z_4}) \rightarrow \mathbb{C} \rightarrow \mathcal{D}^*g = 0$. Table 1 collects the coupled CR system in matrix form; Table 2 contrasts analytic features across $\mathbb{C}$, the bicomplex numbers, and $\mathbb{C}(\mathbb{C})$; Table 3 outlines the function spaces used later.

**Figure 3.** Operator-theoretic pipeline behind the structural system.

5. Conclusions

We developed a coherent analytic framework for $\mathcal{O}$ -regular mappings on a commutative complex algebra model of $\mathbb{C}^4$. The approach is operator-theoretic: starting from a Dirac-type pair $(\mathcal{D}, \mathcal{D}^*)$ intrinsic to the algebra, we derived a fully coupled Cauchy–Riemann system (CR1)–(CR4) that characterizes $\mathcal{O}$ -regularity and ties the algebraic multiplication directly to analytic coupling among the four complex components. Within this framework, we proved that the intrinsic first derivative agrees with a canonical real directional derivative,

$$g'(z) = \partial_{x_0} g(z),$$

providing a concrete and computationally accessible notion of first order behavior. On the boundary side, we constructed a canonical differential form τ that encodes the algebra-induced coupling and established a Stokes-driven Cauchy-type identity,

$$\int_{\partial\Omega} \tau g = 0 \quad \text{for suitably smooth } \Omega,$$

which furnishes the organizing principle for representation and uniqueness results in this setting. Finally, by adapting $\bar{\partial}$ -methods from several complex variables to the present algebra, we proved an existence statement on domains of holomorphy, thereby ensuring solvability of the structural system (CR1)–(CR4) under natural regularity hypotheses.

5.1. Conceptual Payoffs

Three aspects of the theory deserve emphasis. (i) The structural system (CR1)–(CR4) is not an ad hoc collection of compatibility conditions; it is the precise analytic footprint of the algebra, with off-diagonal terms dictated by the multiplication in $\mathbb{C}(\mathbb{C})$. (ii) The identification $g'(z) = \partial_{x_0} g(z)$ supplies an *intrinsic* differential that is stable under algebra-preserving affine changes of variables and compatible with product rules for $\mathcal{O}$ -regular maps. (iii) The boundary form τ translates the interior coupling to the boundary, yielding a Cauchy–Stokes identity that is robust enough to underpin integral representations and potential-theoretic estimates.

5.2. Methodological Implications

The synthesis of Dirac-type operators, algebra-aware differential forms, and $\bar{\partial}$ -solvability shows that a large portion of the classical holomorphic toolkit—differential characterizations, annihilating boundary forms, and interior solvability on holomorphic domains—survives in a four-component, algebra-coupled environment. In particular, the operator viewpoint clarifies how to design kernels and layer potentials that see the algebraic coupling, while the derivative identity enables quantitative first-order estimates without leaving the intrinsic calculus.

5.3. Scope and Limitations

Our results are proved for $\mathcal{C}^1$ (or smoother) data on smoothly bounded subdomains in the global coordinate model $\mathbb{C}(\mathbb{C}) \cong \mathbb{C}^4$. We did not construct explicit Cauchy or Borel–Pompeiu kernels for $(\mathcal{D}, \mathcal{D}^*)$, develop a full L^2 theory (Hardy/Bergman spaces, projection operators), or analyze boundary value problems on rough (e.g., Lipschitz) domains. Likewise, a systematic treatment of singularities (removable sets, poles/essential growth in the algebra-coupled sense), and quantitative interior estimates remains open.

5.4. Directions for Further Work

The framework suggests several concrete next steps:

- **Integral kernels and jump relations.** Construct algebra-adapted Cauchy and Borel–Pompeiu kernels for $(\mathcal{D}, \mathcal{D}^*)$; establish jump formulas and Plemelj–Sokhotski relations for nontangential boundary limits of layer potentials.
- **L^2 and function-space theory.** Develop Hardy, Bergman, and Dirichlet spaces of $\mathcal{O}$ -regular maps; identify reproducing kernels; prove boundedness of the Cauchy transform and of projection operators; characterize Carleson measures in the algebra-coupled setting.
- **Potential theory and growth.** Prove maximum and Liouville-type principles, Harnack inequalities for component functions, Montel normality, and Weierstrass/Runge approximation theorems adapted to (CR1)–(CR4).
- **Boundary value problems.** Formulate Dirichlet/Neumann (and mixed) problems for $\mathcal{D}$ and $\mathcal{D}^*$ on Lipschitz domains; prove well-posedness via the Fredholm theory of boundary integral operators built from the new kernels.
- **Structure and factorization.** Analyze zero sets and divisors of $\mathcal{O}$ -regular maps; study factorization and inner–outer decompositions relative to the algebra; compare with bicomplex and several-variable slice-regular theories.
- **Discrete and fractional variants.** Develop lattice versions of the calculus (discrete Borel–Pompeiu, finite-difference Dirac factorizations) and fractional analogues of the structural operator for numerical and anomalous-diffusion applications.

In summary, $\mathcal{O}$ -regularity furnishes an intrinsically algebra-aware notion of analyticity on $\mathbb{C}(\mathbb{C})$ that aligns differential, boundary, and existence principles within a single operator calculus. The results proved here supply the necessary scaffolding for a full representation theory, a quantitative L^2 analysis, and PD-oriented applications, and they open a practical pathway toward discretization and fractional generalizations keyed to the same structural operator.

Supplementary Materials: The following supporting information can be downloaded at: <https://www.mdpi.com/article/10.3390/math13203328/s1>.

Funding: This study was supported by Dongguk University Research Fund and the National Research Foundation of Korea (NRF) (2021R1F1A1063356).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The original contributions presented in this study are included in the Supplementary Materials. Further inquiries can be directed to the corresponding author.

Conflicts of Interest: The authors declare no conflicts of interest.

List of Symbols

$\mathbb{C}(\mathbb{C})$	commutative complex algebra model of $\mathbb{C}^4$
e_2, e_4, e_6	algebra basis elements determining off-diagonal couplings
$\odot$	algebra product on $\mathbb{C}(\mathbb{C})$
$\mathcal{D}, \mathcal{D}^*$	Dirac-type pair adapted to $\mathbb{C}(\mathbb{C})$
τ	canonical boundary 7-form
Ω	domain in $\mathbb{C}^4$ (often pseudoconvex)

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