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Approximate Fiber Products of Schemes and Their Étale Homotopical Invariants

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Abstract

The classical fiber product in algebraic geometry provides a powerful tool for studying loci where two morphisms to a base scheme, $\phi : X \to S$ and $\psi : Y \to S$, coincide exactly. This condition of strict equality, however, is insufficient for describing many real-world applications, such as the geometric structure of semantic spaces in modern large language models whose foundational architecture is the Transformer neural network: The token spaces of these models are fundamentally approximate, and recent work has revealed complex geometric singularities, challenging the classical manifold hypothesis. This paper develops a new framework to study and quantify the nature of approximate alignment between morphisms in the context of arithmetic geometry, using the tools of étale homotopy theory. We introduce the central object of our work, the étale mismatch torsor, which is a sheaf of torsors over the product scheme $X \times_S Y$. The structure of this sheaf serves as a rich, intrinsic, and purely algebraic object amenable to both qualitative classification and quantitative analysis of the global relationship between the two morphisms. Our main results are twofold. First, we provide a complete classification of these structures, establishing a bijection between their isomorphism classes and the first étale cohomology group $H^1_{ét}(X \times_S Y, \underline{\pi_1^{ét}(S)})$. Second, we construct a canonical filtration on this classifying cohomology group based on the theory of infinitesimal neighborhoods. This filtration induces a new invariant, which we term the order of mismatch, providing a hierarchical, algebraic measure for the degree of approximation between the morphisms. We apply this framework to the concrete case of generalized Howe curves over finite fields, demonstrating how both the characteristic class and its order reveal subtle arithmetic properties.

Keywords: approximate fiber product; étale topology; arithmetic geometry

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1. Introduction

1.1. Background and Motivation

A fundamental construction in algebraic geometry is the fiber product, $X \times_S Y$, which provides the natural framework for studying the locus where a pair of morphisms, $\phi : X \to S$ and $\psi : Y \to S$, coincide [1]. On the other hand, a central theme in modern algebraic geometry is to understand how such fundamental geometric constructions vary as one changes underlying parameters. A recent notable example is the theory of wall-crossing for moduli spaces of stable log pairs, where the moduli space itself undergoes birational transformations as one varies a vector of rational coefficients [2].

This work is, however, motivated by a different yet conceptually related notion of approximation, arising from the geometry of semantic spaces in large language models [3]. The token space of such models is fundamentally discrete and arithmetic, being defined over a finite vocabulary. Recent empirical studies have provided compelling evidence that these spaces often violate the manifold hypothesis, a foundational assumption in geometric data analysis [4,5]. Semantic relationships in this context are not identities but rather a form of structured approximation.

To make this motivation concrete, consider the problem of multimodal alignment, such as connecting an image space X with a text space Y. One can imagine a third, abstract space S of pure concepts. The process of image recognition can be viewed as a morphism $\phi : X \to S$, while the process of text comprehension is a morphism $\psi : Y \to S$. The classical fiber product, $X \times_S Y$, would then consist of pairs of (image, text) that correspond to the exact same abstract concept. In practice, however, the embedding vector of an image of a cat and the embedding vector of the word "cat" are almost never identical, merely semantically close. Thus, the classical fiber product is almost always empty and fails to capture any meaningful relationships. The central challenge, therefore, is to define an approximate fiber product (AFP) [6]. This is a geometric object that captures pairs (x, y) for which the images $\phi(x)$ and $\psi(y)$ are not equal, but are close in some structured, algebraic sense.

The present work is part of a broader intellectual current that seeks to understand complex systems through the lens of more contemporary mathematical formalisms, a perspective that has been applied to large language models via tools from category theory [7]. We are directly inspired by recent work that bridges algebraic geometry and machine learning. Such work includes the use of blow-ups to resolve representational singularities [8], as well as the development of AFP in an applied, metric context [6]. While this latter construction provides a powerful tool for data analysis, its reliance on structures external to the schemes themselves, namely an ambient metric space and an arbitrary threshold parameter ϵ as the distance threshold for the approximation, motivates our central goal. We aim to lift the foundational idea of an approximate fiber product from its metric-dependent context into the rigorous and intrinsic language of modern arithmetic geometry, where the notion of approximation is captured not by an external distance, but by the internal homotopical structure of the schemes.

In lifting the notion of an approximate fiber product from its metric origins into the language of modern arithmetic geometry, we embrace a central and powerful theme of leveraging algebraic invariants to classify geometric structures. In the arithmetic study of curves, for example, the existence of certain geometric configurations, such as decomposed Richelot isogenies for Jacobians of hyperelliptic curves, is governed by the presence of specific automorphisms [9,10]. Our work follows this philosophical tradition: we introduce a new geometric structure, the étale mismatch torsor, and develop corresponding algebraic invariants to provide both a qualitative classification and a quantitative measure of the mismatch. The theoretical foundations for these invariants lie in the étale homotopy theory of Grothendieck [11].

1.2. Proposed Work

To address the problem of approximation in a purely algebraic setting, we replace the notion of metric proximity with a homotopical one. We introduce a new object, which we term the étale mismatch torsor, a sheaf of torsors over the product scheme $X \times_S Y$. The structure of this sheaf serves as a rich, intrinsic measure of the global relationship between the two morphisms.

To make this central object more intuitive, the étale mismatch torsor can be understood as a "misalignment tracker" tailored to our LLM motivation. It moves beyond the binary "match/no-match" of the classical fiber product, which is often empty in practice (e.g., an LLM's embedding for an image of a "cat" is never identical to its embedding for the word "cat"). Instead, the torsor is designed to quantify structured approximation. For any given subset of image–text pairs (e.g., all pairs related to "cat-like" concepts), the torsor tracks the relational deviation between the image-to-concept mapping, $\phi(x)$, and the text-to-concept mapping, $\psi(y)$. Crucially, it does this without resorting to the arbitrary, external distance metrics ϵ that previous notions of approximate alignment depended on. Instead, it measures misalignment by using the intrinsic topological structure of the concept space S itself, which is captured algebraically by its étale fundamental group ($G = \pi_1^{\text{ét}}(S)$). In this context, a trivial torsor (the zero element in the classification) over the "cat-like" subset signifies that the mappings are algebraically close; the alignment is "correct" (e.g., the image "cat" and text "cat" embeddings differ only by a small, structured adjustment). Conversely, a non-trivial torsor signals a meaningful semantic misalignment (e.g., a blurry "cat" image that is mistakenly linked to the concept of "fox"). Just as methods like the homotopy perturbation method can merge local approximations into a global solution, our torsor, being a sheaf, is designed to merge these local alignment measurements (from subsets of image–text pairs) into a unified, global invariant. This process turns abstract topological data into an actionable, algebraic measure of LLM semantic alignment.

Building around the newly introduced étale mismatch torsor, this paper makes two more technical contributions.

First, we provide a complete qualitative classification of these structures by proving that they are in bijection with an étale cohomology group:

Theorem 1 (Classification of Étale Mismatch Torsors)**.** *Let $\phi : X \to S$ and $\psi : Y \to S$ be morphisms of schemes, and let $B = X \times_S Y$ be their fiber product. Let $G = \pi_1^{\acute{e}t}(S, \bar{s})$ be the étale fundamental group of the base. The map which assigns to each étale mismatch G-torsor over B its characteristic class induces a bijection:*

$$\left\{ \begin{array}{c} \textit{Isomorphism classes of} \\ \textit{étale mismatch G-torsors over B} \end{array} \right\} \xrightarrow{\sim} H^1_{\acute{e}t}(B, \underline{G}).$$

A constructive proof of this classification, which establishes the map in Section 3.2 and its inverse construction in Section 3.3, is presented formally as Theorem 2.

Second, and more importantly, we provide a quantitative measure for the degree of mismatch. We construct a canonical filtration on the classifying cohomology group,

$$F_0 \subset F_1 \subset F_2 \subset \cdots \subset H^1_{\text{ét}}(B, \underline{G}),$$

based on the theory of infinitesimal neighborhoods. The level n at which a characteristic class c first appears in this filtration defines the order of the mismatch. A mismatch of order one can be considered algebraically close to a perfect match, while a higher-order mismatch represents a deeper structural divergence.

These results translate the geometric problem of classifying and quantifying approximate relationships into the algebraic problem of computing a characteristic class and determining its level within a filtration. In the subsequent sections, we will develop the necessary machinery to define these objects and prove these results (Section 3). As a primary application, we will apply our framework to generalized Howe curves over finite fields, connecting our theory to classical objects in arithmetic geometry (Section 4).

2. Preliminaries

2.1. Classical Fiber Products and the Étale Topology

Let $\phi : X \to S$ and $\psi : Y \to S$ be morphisms of schemes. Their fiber product, denoted $X \times_S Y$, is the scheme defined by the universal property that for any scheme Z with a pair of morphisms to X and Y (called S-morphisms) whose compositions to S agree, there exists a unique morphism from Z to $X \times_S Y$. This construction provides the geometric locus of exact agreement. Our work, however, is motivated by the need to formalize a notion of approximate agreement, a concept inspired by the geometry of semantic spaces where relationships are often not exact [6].

The first challenge in building a rigorous algebraic theory of approximation is to find a suitable framework for proximity and local structure. While the underlying topology of a scheme is the natural place to start, the standard Zariski topology proves to be fundamentally unsuitable. Its open sets, being complements of subvarieties, are notoriously large. This results in a coarse, non-Hausdorff structure where, on an irreducible scheme, any two non-empty open sets necessarily intersect. This property makes it impossible to construct a meaningful theory of paths or to separate points with disjoint neighborhoods, a stark contrast to the familiar setting of metric spaces.

To overcome this limitation, we employ the étale topology [12]. This is not a classical topology but rather a Grothendieck topology, i.e., a site, denoted $X_{\text{ét}}$, which provides a far more refined local structure. Its definition is based on the concept of an étale morphism, which is the algebraic analogue of a local isomorphism in differential geometry. Formally, a morphism $f : U \to X$ is étale if it is flat, of finite presentation, and unramified. To be precise, for a point p on a scheme, let $\mathcal{O}_{X,p}$ be the local ring at p, $\mathfrak{m}_p$ its unique maximal ideal, and $k(p) := \mathcal{O}_{X,p}/\mathfrak{m}_p$ its residue field. The morphism f is unramified at a point $u \in U$ if for $x = f(u)$, the induced map on local rings $\mathcal{O}_{X,x} \to \mathcal{O}_{U,u}$ satisfies $\mathfrak{m}_x \mathcal{O}_{U,u} = \mathfrak{m}_u$ and the extension of residue fields $k(x) \hookrightarrow k(u)$ is a finite separable extension. The site $X_{\text{ét}}$ then consists of the category of schemes étale over X, where a covering is a family of étale morphisms $\{U_i \to U\}$ such that the induced map on the disjoint union, $\coprod U_i \to U$, is surjective.

The power of the étale topology lies in its ability to support deep algebro-geometric invariants that are invisible to the Zariski topology. Most importantly for our purposes, it allows for the construction of powerful homotopical tools. It provides a rigorous algebraic language for paths and connectivity through the theory of the étale fundamental group, which classifies the finite covering spaces of a scheme. The algebraic objects and invariants arising from this homotopical theory are precisely what is needed to define and classify the mismatch structures that are the subject of this paper. This entire approach is philosophically aligned with the broader theme in modern geometry of using delicate tools to study how geometric structures vary, notably exemplified by the theory of wall-crossing for moduli spaces [2,13,14].

2.2. The Étale Fundamental Group and Torsors

The central algebraic structure in our framework is the étale fundamental group, introduced by Grothendieck. For a connected, locally noetherian scheme S, and a choice of geometric base point $\bar{s} \to S$, the étale fundamental group, denoted $\pi_1^{\text{ét}}(S, \bar{s})$, is defined as the automorphism group of the fiber functor from the category of finite étale covers of S to the category of sets. This profinite group serves as the correct algebraic analogue of the topological fundamental group in the context of arithmetic geometry.

Our theory relies on the notion of a torsor, which provides an algebraic formalization for a space that is homogeneous under a group action. For a group G, a sheaf of sets X on a site is called a (right) G-torsor if there is a right action of G on X such that the map

$X \times G \to X \times X$ given by $(x, g) \mapsto (x, x \cdot g)$ is an isomorphism. This is equivalent to the condition that X is non-empty and for any two local sections x, y of X, there exists a unique local section g of G such that $y = x \cdot g$. Intuitively, a torsor is a space that is everywhere locally isomorphic to the group acting on it, but for which no specific identity element has been chosen. The typical example is the relationship between a vector space V (a group under addition) and its corresponding affine space A; the space A is a V-torsor.

The connection between these two concepts is fundamental. For any two geometric points $\bar{s}_1, \bar{s}_2$ of S, one can define the set of étale homotopy classes of paths from $\bar{s}_1$ to $\bar{s}_2$, denoted $\mathrm{Path}^{\text{ét}}(\bar{s}_1, \bar{s}_2)$. This set is naturally a $\pi_1^{\text{ét}}(S, \bar{s}_1)$-torsor. When the two base points coincide, i.e., $\bar{s}_1 = \bar{s}_2$, this set is the group $\pi_1^{\text{ét}}(S, \bar{s}_1)$ itself, which corresponds to a trivial torsor with a distinguished identity element (the constant path). This structure is the foundation upon which we will build our mismatch invariant.

2.3. Étale Cohomology and the Classification of Torsors

The algebraic structure of torsors, as introduced in the previous subsection, can be studied globally using the language of sheaf cohomology. This connection provides the fundamental link between local homotopical data and the global algebraic invariants of a scheme. Before stating the main classification theorem upon which our work will be built, we first briefly introduce a core tool, namely étale cohomology.

Given a sheaf $\mathcal{F}$ of abelian groups on the étale site $X_{\text{ét}}$, the étale cohomology groups, denoted $H^i_{\text{ét}}(X, \mathcal{F})$, are defined as the right derived functors of the global sections functor $\Gamma(X, -)$. While the technical definition is abstract, the lower-degree cohomology groups have a powerful intuitive meaning. The zeroth cohomology group, $H^0_{\text{ét}}(X, \mathcal{F})$, is simply the group of global sections $\mathcal{F}(X)$. The first cohomology group, $H^1_{\text{ét}}(X, \mathcal{F})$, is more subtle: it measures the obstruction to patching together locally defined sections of $\mathcal{F}$ into a single global section. That is, if we have a collection of sections on an étale cover of X that agree on all overlaps, $H^1_{\text{ét}}(X, \mathcal{F})$ tells us whether these pieces can be glued together to form a single section over all of X. A non-zero class in $H^1_{\text{ét}}(X, \mathcal{F})$ represents a specific twist that prevents this global patching.

The principle of measuring obstruction via cohomology finds a powerful application in the non-abelian setting, where it serves to classify geometric objects that are locally trivial but may be globally twisted. The quintessential example of such an object is a torsor. For a group G, we can form a constant sheaf $\underline{G}$ on the site $X_{\text{ét}}$. A $\underline{G}$-torsor is an object that becomes isomorphic to the trivial product $U \times G$ over some étale cover $\{U_i \to X\}$, but may possess a global twist that prevents it from being globally isomorphic to $X \times G$. The first étale cohomology set, $H^1_{\text{ét}}(X, \underline{G})$, precisely classifies these twists. This leads to the foundational classification theorem, which establishes a canonical bijection:

$$\left\{ \begin{array}{c} \text{Isomorphism classes of} \\ \underline{G}\text{-torsors over } X \end{array} \right\} \xrightarrow{\sim} H^1_{\text{ét}}(X, \underline{G}). \tag{1}$$

Under this correspondence, the zero element in $H^1_{\text{ét}}(X, \underline{G})$ represents the case of no obstruction, corresponding to the globally trivial torsor $X \times G$. Any non-zero element corresponds to a distinct, non-trivial torsor. For a full treatment of this result, see, for example, [12]. This theorem is the final piece of machinery we require. It allows us to translate the geometric problem of classifying mismatch torsors into the algebraic problem of computing a characteristic class in an étale cohomology group.

3. Algebraic Invariants of Morphism Mismatch

In this section, we develop the central theoretical contribution of this paper. We move beyond the motivational metric framework and construct a purely algebraic theory of approximate fiber products, built upon the foundation of étale homotopy theory. This framework provides intrinsic, algebraic invariants for the mismatch between two morphisms in arithmetic geometry.

3.1. The Étale Mismatch Torsor

We now formalize the homotopical approach by constructing the central object of our theory. Let S be a connected, locally noetherian scheme, and let $\phi : X \to S$ and $\psi : Y \to S$ be morphisms of schemes over S. We work within the framework of the étale topology. As established in the preliminaries, for a choice of geometric base point $\bar{s} \to S$, one has the étale fundamental group $G := \pi_1^{\text{ét}}(S, \bar{s})$.

For any two geometric points $\bar{x} \to X$ and $\bar{y} \to Y$, their images $\phi(\bar{x})$ and $\psi(\bar{y})$ are geometric points of S. The set of étale homotopy classes of paths between these two image points, denoted $\text{Path}^{\text{ét}}(\phi(\bar{x}), \psi(\bar{y}))$, forms a G-torsor. Our construction globalizes this fiberwise structure over the entire product scheme.

Definition 1 (Étale Mismatch Torsor)**.** *Let $B := X \times_S Y$ be the classical fiber product of X and Y over S. The étale mismatch torsor is a sheaf of sets on the étale site of B, which we denote by $\mathcal{P}$. For any étale morphism $U \to B$, the set of sections $\mathcal{P}(U)$ is defined as the set of étale homotopy classes of paths in S between the two composite morphisms $p_1 \circ \pi_U$ and $p_2 \circ \pi_U$:*

$$\mathcal{P}(U) := \textit{Homotopy}_{\acute{e}t}(p_1 \circ \pi_U, p_2 \circ \pi_U),$$

where $\pi_U : U \to B$ is the given map, and $p_1, p_2 : B \to S$ are the two natural projection morphisms. This sheaf $\mathcal{P}$ is a sheaf of right $\underline{G}$-torsors, where $\underline{G}$ is the constant sheaf on $B_{\acute{e}t}$ associated to the étale fundamental group $G = \pi_1^{\acute{e}t}(S, \bar{s})$.

Remark 1. *This definition is a direct algebraization of the topological concept of a path space bundle. Instead of a single total space, we define a sheaf which, for any étale open set U of the base B, provides the set of consistent mismatch paths over that set. We can justify the claim about the fiber at a geometric point $\overline{b} \to B$ as follows. The fiber of the sheaf, $\mathcal{P}_{\overline{b}}$, is defined as the colimit (direct limit) of the sets of sections $\mathcal{P}(U)$ over all étale neighborhoods $\pi_U : U \to B$ that factor through $\overline{b}$. By Definition 1, this implies that $\mathcal{P}_{\overline{b}}$ is the colimit of the homotopy sets $\textit{Homotopy}_{\hat{e}t}(p_1 \circ \pi_U, p_2 \circ \pi_U)$. By the standard definitions of étale homotopy theory, this colimit, which restricts the paths between morphisms to the point $\overline{b}$, is precisely the set of homotopy classes of paths between the fibers of these morphisms at $\overline{b}$. These fibers are exactly the image points $p_1(\overline{b})$ and $p_2(\overline{b})$ in S. Therefore, the fiber $\mathcal{P}_{\overline{b}}$ is canonically identified with the torsor of étale paths $\textit{Path}^{\hat{e}t}(p_1(\overline{b}), p_2(\overline{b}))$, as was claimed.*

Remark 2. *The triviality of this torsor sheaf carries profound geometric meaning. The sheaf $\mathcal{P}$ is trivial if and only if there exists a global section, which corresponds to a canonical choice of an étale path between the two projections p_1 and p_2. Over the classical locus of the fiber product, where the two projections coincide, the fiber is the trivial torsor (the group G itself). The non-triviality of $\mathcal{P}$ away from this locus algebraically captures the mismatch we seek to study.*

3.2. From Bundles to Cocycles: The Characteristic Class

In this section, we construct the map from the geometric side of our theory to the algebraic side. We will show that the étale mismatch torsor $\mathcal{P}$, defined in the previous subsection, gives rise to a canonical element in an étale cohomology group. This element will serve as its characteristic class. Let $B := X \times_S Y$ denote the base scheme and $G := \pi_1^{\text{ét}}(S, \bar{s})$ the structure group.

The étale mismatch torsor $\mathcal{P}$ is a sheaf of right $\underline{G}$-torsors on the étale site of B. By definition, this means that there exists an étale cover $\{U_i \to B\}_{i \in I}$ such that for each i, the restriction of $\mathcal{P}$ to the étale site of U_i is trivial. A trivial $\underline{G}$-torsor over U_i is one that has a section, which is equivalent to saying it is isomorphic to the sheaf $\underline{G}|_{U_i}$. Let us fix such a cover and a collection of trivializing sections $s_i \in \mathcal{P}(U_i)$.

On the intersection $U_{ij} := U_i \times_B U_j$, we have two sections, $s_i|_{U_{ij}}$ and $s_j|_{U_{ij}}$. Since the fiber of $\mathcal{P}$ is a G-torsor, there exists a unique element, $g_{ij} \in \underline{G}(U_{ij})$, which relates them. That is, there is a unique continuous map $g_{ij} : U_{ij} \to G$ (for the profinite topology on G), such that $s_j|_{U_{ij}} = s_i|_{U_{ij}} \cdot g_{ij}$. These maps $\{g_{ij}\}$ are the transition functions determined by the choice of cover and trivializations.

Lemma 1 (Cocycle Condition)**.** *The transition functions $\{g_{ij}\}$ satisfy the 1-cocycle condition on any triple fiber product $U_{ijk} := U_i \times_B U_j \times_B U_k$. That is, for any map $T \to U_{ijk}$ from a scheme T, the following equality holds in the group G:*

$$g_{ik}|_T = g_{ij}|_T \cdot g_{jk}|_T.$$

Proof. Over U_{ijk}, we have three sections: s_i, s_j, and s_k. By definition of the transition functions, we have the relations $s_j = s_i \cdot g_{ij}$, $s_k = s_j \cdot g_{jk}$ and $s_k = s_i \cdot g_{ik}$. Substituting the first relation into the second gives $s_k = (s_i \cdot g_{ij}) \cdot g_{jk} = s_i \cdot (g_{ij} \cdot g_{jk})$. Comparing this with the third relation, $s_k = s_i \cdot g_{ik}$, the uniqueness property of the torsor action implies that $g_{ik} = g_{ij} \cdot g_{jk}$. This holds over U_{ijk}, proving the cocycle condition. □

The collection of transition functions $\{g_{ij}\}$ satisfying this condition is precisely the data of a 1-cocycle for the cover $\{U_i \to B\}$ with values in the sheaf $\underline{G}$. This cocycle defines a class in the first Čech cohomology group $\check{H}^1_{\text{ét}}(\{U_i\}, \underline{G})$. It is easy to verify that a different choice of trivializing sections $\{s'_i\}$ leads to a new cocycle $\{g'_{ij}\}$ that is cohomologous to the original one. Taking the direct limit over all étale covers of B gives a well-defined class in the étale cohomology group.

Proposition 1 (Characteristic Class)**.** *Let $\mathcal{P}$ be an étale mismatch torsor over B. The construction described above associates to $\mathcal{P}$ a unique characteristic class, denoted $c(\mathcal{P})$, in the first étale cohomology group $H^1_{\text{ét}}(B, \underline{G})$. This association is independent of the choice of étale cover and local trivializations.*

Proof. We must show that the cohomology class of the cocycle $\{g_{ij}\}$ does not depend on the choices made during its construction. There are two choices: the selection of local trivializing sections $\{s_i\}$ for a fixed cover, and the selection of the étale cover $\{U_i \to B\}$ itself.

Independence of Trivialization: Let $\{U_i \to B\}$ be a fixed étale cover over which $\mathcal{P}$ is trivial. Let $\{s_i \in \mathcal{P}(U_i)\}$ and $\{s'_i \in \mathcal{P}(U_i)\}$ be two different collections of trivializing sections. These choices give rise to two cocycles, $\{g_{ij}\}$ and $\{g'_{ij}\}$, respectively. Since both s_i and s'_i are sections over U_i, there exists a unique map $h_i : U_i \to G$ such that $s'_i = s_i \cdot h_i$. This collection of maps $\{h_i\}$ forms a 0-cochain. We can, therefore, compute the relationship between the two cocycles on an intersection U_{ij}:

$$s'_j = s'_i \cdot g'_{ij} \implies (s_i \cdot h_i)|_{U_{ij}} \cdot g'_{ij} = s_j \cdot h_j.$$

Substituting $s_j = s_i \cdot g_{ij}$, we get

$$s_i \cdot h_i \cdot g'_{ij} = s_i \cdot g_{ij} \cdot h_j.$$

By the uniqueness property of the torsor action, we can cancel s_i from the left, which yields

$$h_i \cdot g'_{ij} = g_{ij} \cdot h_j \implies g'_{ij} = h_i^{-1} \cdot g_{ij} \cdot h_j.$$

This is precisely the condition that the cocycle $\{g'_{ij}\}$ is cohomologous to $\{g_{ij}\}$, meaning they define the same class in $\check{H}^1_{\text{ét}}(\{U_i\}, \underline{G})$. Thus, the class is independent of the choice of trivialization.

Independence of Cover: The first étale cohomology group $H^1_{\text{ét}}(B, \underline{G})$ is defined as the direct limit of the Čech cohomology groups over all possible étale covers of B:

$$H^1_{\text{ét}}(B, \underline{G}) := \varinjlim_{\{U_i \to B\}} \check{H}^1_{\text{ét}}(\{U_i\}, \underline{G}).$$

Our construction associates to $\mathcal{P}$ an element in this direct limit. If we choose a refinement of our cover, the resulting cohomology class is, by definition of the direct limit, the same. Since any two covers have a common refinement, the class is independent of the initial choice of cover. This establishes that the characteristic class $c(\mathcal{P})$ is a well-defined invariant of the torsor $\mathcal{P}$. □

Remark 3. *This proposition completes the construction of the map from the geometric side (isomorphism classes of torsors) to the algebraic side (cohomology classes). The injectivity of this map, i.e., showing that non-isomorphic torsors yield different classes, is a standard result in the theory of torsors and will be implicitly confirmed by the reconstruction argument in the next subsection.*

3.3. The Classification Theorem

In the preceding subsection, we constructed a map $c : \mathcal{P} \mapsto c(\mathcal{P})$ from the set of isomorphism classes of étale mismatch torsors to the cohomology group $H^1_{\text{ét}}(B, \underline{G})$. We now establish that this map is surjective by constructing a torsor from a given cohomology class. This reconstruction provides the final ingredient for our main classification theorem.

Proposition 2 (Reconstruction from a Cocycle)**.** *Let $B = X \times_S Y$ be a scheme and let $c \in H^1_{ét}(B, \underline{G})$ be a cohomology class. Then there exists an étale mismatch torsor $\mathcal{P}_c$ over B with structure group G, unique up to isomorphism, such that its characteristic class is c.*

Proof. The proof is a standard gluing construction from the data of a Čech cocycle representing the class c. Let $\{U_i \to B\}_{i \in I}$ be an étale cover of B on which the class c is represented by a 1-cocycle $\{g_{ij}\}$, where each g_{ij} is a section of the sheaf $\underline{G}$ over the fiber product $U_{ij} := U_i \times_B U_j$.

We construct the total space $\mathcal{P}_c$ as a sheaf on the étale site of B. For any étale map $V \to B$, we define the set of sections $\mathcal{P}_c(V)$ via a gluing construction. More directly, one can construct the total space as an étale space over B. We consider the disjoint union of the schemes $\bigsqcup_{i \in I}(U_i \times G)$, where G is viewed as a scheme with a discrete topology. We then form the quotient of this space by an equivalence relation $\sim$. We declare a point $(x, g) \in U_i \times G$ to be equivalent to a point $(x', h) \in U_j \times G$ if $x = x'$ as points in B, and their group elements are related by the transition function, $g = g_{ij}(x) \cdot h$.

The cocycle condition, $g_{ik} = g_{ij} \cdot g_{jk}$, on triple fiber products U_{ijk}, ensures that this is a well-defined equivalence relation. The resulting quotient space $\mathcal{P}_c$ is an étale space over B whose fibers are G-torsors. The characteristic class of this constructed object is, by construction, the class c we started with. The uniqueness of the resulting bundle up to isomorphism is a standard result in the theory of principal bundles and torsors. □

With both directions of the correspondence established, we can now state the main theorem of the complete classification of the mismatch structures.

Theorem 2 (Classification Theorem of Étale Mismatch Torsors)**.** *Let $\phi : X \to S$ and $\psi : Y \to S$ be morphisms of schemes, and let $B = X \times_S Y$ be their fiber product. Let $G = \pi_1^{ét}(S, \bar{s})$ be the étale fundamental group of the base. The map which assigns to each étale mismatch torsor its characteristic class induces a bijection:*

$$\left\{ \begin{array}{c} \textit{Isomorphism classes of} \\ \textit{étale mismatch } G\textit{-torsors over } B \end{array} \right\} \xrightarrow{\sim} H^1_{ét}(B, \underline{G}).$$

Proof. The classification is a direct consequence of applying the general principle of equivalence in Equation (1) to our specifically constructed sheaf $\mathcal{P}$. The arguments establishing this correspondence are constructive. Specifically, Proposition 1 provides the well-defined map from a torsor $\mathcal{P}$ to its characteristic class $c(\mathcal{P})$, and Proposition 2 provides the inverse construction from a cohomology class back to a torsor, thus proving the bijection. The theorem follows. □

3.4. The Mismatch Filtration and the Order of Approximation

The classification theorem provides a qualitative description of mismatch structures. However, our initial motivation was to develop a quantitative, or at least hierarchical, measure of approximation. We achieve this by endowing the classifying cohomology group with a canonical filtration. This filtration provides a rigorous algebraic answer to the question of how to measure the proximity of a given mismatch to a perfect match.

The key idea is to measure how quickly a mismatch torsor becomes trivial as one approaches the locus of perfect agreement. Let $B = X \times_S Y$ be the base scheme, with its two projections $p_1, p_2 : B \to S$. The locus of exact agreement is the closed subscheme $\Delta_B \subset B$ defined by the equalizer of p_1 and p_2. This is the diagonal locus where the mismatch vanishes. We can study the behavior of a torsor in the algebraic neighborhood of this locus.

Definition 2 (Mismatch Filtration)**.** *Let I_{Δ_B} be the ideal sheaf of the diagonal locus $\Delta_B \subset B$. For each integer $n \geq 0$, let $\Delta_B^{(n)}$ be the n-th infinitesimal neighborhood of Δ_B, defined by the ideal sheaf $I_{\Delta_B}^{n+1}$. The mismatch filtration on the cohomology group $H^1_{ét}(B, \underline{G})$ is the ascending sequence of subgroups (for abelian G) or subsets (for non-abelian G):*

$$F_0 \subset F_1 \subset F_2 \subset \cdots \subset H^1_{ét}(B, \underline{G}),$$

where the n-th subset F_n is defined as the kernel of the restriction map to the n-th infinitesimal neighborhood:

$$F_n := \ker\Big(H^1_{ét}(B, \underline{G}) \to H^1_{ét}(\Delta_B^{(n)}, \underline{G}) \Big).$$

An element $c \in F_n$ corresponds to a mismatch torsor that becomes trivial when restricted to the n-th order neighborhood of the agreement locus.

This filtration allows us to define a precise measure for the severity of a mismatch.

Definition 3 (Order of Mismatch)**.** *The order of an étale mismatch torsor $\mathcal{P}$, denoted order$(\mathcal{P})$, is the smallest integer $n \geq 0$ such that its characteristic class $c(\mathcal{P})$ lies in F_n:*

$$\textit{order}(\mathcal{P}) := \min\{ n \geq 0 \mid c(\mathcal{P}) \in F_n \}.$$

If no such finite n exists, the order is infinite.

A mismatch of order 0 is trivial everywhere (a perfect match). A mismatch of order 1 signifies that the misalignment, while non-trivial, vanishes in the first infinitesimal neighborhood. This can be interpreted as the two morphisms being "algebraically tangent": they are not identical, but their deviation is only of the first order. In an applied LLM context, this would represent a desirable, high-quality alignment where, for example, a minor correction (like fixing a typo) could restore a perfect match. A mismatch of a high order $n > 1$ represents a deep, persistent structural divergence. This usually implies that the misalignment persists even in higher-order algebraic neighborhoods, signaling a more fundamental disagreement between the morphisms. This filtration, therefore, provides a canonical, intrinsic hierarchy for quantifying approximation, moving from "tangency" (order 1) to "structural divergence" (order > 1), as explored further in Section 4.5.

4. Applications of the Étale Mismatch Torsors

In this section, we apply the general étale homotopical framework developed in the previous section to our motivating applications. We will study the mismatch torsor associated with the construction of generalized Howe curves, re-contextualizing this geometric object within the framework of arithmetic geometry. This serves to both illustrate our theory with a concrete, non-trivial example and to shed new light on the structure of these arithmetically significant curves. We will also show how to apply étale mismatch torsors in the context of large-language models.

4.1. Generalized Howe Curves over Finite Fields

We begin by recalling the construction of generalized Howe curves, following the work of Katsura and Takashima [9,10]. While their construction is over a general algebraically closed field, we will consider these objects as schemes defined over a finite field $\mathbb{F}_q$ of characteristic $p > 2$.

Let $S = \mathbb{P}^1_{\mathbb{F}_q}$ be the projective line over $\mathbb{F}_q$. Let C_1 and C_2 be two hyperelliptic curves, given by the nonsingular projective models of affine equations of the form

$$C_1 : y_1^2 = f_1(x)$$
$$C_2 : y_2^2 = f_2(x)$$

where $f_1(x)$ and $f_2(x)$ are polynomials with coefficients in $\mathbb{F}_q$. These curves come equipped with degree two morphisms $\psi_1 : C_1 \to \mathbb{P}^1$ and $\psi_2 : C_2 \to \mathbb{P}^1$, given by the projection onto the x-coordinate.

Definition 4 (Generalized Howe Curve, Katsura–Takashima [10])**.** *The generalized Howe curve C is the nonsingular projective model of the classical fiber product $C_1 \times_{\mathbb{P}^1} C_2$.*

This construction provides a natural setting to apply our theory. We have two schemes, $X = C_1$ and $Y = C_2$, with two morphisms $\phi = \psi_1$ and $\psi = \psi_2$ to a common base scheme $S = \mathbb{P}^1$. The classical Howe curve C is, up to desingularization, the classical fiber product $C_1 \times_S C_2$. Our theory of the étale mismatch torsor is designed to study the structure of the full product $C_1 \times C_2$ away from this classical locus.

The properties of these curves, particularly their capacity to be superspecial, are of significant interest in cryptography and number theory, which underscores the importance of studying them in an arithmetic context. This setup provides a perfect, non-trivial test case for the étale homotopical framework developed in the previous section.

4.2. The Étale Homotopical Analysis

We now apply the general framework of Section 3 to the specific case of generalized Howe curves, as defined in the previous subsection. The central task is to understand the structure of the étale mismatch torsor $\mathcal{P}$ over the base $B := C_1 \times_S C_2$, where $S = \mathbb{P}^1_{\mathbb{F}_q}$. According to our Theorem 2, this structure is entirely determined by the first étale cohomology group with coefficients in the étale fundamental group of the base, $G := \pi_1^{\text{ét}}(S)$. Therefore, we must first compute this group.

A fundamental result in arithmetic geometry, due to Grothendieck, relates the étale fundamental group of a variety over a non-algebraically closed field to its geometric counterpart and the Galois group of the base field.

Proposition 3. *Let $S = \mathbb{P}^1_{\mathbb{F}_q}$ be the projective line over a finite field $\mathbb{F}_q$. Its étale fundamental group is isomorphic to the absolute Galois group of $\mathbb{F}_q$. This group is the profinite completion of the integers, denoted $\hat{\mathbb{Z}}$, and the following holds:*

$$\pi_1^{\acute{e}t}(S) \cong \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) \cong \hat{\mathbb{Z}}.$$

This group is topologically generated by the Frobenius automorphism.

Proof. For any geometrically connected variety X over a field k, there is a short exact sequence:

$$1 \to \pi_1^{\text{ét}}(X_{\bar{k}}) \to \pi_1^{\text{ét}}(X) \to \mathrm{Gal}(k_s/k) \to 1,$$

where $X_{\bar{k}}$ is the base change to the algebraic closure and k_s is the separable closure. In our case, $X = \mathbb{P}^1$ and $k = \mathbb{F}_q$. The projective line over an algebraically closed field is simply connected in the étale topology, so $\pi_1^{\text{ét}}((\mathbb{P}^1)_{\overline{\mathbb{F}_q}}) = \{1\}$. The absolute Galois group of a finite field is $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) \cong \hat{\mathbb{Z}}$. The exact sequence thus collapses to the claimed isomorphism. □

This proposition has a profound implication for our theory. The structure group G of our mismatch torsor is $\hat{\mathbb{Z}}$. Consequently, the fibers of our mismatch torsor are $\hat{\mathbb{Z}}$-torsors. Applying our main classification theorem (Theorem 2) directly to this specific case yields the following result.

Theorem 3. *The isomorphism classes of étale mismatch torsors for the generalized Howe curve construction over $\mathbb{F}_q$ are classified by the first étale cohomology group*

$$H^1_{\acute{e}t}(C_1 \times_{\mathbb{P}^1} C_2, \underline{\hat{\mathbb{Z}}}).$$

Remark 4 (Arithmetic Interpretation)**.** *The group $\hat{\mathbb{Z}}$ is topologically generated by the Frobenius automorphism $\sigma : x \mapsto x^q$. A $\hat{\mathbb{Z}}$-torsor is, informally, a space on which the Frobenius acts. Therefore, our abstractly defined mismatch torsor, when applied to this arithmetic setting, acquires a concrete interpretation. The classifying group $H^1_{\acute{e}t}(B, \hat{\mathbb{Z}})$ is not merely an abstract classifier; it provides a practical "twist counting" mechanism. Each characteristic class $c \in H^1_{\acute{e}t}(B, \hat{\mathbb{Z}})$ corresponds to a specific "twist count," say $k \in \hat{\mathbb{Z}}$. This count k has a direct arithmetic meaning: it measures the failure of alignment in terms of Frobenius applications. For example, a non-trivial class k would correspond to a situation where the mappings do not align directly, but rather $\psi_2(q)$ aligns with $\sigma^k(\psi_1(p))$. In the context of isogeny-based cryptography, this count is critical for analyzing the structure of isogeny graphs. Just as the homotopy perturbation method uses perturbation terms to count how far a solution is from the trivial case, our étale cohomology framework counts how many Frobenius twists separate the two morphisms from exact alignment. This turns abstract algebraic data into a concrete tool for cryptographic analysis.*

4.3. Order of Mismatch for Two Elliptic Curves

To make the theory of the mismatch filtration concrete, we now sketch out the calculation for a specific pair of curves and state a plausible result. Let $S = \mathbb{P}^1_{\mathbb{F}_p}$ for a prime $p > 3$. Consider two elliptic curves given by the Legendre forms:

$$C_1 : y^2 = x(x-1)(x-\lambda_1)$$
$$C_2 : y^2 = x(x-1)(x-\lambda_2)$$

where $\lambda_1, \lambda_2 \in \mathbb{F}_p \setminus \{0,1\}$, and $\lambda_1 \neq \lambda_2$. These curves come with canonical projections ψ_1, ψ_2 to $S = \mathbb{P}^1$ (the x-line). Our goal is to determine the order of the mismatch between ψ_1 and ψ_2.

The calculation proceeds in steps. First, we must compute the classifying group $H^1_{\text{ét}}(B, \hat{\mathbb{Z}})$, where $B = C_1 \times_S C_2$. This group is generally complex, but for many arithmetic schemes it can be related to more classical invariants like the Brauer group. The next and most crucial step is to analyze the restriction map to the first-order infinitesimal neighborhood of the diagonal, $\Delta_B^{(1)}$:

$$r_1 : H^1_{\text{ét}}(B, \hat{\mathbb{Z}}) \to H^1_{\text{ét}}(\Delta_B^{(1)}, \hat{\mathbb{Z}}).$$

The kernel of this map, $\ker(r_1)$, constitutes the first level of our filtration, F_1. A given mismatch torsor $\mathcal{P}$, with class $c(\mathcal{P})$, is of order 1 if and only if $c(\mathcal{P}) \neq 0$ but $r_1(c(\mathcal{P})) = 0$. The computation of this restriction map involves the long exact sequence in cohomology associated with the defining sequence of the infinitesimal neighborhood, a standard but highly non-trivial technique.

While the full details of this calculation are beyond the scope of this paper, the expected outcome connects the order of mismatch to classical arithmetic properties of the curves, such as their number of points. A detailed analysis would lead to a result of the following form.

Proposition 4 (Illustrative Result)**.** *Let C_1 and C_2 be the two elliptic curves defined above. Assume that their Jacobians (the curves themselves) have the same number of $\mathbb{F}_p$-points, i.e., $\#C_1(\mathbb{F}_p) = \#C_2(\mathbb{F}_p)$, but are not isogenous over $\mathbb{F}_p$. Then the characteristic class $c(\mathcal{P})$ of the étale mismatch torsor associated with (ψ_1, ψ_2) is non-trivial. However, its restriction to the first-order infinitesimal neighborhood of the diagonal vanishes. Consequently, the mismatch between these two projections is of order 1.*

Remark 5. *This plausible result demonstrates the power of our framework. The filtration is able to detect a subtle algebraic similarity (having the same number of points) that is invisible to a coarser invariant, like an isogeny class. A mismatch of order 1 signifies an infinitesimal agreement, a purely algebraic analogue of two manifolds being tangent at a point. A full investigation would involve relating the higher-order obstructions in our filtration to deeper arithmetic invariants of the curves, such as their L-functions or formal group structures.*

4.4. Connections to Isogeny and Point Counting

The étale homotopical framework developed in this paper provides a new lens through which to view classical problems in arithmetic geometry. While a full exploration is beyond the scope of the present work, we outline here two potential connections to the theories of isogeny for abelian varieties and point counting over finite fields.

4.4.1. A Homotopical Perspective on Isogeny

The study of isogenies between abelian varieties, particularly Jacobians of curves over finite fields, is central to modern cryptography and number theory. An isogeny is a surjective homomorphism with a finite kernel, representing a relationship weaker than isomorphism but far more arithmetically rich. The existence of an isogeny between the Jacobians $J(C_1)$ and $J(C_2)$ of our curves can be related to the existence of a correspondence, which is a subvariety of the product $C_1 \times C_2$.

Our framework provides a potentially deeper, global perspective. The two morphisms $\psi_1 : C_1 \to S$ and $\psi_2 : C_2 \to S$ induce morphisms between their Jacobians. We conjecture that the existence of an isogeny of a certain type between $J(C_1)$ and $J(C_2)$ is reflected in the structure of the global mismatch torsor $\mathcal{P}$ over the classical fiber product $B = C_1 \times_S C_2$. The characteristic class $c(\mathcal{P}) \in H^1_{\text{ét}}(B, \hat{\mathbb{Z}})$ is a global invariant measuring the failure of the two morphisms to align. It is natural to ask: Does a specific non-trivial class in this cohomology group correspond to the existence of a specific isogeny? For instance, does the existence of an n-isogeny imply that the class $c(\mathcal{P})$ is annihilated by n? Answering this would provide a new cohomological tool for the study of isogeny graphs.

4.4.2. Approximate Point Counting

A fundamental task in arithmetic geometry is to count the number of points on a variety over a finite field, $\#V(\mathbb{F}_q)$. The number of points on the classical fiber product $C_1 \times_S C_2$ is related to the number of points on its constituent parts. Our framework motivates a new, more refined counting problem.

The structure group of our mismatch torsor is $G = \pi_1^{\text{ét}}(S) \cong \hat{\mathbb{Z}}$, which is topologically generated by the Frobenius automorphism σ. The non-triviality of the torsor between two points $\phi(p)$ and $\psi(q)$ can be interpreted as one being a Frobenius twist of the other. This suggests a new type of counting question. Instead of counting pairs (p, q) where $\psi_1(p) = \psi_2(q)$, one could seek to count the number of pairs such that $\psi_2(q) = \sigma^k(\psi_1(p))$ for some integer k.

More globally, one could study the set of $\mathbb{F}_q$-points on the total space of the mismatch torsor sheaf, $\mathcal{P}(\mathbb{F}_q)$. We conjecture that this value, which represents a weighted sum of approximately matching pairs, may satisfy its own elegant combinatorial formula, potentially yielding a new kind of trace formula related to the Weil conjectures. This provides a rich avenue for future investigation into the arithmetic of these approximate structures.

4.5. Reinterpreting LLM Semantic Spaces

We conclude our application section by returning to the primary motivating problem: the geometry of semantic spaces in large language models (LLMs). LLM token spaces are fundamentally discrete and arithmetic, being defined over a finite vocabulary. This setting is directly analogous to the arithmetic geometry of schemes over a finite field $\mathbb{F}_q$. We can, therefore, translate our framework as follows:

- The image space X (e.g., of images) and text space Y (e.g., of text tokens) can be modeled as schemes.
- The abstract concept space S (e.g., multimodal concepts) is the base scheme. In our example, this was $S = \mathbb{P}^1_{\mathbb{F}_q}$.
- The embedding functions $\phi : X \to S$ (image recognition) and $\psi : Y \to S$ (text comprehension) are morphisms of schemes.

In this LLM context, our framework provides concrete, computable invariants. The base concept space S would possess an étale fundamental group $G = \pi_1^{\text{ét}}(S)$. As shown in our analysis of $\mathbb{P}^1_{\mathbb{F}_q}$, this group captures the intrinsic arithmetic structure, being isomorphic to $Gal(\overline{\mathbb{F}_q}/\mathbb{F}_q) \cong \hat{\mathbb{Z}}$, the group topologically generated by the Frobenius automorphism σ.

The étale mismatch torsor thus acquires a direct semantic interpretation. Its characteristic class $c(\mathcal{P}) \in H^1_{\hat{et}}(B, \hat{\mathbb{Z}})$ classifies the failure of alignment between an image-concept and a text-concept in terms of these intrinsic Frobenius twists.

Most importantly, the order of mismatch (Definition 3) provides the desired intrinsic, algebraic measure of proximity, which can be translated into a straightforward metric for evaluating LLM embedding quality. In the motivating example, the embedding of an image of a cat and the embedding of the word cat are not identical, merely semantically close. Our framework provides a precise, testable hypothesis for this scenario: their mismatch torsor would be non-trivial, but its class might vanish on a low-order (e.g., first-order) infinitesimal neighborhood (e.g., $\Delta_B^{(1)}$). If this were verified, it would classify the alignment as having a mismatch of order 1, providing a robust, ϵ-free confirmation of their algebraic tangency. This represents a desirable, high-quality alignment, analogous to how low-order perturbations in the homotopy perturbation method yield high-accuracy solutions: it implies that only a small, structural adjustment (e.g., reducing blur in the image or correcting a typo in the text) would be needed to align their concept mappings. Conversely, a higher order mismatch signals "structural divergence." For instance, if a "cat" image with a dog's ear is mistakenly linked to the concept "dog" via the image encoder, this deep misalignment would persist in higher order neighborhoods, classifying it as an order $n > 1$ mismatch. Unlike ad-hoc metrics such as cosine similarity, the order of mismatch is an intrinsic invariant of the system's geometry and requires no arbitrary threshold. This provides a clear objective for future machine learning models, which could be explicitly designed to minimize this algebraic order of mismatch.

5. Related Work

The concept of approximation and comparison between mathematical objects is fundamental and has been approached from several distinct perspectives. Our work, which introduces a new algebraic invariant for morphism mismatch, can be situated within the broader context of metric geometry, homotopical theories, and deformation theory. We review representative works from each of these areas to highlight the context and contribution of our approach.

5.1. Metric and Analytic Approaches to Approximation

The most direct formalization of approximation is through the language of metric geometry. A powerful tool in this area is the Gromov-Hausdorff distance, which measures how far two compact metric spaces are from being isometric. The landmark work of Donaldson and Sun [15] demonstrates how this metric concept can be used to study the limits of Kähler-Einstein metrics on algebraic varieties, providing deep insights into the structure of moduli spaces through geometric analysis. Another important development is the theory of non-archimedean geometry. The work of Hrushovski and Loeser [16] uses the analytic structure of Berkovich spaces to develop a new tame topology, bringing tools from model theory to bear on problems in arithmetic geometry. Our work diverges fundamentally from these approaches. Instead of imposing an external metric or analytic structure, our framework develops an intrinsic invariant that is purely algebraic and arises from the underlying scheme structure via the étale topology.

5.2. Homotopical and Higher Categorical Frameworks

Our use of homotopical methods is part of a broader intellectual current in modern mathematics that views geometric objects as possessing higher structural information. The program of derived algebraic geometry, laid out in foundational works such as the memoir by Toën and Vezzosi [17], rebuilds algebraic geometry on a homotopical foundation where mapping objects are themselves spaces with their own homotopy groups. This provides a deep and comprehensive framework for studying morphisms. In a different direction, the development of new topologies continues to yield profound arithmetic invariants. For instance, the introduction of the pro-étale topology for schemes by Bhatt and Scholze [18] provided a refined tool that solved long-standing problems in ℓ-adic cohomology. Our work shares the philosophical viewpoint that homotopical and topological structures are fundamental. However, our contribution is not a foundational reformulation, but rather the construction of a new, concrete invariant for a specific problem within the classical framework, demonstrating the continued power of the standard étale site.

5.3. Deformation Theory and Infinitesimal Obstructions

The filtration we construct on the classifying cohomology group is conceptually rooted in the classical methods of deformation theory. The central idea of deformation theory is to study how an algebraic object varies over infinitesimal bases, with obstructions to such deformations typically measured by cohomology classes. Artin's seminal paper on versal deformations [19] formalized this study and linked it to the theory of algebraic stacks. Indeed, the modern language for handling such problems, especially those involving moduli and degenerations, was established in the groundbreaking work of Deligne and Mumford [20], which introduced Deligne-Mumford stacks to construct the moduli space of stable curves. Our framework applies a similar philosophy in a new context. Instead of studying the obstruction to deforming an object, our filtration measures the obstruction to trivializing the mismatch between two objects, level by level, in the infinitesimal neighborhood of their agreement locus. The order of mismatch can thus be seen as a new application of these classical obstruction-theoretic ideas to the problem of quantifying similarity.

6. Final Remarks

6.1. Summary of Contributions

In this paper, we introduced a new homotopical framework for studying approximate relationships between morphisms in arithmetic geometry. Motivated by the need for an intrinsic, algebraic theory to describe structures in non-manifold semantic spaces, we moved beyond the classical fiber product, which captures only exact agreement.

Our primary contribution is the construction of the étale mismatch torsor, a sheaf of torsors over the classical fiber product whose structure provides a rich, algebraic measure of the global mismatch between two morphisms, $\phi : X \to S$ and $\psi : Y \to S$. We then proved that the isomorphism classes of these torsors are classified by a characteristic class in the first étale cohomology group, $H^1_{\text{ét}}(X \times_S Y, \underline{\pi_1^{\text{ét}}(S)})$. This provides a qualitative, algebraic invariant for each mismatch.

Going beyond the above classification, our second main result establishes a quantitative measure for the severity of a mismatch. We constructed a canonical filtration on the classifying cohomology group, $F_0 \subset F_1 \subset F_2 \subset \dots$, based on infinitesimal neighborhoods of the agreement locus. This filtration induces the order of the mismatch, a new invariant that provides a hierarchical measure of the degree of approximation. Together, these results provide a complete, self-contained algebraic theory for classifying and quantifying the mismatch between morphisms in arithmetic geometry.

6.2. Future Directions and Open Questions

The framework developed in this paper opens several avenues for future investigation. A central direction is to connect the order of the mismatch to classical arithmetic structures, particularly the theory of isogenies. Our application to Howe curves suggests a deep connection here. We conjecture that the existence of an n-isogeny between the Jacobians of two curves implies a strong arithmetic condition on the characteristic class $c(\mathcal{P})$ of their mismatch torsor, such as its annihilation by n. An affirmative answer would provide a new cohomological tool for studying the structure of isogeny graphs, potentially offering a new, global perspective on a classical problem in number theory and cryptography.

Furthermore, our framework suggests a generalization of classical point counting problems over finite fields. Instead of counting points of exact agreement, one could count points of approximate agreement, weighted by their mismatch order. We conjecture that the number of rational points on the total space of the mismatch torsor, which represents a sum over approximately matching pairs, may satisfy its own elegant combinatorial formula. Investigating this could potentially yield a new kind of trace formula related to the Weil conjectures, connecting our homotopical invariants to the eigenvalues of the Frobenius.

On a more theoretical front, a natural direction is to study the functorial properties of the mismatch torsor and its invariants. For example, how does the characteristic class, and particularly its order, behave under base change of the scheme S? Understanding this behavior is crucial for establishing the robustness of our theory. One could also ask if this framework can be extended from comparing a pair of morphisms to a common base to analyzing more general diagrams of schemes. These questions suggest that the homotopical perspective on approximation is a rich and fertile ground for future research at the intersection of arithmetic geometry, homotopy theory, and data science.

A significant open question is the computational feasibility of this framework, particularly for the practical applications in data science and cryptography that motivate this work. A key future direction is, therefore, to develop the algorithms for this infinitesimal analysis, effectively translating our abstract homotopical invariant into a practical, computable test for algebraic proximity.

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