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A Caputo-Type Fractional Derivative of a Function with Respect to Power Functions and Solutions to Non-Local Problems of Fractional Differential Equations

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Abstract

This paper investigates the uniform continuity and strong continuity of the semigroups of the fractional integral operators of power functions. Using the Krasnoselskii's fixed-point theorem, we have studied the non-local problem related to fractional differential equations involving power functions with multi-point integral boundary conditions and obtain the existence of the solution.

Keywords: fractional calculus; fractional differential equations; non-local conditions; strongly continuous semigroup; Krasnoselskii's fixed-point

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1. Introduction

The fractional derivative originated from an initial discussion between L'Hospital and Leibnitz in 1695, but it did not attract enough attention at that time, and was considered a paradox for a long time. Many researchers have cited fractional calculus as the most useful in characterizing materials and processes with memory genetic properties up until the 2000s. Until recent decades, many researchers pointed out that fractional calculus is the most effective in characterizing materials and processes with memory genetic properties. For example, the transport of chemical pollutants around rocks through water, viscoelastic material dynamics, cell diffusion processes, and network flow. Fractional-order equations can be more accurate than integer-order differential equation while describing the physical change process (cf. [1–3]). As a branch of calculus theory, fractional differential equations have been developed in both theory and application (cf. [4–10]), especially in the modeling of abnormal phenomena [11]. There are many forms of fractional calculus, such as the Riemann–Liouville, Caputo, and Hadamard fractional calculus. In ref. [12], Erdelyi also defined fractional integration with respect to x^n for any non-zero real n . Recently, a generalized derivative has been considered in [13,14] by Katugampola, which unifies the Riemann–Liouville and Hadamard integrals into a single form. Ref. [15] presents the existence and uniqueness results for the solutions to initial value problems of the fractional differential equation with respect to a power function of order $0 < \alpha < 1$.

Usually, initial and boundary conditions cannot describe some information of physical or other processes happening inside the whole area. In order to cope with this situation, non-local conditions are found to be more valuable in modeling many physical change

processes and others (cf. [16–24]). In ref. [17], by the use of some fixed-point index theory on the cone, Bai obtain the existence of positive solutions for the equation

$$D_{0+}^{\alpha}x(t) = f(t, x(t))$$

by employing a fixed-point index theory on the cone with non-local boundary value conditions

$$x(0) = 0, \quad x(1) = \beta x(\eta),$$

where $0 < \alpha < 1$, $0 < \eta < 1$, $0 < \beta\eta^{\alpha-1} < 1$, D_{0+}^{α} is the Riemann–Liouville fractional differential operator. N’Guerekata considered the solution to the above problem when the boundary condition becomes

$$x(0) + g(x) = x_0$$

in a Banach space [18]. He proved that if f is a jointly continuous function and g is a Lipschitzian function, then the problem has a unique solution. Deng’s paper indicated that the above non-local condition is better than the initial condition $x(0) = x_0$ in physics [20].

Recently, Ahmad et al. [25] obtained the uniqueness of solutions for a boundary value problem

$$\begin{cases} ({}^{\rho}D_{0+}^{\alpha}u)(t) = f(t, u(t)), & 0 \leq t \leq T, \\ u(0) = 0, \quad \int_0^T u(s)dH(s) = \lambda({}^{\rho}I^{\beta}u)(\xi), & \xi \in (0, T), \end{cases}$$

where ${}^{\rho}D_{0+}^{\alpha}$ is the fractional differential operator with respect to a power function of order $1 < \alpha \leq 2$, ${}^{\rho}I_{0+}^{\beta}$ is the fractional differential operator with respect to a power function of order β , $\int_0^T u(s)dH(s)$ is the Stieltjes integral with respect to the function H , and H is a bounded variation function on $[0, T]$.

In 2015, Chatthai et al. [21] considered the existence and uniqueness of solutions for a problem consisting of the nonlinear Langevin equation of Riemann–Liouville-type fractional derivatives with the non-local Katugampola fractional integral conditions

$$x(0) = 0, \quad x(\eta) = \sum_{i=1}^n \alpha_i^{\rho_i} I^{\rho_i} x(\xi_i).$$

In this paper, we initiate the study of non-local boundary value problems of generalized fractional differential equations supplemented with generalized fractional integral boundary conditions

$$({}^{\rho}D_{0+}^{\alpha}u)(t) = f(t, u(t)), \quad 0 \leq t \leq 1, \quad (1)$$

$$u(0) = 0, \quad u(1) = \sum_{i=1}^n k_i ({}^{\rho}I_{0+}^{\alpha_i}u)(\eta_i). \quad (2)$$

where $\rho > 0$, $J = [0, 1]$, $f \in C(J \times \mathbb{R}, \mathbb{R}) \cap X_v^p(0, 1)$, $v \in \mathbb{R}$, $1 \leq p < \infty$, $1 < \alpha \leq 2$ is a real number, ${}^{\rho}D_{0+}^{\alpha}$ is a criterion fractional differential operator with respect to a power function of order α , ${}^{\rho}I_{0+}^{\alpha_i}$ is the fractional integral with respect to a power function of order α_i , $\alpha_i > 0$, $\eta_i \in (0, 1)$, and $k_i \in \mathbb{R}$, $i = 1, 2, \dots, n$ are real constants such that

$$\sum_{i=1}^n k_i \frac{\Gamma(\alpha)}{\rho^{\alpha_i} \Gamma(\alpha + \alpha_i)} \eta_i^{\rho(\alpha + \alpha_i - 1)} \neq 1.$$

The objective of this paper is to investigate a class of Caputo-type fractional derivatives defined with respect to power functions, focusing on their analytical properties and behavior in non-local differential equations. Distinct from classical Riemann–Liouville fractional integrals and Hadamard-integrals, this family of operators converge to classical

forms under variations of the power exponent. The presence of an additional parameter introduces structural dependence in the operator, which may lead to fundamental changes in the existence of solutions to associated differential equations. The structure of this paper is organized as follows: In Section 2, we describe the necessary background material related to our problem, prove operator semigroup ${}^{\rho}I_{a+}^{\alpha}$ uniform continuous and strongly continuous, and prove an auxiliary lemma. In Section 3, using fixed-point theory, we establish existence conditions for solutions to this class of nonlinear fractional differential equations under different non-local conditions, and determine the value ranges of certain parameters when solutions exist. To demonstrate the validity of the Theorems, Section 4 presents three examples.

2. Caputo-Type Fractional Derivative

In this section, let us review the definitions and certain related theorems regarding the fractional calculus of a function with respect to power functions, and give some lemmas which are helpful in the next section. In ref. [1], Samko et al. provided the definitions of fractional integrals of a function f with respect to another function g on $[a, b]$,

$$({}^{\rho}I_{a+;g}^{\alpha}f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(y)}{(g(x) - g(y))^{1-\alpha}} f(y) dy, \quad (x > a; \alpha > 0),$$

where g is an increasing and positive monotone function on $(a, b]$, having a continuous derivative g' on (a, b) , and Γ is the gamma function defined by $\Gamma(x) = \int_0^{\infty} e^{-s} s^{x-1} ds$.

For $\nu \in \mathbb{R}$, $1 \leq p < \infty$. Let $X_{\nu}^p(a, b)$ denote the space of all Lebesgue measurable functions $f : (a, b) \rightarrow \mathbb{R}$ for which $\|f\|_{X_{\nu}^p} < \infty$, where the norm is defined by

$$\|f\|_{X_{\nu}^p} = \left(\int_a^b |t^{\nu} f(t)|^p \frac{dt}{t} \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.$$

$$\|f\|_{X_{\nu}^p} = \text{ess sup}_{t \in (a,b)} |t^{\nu} f(t)|, \quad p = \infty.$$

In particular, when $\nu = \frac{1}{p}$, the space $X_{\nu}^p(a, b) = L_p(a, b)$.

In the above definition of the fractional integral of a function with respect to another function, when selecting $g(x) = \frac{x^{\rho}}{\rho}$, we can obtain the following definitions of the generalized fractional differential and fractional integral.

Definition 1. Let $\rho > 0$, $\alpha > 0$. $-\infty < a < x < b < \infty$, and $f \in X_{\nu}^p(a, b)$. The fractional integral operator ${}^{\rho}I_{a+}^{\alpha}$ with respect to a power function $g(x)$ of order α is defined by

$$({}^{\rho}I_{a+}^{\alpha}f)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{y^{\rho-1}}{(x^{\rho} - y^{\rho})^{1-\alpha}} f(y) dy. \quad (3)$$

This integral is called the left-sided fractional integral. The right-sided fractional integral ${}^{\rho}I_{b-}^{\alpha}$ is defined by

$$({}^{\rho}I_{b-}^{\alpha}f)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{y^{\rho-1}}{(y^{\rho} - x^{\rho})^{1-\alpha}} f(y) dy. \quad (4)$$

Definition 2. Let $\rho > 0$, $\alpha > 0$, $n = [\alpha] + 1$. $0 \leq a < x < b \leq \infty$, and $g \in X_{\nu}^p(a, b)$. The left-sided fractional derivatives operator ${}^{\rho}D_{a+}^{\alpha}$ with respect to power function

$g(x)$ and right-sided fractional derivatives operator ${}^{\rho}D_{b-}^{\alpha}$ with respect to power function $g(x)$ are defined by

$$\begin{aligned} ({}^{\rho}D_{a+}^{\alpha}g)(x) &= \left(x^{1-\rho}\frac{d}{dx}\right)^n ({}^{\rho}I_{a+}^{n-\alpha}g)(x) \\ &= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \left(x^{1-\rho}\frac{d}{dx}\right)^n \int_a^x \frac{y^{\rho-1}}{(x^{\rho}-y^{\rho})^{\alpha-n+1}} g(y) dy, \end{aligned} \quad (5)$$

and

$$\begin{aligned} ({}^{\rho}D_{b-}^{\alpha}g)(x) &= \left(-x^{1-\rho}\frac{d}{dx}\right)^n ({}^{\rho}I_{b-}^{n-\alpha}g)(x) \\ &= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \left(-x^{1-\rho}\frac{d}{dx}\right)^n \int_x^b \frac{y^{\rho-1}}{(y^{\rho}-x^{\rho})^{\alpha-n+1}} g(y) dy. \end{aligned} \quad (6)$$

The physical basis for using the power function x^{α} as the kernel in fractional-order differentiation primarily stems from its mathematical ability to describe memory effects and hereditary properties. It captures the phenomenon of long-range memory or non-locality in many physical and engineering systems, where the current state depends on the entire historical process. In many physical processes, the current state of a system depends not only on its recent state but also on its past states; however, this dependence decays over time. The power function $(t - \tau)^{-\alpha}$ perfectly describes this decay. When τ is close to the current time t , the kernel function value is large, meaning recent history has a strong influence on the current state. When τ is far from the current time t , the kernel function value is small, meaning distant history has a weak influence on the current state, but the influence never completely vanishes (as long as $\alpha < 1$). Furthermore, power functions are the hallmark of fractal structures and scale-invariant systems. In many materials with complex microstructures (such as porous media, rough surfaces, colloids), their physical behavior is statistically scale-invariant. The fractional differential operator, due to its power-law kernel, itself possesses scale invariance. Performing a scale transformation on a fractional derivative only yields an additional constant factor. This mathematical property matches the fractal nature of physical systems, enabling fractional models to more fundamentally describe transport processes (such as percolation, diffusion) in such systems.

From the pure mathematical theory of the 19th century to its successful application in viscoelastic mechanics in the mid-20th century and to today's extensive exploration in numerous scientific and engineering fields, the power-law kernel-based fractional model has established a solid mathematical foundation, physical interpretation, and a wealth of successful application cases. It has become an indispensable tool for modeling and analyzing complex systems.

The properties and related theorems concerning generalized fractional differential operators and generalized fractional integral operators were introduced by Katugampola in 2014 [13].

The generalized differential operators depend on parameter ρ compared with classical fractional derivatives. Most of the characteristics of generalized fractional derivatives depend on the value of ρ [13]. In fact, we have $({}^{\rho}D_{a+}^{\alpha}f)(x) \rightarrow ({}^LD_{a+}^{\alpha}f)(x) (\rho \rightarrow 1)$, where ${}^LD_{a+}^{\alpha}$ is a Riemann–Liouville fractional differential operator, and $({}^{\rho}D_{a+}^{\alpha}f)(x) \rightarrow ({}^HD_{a+}^{\alpha}f)(x) (\rho \rightarrow 0^+)$, ${}^HD_{a+}^{\alpha}$ is a Hadamard differential fractional operator.

From Definition 1, and by direct computation with respect to z^{λ} , we can find the following proposition:

Proposition 1 ([14]). Let $\alpha > 0$ and $\rho \in \mathbb{R}$. We have

$${}^{\rho}I_{0+}^{\alpha} z^{\lambda} = \frac{\Gamma(\frac{\lambda}{\rho} + 1)}{\rho^{\alpha} \Gamma(\frac{\lambda}{\rho} + \alpha + 1)} z^{\alpha\rho + \lambda}. \quad (7)$$

where $\lambda \in \mathbb{C}$.

Let $AC[0, 1]$ be the space of an absolutely continuous function on $[0, 1]$. In addition, the space $AC_{\rho}^2[0, 1]$ consists of those functions g that have an absolutely continuous $x^{1-\rho} \frac{d}{dx}$ derivative.

$$AC_{\rho}^2[0, 1] = \left\{ g : [0, 1] \rightarrow \mathbb{R} : (x^{1-\rho} \frac{d}{dx})g(x) \in AC[0, 1] \right\}.$$

The conclusions regarding the simple properties of the generalized differential operators [13,14] are as follows. Unless otherwise stated, we suppose throughout that $\rho > 0$ and $\alpha > 0$. For $\rho > 0$, $\alpha > 0$ and $h \in X_{\nu}^p(a, b)$, we have

$$({}^{\rho}D_{a+}^{\alpha} {}^{\rho}I_{a+}^{\alpha})h(t) = h(t). \quad (8)$$

In particular, the solution of differential equation

$${}^{\rho}D_{a+}^{\alpha} f(x) = 0$$

has the form

$$f(x) = \sum_{i=1}^n a_i \left(\frac{x^{\rho} - a^{\rho}}{\rho} \right)^{\alpha-1}, \quad (9)$$

where $n = [\alpha] + 1$, $a_i (i = 1, 2, \dots, n)$ are real constants.

For $\rho > 0$, $1 < \alpha \leq 2$, $\nu \in \mathbb{R}$ and $1 \leq p \leq \infty$. If $h \in X_{\nu}^p(0, 1)$ and ${}^{\rho}I_{0+}^{2-\alpha} h \in AC_{\rho}^2[0, 1]$, then we have

$$({}^{\rho}I_{0+}^{\alpha} {}^{\rho}D_{0+}^{\alpha} h)(x) = h(x) + a_1 x^{\rho(\alpha-1)} + a_2 x^{\rho(\alpha-2)}.$$

On the other hand, we can estimate the $\|{}^{\rho}I_a^{\alpha}\|_{X_{\nu}^p}$ in ref. [14]. For $0 < a < b < \infty$, $\rho > 0$ and $\nu \in \mathbb{R}$ such that $\rho - 1 \geq \nu$. For any $h \in X_{\nu}^p(a, b)$, we have

$$\|{}^{\rho}I_a^{\alpha} h\|_{X_{\nu}^p} \leq M_0 \|h\|_{X_{\nu}^p}, \quad (10)$$

where

$$M_0 = \frac{b^{\alpha\rho-1}}{\rho^{\alpha-1}\Gamma(\alpha)} \int_1^{\frac{b}{a}} x^{\nu-\alpha\rho-1} (x^{\rho} - 1)^{\alpha-1} dx.$$

In order to prove Theorem 2, we need the following Theorem 1, which is a fundamental result of the fractional integration operator ${}^{\rho}I_{a+}^{\alpha}$ [13].

Theorem 1 ([13]). Let $\alpha, \beta \in \mathbb{C}$, $1 \leq p < \infty$, $0 < a < b < \infty$ and let $\rho > 0$, $\nu \in \mathbb{R}$. Then, for $f \in X_{\nu}^p(a, b)$, the semigroup property holds,

$${}^{\rho}I_{a+}^{\alpha} {}^{\rho}I_{a+}^{\beta} f = {}^{\rho}I_{a+}^{\alpha+\beta} f. \quad (11)$$

For all $g, h \in X_{\nu}^p(a, b)$,

$${}^{\rho}I_{a+}^{\alpha} (c_1 g + c_2 h) = c_1 {}^{\rho}I_{a+}^{\alpha} g + c_2 {}^{\rho}I_{a+}^{\alpha} h, \quad (12)$$

where c_1, c_2 are arbitrary constants.

Theorem 2. If $\alpha > 0$, $p \geq 1$ such that $\min(\rho, \alpha + \frac{1}{p}) \geq \nu + 1$, then the fractional integration operator ${}^\rho I_{a+}^\alpha$ is a uniform continuous semigroup in $X_\nu^p(a, b)$ which is strongly continuous for all $\alpha \geq 0$.

Proof. By (10) and (12), ${}^\rho I_{a+}^\alpha$ is the boundary linear operator in $X_\nu^p(a, b)$. Let $\alpha_0 > 0$, we have

$$\begin{aligned} {}^\rho I_{a+}^\alpha f - {}^\rho I_{a+}^{\alpha_0} f &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}} f(s) ds - \frac{\rho^{1-\alpha_0}}{\Gamma(\alpha_0)} \int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha_0}} f(s) ds \\ &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \left(\frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}} - \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha_0}} \right) f(s) ds \\ &\quad + \left(\frac{\rho^{1-\alpha}}{\Gamma(\alpha)} - \frac{\rho^{1-\alpha_0}}{\Gamma(\alpha_0)} \right) \int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha_0}} f(s) ds \\ &= \mathcal{P}f + \mathcal{L}f. \end{aligned}$$

First, let us estimate the operator norm $\|\mathcal{P}f\|_{X_\nu^p}$ and $\|\mathcal{L}f\|_{X_\nu^p}$. In view of (10)

$$\|\mathcal{L}f\|_{X_\nu^p} \leq M_0 \left| \frac{\Gamma(\alpha_0)}{\Gamma(\alpha)} \rho^{-\alpha+\alpha_0} - 1 \right| \|f\|_{X_\nu^p}. \quad (13)$$

Next, since $f(s) \in X_\nu^p$, then $s^{\nu-\frac{1}{p}} f(s) \in L^p(a, b)$, we have

$$\begin{aligned} \mathcal{P}f &\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t (t^\rho - s^\rho)^{\alpha-1} |1 - (t^\rho - s^\rho)^{\alpha_0-\alpha}| s^{\rho-1} f(s) ds \\ &\leq \frac{\rho^{-\alpha}}{\Gamma(\alpha)} \int_0^{b^\rho - a^\rho} x^{\alpha-1} |1 - x^{\alpha_0-\alpha}| f((t^\rho - x)^{\frac{1}{\rho}}) dx. \end{aligned}$$

Consequently, applying the generalized Minkowski inequality

$$\begin{aligned} \|\mathcal{P}f\|_{X_\nu^p} &\leq \frac{\rho^{-\alpha}}{\Gamma(\alpha)} \int_0^{b^\rho - a^\rho} x^{\alpha-1} |1 - x^{\alpha_0-\alpha}| x^{\frac{1}{p}-\nu} dx \left(\int_a^b |t^\nu f((t^\rho - x)^{\frac{1}{\rho}})|^p \frac{dt}{t} \right)^{\frac{1}{p}} \\ &\leq \frac{\rho^{-\alpha}}{\Gamma(\alpha)} \int_0^{b^\rho - a^\rho} x^{\alpha-1} |1 - x^{\alpha_0-\alpha}| x^{\frac{1}{p}-\nu} dx \|f\|_{X_\nu^p}. \end{aligned} \quad (14)$$

Combining (13) and (14), we can obtain

$$\frac{\|({}^\rho I_{a+}^\alpha - {}^\rho I_{a+}^{\alpha_0})f\|_{X_\nu^p}}{\|f\|_{X_\nu^p}} \leq M_0 \left| \frac{\Gamma(\alpha_0)}{\Gamma(\alpha)} \rho^{-\alpha+\alpha_0} - 1 \right| + \frac{\rho^{-\alpha}}{\Gamma(\alpha)} \int_0^{b^\rho - a^\rho} x^{\alpha-1+\frac{1}{p}-\nu} |1 - x^{\alpha_0-\alpha}| dx.$$

Letting $\alpha \rightarrow \alpha_0$, taking into account that $\Gamma(\alpha)$ is continuous for $\alpha > 0$ and $\Gamma(\alpha) \neq 0$, it follows that

$$\lim_{\alpha \rightarrow \alpha_0} \|{}^\rho I_{a+}^\alpha - {}^\rho I_{a+}^{\alpha_0}\|_{X_\nu^p} = 0.$$

Let $\alpha_0 = 0$, define the identity integration operator ${}^\rho I_{a+}^0 f = f$. Let us prove that

$$\lim_{\alpha \rightarrow 0} \|{}^\rho I_{a+}^\alpha f - f\|_{X_\nu^p} = 0. \quad (15)$$

We have

$$\begin{aligned} {}^{\rho}I_{a+}^{\alpha}f - f &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{s^{\rho-1}}{(t^{\rho}-s^{\rho})^{1-\alpha}} f(s) ds - f(t) \\ &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{s^{\rho-1}}{(t^{\rho}-s^{\rho})^{1-\alpha}} f(s) ds - \frac{\alpha\rho}{(t^{\rho}-a^{\rho})^{\alpha}} \int_a^t \frac{s^{\rho-1}}{(t^{\rho}-s^{\rho})^{1-\alpha}} f(t) ds \\ &\leq \left(\frac{\rho^{1-\alpha}}{\Gamma(\alpha)} - \frac{\alpha\rho}{(b^{\rho}-a^{\rho})^{\alpha}} \right) \int_a^t \frac{s^{\rho-1}}{(t^{\rho}-s^{\rho})^{1-\alpha}} |f(s) - f(t)| ds \\ &\leq 2 \left(\frac{\rho^{1-\alpha}}{\Gamma(\alpha)} - \frac{\alpha\rho}{(b^{\rho}-a^{\rho})^{\alpha}} \right) \int_a^t \frac{s^{\rho-1}}{(t^{\rho}-s^{\rho})^{1-\alpha}} |f(s)| ds. \end{aligned}$$

Thus, we have

$$\begin{aligned} \| {}^{\rho}I_{a+}^{\alpha}f - f \|_{X_v^p} &\leq 2 \left(\frac{\rho^{1-\alpha}}{\Gamma(\alpha)} - \frac{\alpha\rho}{(b^{\rho}-a^{\rho})^{\alpha}} \right) \left(\int_a^b t^{\nu p} \left| \int_a^t \frac{s^{\rho-1}}{(t^{\rho}-s^{\rho})^{1-\alpha}} |f(s)| ds \right|^p \frac{dt}{t} \right)^{\frac{1}{p}} \\ &= 2\mathcal{Q}(\alpha)\mathcal{N}(f). \end{aligned}$$

Applying the generalized Minkowski inequality in the right-hand side integral

$$\begin{aligned} \mathcal{N}(f) &= \left(\int_a^b \left| \int_a^t t^{\nu-\frac{1}{p}} \frac{s^{\rho-1}}{(t^{\rho}-s^{\rho})^{1-\alpha}} |f(s)| ds \right|^p dt \right)^{\frac{1}{p}} \\ &= \left(\int_a^b \left| \int_1^{\frac{b}{a}} t^{\nu-\frac{1}{p}} \left(\frac{t}{u} \right)^{\alpha\rho-1} (u^{\rho}-1)^{\alpha-1} |f(\frac{t}{u})| \frac{t}{u^2} du \right|^p dt \right)^{\frac{1}{p}} \\ &\leq \int_1^{\frac{b}{a}} \frac{(u^{\rho}-1)^{\alpha-1}}{u^{\alpha\rho+1}} b^{\alpha\rho} \left(\int_{ua}^b t^{\nu-\frac{1}{p}} |f(\frac{t}{u})|^p \frac{dt}{t} \right)^{\frac{1}{p}} du \\ &\leq \int_1^{\frac{b}{a}} b^{\alpha\rho} (u^{\rho}-1)^{\alpha-1} u^{\nu-\alpha\rho-1} du \|f\|_{X_v^p}. \end{aligned}$$

By the Lebesgue-dominated convergence theorem, we obtain

$$\int_1^{\frac{b}{a}} b^{\alpha\rho} (u^{\rho}-1)^{\alpha-1} u^{\nu-\alpha\rho-1} du \|f\|_{X_v^p} < \infty,$$

And, we have

$$\lim_{\alpha \rightarrow 0} \mathcal{N}(f) = 0. \quad (16)$$

Since $\Gamma(\alpha)$ is a continuous function for $\alpha > 0$, and $\Gamma(\alpha) \rightarrow +\infty$ when $\alpha \rightarrow 0^+$; therefore, we have $\lim_{\alpha \rightarrow 0} \mathcal{Q}(\alpha) = 0$. Combining the above argument, (15) is held, which completes the estimation and the proof.

□

Remark 1. When $\nu = \frac{1}{p}$, the operator ${}^{\rho}I_{a+}^{\alpha}$ is a semigroup in $L_p(a, b)$. This is the same as the standard Riemann–Liouville fractional integration operator I^{α} (see [7]).

Lemma 1. Assume $\{T_s | s > 0\}$ is a strongly continuous operator semigroup in Banach space X , $0 < a < b < +\infty$, then constant $M_1 > 0$ exists such that

$$\|T_s\| \leq M_1, \quad s \in [a, b].$$

3. Non-Local Boundary Value Problems

Let $1 < \alpha \leq 2$, $\rho > 0$, $C^{\alpha}(J, \mathbb{R}) = \{u \in C([0, 1], \mathbb{R}) : {}^{\rho}I_{0+}^{2-\alpha}u \in AC[0, 1]\}$. For $u \in C^{\alpha}(J, \mathbb{R})$, define the norm $\|u\|_{C^{\alpha}} = \sup_{t \in [0, 1]} |u(t)|$. When $\rho(2 - \alpha) > 1$, $C^{\alpha}(J, \mathbb{R})$ is a Banach space.

Proposition 2. Let $\alpha_i > 0$, $1 < \alpha \leq 2$ and $\rho > 0$, $\eta_i \in (0, 1)$, $k_i \in \mathbb{R}$, $i = 1, 2, \dots, n$. Let

$$\Delta = 1 - \sum_{i=1}^n k_i \frac{\Gamma(\alpha)}{\rho^{\alpha_i} \Gamma(\alpha + \alpha_i)} \eta_i^{\rho(\alpha + \alpha_i - 1)} \neq 0.$$

For any $\varphi \in C(0, 1)$ and $x \in C([0, 1], \mathbb{R})$ with ${}^\rho I_{0+}^{2-\alpha} x \in AC_\rho^2[0, 1]$, Then, the function x is the solution of the non-local fractional differential equation boundary-value problem

$$\begin{cases} ({}^\rho D_{0+}^\alpha x)(t) = \varphi(t), & 0 < t < 1, \\ x(0) = 0, \quad x(1) = \sum_{i=1}^n k_i ({}^\rho I_{0+}^{\alpha_i} x)(\eta_i), \end{cases} \quad (17)$$

if, and only if,

$$x(t) = {}^\rho I_{0+}^\alpha \varphi(t) + \frac{1}{\Delta} \left(\sum_{i=1}^n k_i {}^\rho I_{0+}^{\alpha + \alpha_i} \varphi(\eta_i) - {}^\rho I_{0+}^\alpha \varphi(1) \right) t^{\rho(\alpha - 1)}. \quad (18)$$

Proof. Applying the operator ${}^\rho I_{0+}^\alpha$ on the linear differential Equation (17), we have

$${}^\rho I_{0+}^\alpha ({}^\rho D_{0+}^\alpha x)(t) = ({}^\rho I_{0+}^\alpha \varphi)(t).$$

Using (8) and (9), we can obtain

$$x(t) = ({}^\rho I_{0+}^\alpha \varphi)(t) + c_1 t^{\rho(\alpha - 1)} + c_2 t^{\rho(\alpha - 2)}, \quad (19)$$

where $c_1, c_2 \in \mathbb{R}$. The condition $x(0) = 0$ implies that $c_2 = 0$. Applying the fractional integral operator with respect to a power function $\frac{x^\rho}{\rho}$ of order $\alpha_i > 0$ on (19) after inserting $c_2 = 0$ in it, and using (19), we get

$${}^\rho I_{0+}^{\alpha_i} x(t) = {}^\rho I_{0+}^{\alpha + \alpha_i} \varphi(t) + c_1 \frac{\Gamma(\alpha)}{\rho^{\alpha_i} \Gamma(\alpha + \alpha_i)} t^{\rho(\alpha + \alpha_i - 1)}.$$

which, together with the second condition $x(1) = \sum_{i=1}^n k_i ({}^\rho I_{0+}^{\alpha_i} x)(\eta_i)$, we have

$${}^\rho I_{0+}^{\alpha_i} \varphi(1) + c_1 = \sum_{i=1}^n k_i ({}^\rho I_{0+}^{\alpha + \alpha_i} \varphi)(\eta_i) + c_1 \sum_{i=1}^n k_i \frac{\Gamma(\alpha)}{\rho^{\alpha_i} \Gamma(\alpha + \alpha_i)} \eta_i^{\rho(\alpha + \alpha_i - 1)}.$$

Thus,

$$c_1 = \frac{1}{\Delta} \left(\sum_{i=1}^n k_i ({}^\rho I_{0+}^{\alpha + \alpha_i} \varphi)(\eta_i) - {}^\rho I_{0+}^{\alpha_i} \varphi(1) \right).$$

Substituting c_1, c_2 into (19), we obtain the solution (18). Conversely, it can easily be shown by direct computation that the integral equation (18) satisfies the boundary value problem (17). Thus, the function x is a solution to the linear boundary value problem (17) if, and only if, it can be expressed in the form (18). $\square$

To prove the main theorems of Section 3, we need the following well-known fixed-point theorem [26].

Theorem 3 ([26]). Let E be a non-empty, closed, convex and bounded subset of the Banach space X and let $A : X \rightarrow X$ and $B : E \rightarrow X$ be two operators such that

- (a) A is a contraction,
- (b) B is completely continuous, and
- (c) $x = Ax + By$ for all $y \in E \Rightarrow x \in E$.

Then, the operator equation $Ax + Bx = x$ has a solution in E .

Lemma 2. The space $C^\alpha(J, \mathbb{R})$ is a Banach space.

Proof. Set $J = [0, 1]$. Let $1 < \alpha \leq 2$ and $\rho > 0$ such that $\rho(2 - \alpha) > 1$. Given a Cauchy sequence $\{u_n\}$ in $C^\alpha(J, \mathbb{R})$, then $\{\rho I_{0+}^{2-\alpha} u_n\}$ is a Cauchy sequence in $AC[0, 1]$. Since $AC[0, 1]$ is complete, there exists a function $u \in AC[0, 1]$ such that

$$\rho I_{0+}^{2-\alpha} u_n \rightarrow u (n \rightarrow \infty).$$

Assume a function $u_0 \in AC[0, 1]$ such that

$$u_n \rightarrow u_0 (n \rightarrow \infty).$$

We will prove that

$$u_0 \in C^\alpha(J, \mathbb{R}), \quad \rho I_{0+}^{2-\alpha} u_0 = u. \quad (20)$$

In order to prove $u_0 \in C^\alpha(J, \mathbb{R})$, we need to prove $\rho I_{0+}^{2-\alpha} u_0(x) \in AC[0, 1]$. Since u_0 is a continuous function on $[0, 1]$, there exists a constant K_α such that $\|u_0\|_{L_p} \leq K_\alpha$ for all $x \in [0, 1]$. For any $\varepsilon > 0$, we take $\delta = \min\{1, \frac{\rho^{1-\alpha}\Gamma(2-\alpha)\varepsilon}{4K_\alpha}\}$, $\{(t_i, s_i) : i = 1, \dots, n\}$ is any finite collection of mutually disjoint subintervals of $[0, 1]$, such that $\sum_{i=1}^n |s_i - t_i| < \delta$ holds.

Based on the above results, we have

$$\begin{aligned} \sum_{i=1}^n |\rho I_{0+}^{2-\alpha} u_0(s_i) - \rho I_{0+}^{2-\alpha} u_0(t_i)| &= \sum_{i=1}^n \frac{\rho^{\alpha-1}}{\Gamma(2-\alpha)} \left| \int_0^{s_i} \frac{t^{\rho-1} u_0(t)}{(s_i^\rho - t^\rho)^{\alpha-1}} dt - \int_0^{t_i} \frac{t^{\rho-1} u_0(t)}{(t_i^\rho - t^\rho)^{\alpha-1}} dt \right| \\ &\leq \sum_{i=1}^n \frac{\rho^{\alpha-2}}{\Gamma(2-\alpha)} \left(\int_0^{s_i^\rho} v^{1-\alpha} |u_0((s_i^\rho - v)^{\frac{1}{\rho}}) - u_0((t_i^\rho - v)^{\frac{1}{\rho}})| dv \right. \\ &\quad \left. + \int_{t_i^\rho}^{s_i^\rho} v^{1-\alpha} |u_0((t_i^\rho - v)^{\frac{1}{\rho}})| dv \right) \\ &\leq \sum_{i=1}^n \frac{4\|u_0\|_{L_p}}{(2-\alpha)\rho^{2-\alpha}\Gamma(2-\alpha)} |s_i^{\rho(2-\alpha)} - t_i^{\rho(2-\alpha)}| \\ &\leq \frac{4\|u_0\|_{L_p}}{\rho^{1-\alpha}\Gamma(2-\alpha)} \sum_{i=1}^n |s_i - t_i| < \varepsilon. \end{aligned}$$

Which implies that $\rho I_{0+}^{2-\alpha} u_0(x) \in AC[0, 1]$.

Furthermore, since $u_n - u_0 \rightarrow 0 (n \rightarrow \infty)$, $N \in \mathbb{N}$ exists such that for $n > N$,

$$\|u_n - u_0\| < \frac{\varepsilon}{K_{\alpha-1}},$$

where $K_{\alpha-1} = \frac{1}{(2-\alpha)\rho^{2-\alpha}\Gamma(2-\alpha)}$, which yields that $\|\rho I_{0+}^{2-\alpha} u_n(x) - \rho I_{0+}^{2-\alpha} u_0(x)\| \rightarrow 0 (n \rightarrow \infty)$ in $C[0, 1]$. Consequently, we have

$$\rho I_{0+}^{2-\alpha} u_0 = \lim_{n \rightarrow \infty} \rho I_{0+}^{2-\alpha} u_n = u.$$

Thus, (20) is valid. Since $\rho I_{0+}^{2-\alpha} u_n \rightarrow \rho I_{0+}^{2-\alpha} u_0$ in $AC[0, 1]$. It shows that $u_n \rightarrow u_0$ in $C^\alpha(J, \mathbb{R})$. Therefore, $C^\alpha(J, \mathbb{R})$ is a Banach space. $\square$

Theorem 4. Let $1 < \alpha \leq 2$. Assume $f \in C(J \times \mathbb{R}, \mathbb{R}) \cap X_V^p(0, 1)$, and the following conditions hold:

(H_1) : constant $0 < \theta_1 < 1$ and $\theta \in X_V^p((0, 1), [0, \infty))$ exist such that

$$\sup_{0 \leq t \leq 1} \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} 2 \int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}} \theta(s) ds \leq 1 - \theta_1, \quad (21)$$

and

$$|f(t, x_1) - f(t, x_2)| \leq \theta(t)|x_1 - x_2|, \quad \forall t \in J, x_1, x_2 \in \mathbb{R}. \quad (22)$$

(H₂): $\alpha_i > 0$, $\eta_i \in (0, 1)$, and $k_i \in \mathbb{R}$, $i = 1, 2, \dots, n$ satisfy

$$0 < r_0 := \frac{\rho^{1-\alpha}}{|\Delta|} \left(\sum_{i=1}^{n+1} \frac{|k_i|}{\rho^{\alpha_i} \Gamma(\alpha + \alpha_i)} \int_0^{\eta_i} \frac{s^{\rho-1}}{(\eta_i^\rho - s^\rho)^{1-\alpha-\alpha_i}} \theta(s) ds \right) < 1, \quad (23)$$

where $\alpha_{n+1} = 0$, $k_{n+1} = \eta_{n+1} = 1$.

Then, problem (1) and (2) has at least one solution.

Proof. Let $1 < \alpha \leq 2$, $\rho > 0$, set $f_1 = \max_{0 \leq s \leq 1} |f(s, u(1))| < \infty$. For $\lambda > \frac{f_1}{\rho^\alpha \Gamma(\alpha+1)}$, define the space S by

$$S = \{u \in C^\alpha(J, \mathbb{R}) : \|u(t)\| \leq \lambda\}.$$

Define an operator $\mathcal{T}$ on S as follows:

$$\mathcal{T}u(t) = {}^\rho I_{0+}^\alpha f(t, u(t)) + \frac{1}{\Delta} \left(\sum_{i=1}^n k_i {}^\rho I_{0+}^{\alpha+\alpha_i} f(\eta_i, u(\eta_i)) - {}^\rho I_{0+}^\alpha f(1, u(1)) \right) t^{\rho(\alpha-1)}. \quad (24)$$

Let

$$\mathcal{T}_1 u(t) = {}^\rho I_{0+}^\alpha f(t, u(t)).$$

$$\mathcal{T}_2 u(t) = \frac{1}{\Delta} \left(\sum_{i=1}^n k_i {}^\rho I_{0+}^{\alpha+\alpha_i} f(\eta_i, u(\eta_i)) - {}^\rho I_{0+}^\alpha f(1, u(1)) \right) t^{\rho(\alpha-1)}.$$

It is clear that $u(t)$ is a solution of (1) if it is a fixed point of the operator $\mathcal{T}$. Then, we will prove $\mathcal{T}_1$ is a completely continuous operator and $\mathcal{T}_2$ is a contractor operator. For $u(t) \in S$,

$$\begin{aligned} \mathcal{T}_1 u(t) &\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}} |f(s, u(s))| ds \\ &\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}} \left(|f(s, u(s)) - f(s, u(1))| + |f(s, u(1))| \right) ds \\ &\leq \lambda(1 - \theta_1) + \frac{1}{\rho^\alpha \Gamma(\alpha+1)} f_1 \\ &< \lambda. \end{aligned} \quad (25)$$

which implies that $\mathcal{T}_1 S \subset S$. In order to show that the operator $\mathcal{T}_1$ is continuous, for any $u_n, u_0 \in S$, $n = 1, 2, \dots$ with $u_n \rightarrow u_0$ ($n \rightarrow \infty$), by the Lebesgue-dominated convergence theorem, we have

$$\begin{aligned} |(\mathcal{T}_1 u_n)(t) - (\mathcal{T}_1 u_0)(t)| &\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}} |f(s, u_n(s)) - f(s, u_0(s))| ds \\ &\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}} \theta(s) |u_n(s) - u_0(s)| ds \\ &\rightarrow 0 \quad (n \rightarrow \infty), \end{aligned} \quad (26)$$

Next, we prove that $\mathcal{T}_1 S$ is equicontinuous. Let $t_1, t_2 \in J$, $t_1 > t_2$. For given $\varepsilon > 0$, we take

$$\delta = \min \left\{ 1, \left(\frac{\varepsilon \rho^\alpha \Gamma(\alpha+1)}{N_0 2^{\alpha\rho+1}} \right)^{\frac{1}{\alpha\rho}} \right\},$$

where $N_0 = \left(\int_0^1 |f(s, u(s))|^p ds \right)^{\frac{1}{p}} < \infty$. Then, when $|t_1 - t_2| < \delta$, for each $u \in S$, we will get

$$\begin{aligned} |(\mathcal{T}_1 u)(t_1) - (\mathcal{T}_1 u)(t_2)| &\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \left| \int_0^{t_1} \frac{s^{\rho-1}}{(t_1^\rho - s^\rho)^{1-\alpha}} f(s, u(s)) ds - \int_0^{t_2} \frac{s^{\rho-1}}{(t_2^\rho - s^\rho)^{1-\alpha}} f(s, u(s)) ds \right| \\ &\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_2} \left(\frac{s^{\rho-1}}{(t_1^\rho - s^\rho)^{1-\alpha}} - \frac{s^{\rho-1}}{(t_2^\rho - s^\rho)^{1-\alpha}} \right) |f(s, u(s))| ds \\ &\quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_2}^{t_1} \frac{s^{\rho-1}}{(t_1^\rho - s^\rho)^{1-\alpha}} |f(s, u(s))| ds \\ &=: \mathcal{I}_1 + \mathcal{I}_2. \end{aligned}$$

For $\mathcal{I}_1$, Consider the function $g(x) = \frac{x^{\rho-1}}{(t_1^\rho - x^\rho)^{1-\alpha}} - \frac{x^{\rho-1}}{(t_2^\rho - x^\rho)^{1-\alpha}}$, $x \in [0, t_2]$, we can obtain that $\|g\|_{L_1} < \frac{1}{\alpha\rho}(t_1^{\alpha\rho} - t_2^{\alpha\rho})$. Case 1: Let $\delta \leq t_2 < t_1 < 1$, then $t_1^{\alpha\rho} - t_2^{\alpha\rho} \leq \alpha\rho\delta^{\alpha\rho}$. Case 2: Let $0 < t_2 < \delta$, $t_1 < 2\delta$, then $t_1^{\alpha\rho} - t_2^{\alpha\rho} < (2\delta)^{\alpha\rho}$. Combining the above two cases, $\|g\|_{L_1} < \frac{1}{\alpha\rho}(2\delta)^{\alpha\rho}$. Apply the generalized Minkowski's inequality

$$\begin{aligned} \mathcal{I}_1 &\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_2} \left(\frac{s^{\rho-1}}{(t_1^\rho - s^\rho)^{1-\alpha}} - \frac{s^{\rho-1}}{(t_2^\rho - s^\rho)^{1-\alpha}} \right) \|f\|_{L_p} ds \\ &< \frac{2^{\alpha\rho}}{\rho^\alpha \Gamma(\alpha+1)} |t_1 - t_2|^{\alpha\rho} \|f\|_{L_p}, \end{aligned} \quad (27)$$

and similarly,

$$\begin{aligned} \mathcal{I}_2 &\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_1-t_2} \frac{(x+t_2)^{\rho-1}}{(t_1^\rho - (x+t_2)^\rho)^{1-\alpha}} \|f\|_{L_p} dx \\ &\leq \frac{2^{\alpha\rho}}{\rho^\alpha \Gamma(\alpha+1)} |t_1 - t_2|^{\alpha\rho} \|f\|_{L_p}. \end{aligned} \quad (28)$$

Consequently, together with (27) and (28), gives

$$|(\mathcal{T}_1 u)(t_1) - (\mathcal{T}_1 u)(t_2)| < \varepsilon. \quad (29)$$

Therefore, $\mathcal{T}_1$ is a completely continuous operator.

Finally, we show that $\mathcal{P}_2$ is a contractive operator. For any $u, v \in S$,

$$\begin{aligned} |(\mathcal{T}_2 u)(t) - (\mathcal{T}_2 v)(t)| &\leq \frac{1}{|\Delta|} \left(\sum_{i=1}^n k_i^\rho I_{0+}^{\alpha+\alpha_i} |f(\eta_i, u(\eta_i)) - f(\eta_i, v(\eta_i))| + {}^\rho I_{0+}^\alpha |f(1, u(1)) - f(1, v(1))| \right) \\ &\leq \frac{1}{|\Delta|} \left(\sum_{i=1}^n k_i^\rho I_{0+}^{\alpha+\alpha_i} \theta(\eta_i) |u(\eta_i) - v(\eta_i)| + {}^\rho I_{0+}^\alpha \theta(1) |u(1) - v(1)| \right) \\ &\leq \frac{1}{|\Delta|} \left(\sum_{i=1}^n |\lambda_i| {}^\rho I_{0+}^{\alpha+\alpha_i} \theta(\eta_i) + {}^\rho I_{0+}^\alpha \theta(1) \right) \|u - v\|_{C^\alpha} \\ &< r_0 \|u - v\|_{C^\alpha}. \end{aligned}$$

Which implies that $\mathcal{T}_2$ is a contraction by using (H₂). Thus, according to Theorem 3, there exists a $u \in S$ such that $u = \mathcal{T}_1 u + \mathcal{T}_2 u$. So, operator $\mathcal{T}$ has a fixed point implies that the problem (1) and (2) has at least one solution on $[0, 1]$. $\square$

Remark 2. If $\theta(t) = L$ is a constant, then condition (21) reduces to

$$L < \rho^\alpha \Gamma(\alpha + 1),$$

where L and λ satisfies $L\lambda > f_1$.

Remark 3. In the case that the generalized fractional integral boundary condition reduces to

$$u(1) = k \int_0^\eta u(s) ds, \quad \eta \in (0, 1).$$

Then, the value Δ is found to be

$$\Delta = 1 - k \frac{\eta^{\rho(\alpha-1)+1}}{\rho(\alpha-1)+1},$$

(24) modifies the form

$$\mathcal{T}u(t) = {}^\rho I_{0+}^\alpha f(t, u(t)) + \frac{1}{\Delta} \left(k \int_0^\eta ({}^\rho I_{0+}^\alpha f(t, u(t))) dt - {}^\rho I_{0+}^\alpha f(1, u(1)) \right) t^{\rho(\alpha-1)}. \quad (30)$$

Then, we consider the existence of a solution for the differential Equation (1) with boundary condition

$$u(0) = 0, \quad ({}^\rho I_{0+}^\beta u)(\zeta) = \sum_{i=1}^n \lambda_i u(\zeta_i), \quad (31)$$

where $\beta > 0$, $1 < \alpha \leq 2$ and $\rho > 0$, $0 < \zeta \leq 1$, $\zeta_i \in (0, 1)$, $\lambda_i \in \mathbb{R}$, $i = 1, 2, \dots, n$.

In Theorem 4, the boundary condition (2) provides information about the unknown function at the two endpoints, with the value at the second endpoint given by the sum of Caputo-type integrals at a finite number of points within the interval. Next, we examine the existence conditions for solutions to Equation (1) when the boundary condition is replaced by (31). In this case, the Caputo-type integral of the unknown function at a certain point within the interval is expressed as a linear combination of the function values at a finite number of points inside the interval.

Proposition 3. Let $\beta > 0$, $1 < \alpha \leq 2$ and $\rho > 0$, $0 < \zeta \leq 1$, $\zeta_i \in (0, 1)$, $\lambda_i \in \mathbb{R}$, $i = 1, 2, \dots, n$. Let

$$\sigma = \frac{\Gamma(\alpha)}{\rho^\beta \Gamma(\alpha + \beta)} \zeta^{\rho(\alpha+\beta-1)} - \sum_{i=1}^n \lambda_i \zeta_i^{\rho(\alpha-1)} \neq 0.$$

For any $\varphi \in C(0, 1)$ and $x \in C([0, 1], \mathbb{R})$ with ${}^\rho I_{0+}^{2-\alpha} x \in AC_\rho^2[0, 1]$, Then, the function x is the solution of the non-local fractional differential equation boundary-value problem

$$\begin{cases} ({}^\rho D_{0+}^\alpha x)(t) = \varphi(t), & 0 < t < 1, \\ x(0) = 0, \quad ({}^\rho I_{0+}^\beta x)(\zeta) = \sum_{i=1}^n \lambda_i x(\zeta_i). \end{cases} \quad (32)$$

if, and only if,

$$x(t) = {}^\rho I_{0+}^\alpha \varphi(t) + \frac{1}{\sigma} \left(\sum_{i=1}^n \lambda_i {}^\rho I_{0+}^\alpha \varphi(\zeta_i) - {}^\rho I_{0+}^{\alpha+\beta} \varphi(\zeta) \right) t^{\rho(\alpha-1)}. \quad (33)$$

Proof. This Proposition is a special case of Proposition 2. So we will not prove it again. $\square$

Theorem 5. Let $1 < \alpha \leq 2$. Assume $f \in C(J \times \mathbb{R}, \mathbb{R}) \cap X_V^p(0, 1)$, and the following conditions hold:

(H₃): constant $0 < \delta < 1$ and $\theta \in X_V^p((0, 1), [0, \infty))$ exist such that

$$\sup_{0 \leq t \leq 1} \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}} \theta(s) ds \leq 1 - \delta, \quad (34)$$

and

$$|f(t, x_1) - f(t, x_2)| \leq \theta(t) |x_1 - x_2|, \quad \forall t \in J, \quad x_1, x_2 \in \mathbb{R}. \quad (35)$$

(H₄): $\beta > 0, 0 < \zeta \leq 1, \zeta_i \in (0, 1)$, and $\lambda_i \in \mathbb{R}, i = 1, 2, \dots, n$ satisfy

$$0 < \varepsilon_0 := \frac{1}{|\sigma|} \left(\sum_{i=1}^n |\lambda_i| + {}^\rho I_{0+}^{\alpha+\beta} \theta(\zeta) \right) < 1. \quad (36)$$

Then, problem (1)–(31) has at least one solution.

Proof. Let $1 < \alpha \leq 2, \rho > 0$, set $f_\zeta = \max_{0 \leq s \leq 1} |f(s, u(\zeta))| < \infty$, λ is a constant, $\lambda > \frac{f_\zeta}{2\rho^\alpha \Gamma(\alpha+1)}$. $S = \{u \in C^\alpha(J, \mathbb{R}) : \|u(t)\| \leq \lambda\}$. Define an operator $\mathcal{P}$ on S as follows:

$$\mathcal{P}u(t) = {}^\rho I_{0+}^\alpha f(t, u(t)) + \frac{1}{\sigma} \left(\sum_{i=1}^n \lambda_i {}^\rho I_{0+}^\alpha f(\zeta_i, u(\zeta_i)) - {}^\rho I_{0+}^{\alpha+\beta} f(\zeta, u(\zeta)) \right) t^{\rho(\alpha-1)}. \quad (37)$$

Let

$$\begin{aligned} \mathcal{P}_1 u(t) &= {}^\rho I_{0+}^\alpha f(t, u(t)). \\ \mathcal{P}_2 u(t) &= \frac{1}{\sigma} \left(\sum_{i=1}^n \lambda_i {}^\rho I_{0+}^\alpha f(\zeta_i, u(\zeta_i)) - {}^\rho I_{0+}^{\alpha+\beta} f(\zeta, u(\zeta)) \right) t^{\rho(\alpha-1)}. \end{aligned}$$

It is clear that $u(t)$ is a solution of (1) and (31) if it is a fixed point of the operator $\mathcal{P}$. Similar to the proof of Theorem 4, we may deduce that $\mathcal{P}_1$ is a completely continuous operator and $\mathcal{P}_2$ is a contractor operator. Therefore, according to Proposition 1, $\mathcal{P}$ has a fixed point in S . $\square$

Remark 4. In Deng's paper [20], the non-local condition

$$x(s, 0) + \sum_{i=1}^n p_i(s)x(s, t_i) = q(x), \quad (38)$$

with $t_i \in (0, T] (i = 1, 2, \dots, n)$ can be applied to describe the diffusion phenomenon of a small amount of gas in a transparent tube. Obviously, the boundary condition (31) in Theorem 5 is a special form of condition (38).

4. Examples

In this section, to illustrate the application of the Theorems, we constructed the following examples.

Example 1. Let us consider the following fractional differential equation boundary value problem

$$\begin{cases} ({}^\rho D_{0+}^{\frac{3}{2}} u)(t) = t^\beta u(t), & 0 \leq t \leq 1, \\ u(0) = 0, \quad u(1) = \int_0^\eta u(s) ds, & \eta \in (0, 1). \end{cases} \quad (39)$$

where $\alpha = \frac{3}{2}, \rho > 0, \beta \in \mathbb{R}$ and $f(t, u) = t^\beta u$. $f(t, u)$ satisfy $|f(t, x_1) - f(t, x_2)| \leq t^\beta |x_1 - x_2|$. By Theorem 4, if the continuous solution to problem (39) exists, λ and β must satisfy certain conditions.

In fact, since

$$\frac{2\rho^{-\frac{1}{2}}}{\Gamma(\frac{3}{2})} \int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{-\frac{1}{2}}} s^\beta ds = \frac{2t^{\frac{3}{2}\rho+\beta}}{\rho^{\frac{3}{2}}\Gamma(\frac{3}{2})} \int_0^1 \tau^{\frac{\beta}{\rho}} (1-\tau)^{\frac{1}{2}} d\tau = \frac{2\Gamma(\frac{\beta}{\rho}+1)}{\rho^{\frac{3}{2}}\Gamma(\frac{\beta}{\rho}+\frac{5}{2})} t^{\frac{3}{2}\rho+\beta}$$

and

$$\frac{1}{\Delta} \left(\int_0^\eta {}^\rho I_{0+}^\alpha t^\beta dt + {}^\rho I_{0+}^\alpha 1 \right) = \frac{1}{1 - \frac{\eta^{\frac{\rho}{2}+1}}{\frac{\rho}{2}+1}} \left(\frac{\Gamma(\frac{\rho}{2}+1)}{\rho^{\frac{3}{2}}\Gamma(\frac{\rho}{2}+\frac{5}{2})} \left(\frac{\eta^{\frac{3}{2}\rho+\beta+1}}{\frac{\rho}{2}+\beta+1} + 1 \right) \right).$$

Thus, conditions H_1 and H_2 now are

$$\frac{\Gamma(\frac{\rho}{2}+1)}{\rho^{\frac{3}{2}}\Gamma(\frac{\rho}{2}+\frac{5}{2})} < \frac{1}{2}, \quad \frac{\Gamma(\frac{\rho}{2}+1)}{\rho^{\frac{3}{2}}\Gamma(\frac{\rho}{2}+\frac{5}{2})} \left(\frac{\eta^{\frac{3}{2}\rho+\beta+1}}{\frac{3}{2}\rho+\beta+1} + 1 \right) < 1 - \frac{\eta^{\frac{\rho}{2}+1}}{\frac{\rho}{2}+1}, \eta \in (0, 1). \quad (40)$$

Combining the above two inequalities, we have

$$\frac{1 - \int_0^\eta t^{\frac{\rho}{2}} dt}{1 + \int_0^\eta t^{\frac{3}{2}\rho+\beta} dt} \leq \frac{1}{2}.$$

Let us choose $\rho = 1, \beta = \frac{1}{2}$, the first inequality in (40) becomes $\sqrt{\pi} < 2.25$. η satisfies inequality $\eta^3 + 4\eta^{\frac{3}{2}} - 3 \geq 0$. By solving this inequality, we get $\eta \geq 0.5867862$. When $\rho \rightarrow 0^+$, the inequality stated above is not true. Therefore, for $\rho = 1, \beta = \frac{1}{2}, \eta \geq 0.5867862$, the boundary value problem (39) has at least one solution on $[0, 1]$.

Example 2. Consider the following fractional differential equation boundary value problem

$$\begin{cases} ({}^{\frac{1}{2}}D_{0+}^{\frac{3}{2}}u)(t) = \frac{1}{7(2e^t+1)} \left(\frac{|u(t)|}{|u(t)|+1} + \sin t \right), & 0 \leq t \leq 1, \\ u(0) = 0, \quad u(1) = ({}^{\frac{1}{2}}I_{0+}^{\frac{1}{2}}u)(\frac{1}{2}) + 2({}^{\frac{1}{2}}I_{0+}^{\frac{5}{2}}u)(\frac{1}{3}). \end{cases} \quad (41)$$

where $\alpha = \frac{3}{2}, \rho = \frac{1}{2}, n = 2, \alpha_1 = \frac{1}{2}, \alpha_2 = \frac{5}{2}, \eta_1 = \frac{1}{2}, \eta_2 = \frac{1}{3}, k_1 = 1, k_2 = 2$ and

$$f(t, u(t)) = \frac{1}{7(2e^t+1)} \left(\frac{|u(t)|}{|u(t)|+1} + \sin t \right), \quad t \in J.$$

Using the given values, we can calculate $|\Delta| \approx 0.2078275$ and $\rho^\alpha \Gamma(\alpha+1) \approx 0.4699928$. It is easy to check that $f(t, u(t))$ is continuous and

$$|f(t, u(t)) - f(t, v(t))| \leq \frac{1}{7(2e^t+1)} |u(t) - v(t)|,$$

select $L = \frac{1}{21}$. Also $r_0 \approx 0.97587238 < 1$ satisfy the condition (H_2) of Theorem 4, for any $0 < \theta_1 < 1, 0.20266529 < 1 - \theta_1$ satisfy the condition (H_1) .

Therefore, by the conclusion of Theorem 5, the non-local boundary value problem (41) has at least one solution on $[0, 1]$.

Example 3. Consider the following problem

$$\begin{cases} ({}^2D_{0+}^{\frac{5}{3}}u)(t) = \zeta(t)(t - u(t)) + \zeta_0, & 0 \leq t \leq 1, \\ u(0) = 0, \quad ({}^2I_{0+}^{\frac{4}{3}}u)(\frac{1}{2}) = \frac{1}{2}u(\frac{1}{4}) + \frac{1}{3}u(\frac{3}{4}) + \frac{1}{8}u(\frac{4}{5}). \end{cases} \quad (42)$$

where $\alpha = \frac{5}{3}, \rho = 2, \beta = \frac{4}{3}, n = 3, \zeta = \frac{1}{2}, \zeta_1 = \frac{2}{3}, \zeta_2 = \frac{3}{4}, \zeta_3 = \frac{4}{5}, \lambda_1 = -\frac{1}{15}, \lambda_2 = \frac{1}{30}, \lambda_3 = \frac{1}{60}$. ζ_0 is a constant, $\zeta(t) \in C^\alpha(J, \mathbb{R})$ such that

$$\zeta(t) \leq t^{-\frac{1}{2}}, \quad t \in J.$$

Set

$$f(t, x) = \xi(t)(t - x) + \xi_0, \quad (t, x) \in J \times \mathbb{R}.$$

Using the given values, we can calculate $|\sigma| \approx 0.1828616$, $\epsilon_0 \approx 0.7959415$. Selecting $\theta(t) = t^{-\frac{1}{2}}$, we have

$$\frac{2^{-\frac{2}{3}}}{\Gamma(\frac{5}{3})} \int_0^t \frac{s}{(t^2 - s^2)^{-\frac{2}{3}}} s^{-\frac{1}{2}} ds = \frac{\Gamma(\frac{3}{4})}{2^{\frac{5}{3}} \Gamma(\frac{29}{12})} t^{\frac{17}{6}}.$$

It is easy to check that $f(t, u(t))$ is continuous and

$$|f(t, u(t)) - f(t, v(t))| \leq t^{-\frac{1}{2}} |u(t) - v(t)|,$$

$\rho^\alpha \Gamma(\alpha + 1) \approx 4.776279$, $\rho^\alpha \Gamma(\alpha + \beta) \approx 6.349604$. The calculation results satisfy condition (H_3) and (H_4) in Theorem 5.

Therefore, by the conclusion of Theorem 5, the non-local boundary value problem (42) has at least one solution on $[0, 1]$.

5. Conclusions and Future Work

In this paper, we investigate the definitions and properties of fractional integrals with respect to a power function. We proved the strong continuity properties of the associated semigroups and obtained an existence Theorem for solutions of differential equations under non-local boundary conditions when the order is $1 < \alpha \leq 2$. Notably, the definition contains a special parameter ρ , which influences the results of the integrals. Furthermore, the existence conditions of solutions to the non-local problem are impacted by the selection of the parameter $\rho^\alpha \Gamma(\alpha + 1)$.

An interesting question is whether, when a considerable amount of experimental data are available, it is possible to adjust the exponent ρ of the power function and identify an appropriate integral kernel. This would allow for fitting the experimental data with specific equations to predict the system's future trends. Alternatively, fractional operators with other special functions as integration kernels can be explored, and their applications in specific problems considered. Of course, the very process of theoretical development is inherently meaningful.

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