

Article

Investigations on the Chaos in the Generalized Double Sine-Gordon Planar System: Melnikov's Approach and Applications to Generating Antenna Factors

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Abstract

Many authors analyze the chaotic motion of the driven and damped double sine-Gordon equations and compute the Melnikov functions by numerical methods, taking an example to verify good agreement between numerical methods and analytical ones. Unfortunately, due to the lack of an explicit presentation of the Melnikov integral, the reader has difficulty navigating and touching upon Melnikov's elegant theory and, in particular, the formulation of the Melnikov criterion for the occurrence of chaos in a dynamical system, based solely on the provided illustrations of dependencies between the main parameters of the model under consideration. In this paper we will try to shed additional light on this important problem. A new planar system corresponding to the generalized double sine-Gordon model with many free parameters is considered. We also look at the modeling of radiation diagrams and antenna factors as potential uses for the Melnikov functions. A number of simulations are created. We also show off a few specific modules for examining the model's behavior. There is also discussion of one use for potential oscillation control.

Keywords: homoclinic and heteroclinic bifurcations; Melnikov function; Melnikov antenna factor; control over oscillations

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1. Introduction

Many authors focus their study on the planar system

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -\sum_{i=1}^M \frac{\lambda_i}{i} \sin \frac{x}{i} + \epsilon(A \cos(\omega t) - ay) \end{cases} \quad (1)$$

corresponding to the so called multiple sine-Gordon equation [1]. There is a dearth of literature on differential models of type (1) for $M > 2$, despite their physical significance. The propagation of strictly resonant sharp-line optical pulses is described by the triple

sine-Gordon equation [2]. A novel transformation for solving the triple sine-Gordon equation is presented in [3]. It is demonstrated that this intermediate transformation approach is effective in resolving complicated nonlinear evolution equations of a particular type. In the case $M = 1$, the reader can find detailed information in the classic studies of Tricomi [4], Stoker [5], Levi, Hoppensteadt and Miranker [6], Perko [7], and Guckenheimer and Holmes [8,9].

Interesting research on the topic of travelling waves and double-periodic structures in two-dimensional sine-Gordon systems can be found in [10–12].

Studies on the sine-Gordon and sinh-Gordon equations in the circle can be found in [13,14].

The sinh-Gordon-type equations for CMC surfaces are considered in [15].

For the stochastic hyperbolic sine-Gordon equation, see [16].

Melnikov-type chaos of planar systems with two discontinuities is considered in [17].

For chaotic solitons in the driven sine-Gordon model, see [18].

The double sine-Gordon equation ($M = 2$) plays an important role. Detailed information can be found in [19–25]. For other results, see [26–31]. The horseshoe chaos in the double sine-Gordon system

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -\lambda_1 \sin x - \frac{\lambda_2}{2} \sin \frac{x}{2} + \epsilon(A \cos(\omega t) - ay) \end{cases} \quad (2)$$

using the Melnikov technique [32] is considered in [33]. Perhaps the most advanced and trustworthy theoretical method for examining the beginning of chaos in dynamical systems is the Melnikov function technique. Specifically, this method allows the requirements for the presence of Smale’s horseshoe chaos to be calculated analytically in perturbed dynamical systems that are “sufficiently close” to an integrable system in a function-space sense (see [34]). The Melnikov approach can be successfully used for systems with complex potentials, for which an analytic calculation of the corresponding integrals is not feasible, as demonstrated by Bruhn and Koch [35]. The authors suggest using the Melnikov approach numerically to identify parameter regions of complex behavior. The resulting integrals must be assessed numerically due to the complex potential (e.g., using a fourth-order Runge–Kutta method). More precisely, the authors in [33] apply the Melnikov method to Equation (2) to study the phase-space portraits of the unperturbed equation (here written as a first-order system)

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -\lambda_1 \sin x - \frac{\lambda_2}{2} \sin \frac{x}{2}, \end{cases} \quad (3)$$

where λ_1, λ_2 are varied in the range $(-\infty, +\infty)$. Evidently, (3) is a Hamiltonian given by

$$H(x, y) = \frac{y^2}{2} + \lambda_1(1 - \cos x) + \lambda_2(1 - \cos \frac{x}{2}).$$

The chaotic motion of the driven and damped double sine-Gordon equation is examined in [36]. The authors compute the Melnikov functions by numerical methods, taking an example to verify the good agreement between numerical methods and analytical ones. Unfortunately, due to the lack of an explicit presentation of the Melnikov integral in the article cited above [33], the reader has difficulty navigating and touching upon Melnikov’s elegant theory and, in particular, the formulation of the Melnikov criterion for the occurrence of chaos in a dynamical system, based solely on the provided illustrations of dependencies between the main parameters of the model under consideration.

We will attempt to provide more insight into this significant issue in this study. We also consider a possible application of the Melnikov functions in modeling antenna factors and radiation diagrams. We suggest a new planar system corresponding to the generalized double sine-Gordon model. A number of simulations are created. We also show off a few specific modules for examining the model's behavior.

The main novelties of our generalization are mainly in three directions. First, the introduction of scaled cosines influences the frequencies of the perturbations. Second, the weights of these cosines have an impact on the amplitude of the oscillations. And third, the combination of both factors may capture some desired complicated behaviors of the spectrum.

Last but not least, we suggest a probabilistic construction that has several significant features. First, the approach we use allows a further improvement of the model both in amplitude and frequency by assuming non-discrete factors as well as specific mass in the different parts of the domain. Second, using a complex exponential presentation of the trigonometric functions, we relate the cosine functions with exponential functionals. Later, summing them (in stochastic terms, taking expectation), we reach the characteristic functions. Thus we can use the related techniques to simplify the oscillator dynamics while at the same time adding the desired behavior of the perturbations through suitable stochastic distributions. Three examples based on some very important distributions are discussed—normal (Gaussian), exponential, and gamma. Furthermore, this technique can be further generalized via Fourier transform theory.

The paper's plan is as follows. In Section 2, we provide our model. Additionally, a few simulations are shown. Section 3 applies the probabilistic techniques based on the characteristic functions to the oscillators. Finally, Section 4 concludes the paper.

2. A Fresh Model—A Few Simulations

We take into consideration the following updated model of the form

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -\sum_{i=1}^M \frac{\lambda_i}{i} \sin \frac{x}{i} + \epsilon \left(\sum_{i=1}^N A_i \cos(i\omega t) - ay \right), \end{cases} \quad (4)$$

where $0 \leq \epsilon < 1$, $a > 0$, $\omega > 0$, $A_i \geq 0$, $i = 1, 2, \dots, N$ and N is an integer.

2.1. The Case $M = 2$, $\lambda_1 = -1$, $\lambda_2 = \lambda$, $0 < \lambda < 4$

More precisely, we consider system (4) in the case $M = 2$, $\lambda_1 = -1$, $\lambda_2 = \lambda$, $0 < \lambda < 4$ and random N . The level set $H(x, y) = 0$ is composed of two homoclinic orbits:

$$\begin{cases} x_{hom}(t) = \pm 4 \arctan \left(\sqrt{\frac{4-\lambda}{\lambda}} \operatorname{sech} \left(\sqrt{\frac{4-\lambda}{4}} t \right) \right) \\ y_{hom}(t) = \mp \frac{2\sqrt{\lambda}(4-\lambda) \operatorname{sech} \left(\sqrt{\frac{4-\lambda}{4}} t \right) \tanh \left(\sqrt{\frac{4-\lambda}{4}} t \right)}{\lambda + (4-\lambda) \operatorname{sech}^2 \left(\sqrt{\frac{4-\lambda}{4}} t \right)}. \end{cases} \quad (5)$$

The heteroclinic orbits are of the form [33]

$$\begin{cases} x_{het}(t) = \pm 4 \arctan \left(\sqrt{\frac{\lambda}{4+\lambda}} \sinh \left(\sqrt{1 + \frac{1}{4}\lambda} t \right) \right) \\ y_{het}(t) = \pm \frac{2\sqrt{\lambda}(4+\lambda) \cosh \left(\sqrt{1 + \frac{1}{4}\lambda} t \right)}{4 + \lambda + \lambda \sinh^2 \left(\sqrt{1 + \frac{1}{4}\lambda} t \right)}. \end{cases} \quad (6)$$

The homo/heteroclinic orbits for $\lambda = 0.001$ are depicted in Figure 1. In [37], numerical techniques were used to establish the existence (and non-existence) of heteroclinic connections between the fixed points of the reduced ordinary differential equation associated with the model partial differential equation. For other results, see [38]. We note that the Melnikov function [32] corresponding to model (4) is of the form

$$M(t_0; \lambda; a; A_i) = \int_{-\infty}^{\infty} \left(y_{hom} \sum_{i=1}^N A_i \cos(i\omega(t + t_0)) - ay_{hom}^2 \right) dt. \quad (7)$$

From

$$\cos(i\omega(t + t_0)) = \cos(i\omega t) \cos(i\omega t_0) + \sin(i\omega t) \sin(i\omega t_0)$$

it is easy to see that

$$M(t_0; \lambda; a; A_i) = \sum_{i=1}^N A_i \sin(i\omega t_0) I_i - aI, \quad (8)$$

where

$$I = \int_{-\infty}^{\infty} \left(\frac{2\sqrt{\lambda}(4-\lambda) \operatorname{sech}\left(\sqrt{\frac{4-\lambda}{4}}.t\right) \tanh\left(\sqrt{\frac{4-\lambda}{4}}.t\right)}{\lambda + (4-\lambda) \operatorname{sech}^2\left(\sqrt{\frac{4-\lambda}{4}}.t\right)} \right)^2 dt,$$

$$I_i = \int_{-\infty}^{\infty} \frac{2\sqrt{\lambda}(4-\lambda) \operatorname{sech}\left(\sqrt{\frac{4-\lambda}{4}}.t\right) \tanh\left(\sqrt{\frac{4-\lambda}{4}}.t\right)}{\lambda + (4-\lambda) \operatorname{sech}^2\left(\sqrt{\frac{4-\lambda}{4}}.t\right)} \sin(i\omega t) dt$$

for $i = 1, 2, \dots, N$.

It is known that if $M(t_0) = 0$ and $\frac{dM(t_0)}{dt_0} \neq 0$ for some t_0 and some sets of parameters, then chaos occurs in the considered differential model.

Remark 1. 1. For fixed $N = 1$, the threshold curve is

$$\left| \frac{aI}{A_1 I_1} \right| \leq 1.$$

This result coincides with that obtained by Bartuccelli et al. in [33]. As already noted, due to the lack of an explicit presentation of the Melnikov integral, the reader encounters difficulties in the formulation of the Melnikov criterion for the occurrence of chaos in a dynamical system, based solely on the provided illustrations of dependencies between the main parameters of the model under consideration and bifurcation diagrams in confidential intervals. Now we will try to shed additional light on this important problem.

2. First, we will present the integrals I and I_i , $i = 1, 2, \dots, N$ in a different way, in order to more conveniently calculate the integrals I_i . As a result, we obtain

$$M(t_0; \lambda; a; A_i) = \sum_{i=1}^N A_i \sin(i\omega t_0) I_i^* - aI^*, \quad (9)$$

where

$$I^* = \int_{-\infty}^{\infty} \left(\frac{2\sqrt{\lambda} \sinh\left(\sqrt{\frac{4-\lambda}{4}}.t\right)}{1 + \frac{\lambda}{4-\lambda} \cosh^2\left(\sqrt{\frac{4-\lambda}{4}}.t\right)} \right)^2 dt,$$

$$I_i^* = \int_{-\infty}^{\infty} \frac{2\sqrt{\lambda} \sinh\left(\sqrt{\frac{4-\lambda}{4}} \cdot t\right)}{1 + \frac{\lambda}{4-\lambda} \cosh^2\left(\sqrt{\frac{4-\lambda}{4}} \cdot t\right)} \sin(i\omega t) dt$$

for $i = 1, 2, \dots, N$.

3. We will note that the integral I^* can be written in the following explicit form

$$I^* = 8\sqrt{4-\lambda} - 4\lambda \operatorname{ArcCoth}\left(\frac{2}{\sqrt{4-\lambda}}\right).$$

4. For any arbitrary N , the reader can define the Melnikov criterion for the occurrence of chaos in the dynamical system under consideration. The integrals I_i^* can be calculated numerically.

We will illustrate what has been said with two relevant examples.

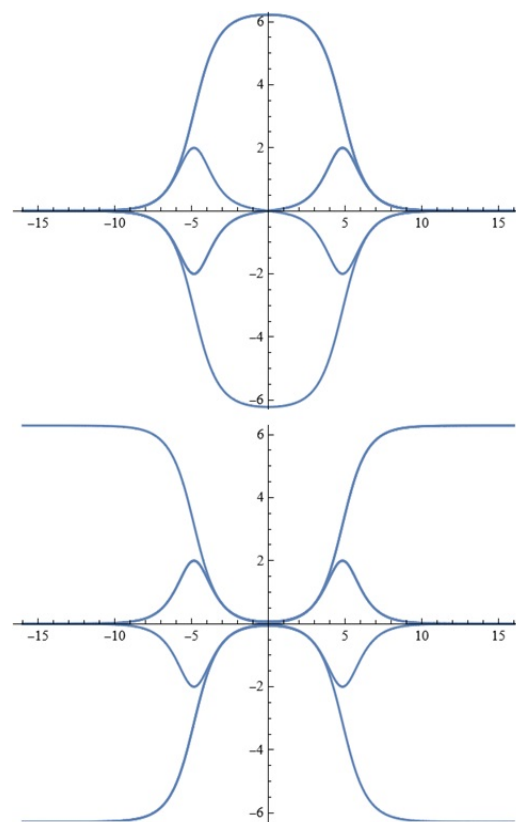


Figure 1. Homo/heteroclinic orbits for $\lambda = 0.001$.

Example 1. Let us fix $M = 2$; $N = 5$; $A_1 = 0.9$; $A_2 = 0.1$; $A_3 = 0.2$; $A_4 = 0.4$; $A_5 = 0.8$; and $a = 0.03$.

(a) For $\omega = 0.22$ and $\lambda = 0.1$, from (9) for the Melnikov function, we obtain

$$\begin{aligned} M(t_0) = & A_1 6.32597 \sin(\omega t_0) + A_2 9.04617 \sin(2\omega t_0) + \\ & A_3 7.7784 \sin(3\omega t_0) + A_4 4.52442 \sin(4\omega t_0) + \\ & A_5 1.34062 \sin(5\omega t_0) - 0.443587. \end{aligned}$$

From Figure 2a, it can be seen that $M(t_0)$ has a unique root in the interval $(0, 15)$.

(b) For $\omega = 0.22$ and $\lambda = 0.363$, from (9) for the Melnikov function, we obtain

$$\begin{aligned} M(t_0) = & A_1 4.92654 \sin(\omega t_0) + A_2 7.48983 \sin(2\omega t_0) + \\ & A_3 7.3189 \sin(3\omega t_0) + A_4 5.52315 \sin(4\omega t_0) + \\ & A_5 3.33141 \sin(5\omega t_0) - 0.376268. \end{aligned}$$

From Figure 2b, it can be seen that $M(t_0)$ has a root $t_0 \approx 5.03$ and $\frac{dM(t_0)}{dt_0} \approx 0.8847$. Some nonstandard methods for solving nonlinear equations of type $M(t_0) = 0$ can be found in [39].

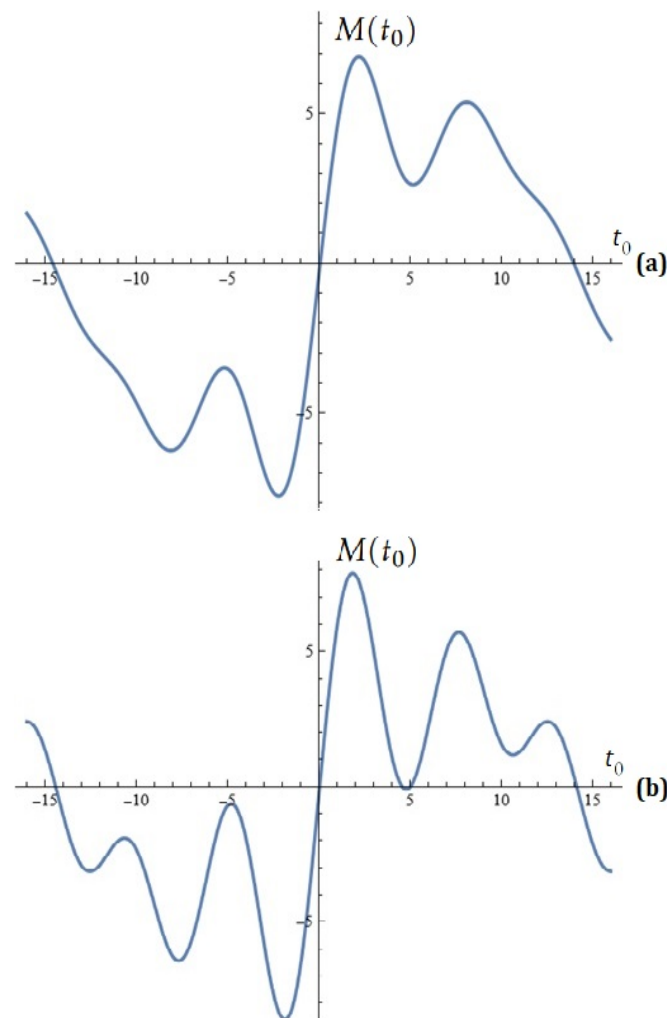


Figure 2. Melnikov function $M(t_0)$: (a) Example 1(a); (b) Example 1(b).

We will look at some interesting simulations of model (4) (for fixed $M = 2$):

Example 2. For given $N = 3$, $a = 0.03$, $\omega = 0.32$, $\lambda = 0.001$, $A_1 = 0.23$, $A_2 = 0.1$, $A_3 = 0.8$, and $\epsilon = 0.1$, the simulations of system (4) for $x_0 = 0.4$ and $y_0 = 0.2$ are depicted in Figure 3.

Example 3. For given $N = 6$, $a = 0.03$, $\omega = 0.25$, $\lambda = 3.1$, $A_1 = 0.9$, $A_2 = 0.1$, $A_3 = 0.2$, $A_4 = 0.6$, $A_5 = 0.6$, $A_6 = 0.9$, and $\epsilon = 0.01$, the simulations of system (4) for $x_0 = 0.6$ and $y_0 = 0.2$ are depicted in Figure 4.

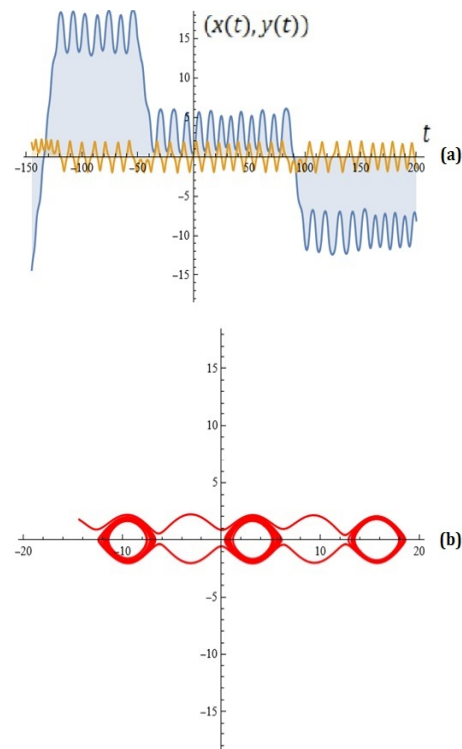


Figure 3. (a) The solutions of the system (4); (b) phase space (Example 2).

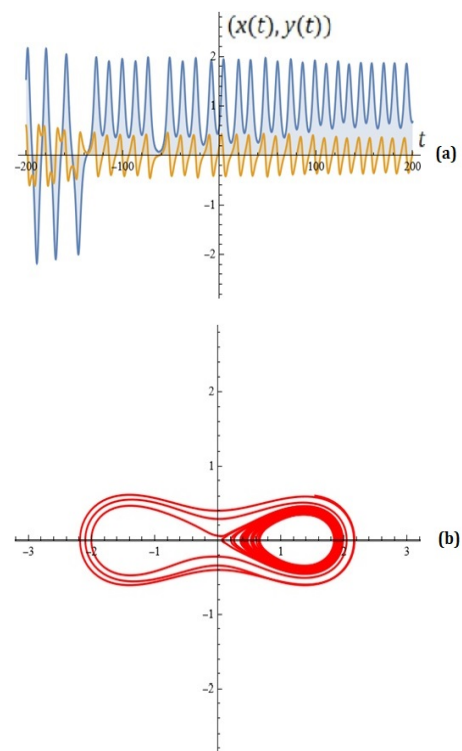


Figure 4. (a) The solutions of the system (4); (b) phase space (Example 3).

2.2. The Case $M = 2$, $\lambda_1 = 1$, $\lambda_2 = \lambda$, $0 < \lambda < 4$

More precisely, we consider system (4) in the case $M = 2$, $\lambda_2 = \lambda$, $0 < \lambda < 4$, and random N .

The heteroclinic orbits are of the form [33]

$$\begin{cases} x_{het}(t) = \pm 4 \arctan\left(\sqrt{\frac{4+\lambda}{4-\lambda}} \tanh\left(\sqrt{\frac{16-\lambda^2}{64}} \cdot t\right)\right) \\ y_{het}(t) = \pm \frac{(16-\lambda^2) \operatorname{sech}^2\left(\sqrt{\frac{16-\lambda^2}{64}} \cdot t\right)}{2\left(4-\lambda+(4+\lambda) \tanh^2\left(\sqrt{\frac{16-\lambda^2}{64}} t\right)\right)}. \end{cases} \quad (10)$$

The heteroclinic orbits for $\lambda = 0.001$ are depicted in Figure 5.

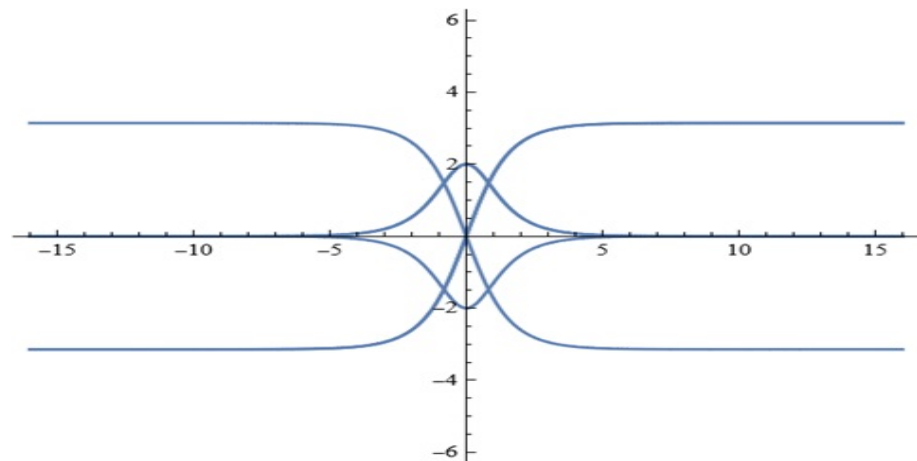


Figure 5. Heteroclinic orbits for $\lambda = 0.001$.

In this case, the Melnikov function corresponding to model (4) is of the form

$$\begin{aligned} M(t_0; \lambda; a; A_i) &= \int_{-\infty}^{\infty} \left(y_{het} \sum_{i=1}^N A_i \cos(i\omega(t+t_0)) - a y_{het}^2 \right) dt \\ &= \sum_{i=1}^N A_i \cos(i\omega t_0) J_i - a J, \end{aligned} \quad (11)$$

where

$$\begin{aligned} J_i &= \int_{-\infty}^{\infty} \frac{(16-\lambda^2) \operatorname{sech}^2\left(\sqrt{\frac{16-\lambda^2}{16}} \cdot t\right)}{2\left(4-\lambda+(4+\lambda) \tanh^2\left(\sqrt{\frac{16-\lambda^2}{64}} t\right)\right)} \cos(i\omega t) dt, \\ J &= \int_{-\infty}^{\infty} \left(\frac{(16-\lambda^2) \operatorname{sech}^2\left(\sqrt{\frac{16-\lambda^2}{16}} \cdot t\right)}{2\left(4-\lambda+(4+\lambda) \tanh^2\left(\sqrt{\frac{16-\lambda^2}{64}} t\right)\right)} \right)^2 dt \end{aligned}$$

for $i = 1, 2, \dots, N$.

We will note that the integral J can be written in the following explicit form:

$$J = 2 \left(\sqrt{16-\lambda^4} + 2\lambda \operatorname{ArcTanh}\left(\frac{4+\lambda}{\sqrt{16-\lambda^2}}\right) \right).$$

Example 4. Let us fix $M = 2$; $N = 5$; $A_1 = 0.9$; $A_2 = 0.1$; $A_3 = 0.2$; $A_4 = 0.4$; $A_5 = 0.8$; and $a = 0.03$.

(a) For $\omega = 0.22$ and $\lambda = 0.1$, from (11) for the Melnikov function, we obtain

$$\begin{aligned}
 M(t_0) = & A_1 6.02374 \cos(\omega t_0) + A_2 5.12469 \cos(2\omega t_0) + \\
 & A_3 4.042082 \cos(3\omega t_0) + A_4 3.04087 \cos(4\omega t_0) + \\
 & A_5 2.22736 \cos(5\omega t_0) - 0.2495.
 \end{aligned} \quad (12)$$

From Figure 6a, it can be seen that $M(t_0)$ has a unique root in the interval $(0, 15)$.

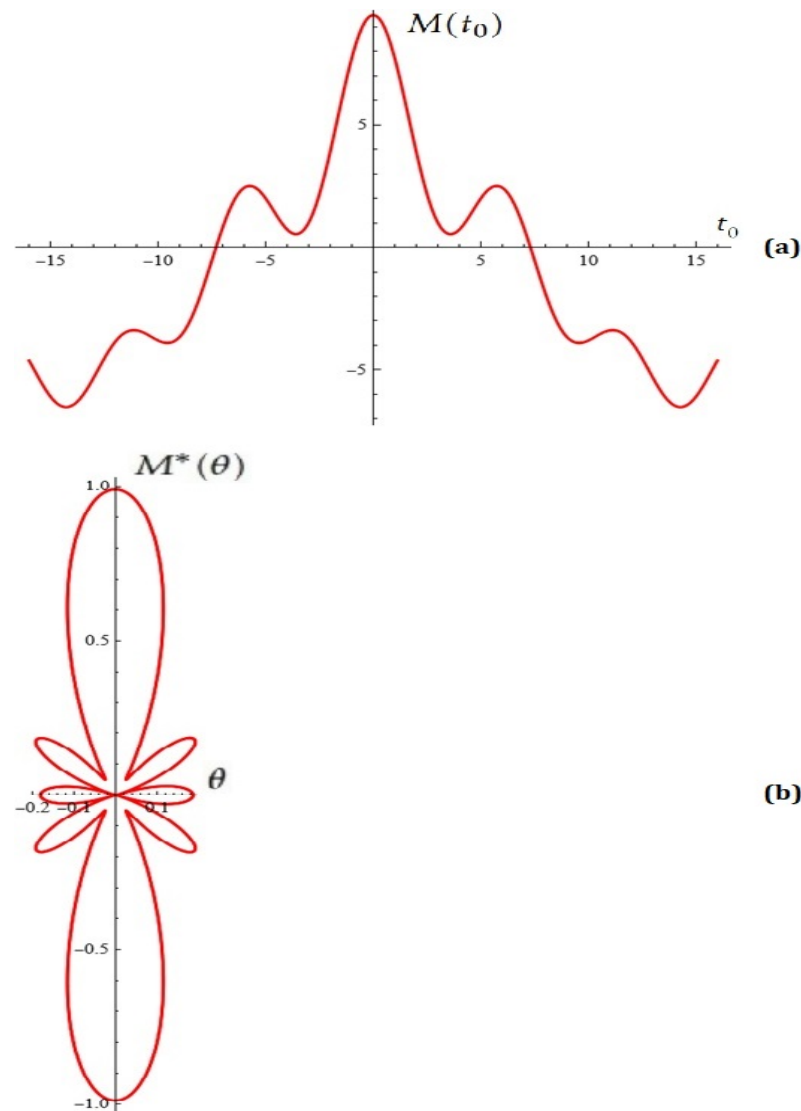


Figure 6. (a) Melnikov function $M(t_0)$ (from Example 6(a)); (b) Melnikov antenna factor (from Example 6(b)).

(b) In our previous publications, we discussed a possible application of the Melnikov functions in the modeling and synthesis of radiation antenna diagrams.

We define the hypothetical normalized antenna factor as follows:

$$M^*(\theta) = \frac{1}{D} |M(K \cos \theta + k_1)|,$$

where θ is the azimuth angle; $K = kd$; and $k = \frac{2\pi}{\lambda}$ (λ is the wave length; d is the distance between emitters; k_1 is the phase difference; and $D = \max M(\theta)$ in the considered interval [40]). Usually d is chosen as a function of wave length, i.e., $d = g(\lambda)$.

The general N -element linear-phased array factor used to find A_k coefficients is

$$AF(\theta) = \sum_{k=1}^{\frac{N}{2}} A_k \cos((2k-1)u) = M(x),$$

where $u = \frac{\pi d}{\lambda} \cos \theta$ and $x = x_0 \cos u$, where x_0 is a design parameter.

We note that the array factor depends on the array geometry, amplitudes, and phase of the excitation of individual antennas. This idea was borrowed from Soltis [41], in his generation of new Gegenbauer-like and Jacobi-like antenna arrays. In the cited article, the reader can find interesting comparisons with the classical Dolph–Chebyshev antenna array.

After determining the amplitudes of the excitation currents, it is necessary to take the so-called energy characteristic. We will note that the Melnikov function corresponding to the considered differential model is in practice a generalized trigonometric polynomial. These studies encouraged us to offer specialists working in the field of antenna feeder analysis tests the possibility of generating Melnikov-type antenna arrays. This will be a part of much wider Web-based application that we plan to develop [42]. We foresee future contacts with specialists from the field of radio-electronics and radio-location and possible development of a prototype with the possibility of applying additional optimization methods for controlling the side radiation.

For $\omega = 0.22$; $\lambda = 0.1$; and $K = 7.96$; $k_1 = 0.01$, the Melnikov antenna factor based on (12) is depicted in Figure 6b.

Example 5. Let us fix $M = 2$; $N = 7$; $A_1 = 0.3$; $A_2 = 0.1$; $A_3 = 0.1$; $A_4 = 0.2$; $A_5 = 0.3$; $A_6 = 0.71$; $A_7 = 0.65$; and $a = 0.03$.

(a) For $\omega = 0.2$ and $\lambda = 0.2$, from (11) for the Melnikov function, we obtain

$$\begin{aligned} M(t_0) = & A_1 6.181794 \cos(\omega t_0) + A_2 5.48496 \cos(2\omega t_0) + \\ & A_3 4.42261 \cos(3\omega t_0) + A_4 3.46242 \cos(4\omega t_0) + \\ & A_5 2.63921 \cos(5\omega t_0) + A_6 1.98139 \cos(6\omega t_0) + \\ & A_7 1.47511 \cos(7\omega t_0) - 0.25915. \end{aligned} \quad (13)$$

Melnikov function $M(t_0)$ is depicted in Figure 7a.

(b) For $\omega = 0.2$; $\lambda = 0$; $K = 11.5$; and $k_1 = 0$, the Melnikov antenna factor based on (13) is depicted in Figure 7b.

Example 6. Let us fix $M = 2$; $N = 11$; $A_1 = 0.18$; $A_2 = 0.01$; $A_3 = 0.05$; $A_4 = 0.01$; $A_5 = 0.26$; $A_6 = 0.1$; $A_7 = 0.1$; $A_8 = 0.86$; $A_9 = 0.1$; $A_{10} = 0.16$; $A_{11} = 0.36$; and $a = 0.01$.

(a) For $\omega = 0.18$ and $\lambda = 0.3$, from (11) for the Melnikov function, we obtain

$$\begin{aligned} M(t_0) = & A_1 6.33567 \cos(\omega t_0) + A_2 5.6794 \cos(2\omega t_0) + \\ & A_3 4.81018 \cos(3\omega t_0) + A_4 3.91328 \cos(4\omega t_0) + \\ & A_5 3.10156 \cos(5\omega t_0) + A_6 2.41932 \cos(6\omega t_0) + \\ & A_7 1.86959 \cos(7\omega t_0) + A_8 1.43699 \cos(8\omega t_0) + \\ & A_9 1.10117 \cos(9\omega t_0) + A_{10} 0.842396 \cos(10\omega t_0) + \\ & A_{11} 0.643827 \cos(11\omega t_0) - 0.0896499. \end{aligned} \quad (14)$$

Melnikov function $M(t_0)$ is depicted in Figure 8a.

- (b) For $\omega = 0.18$; $\lambda = 0.3$; $K = 16.5$; and $k_1 = 0$ the Melnikov antenna factor based on (14) is depicted in Figure 8b.

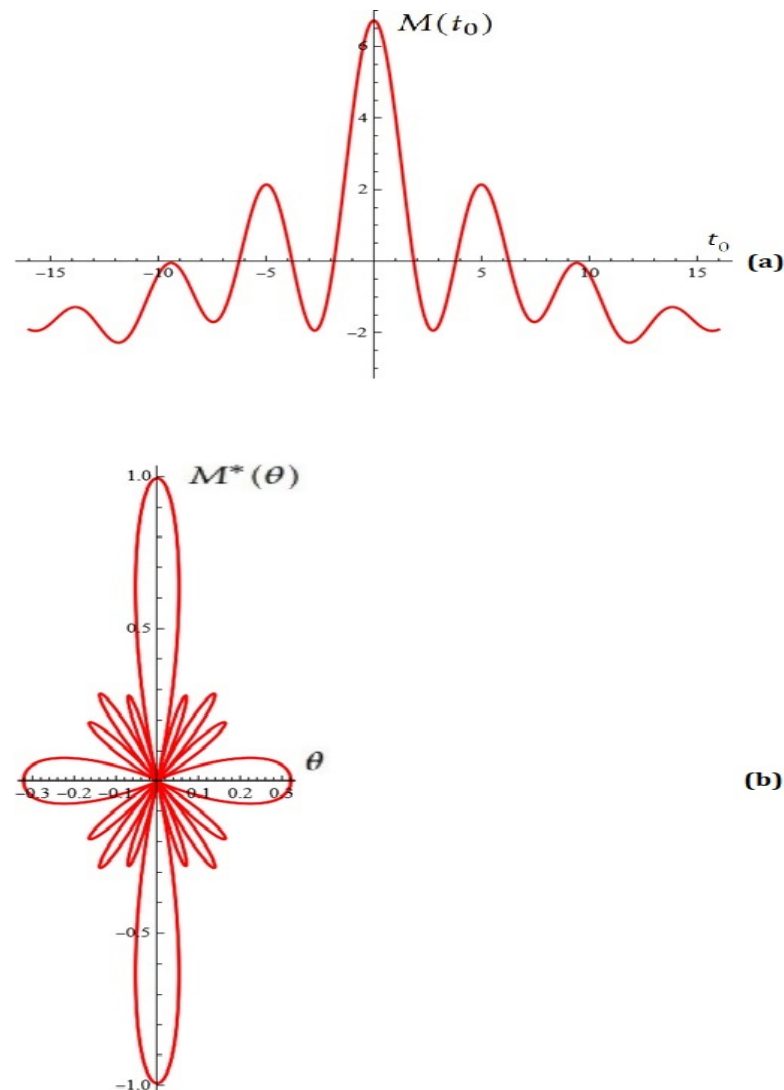


Figure 7. (a) Melnikov function $M(t_0)$ (from Example 5(a)); (b) Melnikov antenna factor (from Example 5(b)).

Some simulations

Example 7. For given $N = 3$, $a = 0.03$, $\omega = 0.32$, $\lambda = 0.001$, $A_1 = 0.23$, $A_2 = 0.1$, $A_3 = 0.8$, and $\epsilon = 0.1$, the simulations of system (4) for $x_0 = 0.4$ and $y_0 = 0.20$ are depicted in Figure 9.

Example 8. For given $N = 5$, $a = 0.03$, $\omega = 0.22$, $\lambda = 0.1$, $A_1 = 0.9$, $A_2 = 0.1$, $A_3 = 0.2$, $A_4 = 0.55$, $A_5 = 0.8$, and $\epsilon = 0.1185$, the simulations of system (4) for $x_0 = 0.5$ and $y_0 = 0.2$ are depicted in Figure 10.

With the help of a dedicated module implemented in our web-based application [42], we will present two more simulations on the triple and quadruple sine-Gordon differential model.

Example 9. For given $N = 6$, $M = 3$, $a = 0.03$, $\omega = 0.2$, $\lambda_1 = 1$, $\lambda_2 = 3$, $\lambda_3 = 0.6$, $A_1 = 0.2$, $A_2 = 0.16$, $A_3 = 0.1$, $A_4 = 0.6$, $A_5 = 0.2$, $A_6 = 0.9$, and $\epsilon = 0.13$, the simulations of system (4) for $x_0 = 0.6$ and $y_0 = 0.2$ are depicted in Figure 11.

Example 10. For given $N = 7$, $M = 4$, $a = 0.03$, $\omega = 0.2$, $\lambda_1 = 1$, $\lambda_2 = 3$, $\lambda_3 = 1.6$, $\lambda_4 = 2$, $A_1 = 0.2$, $A_2 = 0.16$, $A_3 = 0.1$, $A_4 = 0.6$, $A_5 = 0.1$, $A_6 = 0.9$, $A_7 = 0.7$, and $\epsilon = 0.13$, the simulations of system (4) for $x_0 = 0.5$ and $y_0 = 0.2$ are depicted in Figure 12.

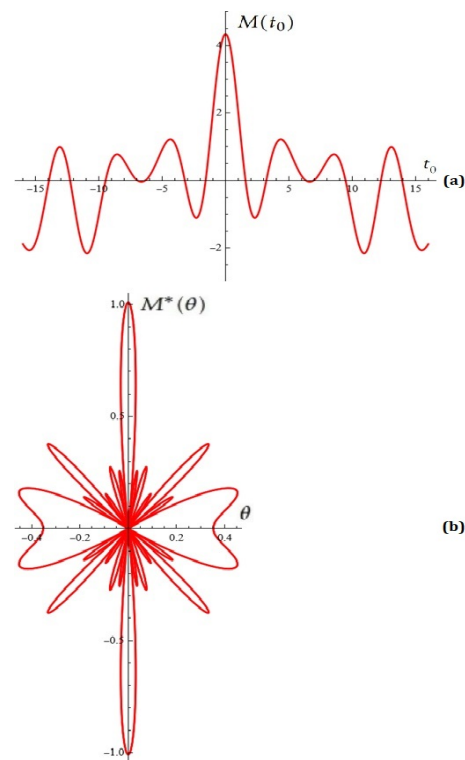


Figure 8. (a) Melnikov function $M(t_0)$ (from Example 6(a)); (b) Melnikov antenna factor (from Example 6(b)).

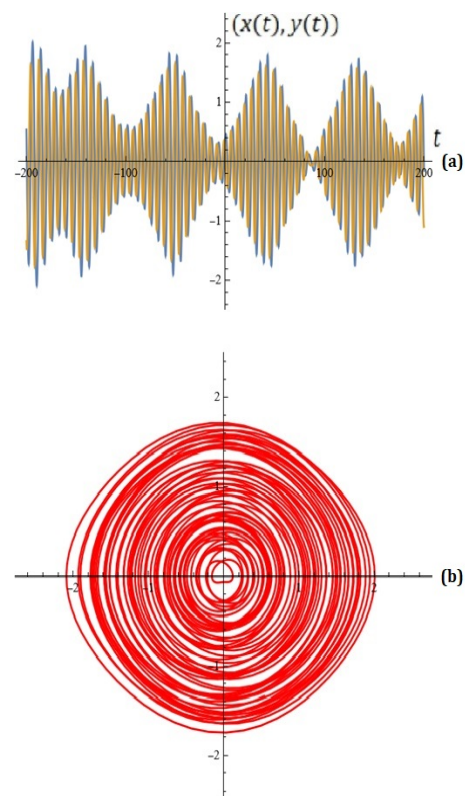


Figure 9. (a) The solutions of the system (4); (b) phase space (Example 3).

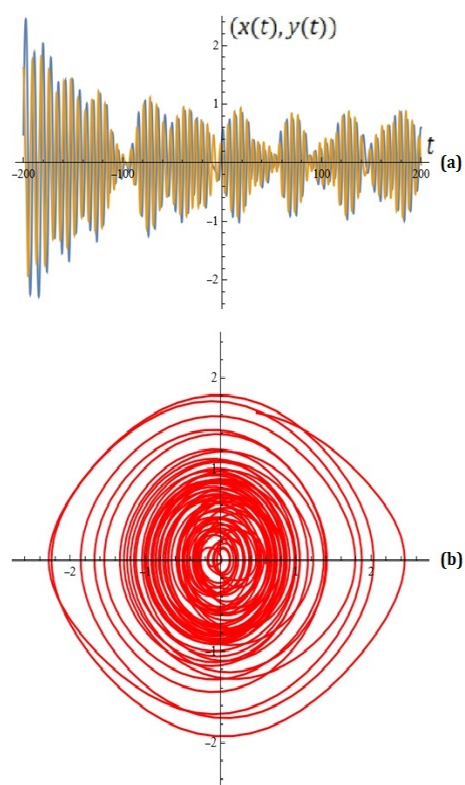


Figure 10. (a) The solutions of the system (4); (b) phase space (Example 4).

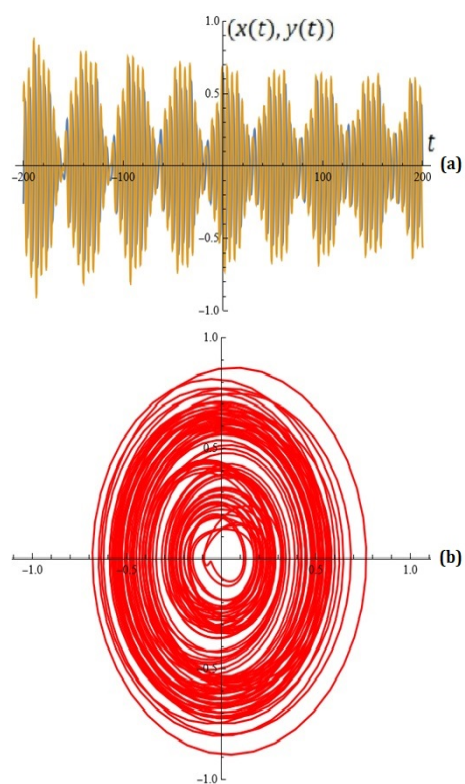


Figure 11. (a) The solutions of the system (4); (b) phase space (Example 5).

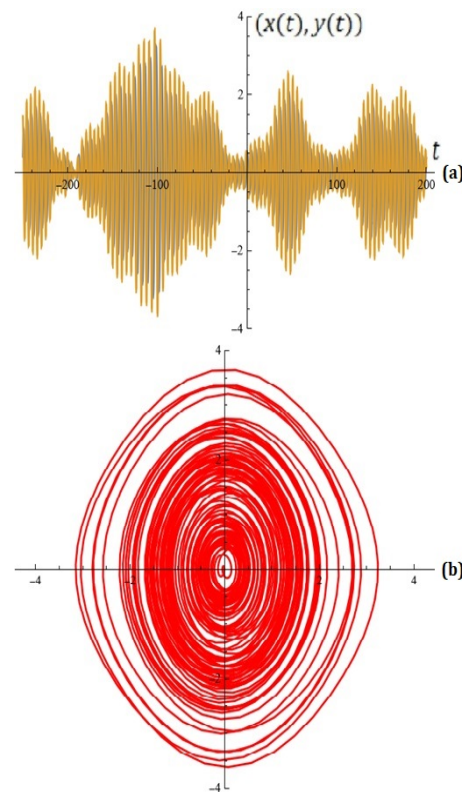


Figure 12. (a) The solutions of the system (4); (b) phase space (Example 6).

3. Probabilistic Constructions for Model Dynamics

We shall rewrite model (4) in terms of the characteristic functions of random variables.

3.1. Reformulation of the Problem

Suppose that $\lambda_j \geq 0$ and let $\lambda = \sum_{j=1,2,\dots} \lambda_j$, $A = \sum_{j=1,2,\dots} A_j$, $\bar{a} = \frac{a}{A}$, $p_j = \frac{\lambda_j}{\lambda}$, and $q_j = \frac{A_j}{A}$. Let us have a measurable space $(\Omega, \mathcal{F})$ with a discrete, possibly infinite, structure of the sample outcomes $\Omega = \{\gamma_1, \gamma_2, \gamma_3, \dots\}$. Let us have two not necessarily equivalent probability measures $\mathbb{P}$ and $\mathbb{Q}$, such that $\mathbb{P}(\gamma_j) = p_j$ and $\mathbb{Q}(\gamma_j) = q_j$. Furthermore, let us define two random variables ζ and η by $\zeta(\gamma_j) = j$ and $\eta(\gamma_j) = \frac{1}{j}$. Thus we can rewrite the second part of dynamics (4) as

$$\begin{aligned} \frac{dy}{dt} &= - \sum_{j=1}^M \frac{\lambda_j}{j} \sin \frac{x}{j} + \epsilon \left(\sum_{i=1}^N A_i \cos(j\omega t) - ay \right) \\ &= -\lambda \sum_{j=1}^M \frac{p_j}{j} \sin \frac{x}{j} + \epsilon A \left(\sum_{j=1}^N q_j \cos(j\omega t) - \bar{a}y \right) \\ &= -\lambda \mathbb{E}^{\mathbb{P}}[\eta \sin(x\eta)] + \epsilon A \left(-\bar{a}y + \mathbb{E}^{\mathbb{Q}}[\cos(\zeta\omega t)] \right). \end{aligned} \quad (15)$$

Using the exponential presentation of the trigonometric functions

$$\begin{aligned} \sin(\alpha) &= \frac{e^{i\alpha} - e^{-i\alpha}}{2i} \\ \cos(\alpha) &= \frac{e^{i\alpha} + e^{-i\alpha}}{2}, \end{aligned} \quad (16)$$

we transform dynamics (15) into

$$\begin{aligned}\frac{dy}{dt} &= \frac{\lambda}{2i} \left(E^{\mathbb{P}} \left[\eta e^{-ix\eta} \right] - E^{\mathbb{P}} \left[\eta e^{ix\eta} \right] \right) + \frac{\epsilon A}{2} \left(-2\bar{a}y + \mathbb{E}^{\mathbb{Q}} \left[e^{i\zeta\omega t} \right] + \mathbb{E}^{\mathbb{Q}} \left[e^{-i\zeta\omega t} \right] \right) \\ &= \frac{\lambda}{2} \left(\Psi_{\eta}^{\mathbb{P}'}(x) - \Psi_{\eta}^{\mathbb{P}'}(-x) \right) + \frac{\epsilon A}{2} \left(-2\bar{a}y + \Psi_{\zeta}^{\mathbb{Q}}(\omega t) + \Psi_{\zeta}^{\mathbb{Q}}(-\omega t) \right).\end{aligned}\quad (17)$$

Above, we denoted by $\Psi_{\eta}^{\mathbb{P}}(x)$ and $\Psi_{\zeta}^{\mathbb{Q}}(x)$ the characteristic functions of the random variables η and ζ w.r.t. the measures $\mathbb{P}$ and $\mathbb{Q}$, respectively. Having in mind presentation (17), we conclude that the kind of outcome set Ω is not important. Thus, the y -dynamics in (4) can be generalized as

$$\frac{dy}{dt} = -\lambda \int_{\gamma \in \Omega} \eta(\gamma) \sin(x\eta(\gamma)) \mathbb{P}(d\gamma) + A\epsilon \left(\int_{\gamma \in \Omega} \cos(\zeta(\gamma)\omega t) \mathbb{Q}(d\gamma) - \bar{a}y \right). \quad (18)$$

Thus the Melnikov function turns into

$$M(t_0) = \int_{-\infty}^{+\infty} y_0(t) \left(-2\bar{a}y_0(t) + \Psi_{\zeta}^{\mathbb{Q}}(\omega(t+t_0)) + \Psi_{\zeta}^{\mathbb{Q}}(-\omega(t+t_0)) \right) dt. \quad (19)$$

We shall now provide a particular example to illustrate this generalization. Assume that the random variable η is normally distributed with parameters (μ, σ) , whereas ζ is exponentially distributed with an intensity β . The characteristic functions are

$$\begin{aligned}\Psi_{\text{Gaussian}}(x) &= e^{i\mu x - \frac{\sigma^2 x^2}{2}} \\ \Psi_{\text{exp}}(x) &= \frac{\beta}{\beta - ix}.\end{aligned}\quad (20)$$

The derivatives of these functions are

$$\begin{aligned}\Psi'_{\text{Gaussian}}(x) &= e^{i\mu x - \frac{\sigma^2 x^2}{2}} (i\mu - \sigma^2 x) \\ \Psi'_{\text{exp}}(x) &= \frac{\beta i}{(\beta - ix)^2}.\end{aligned}\quad (21)$$

Thus the oscillator's dynamics (4) can be written as

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -\lambda e^{-\frac{\sigma^2 x^2}{2}} \left(\mu \sin \mu x + \sigma^2 x \cos \mu x \right) + \epsilon A \left(-\bar{a}y + \frac{\beta^2}{\beta^2 + \omega^2 t^2} \right).\end{aligned}\quad (22)$$

On the other hand, Melnikov function (19) turns into

$$M(t_0) = \int_{-\infty}^{+\infty} y_0(t) \left(-\bar{a}y_0(t) + \frac{\beta^2}{\beta^2 + \omega^2(t+t_0)^2} \right) dt. \quad (23)$$

A particular example is depicted in the first column of Figure 13. The used parameters are $a = 0.2$, $A = 1$, $\epsilon = 0.015$, $\mu = 0.1$, $\sigma = 1$, $\lambda = 1$, $\beta = 10$, $\omega = 10$, and $x_0 = y_0 = -0.2$.

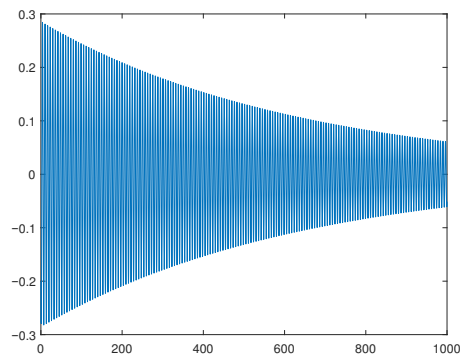
Let us consider the symmetric dynamics assuming that ζ is normally distributed whereas η is exponentially distributed. Thus, dynamics (4) turns into

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -2\frac{\lambda\beta^2x}{(\beta^2+x^2)^2} + \epsilon A\left(-\bar{a}y + \cos(\mu\omega t)e^{-\frac{\sigma^2\omega^2t^2}{2}}\right).\end{aligned}\quad (24)$$

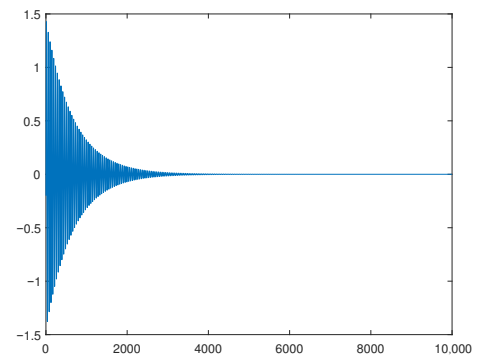
The Melnikov function (19) is

$$M(t_0) = \int_{-\infty}^{+\infty} y_0(t) \left(-\bar{a}y_0(t) + \cos(\mu\omega(t+t_0))e^{-\frac{\sigma^2\omega^2(t+t_0)^2}{2}} \right) dt. \quad (25)$$

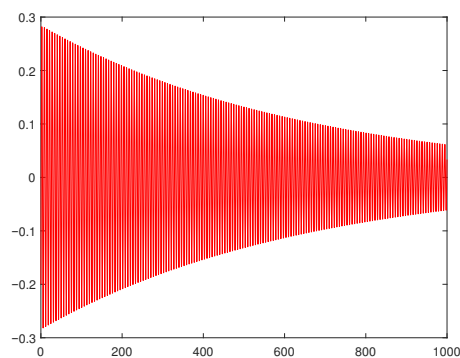
The results based on the same parameters as above are presented in the second column of Figure 13.



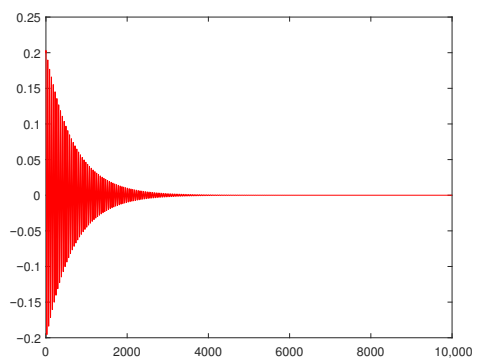
(a) Normal vs. Exponential— $x(t)$



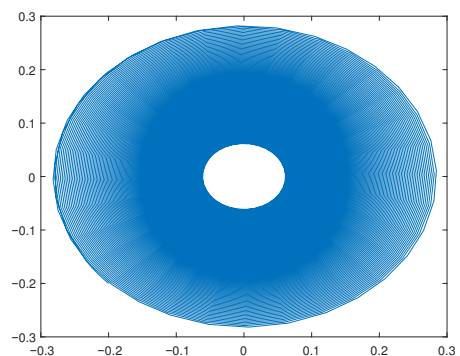
(b) Exponential vs. Normal— $x(t)$



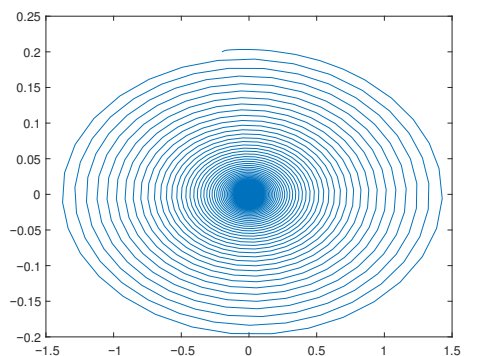
(c) Normal vs. Exponential— $y(t)$



(d) Exponential vs. Normal— $y(t)$



(e) Normal vs. Exponential—phase portrait



(f) Exponential vs. Normal—phase portrait

Figure 13. Dynamics.

In addition to the previous examples, we present the behavior of both models w.r.t. to their parameters. We vary β and σ in the interval $(0.01, 10)$, and μ in $(-10, 10)$. The used parameters are as follows: $a = 0.2$, $A = 1$, $\epsilon = 0.015$, $\mu = 1.1$, $\sigma = 1$, $\lambda = 1.5$, $\beta = 10$, $\omega = 1.6$, $x_0 = -0.01$, and $y_0 = 0$. When some of the parameters, β , σ , and μ , are fixed, then we use the values $\beta = 10$, $\sigma = 1$, and $\mu = 1.1$. The derived results are illustrated by the phase portraits in Figure 14—the first column is for dynamics (22), whereas the second one is for (24).

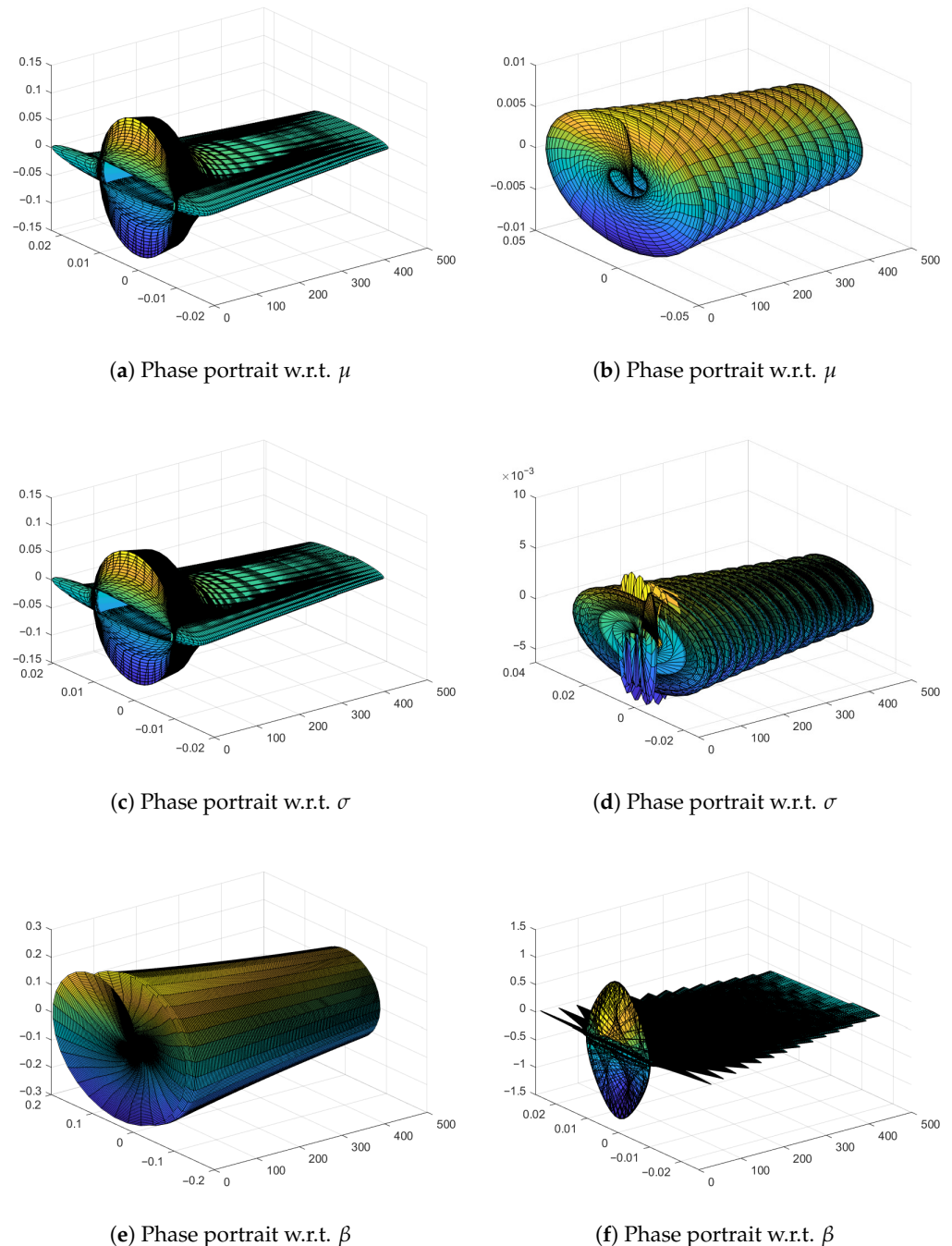


Figure 14. Dynamics.

3.2. Discussion on the Possible Control on the Perturbation

Note that the perturbation in dynamics (17) depends on the characteristic function of the random variable ξ through the term $\Psi_{\xi}^{\mathbb{Q}}(\omega t) + \Psi_{\xi}^{\mathbb{Q}}(-\omega t)$. We indicate above that its form for the normal and exponential distributions is

$$\begin{aligned}\Psi_{\xi}^{\mathbb{Q}}(\omega t) + \Psi_{\xi}^{\mathbb{Q}}(-\omega t) &= \cos(\mu\omega t)e^{-\frac{\sigma^2\omega^2 t^2}{2}} \\ \Psi_{\xi}^{\mathbb{Q}}(\omega t) + \Psi_{\xi}^{\mathbb{Q}}(-\omega t) &= \frac{\beta^2}{\beta^2 + \omega^2 t^2},\end{aligned}\quad (26)$$

respectively. We conclude that both alternatives lead to decay of the perturbations when t tends to infinity. In the exponential case, this decay is of order $\frac{1}{t^2}$, whereas it is significantly faster for the Gaussian alternative – the order is $e^{-\frac{\sigma^2\omega^2 t^2}{2}}$. Furthermore, it is important to note that the periodic term exists in the Gaussian case, whereas it vanishes for the exponential one. On the other hand, the original model (4), defined on the distribution with atoms in the points $1, 2, \dots, N$, leads to the existence of a periodicity without the time decay. It begs the question whether there exists a distribution that leads to a growth dependence. To answer, we consider one more example based on the gamma distribution with parameters θ and α . Its characteristic function is

$$\Psi_{\xi}^{\mathbb{Q}}(x) = (1 - i\theta x)^{-\alpha} \quad (27)$$

and thus

$$\begin{aligned}\Psi_{\xi}^{\mathbb{Q}}(x) + \Psi_{\xi}^{\mathbb{Q}}(-x) &= (1 - i\theta x)^{-\alpha} + (1 + i\theta x)^{-\alpha} \\ &= \sqrt{(1 + \theta^2 x^2)^{-\alpha}} \left(\frac{1}{\sqrt{1 + \theta^2 x^2}} - i \frac{\theta x}{\sqrt{1 + \theta^2 x^2}} \right)^{-\alpha} \\ &\quad + \sqrt{(1 + \theta^2 x^2)^{-\alpha}} \left(\frac{1}{\sqrt{1 + \theta^2 x^2}} + i \frac{\theta x}{\sqrt{1 + \theta^2 x^2}} \right)^{-\alpha} \\ &= \sqrt{(1 + \theta^2 x^2)^{-\alpha}} \left(\cos\left(-\arccos\left(\frac{1}{\sqrt{1 + \theta^2 x^2}}\right)\right) + i \sin\left(-\arccos\left(\frac{1}{\sqrt{1 + \theta^2 x^2}}\right)\right) \right)^{-\alpha} \\ &\quad + \sqrt{(1 + \theta^2 x^2)^{-\alpha}} \left(\cos\left(\arccos\left(\frac{1}{\sqrt{1 + \theta^2 x^2}}\right)\right) + i \sin\left(\arccos\left(\frac{1}{\sqrt{1 + \theta^2 x^2}}\right)\right) \right)^{-\alpha} \\ &= \sqrt{(1 + \theta^2 x^2)^{-\alpha}} \left(e^{i\alpha \arccos\left(\frac{1}{\sqrt{1 + \theta^2 x^2}}\right)} + e^{-i\alpha \arccos\left(\frac{1}{\sqrt{1 + \theta^2 x^2}}\right)} \right) \\ &= 2\sqrt{(1 + \theta^2 x^2)^{-\alpha}} \cos\left(\alpha \arccos\left(\frac{1}{\sqrt{1 + \theta^2 x^2}}\right)\right).\end{aligned}\quad (28)$$

We can conclude that the perturbation term increases w.r.t. the time as t . To complete our example, we assume that the first random variable η is again gamma distributed with parameters $\bar{\theta}$ and $\bar{\alpha}$. Similar arguments as in (28) lead to

$$\Psi_{\eta}^{\mathbb{P}'}(x) - \Psi_{\eta}^{\mathbb{P}'}(-x) = -2\text{sign}(x)\bar{\alpha}\bar{\theta}\sqrt{(1 + \bar{\theta}^2 x^2)^{-\bar{\alpha}-1}} \sin\left((\bar{\alpha} + 1) \arccos\left(\frac{1}{\sqrt{1 + \bar{\theta}^2 x^2}}\right)\right). \quad (29)$$

Thus the y -dynamics of model (4) can be written as

$$\begin{aligned} \frac{dy}{dt} = & -\lambda \operatorname{sign}(x) \bar{\alpha} \bar{\theta} \sqrt{(1 + \bar{\theta}^2 x^2)^{-\bar{\alpha}-1}} \sin \left((\bar{\alpha} + 1) \arccos \left(\frac{1}{\sqrt{1 + \bar{\theta}^2 x^2}} \right) \right) \\ & + \epsilon A \left(-\bar{a} y + \sqrt{(1 + \theta^2 x^2)^{-\alpha}} \cos \left(\alpha \arccos \left(\frac{1}{\sqrt{1 + \theta^2 \omega^2 t^2}} \right) \right) \right). \end{aligned} \quad (30)$$

The Melnikov integral turns into

$$M(t_0) = \int_{-\infty}^{+\infty} y_0(t) \left(-\bar{a} y_0(t) + \sqrt{(1 + \theta^2 (t + t_0)^2)^{-\alpha}} \cos \left(\alpha \arccos \left(\frac{1}{\sqrt{1 + \theta^2 \omega^2 (t + t_0)^2}} \right) \right) \right) dt. \quad (31)$$

The generated behavior for parameter values $\bar{\theta} = 1$, $\bar{\alpha} = 2$, $\theta = 1$, $\alpha = 0.2$, $a = 0.2$, $A = 1$, $\epsilon = 0.01$, $\omega = 1.6$, $x_0 = -0.01$, and $y_0 = 0$ can be seen in Figure 15.

Last but not least, similar conclusions can be made for the periodic behavior of the main part of the y -dynamics w.r.t. the chosen distribution for the random variable η .

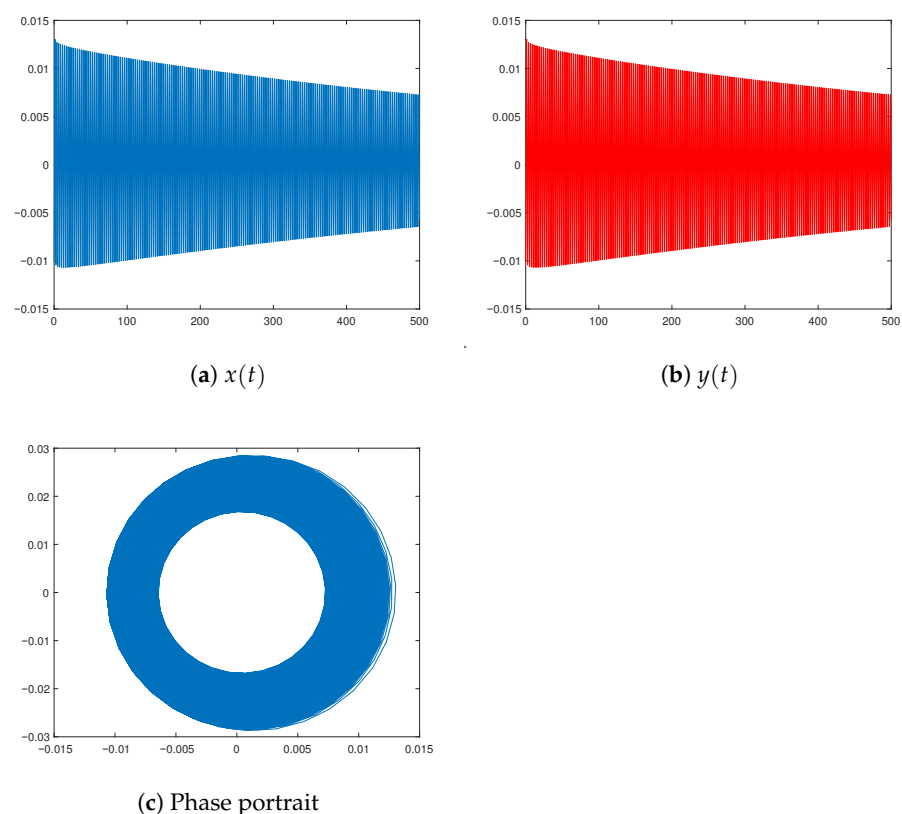


Figure 15. Dynamics based on the gamma distribution.

4. Conclusions

A new planar system corresponding to the generalized double sine-Gordon model is considered. Several simulations are composed. We demonstrate also some specialized modules for investigating the dynamics of the model. One application for possible control over oscillations is also discussed. Some two-sided methods for simultaneous determination of all roots of trigonometric and exponential polynomials are considered in [43]. Regarding other differential models proposed and analyzed by us using the correction $\sum_i A_i \cos(i\omega t)$, see, for example, [44]. The new model has many free parameters, which makes it attractive for engineering calculations. The proposed extended model is especially useful in an

important field of decision making, namely the antenna array theory. This is due to the possibility of generating high-order Melnikov polynomials and their corresponding Melnikov antenna factors. Of course, this requires the use of various optimization and approximation techniques in order to minimize the level of side radiation in the studied diagram. We will not dwell on these issues here. The reader can find the necessary information in [45]. For other basic results, see [41,46–58].

Challenges for Learners

We provide an efficient study strategy that emphasizes learning and challenges our PhD students to consider the triangle of enigmas, creativity, and acmeology.

After providing curriculum modules, we will set the following tasks for self-learning:

Task 1 (self-learning). Study the behavior of model (4) at fixed values $N = 7$; $M = 2$; $\lambda_1 = 1$; $\lambda_2 = 0.991$; $a = 0.2$; $\omega = 0.32$; $\epsilon = 0.1$; $x_0 = 0.5$; $y_0 = 0.2$:

- For what values of the parameters $A_i, i = 1, 2, \dots, 7$ are the dynamics depicted in Figure 16 obtained?
- Draw the corresponding conclusions.

(Answer: when performing the task correctly, you should get the following approximate values: $A_1 = 2.1$; $A_2 = 1.2$; $A_3 = 0.1$; $A_4 = A_5 = A_6 = 0$; $A_7 = 0.05$).

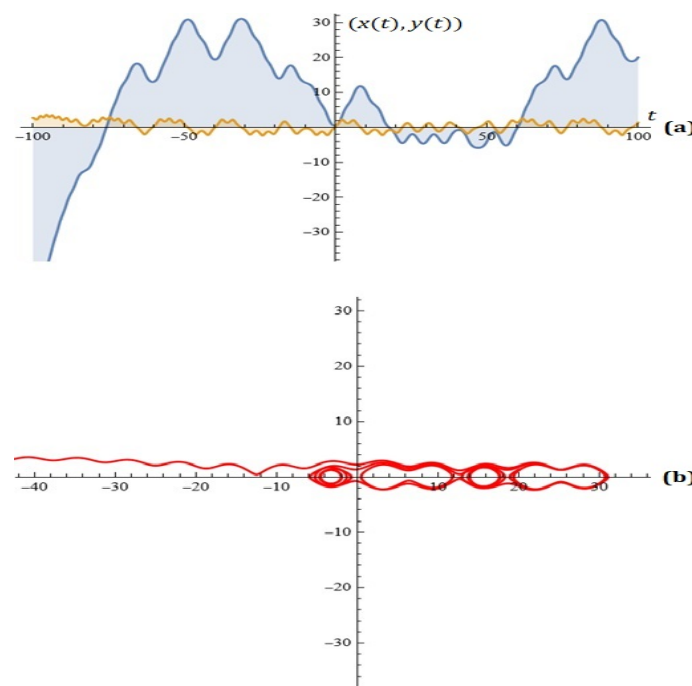


Figure 16. (a) The solutions of the system (4); (b) phase space (Task 1 (self-learning)).

Task 2 (self-learning). Study the behavior of model (4) at fixed values $N = 3$; $M = 2$; $\lambda_1 = 1$; $\lambda_2 = 2.1$; $a = 0.03$; $\omega = 0.962$; $A_1 = 0.9$; $A_2 = 0.1$; $A_3 = 0.9$:

- For what value of the parameter ϵ ($0 < \epsilon \leq 1$) is the phase portrait depicted in Figure 17 obtained?
- Show a chaotic phase portrait/sensitive dependence for parameters above this threshold;
- Draw the corresponding conclusions.

Our conclusion as teachers is that with the proposed methodology, students adapt relatively well and touch upon the elegant theory of Andronov–Melnikov for the possible occurrence of chaos in the dynamical system under consideration.

The reader can find basic research on double and triple sinh-Gordon equations, for example, in [59–63]. We anticipate future research on this interesting topic in light of the

considerations in this article. This analysis will include the computation of Lyapunov exponents and the construction of bifurcation diagrams, as well as the use of Poincaré maps and long-term trajectory simulations.

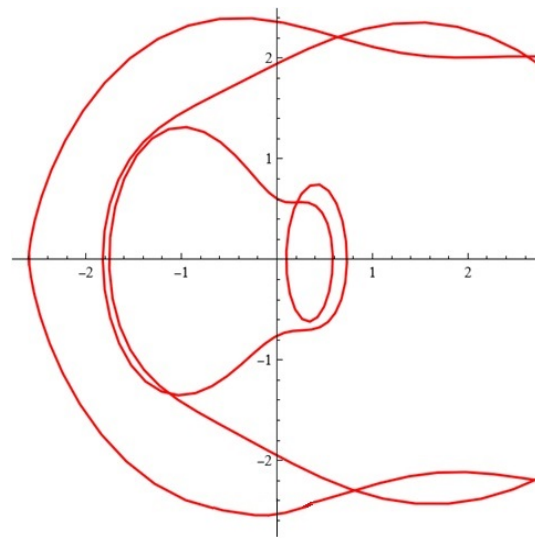


Figure 17. Phase portrait (Task 2 (self-learning)).

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Conflicts of Interest: The authors declare no conflicts of interest.

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