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Existence and Approximate Controllability for Higher-Order Hilfer Fractional Evolution Equations via Fixed-Point and Sequence Approaches

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Abstract

This study addresses the existence and approximate controllability of a type of higher-order Hilfer fractional evolution differential (HOHFED) system with time delays in Banach spaces. Using the properties of the Mittag–Leffler function, cosine families, and Hilfer-type fractional differential operators, we first demonstrate the existence and uniqueness of mild solutions using a fixed-point method. Furthermore, a sequential technique is proposed to establish adequate conditions for approximate controllability. A detailed example is provided to illustrate the applicability and effectiveness of the theoretical results.

Keywords: Hilfer fractional derivative; approximate controllability; mild solutions; cosine operators; fixed point approach

MSC: 34H05; 34A08; 47D09; 34A45; 47H10



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1. Introduction

Fractional dynamical systems have gained attention in recent years due to their enhanced ability to model complex dynamical processes that traditional integer-order frameworks cannot fully capture. Fractional-order operators, unlike classical differential equations, inherently account for memory and hereditary effects, making them particularly suitable for describing phenomena whose future evolution depends on the entire past trajectory of the system. Although fractional calculus presents analytical difficulties, it also provides an effective mathematical framework for modeling processes with temporal, structural, or genetic memory properties. Fractional-order models have effectively represented a wide range of physical, biological, engineering phenomena, viscoelastic materials, anomalous diffusion, porous media flows, signal processing, image processing, and control systems, where integer-order formulations fail to capture important dynamics. This has led to a growing topic of study and important improvements in theory and applications, as shown in [1–7].

The Hilfer fractional derivative is a generalized operator that bridges the gap between the classical Riemann–Liouville (R-L) and Caputo fractional derivatives. It first appeared in [8,9]. Two parameters, the order q ($1 < q < 2$) and the type φ ($0 \leq \varphi \leq 1$), define this derivative and together provide a versatile framework for modeling systems with intermediate memory characteristics. In particular, the Hilfer derivative simplifies to the Caputo derivative when $\varphi = 1$ and to the R-L derivative when $\varphi = 0$. A range of fractional dynamics, from the simply historical (R-L type) to initial value-dependent (Caputo type) behavior,

can be captured by the Hilfer operator due to this formulation. The Hilfer derivative has been effectively used in the study of anomalous diffusion, viscoelastic materials, relaxation processes, and different control systems because of its unifying characteristics. Its adaptability and analytical depth have been further demonstrated by recent studies that have expanded its applicability to differential inclusions, evolution equations, and stochastic systems (see [10–13]). The Hilfer fractional derivative has recently been applied to a wide range of problems, including integro-differential equations, neutral systems, stochastic processes, ψ -Hilfer fractional derivatives in the Caputo sense, Ulam-Hyers-Rassias stability analysis, and systems with almost sectorial operators [14–16].

The concept of approximate controllability is important in the study of fractional-order control systems, especially when exact controllability is difficult to obtain due to the compactness of the related semigroup or the infinite dimensionality of the underlying state space. A control system is said to be approximately controllable if, given any initial state and any desired final state, the system's state may be arbitrarily close to the target in a finite amount of time using admissible control functions. Many scholars have looked on the approximate controllability of systems that are represented by fractional differential systems with some sufficient conditions in recent years [4,13,17–19]. In Hilfer fractional differential systems, approximate controllability becomes more complicated due to the presence of memory effects and the dual dependency of the system dynamics on the fractional order ϱ and the type parameter $\wp$. The interplay of these characteristics has a substantial impact on the system's behavior and controllability. To address such issues, one frequently employs operator-theoretic approaches such as compactness arguments, sequence methods, and fixed-point theorems to establish sufficient requirements for approximate controllability outcomes. Furthermore, the utilization of Mittag-Leffler functions and cosine families, when combined with fractional evolution equations, provides an important analytical foundation for obtaining controllability results in the Hilfer fractional derivative.

Recent work on the existence and controllability problem for mild solutions to Caputo fractional differential equations and inclusions of $(1, 2)$ has been undertaken in [20,21]. Researchers in [22] used sectorial operators and nonlocal situations to develop the existence and uniqueness of the Caputo fractional derivative of $(1, 2)$. Earlier works have also explored Caputo's concept of fractional order $(1, 2)$ in [23]. Furthermore, using Lipschitz continuity, optimal control framework, and Gronwall's inequality, researchers in [24] explored the semilinear fractional order $(1, 2)$. In [25], the authors employed differential equations, existence outcomes, and nonlocal circumstances to obtain fractional evolution differential equations of $(1, 2)$ with control terms. Additionally, the sectorial operator is used in the optimal control results for Caputo fractional impulsive integro-differential equations of order $1 < r < 2$ in [26]. Likewise, the authors in [27] utilized the sequence approach and fixed-point theorem for approximate controllability of $(1, 2)$ fractional differential equations.

In [28], the authors used analytical methods such as the Schauder fixed-point theorem, the Kuratowski measure of noncompactness, and the mild solutions to study stochastic Hilfer fractional systems of order $\mu \in (1, 2)$. In [29], Sobolev-type, stochastic processes, nonlocal conditions, and multivalued maps were used to demonstrate the existence and controllability results for Hilfer fractional systems of order $1 < \mu < 2$. Additionally, the researchers in [30] used hemivariational inequalities and Clarke's subdifferential framework to analyze the optimal control problems for Hilfer-type fractional systems of order $1 < r < 2$. More recently, the continuous dependence and optimal control of Hilfer fractional Volterra–Fredholm integro-differential equations of order $1 < \varrho < 2$ have been studied using the Mönch fixed-point technique, the Laplace transform, finite delay, and stochastic analysis [31].

Sequence approaches are essential for analyzing and studying Hilfer fractional differential equations with approximate controllability. The study described in [32] focuses on fractional differential systems with delay, using C_0 -semigroups to address mild solutions and control problems. In [33,34], the authors examined the approximate controllability of several differential systems, including elastic systems, discrete-time delay models, damping systems, and systems with the R-L fractional derivative. In [35,36], the researchers addressed the approximate controllability results for systems with integer and non-integer order derivatives, mild solutions, state-dependent delays, sequence methods, and impulsive effects. Despite the various differences concerning mild solutions for fractional nonlinear systems, this research is motivated by advances in fractional calculus theory and its numerous applications.

The key contributions of the current paper are outlined below:

- (i) First, we use semigroup theory and the Laplace transform technique to derive the mild solutions to the HOHFED systems (1) and (2).
- (ii) Next, we use fixed-point methods to determine the existence and uniqueness of mild solutions to the given Hilfer fractional system.
- (iii) A sequence approach is employed to establish sufficient conditions for approximate controllability, providing an advancement over the existing results reported in [27,33–36].
- (iv) Finally, an example is provided to demonstrate the application and validity of the theoretical results, using the Hilfer fractional derivative problem as an illustration.

Suppose that HOHFED equations have the framework shown below:

$${}^H D_{0+}^{\varrho, \wp} z(\mu) = \mathcal{A}z(\mu) + \mathcal{E}z(\mu - \varkappa) + \mathfrak{B}x(\mu) + g(\mu, z(\mu - \varkappa)), \quad \mu > 0, \quad (1)$$

$$({}^I_{0+}^{2-\delta} z)(\mu) = \varphi(\mu), \quad \mu \in [-\varkappa, 0], \quad ({}^I_{0+}^{2-\delta} z)'(0) = z_1. \quad (2)$$

The Hilfer fractional differential equation of order $\varrho \in (1, 2)$ is expressed by ${}^H D_{0+}^{\varrho, \wp}$ and type $\wp$ in $[0, 1]$. Here, $\delta = \varrho + \wp(2 - \varrho)$, and $I_{0+}^{2-\delta}$ denotes the R-L integral operator with order $(2 - \delta)$. Let $\mathcal{A}$ be the infinitesimal generator of the strongly continuous cosine function $\{\mathbb{K}(\mu)\}_{\mu \geq 0}$ on Hilbert space $\mathcal{Z}$. The control is x in Hilbert space $\mathcal{X}$. The function $g : V \times \mathcal{C}([-\varkappa, 0], \mathcal{Z}) \rightarrow \mathcal{Z}$ represents the nonlinear part of the system. The linear function $\mathfrak{B} : \mathcal{X} \rightarrow \mathcal{Y}$ denotes a bounded operator. For $\varkappa \in (0, \vartheta)$, the initial conditions are given by $\varphi \in \mathcal{C}([-\varkappa, 0], \mathcal{Z})$ and $z_1 \in \mathcal{Z}$. Finally, $\mathcal{E}$ denotes a bounded and linear operators on $\mathcal{Z}$.

This work is broken down into several sections: Section 2 presents the preliminary findings, assumptions, main findings on definitions, remarks, mild solutions, and lemmas. We investigate the existence and uniqueness results for the problems presented in Section 3. In Section 4, the sequence technique is used to provide approximate controllability outcomes for HOHFED systems (1) and (2). Finally, an application is used to illustrate the principle of the major outcomes.

2. Primitive Results

In this section, will provide valuable foundations, basic definitions, lemmas, and key findings. Consider the Banach space of continuous function $\widehat{\mathcal{C}}(V, \mathcal{Z}) : V = [0, \vartheta] \rightarrow \mathcal{Z}$ with $\|z\|_{\widehat{\mathcal{C}}}^* = \sup_{\mu \in V} e^{-\iota \mu} \{\|z(\mu)\|_{\mathcal{Z}}\}$, where ι is a fixed positive constant. Let $\widehat{\mathcal{C}}(V, \mathcal{Z}) = \{z \in \widehat{\mathcal{C}}(V, \mathcal{Z}) : \lim_{\eta \rightarrow 0^+} \eta^{2-\delta} z(\eta) \text{ exists and finite}\}$, with $\|z\|_{\mathcal{C}} = (\sup_{\eta \in V} \|\eta^{2-\delta} z(\eta)\|)$, where $\mathcal{C} = \widehat{\mathcal{C}}_{2-\delta}(V, \mathcal{Z})$ and $2 - \delta = (1 - \wp)(2 - \varrho)$. Assume that $\mathcal{X} = L^2([0, \vartheta], \mathcal{H})$ is a function space connected to Hilbert space $\mathcal{H}$. Consider $\mathcal{Y} = L^2([0, \vartheta], \mathcal{Z})$ as the function space according to $\mathcal{Z}$.

2.1. Fractional Calculus

Definition 1 ([37]). The R-L fractional integral is defined by

$$I^q g(\mu) = \frac{1}{\Gamma(q)} \int_0^\mu \frac{g(\eta)}{(\mu - \eta)^{1-q}} d\eta, \quad \mu > 0, q \in \mathbb{R}^+.$$

Definition 2 ([37]). The R-L fractional derivative of order $q \in \mathbb{R}^+$ is given by

$${}^L D^q g(\mu) = \frac{1}{\Gamma(m-q)} \frac{d^m}{d\mu^m} \int_0^\mu (\mu - \eta)^{m-q-1} g(\eta) d\eta, \quad \mu > 0, q \in (m-1, m).$$

Definition 3 ([37]). The Caputo fractional derivative of order $q \in \mathbb{R}^+$ is provided by

$${}^C D^q g(\mu) = {}^L D^q \left(g(\mu) - \sum_{i=0}^{m-1} \frac{g^{(i)}(0)}{i!} \mu^i \right), \quad \mu > 0, q \in (m-1, m).$$

Definition 4 ([37]). The Hilfer fractional derivative of order $q \in (m-1, m)$ and type $\wp \in [0, 1]$ for a function $g : [0, \infty) \rightarrow \mathbb{R}$ is expressed as

$${}^H D_{0+}^{q,\wp} g(\mu) = I_{0+}^{\wp(m-q)} \frac{d^m}{d\mu^m} I_{0+}^{(1-\wp)(m-q)} g(\mu), \quad \mu > 0.$$

Remark 1 (Foundational results on Hilfer fractional operators). For the Hilfer fractional derivative of order q and type $\wp$, the following evaluation holds:

(i) R-L case: if $\wp = 0$, and $m-1 < q < m$, then

$${}^H D_{0+}^{q,0} g(\mu) = \frac{d^m}{d\mu^m} I_{0+}^{(m-q)} g(\mu) = {}^{RL} D_{0+}^q g(\mu);$$

(ii) Caputo case: if $\wp = 1$, and $q \in (m-1, m)$, we obtain

$${}^H D_{0+}^{q,1} g(\mu) = I_{0+}^{(m-q)} \frac{d^m}{d\mu^m} g(\mu) = {}^C D_{0+}^q g(\mu).$$

Definition 5 ([28]). The Wright function $W_\wp(\mu)$ is presented as

$$W_\wp(\mu) = \sum_{i=1}^{\infty} \frac{(-\mu)^{i-1}}{(i-1)! \Gamma(1-\wp i)}, \quad \wp \in (0, 1), \mu \text{ in } \mathbb{C},$$

with the subsequent condition

$$\int_0^\infty \mu^\omega W_\wp(\mu) d\mu = \frac{\Gamma(1+\omega)}{\Gamma(1+\wp\omega)}, \text{ for all } \omega \geq 0.$$

Definition 6 ([37]). A two-parameter Mittag-Leffler function is identified as

$$E_{q,\gamma}(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(qi + \gamma)},$$

where $q > 0$, $\gamma > 0$, and $z \in \mathbb{C}$. For $\gamma = 1$, the one-parameter form is shown by

$$E_q(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(qi + 1)}.$$

2.2. Cosine and Sine Function Operators

Definition 7 ([38]). The bounded linear operators $\{\mathbb{K}(\mu)\}_{\mu \in \mathbb{R}} : \mathcal{Z} \rightarrow \mathcal{Z}$ is said to be a strongly continuous cosine family iff

- (i) $\mathbb{K}(0)$ is an identity operator I ;
- (ii) $\mathbb{K}(\mu'' + \mu') + \mathbb{K}(\mu'' - \mu') = 2\mathbb{K}(\mu'')\mathbb{K}(\mu')$, for $\mu'', \mu' \in \mathbb{R}$;
- (iii) For any fixed-point $z \in \mathcal{Z}$, $\mathbb{K}(\vartheta)z$ is continuous in μ on $\mathbb{R}$.

In relation to the sine operator $\{\mathbb{S}(\mu)\}_{\mu \in \mathbb{R}}$, there is a cosine operator $\{\mathbb{K}(\mu)\}_{\mu \in \mathbb{R}}$. $\mathbb{S}(\mu)z = \int_0^\mu \mathbb{K}(\eta)z d\eta$, $z \in \mathcal{Z}$, $\mu \in \mathbb{R}$. Also, $Az = D^2\mathbb{K}(\mu)z \Big|_{\mu=0}$ for all z in $D(\mathcal{A})$, which is

$$D(\mathcal{A}) = \{z \in \mathcal{Z} : \mathbb{K}z, \text{ a continuously differentiable function as for } \mu\}.$$

2.3. Mild Solution

Lemma 1. The HOHFED systems (1) and (2) is examined in the corresponding integral form in the following way:

$$\begin{aligned} z(\mu) = & \frac{\varphi(0)}{\Gamma(\delta-1)}\mu^{\delta-2} + \frac{z_1}{\Gamma(\delta)}\mu^{\delta-1} + \frac{1}{\Gamma(q)} \int_0^\mu (\mu-\eta)^{q-1} [\mathcal{A}z(\eta) + \mathcal{E}z(\eta-\varkappa) \\ & + \mathfrak{B}x(\eta) + g(\eta, z(\eta-\varkappa))] d\eta, \quad \mu \in V. \end{aligned} \quad (3)$$

Definition 8. The function $z(\mu) \in \mathcal{C}([-\varkappa, \mu], \mathcal{Z})$ indicates a mild solution for HOHFED systems (1) and (2), provided that $(I_{0+}^{2-\delta}z)(0) = \varphi(0)$, $(I_{0+}^{2-\delta}z)'(0) = z_1$ such that

$$\begin{aligned} z(\mu) = & {}^{RL}D_{0+}^{1-\alpha}(\mu^{q-1}\mathcal{G}_q(\mu)\varphi(0)) + I_{0+}^\alpha(\mu^{q-1}\mathcal{G}_q(\mu)z_1) \\ & + \int_0^\mu (\mu-\eta)^{q-1}\mathcal{G}_q(\mu-\eta)\mathcal{E}z(\eta-\varkappa)d\eta + \int_0^\mu (\mu-\eta)^{q-1}\mathcal{G}_q(\mu-\eta)\mathfrak{B}x(\eta)d\eta \\ & + \int_0^\mu (\mu-\eta)^{q-1}\mathcal{G}_q(\mu-\eta)g(\eta, z(\eta-\varkappa))d\eta, \quad \mu \in V, \end{aligned} \quad (4)$$

$$z(\mu) = \varphi(\mu), \quad \mu \in [-\varkappa, 0). \quad (5)$$

In the above, $\alpha = \wp(2-q)$ in $(0, 1)$, $q = 2q$, then

$$\begin{aligned} \mathcal{G}_q(\mu) = & \int_0^\infty q\xi\mathbb{N}_q(\xi)\mathcal{S}(\mu^q\xi)d\xi, \quad \mathbb{N}_q(\xi) = \frac{1}{q}\xi^{-\frac{1}{q}-1}\mathfrak{I}_q(\xi^{-\frac{1}{q}}), \quad \xi \in (0, \infty), \text{ and} \\ \mathfrak{I}_q(\xi) = & \frac{1}{\pi} \sum_{k=1}^\infty (-1)^{k-1}\xi^{-qk-1} \frac{\Gamma(qk+1)}{k!} \sin(qk\pi), \quad \xi \in (0, \infty). \end{aligned}$$

2.4. Necessary Conditions

Lemma 2. The operator $\mathcal{G}_q(\mu)$ satisfies the following conditions:

- (i) For $\mu > 0$, the operator $\mathcal{G}_q(\mu)$ denotes the linear and bounded, i.e., for any z in $\mathcal{Z}$,

$$\begin{aligned} \|{}^{RL}D_{0+}^{1-\alpha}(\mu^{q-1}\mathcal{G}_q(\mu)z)\| & \leq \frac{P\mu^{2q+\alpha-2}}{(2q-1)\Gamma(\alpha+2q-1)}\|z\|, \\ \|I_{0+}^\alpha(\mu^{q-1}\mathcal{G}_q(\mu)z)\| & \leq \frac{P\mu^{2q+\alpha-1}}{\Gamma(\alpha+2q)}\|z\|, \\ \|\mathcal{G}_q(\mu)z\| & \leq \frac{P\mu^q}{\Gamma(2q)}\|z\|. \end{aligned}$$

(ii) The operator $\mathcal{G}_q(\mu)$ is uniformly continuous, that is, for all $\mu_2, \mu_1 \geq 0$, such that

$$\|\mathcal{G}_q(\mu_2) - \mathcal{G}_q(\mu_1)\| \rightarrow 0, \text{ as } \mu_2 \rightarrow \mu_1.$$

Remark 2. Classical differential equations describe systems whose evolution depends only on their instantaneous state. However, many dynamical processes particularly those governed by hereditary effects, nonlocal interactions, or long-range memory cannot be represented accurately in this framework. Traditional existence and uniqueness results for integer-order systems derived by semigroup theory (see [36,39]) depends on local operators and fails to account for historical effects. Fractional derivatives naturally account for these features because their operators depend on the entire past history of the state. In this work, we use the Hilfer derivative, which interpolates between the R-L and Caputo derivatives, introducing a nonlocal structure that depends on the entire past trajectory of the state. This feature provides a more general framework for describing higher-order dynamics generated by cosine families and allows the controllability analysis to include systems where present evolution is influenced by historical states. Hence, the Hilfer fractional derivative is more flexible and appropriate than the integer-order model (see [14,15,28]).

Remark 3. This work is different from previous research in several significant respects. Earlier papers, such as [33,36], focused on approximate controllability for classical integer-order differential equations, utilizing fixed-point theorems and the sequence approach, without any fractional operators. The authors in [34] expanded the sequence approach to R-L fractional systems, and the authors in [27,35] investigated Caputo fractional equations of orders $(0,1)$ and $(1,2)$. However, none of these papers addressed the Hilfer fractional derivative, which is more general and occurs between the R-L and Caputo situations. In addition, previous research depended primarily on C_0 -semigroup theory, whereas our study focuses on higher-order abstract systems formed by the cosine family. This affects both the structure of the solution and the controllability analysis. To our knowledge, the approximate controllability of Hilfer fractional equations of order $q \in (1,2)$ using the sequence approach has not yet been explored. As a result, our findings represent an effective extension of the current research and point in an alternative direction for approximate controllability analysis using the Hilfer fractional derivative.

3. Theoretical Insights into Existence of Mild Solution

We will establish and show the existence of HOHFED systems (1) and (2) in this section. We present the following hypotheses:

($\mathcal{O}_1$) For any $z \in \mathcal{Z}$, the function $g(\mu, z)$ indicates continuous in μ and continuous with respect to z for all μ in V , and there exists $\mathcal{L} > 0$ such that

$$\|g(\mu, z) - g(\mu, u)\| \leq \mathcal{L}\mu^{2-\delta}\|z - u\|_{\mathcal{Z}}, \forall z, u \in \mathcal{Z}.$$

Further, $\mathcal{L}_g = \max_{\mu \in [0, \vartheta]} \|g(\mu, 0)\|_{\mathcal{Z}}$.

($\mathcal{O}_2$) For convenience,

$$\|\mathcal{E}\| = \widehat{P}_1, \aleph = P(\widehat{P}_1 + \mathcal{L}\vartheta^{2-\delta}), \text{ and } \sqrt{\frac{\vartheta^{4q-1}}{4q-1}} = \widehat{P}_2.$$

Theorem 1. Consider ($\mathcal{O}_1$)–($\mathcal{O}_2$) to be fulfilled. For every control function $x(\mu)$ in $\mathcal{X}$, HOHFED systems (1) and (2) have a unique mild solution on $\mathcal{C}([-\varkappa, \vartheta], \mathcal{Z})$.

Proof. Assume that Ψ is determined by

$$\begin{aligned}
(\Psi z)(\mu) &= \varphi(\mu), \mu \in [-\varkappa, 0), \\
(\Psi z)(\mu) &= {}^{RL}D_{0+}^{1-\alpha}(\mu^{q-1}\mathcal{G}_q(\mu)\varphi(0)) + I_{0+}^{\alpha}(\mu^{q-1}\mathcal{G}_q(\mu)z_1) \\
&\quad + \int_0^{\mu}(\mu-\eta)^{q-1}\mathcal{G}_q(\mu-\eta)\mathcal{E}z(\eta-\varkappa)d\eta \\
&\quad + \int_0^{\mu}(\mu-\eta)^{q-1}\mathcal{G}_q(\mu-\eta)\mathfrak{B}x(\eta)d\eta \\
&\quad + \int_0^{\mu}(\mu-\eta)^{q-1}\mathcal{G}_q(\mu-\eta)g(\eta, z(\eta-\varkappa))d\eta, \mu \geq 0.
\end{aligned}$$

For every w in $\mathcal{C}([-\varkappa, \vartheta], \mathcal{Z})$, take $z(\mu) = \mu^{\delta-2}w(\mu)$, $\mu \in V$, and we define the operator $F : \mathcal{C}([-\varkappa, \vartheta], \mathcal{Z}) \rightarrow \mathcal{C}([-\varkappa, \vartheta], \mathcal{Z})$ such that

$$(Fw)(\mu) = \begin{cases} \mu^{2-\delta}(\Psi z)(\mu), & \text{for } \mu \in (0, \vartheta], \\ \frac{\varphi(0)}{\Gamma(\alpha+2q-1)}, & \text{for } \mu = 0. \end{cases}$$

The definition of F provides a basis for comparing the technique used to establish mild solutions to the HOHFED systems (1) and (2) with the fixed-point approach applied to F .

Consider the control $x(\mu)$ in $\mathcal{X}$. To clarify that F has a fixed point on $[-\varkappa, \vartheta]$, the constant $\iota > 0$ established in the definition of $\|\cdot\|_{\mathcal{C}}^*$ is selected from the following inequality:

$$\iota \geq \frac{\mu^{4q-1}}{4q-1} \left(\frac{\aleph}{\Gamma(2q)} \right)^2 > 0,$$

and Π_j is determined as

$$\begin{aligned}
j \geq \max \Bigg\{ & \|\varphi\|_{\mathcal{C}}^*, \frac{2P}{(2q-1)\Gamma(\alpha+2q-1)}\|\varphi(0)\| + \frac{2P\mu}{\Gamma(\alpha+2q)}\|z_1\| \\
& + \frac{2P\hat{P}_2\vartheta^{2-\delta}e^{-\iota\mu}}{\Gamma(2q)}\|\mathfrak{B}x\|_{\mathcal{Y}} + \frac{2P\mathcal{L}_g\vartheta^{2-\delta+2q}e^{-\iota\mu}}{\Gamma(2q+1)} \Bigg\}.
\end{aligned}$$

We now verify that F possesses a fixed point. To facilitate this, we break our argument into several steps:

Step 1: Assume that $\Pi_j = \{w(\mu) \in \mathcal{C}([-\varkappa, \vartheta], \mathcal{Z}) : \|w\|_{\mathcal{C}}^* \leq j\}$, we prove $F(\Pi_j) \subset \Pi_j$.

If μ in $[-\varkappa, 0)$, then $\|Fw\|_{\mathcal{C}}^* = \|\varphi\|_{\mathcal{C}}^* \leq j$. For any z in Π_j , if $\mu \in [0, \vartheta]$, by $(\mathcal{O}_1)$ and Lemma 2, we obtain

$$\begin{aligned}
e^{-\iota\mu}\|(Fw)(\mu)\| &\leq e^{-\iota\mu} \sup_{\mu \in V} \mu^{2-\delta}\|(\Psi z)(\mu)\|_{\mathcal{Z}} \\
&\leq e^{-\iota\mu}\vartheta^{2-\delta}\|{}^{RL}D_{0+}^{1-\alpha}(\mu^{q-1}\mathcal{G}_q(\mu)\varphi(0))\| + e^{-\iota\mu}\vartheta^{2-\delta}\|I_{0+}^{\alpha}(\mu^{q-1}\mathcal{G}_q(\mu)z_1)\| \\
&\quad + e^{-\iota\mu}\vartheta^{2-\delta}\left\|\int_0^{\mu}(\mu-\eta)^{q-1}\mathcal{G}_q(\mu-\eta)\mathcal{E}z(\eta-\varkappa)d\eta\right\| \\
&\quad + e^{-\iota\mu}\vartheta^{2-\delta}\left\|\int_0^{\mu}(\mu-\eta)^{q-1}\mathcal{G}_q(\mu-\eta)\mathfrak{B}x(\eta)d\eta\right\| \\
&\quad + e^{-\iota\mu}\vartheta^{2-\delta}\left\|\int_0^{\mu}(\mu-\eta)^{q-1}\mathcal{G}_q(\mu-\eta)g(\eta, z(\eta-\varkappa))d\eta\right\| \\
&\leq \frac{P}{(2q-1)\Gamma(\alpha+2q-1)}\|\varphi(0)\| + \frac{P\mu}{\Gamma(\alpha+2q)}\|z_1\| \\
&\quad + \frac{P\vartheta^{2-\delta}e^{-\iota\mu}}{\Gamma(2q)}\int_0^{\mu}(\mu-\eta)^{2q-1}\|\mathcal{E}z(\eta-\varkappa)\|_{\mathcal{Z}}d\eta \\
&\quad + \frac{P\vartheta^{2-\delta}e^{-\iota\mu}}{\Gamma(2q)}\int_0^{\mu}(\mu-\eta)^{2q-1}\|\mathfrak{B}x(\eta)\|_{\mathcal{Y}}d\eta + \frac{P\vartheta^{2-\delta}e^{-\iota\mu}}{\Gamma(2q)}
\end{aligned}$$

$$\begin{aligned}
& \times \int_0^\mu (\mu - \eta)^{2q-1} [\|g(\eta, z(\eta - \varkappa)) - g(\eta, 0)\|_Z + \|g(\eta, 0)\|_Z] d\eta \\
& \leq \frac{P}{(2q-1)\Gamma(\alpha+2q-1)} \|\varphi(0)\| + \frac{P\mu}{\Gamma(\alpha+2q)} \|z_1\| \\
& \quad + \frac{P\hat{P}_1\vartheta^{2-\delta}e^{-\iota\mu}}{\Gamma(2q)} \int_0^\mu (\mu - \eta)^{2q-1} \|z(\eta - \varkappa)\|_Z d\eta + \frac{P\vartheta^{2-\delta}e^{-\iota\mu}}{\Gamma(2q)} \hat{P}_2 \|\mathfrak{B}x\|_Y \\
& \quad + \frac{P\vartheta^{2-\delta}e^{-\iota\mu}}{\Gamma(2q)} \int_0^\mu (\mu - \eta)^{2q-1} [\mathcal{L}\eta^{2-\delta} \|z(\eta - \varkappa)\|_Z + \mathcal{L}_g] d\eta \\
& \leq \frac{P\|\varphi(0)\|}{(2q-1)\Gamma(\alpha+2q-1)} + \frac{P\mu\|z_1\|}{\Gamma(\alpha+2q)} + \frac{P\hat{P}_2\vartheta^{2-\delta}e^{-\iota\mu}}{\Gamma(2q)} \|\mathfrak{B}x\|_Y \\
& \quad + \frac{\aleph e^{-\iota\mu}}{\Gamma(2q)} \int_0^\mu (\mu - \eta)^{2q-1} \eta^{2-\delta} \|z(\eta - \varkappa)\|_Z d\eta + \frac{P\mathcal{L}_g\vartheta^{2-\delta+2q}e^{-\iota\mu}}{\Gamma(2q+1)} \\
& \leq \nu + \frac{\aleph e^{-\iota\mu}}{\Gamma(2q)} \left(\int_0^\mu (\mu - \eta)^{2(2q-1)} d\eta \right)^{\frac{1}{2}} \left(\int_0^\mu e^{2\iota(\eta-\varkappa)} d\eta \right)^{\frac{1}{2}} \mu^{2-\delta} \|z\|_{\mathcal{C}}^* \\
& \leq \nu + \frac{\hat{P}_2\aleph e^{-\iota\mu}}{\Gamma(2q)} \sqrt{\frac{e^{-2\iota\varkappa}(e^{2\iota\mu}-1)}{2\iota}} J \\
& \leq \nu + \frac{\aleph}{\Gamma(2q)} \sqrt{\frac{\vartheta^{4q-1}}{2\iota(4q-1)}} J,
\end{aligned}$$

where

$$\begin{aligned}
\nu = & \frac{P}{(2q-1)\Gamma(\alpha+2q-1)} \|\varphi(0)\| + \frac{P\mu}{\Gamma(\alpha+2q)} \|z_1\| \\
& + \frac{P\hat{P}_2\vartheta^{2-\delta}e^{-\iota\mu}}{\Gamma(2q)} \|\mathfrak{B}x\|_Y + \frac{P\mathcal{L}_g\vartheta^{2-\delta+2q}e^{-\iota\mu}}{\Gamma(2q+1)}.
\end{aligned}$$

By the definition of ι and J , we obtain $\|Fw\|_{\mathcal{C}}^* \leq J$, proving the assertion.

Step 2: On $\mathcal{C}([-\varkappa, \vartheta], \mathcal{Z})$, we prove that F represents a contraction operator. If μ is $[-\varkappa, 0)$, the assumption is clearly true.

If $\mu \in [0, \vartheta]$, for any $z, y \in \mathcal{C}([-\varkappa, \vartheta], \mathcal{Z})$ and utilizing $(\mathcal{O}_1)$, we obtain

$$\begin{aligned}
& \|(Fw)(\mu) - (F\tilde{w})(\mu)\|_Z \\
& \leq \vartheta^{2-\delta} \left\| \int_0^\mu (\mu - \eta)^{q-1} \mathcal{G}_q(\mu - \eta) \mathcal{E}z(\eta - \varkappa) d\eta - \int_0^\mu (\mu - \eta)^{q-1} \mathcal{G}_q(\mu - \eta) \mathcal{E}\tilde{z}(\eta - \varkappa) d\eta \right\| \\
& \quad + \vartheta^{2-\delta} \left\| \int_0^\mu (\mu - \eta)^{q-1} \mathcal{G}_q(\mu - \eta) g(\eta, z(\eta - \varkappa)) d\eta \right. \\
& \quad \left. - \int_0^\mu (\mu - \eta)^{q-1} \mathcal{G}_q(\mu - \eta) g(\eta, \tilde{z}(\eta - \varkappa)) d\eta \right\| \\
& \leq \vartheta^{2-\delta} \int_0^\mu (\mu - \eta)^{q-1} \|\mathcal{G}_q(\mu - \eta) \mathcal{E}[z(\eta - \varkappa) - \tilde{z}(\eta - \varkappa)]\| d\eta \\
& \quad + \vartheta^{2-\delta} \int_0^\mu (\mu - \eta)^{q-1} \|\mathcal{G}_q(\mu - \eta) [g(\eta, z(\eta - \varkappa)) - g(\eta, \tilde{z}(\eta - \varkappa))]\| d\eta \\
& \leq \frac{P\vartheta^{2-\delta}\hat{P}_1}{\Gamma(2q)} \int_0^\mu (\mu - \eta)^{2q-1} \|z(\eta - \varkappa) - \tilde{z}(\eta - \varkappa)\| d\eta \\
& \quad + \frac{P\mathcal{L}\vartheta^{2-\delta}}{\Gamma(2q)} \int_0^\mu (\mu - \eta)^{2q-1} \eta^{2-\delta} \|z(\eta - \varkappa) - \tilde{z}(\eta - \varkappa)\| d\eta \\
& \leq \frac{\aleph\vartheta^{2-\delta}}{\Gamma(2q)} \int_0^\mu (\mu - \eta)^{2q-1} e^{\iota(\eta-\varkappa)} e^{-\iota(\eta-\varkappa)} \|z(\eta - \varkappa) - \tilde{z}(\eta - \varkappa)\| d\eta \\
& \leq \frac{\aleph\vartheta^{2-\delta}e^{\iota\mu}}{\Gamma(2q)} \sqrt{\frac{\vartheta^{4q-1}}{2\iota(4q-1)}} \|z - \tilde{z}\|_{\mathcal{C}}^*.
\end{aligned}$$

From the definition of ι , which follows simply that

$$\|Fw - F\tilde{w}\|_{\mathcal{C}}^* \leq \frac{\aleph \vartheta^{2-\delta}}{\Gamma(2q)} \sqrt{\frac{\vartheta^{4q-1}}{2\iota(4q-1)}} \|z - \tilde{z}\|_{\mathcal{C}}^* \leq \frac{1}{2} \|z - \tilde{z}\|_{\mathcal{C}}^*.$$

As a result, F on $\mathcal{C}([-\varkappa, \vartheta], \mathcal{Z})$ is a contraction operator. By applying Banach's fixed-point approach, we can determine that F has a unique fixed point $z(\mu)$ on $\mathcal{C}([-\varkappa, \vartheta], \mathcal{Z})$, corresponding to the mild solution of HOHFED systems (1) and (2). $\square$

4. Controllability Analysis

This section, we look at the approximate controllability outcomes for HOHFED systems (1) and (2) required the subsequent definitions.

4.1. Definitions of Approximate Controllability

Definition 9 ([32]). Assume that $z(\mu; x)$ is the state value of HOHFED systems (1) and (2) at time μ according to $x(\mu) \in \mathcal{X}$. At terminal time ϑ ($\vartheta > \varkappa$), the set $\Sigma_{\vartheta}(g) = \{z(\vartheta; x) : x(\mu) \in \mathcal{X}\}$ is known as the reachable set of HOHFED systems (1) and (2). If $g \equiv 0$, then HOHFED systems (1) and (2) are said to correspond to linear equations, as indicated by HOHFED systems (1)* and (2)*. The reachable set of the linear systems (1)* and (2)* is denoted by $\Sigma_{\vartheta}(0)$ in this case.

Definition 10 ([32]). HOHFED systems (1) and (2) are approximately controllable at time ϑ ($\vartheta > \varkappa$), provided that $\overline{\Sigma_{\vartheta}(g)} = \mathcal{Z}$, where $\overline{\Sigma_{\vartheta}(g)}$ indicates the closure of $\Sigma_{\vartheta}(g)$. The associated linear control problems (1)* and (2)* are approximately controllable if $\overline{\Sigma_{\vartheta}(0)} = \mathcal{Z}$.

In order to investigate the approximate controllability of HOHFED systems (1) and (2), we establish a bounded and linear operator $\Omega : \mathcal{Y} \rightarrow \mathcal{C}(V, \mathcal{Z})$ as

$$\Omega \varkappa = \int_0^{\vartheta} (\vartheta - \eta)^{q-1} \mathcal{G}_q(\vartheta - \eta) \varkappa(\eta) d\eta, \text{ for } \varkappa(\vartheta) \text{ in } \mathcal{Y}.$$

According to Definition 10, HOHFED systems (1) and (2) are approximately controllable at time ϑ ($\vartheta > \varkappa$) if there is a control $x_{\epsilon} \in \mathcal{X}$ for every intended final state ξ in $\mathcal{Z}$ and any $\epsilon > 0$ such that

$$\|\xi - \mathcal{G}_q(\vartheta)\varphi(0) - \mathcal{G}_q(\vartheta)z_1 - \Omega \mathcal{E}z_{\epsilon}^{\varkappa} - \Omega \mathbb{G}z_{\epsilon}^{\varkappa} - \Omega \mathfrak{B}x_{\epsilon}\| < \epsilon.$$

In the above, $(\mathbb{G}z^{\varkappa})(\mu) = g(\mu, z(\mu - \varkappa))$, $\mathcal{E}z^{\varkappa} = \mathcal{E}z(\mu - \varkappa)$ and $z_{\epsilon}(\mu) = z(\mu; x_{\epsilon})$ represents a mild solution to HOHFED systems (1) and (2) according to $x_{\epsilon}(\mu)$ in $\mathcal{X}$. Regarding the approximate controllability, relevant results may be found in [4,32,35,40,41].

Remark 4 ([32]). We require that $\vartheta > \varkappa$ in Definition 10. This is reasonable since the value of $z(\eta)$ for $[\vartheta - \varkappa, 0]$ cannot be controlled if $\vartheta < \varkappa$. It can clearly be seen that $z(\eta) = \varphi(\eta)$ in $[-\varkappa, 0]$ is independent of $x(\mu)$.

To present our main argument, we further suppose the following:
($\mathcal{O}_3$) $\zeta(\mu)$ in $\mathcal{Y}$, for every $\epsilon > 0$, and there exists an $x(\mu)$ in $\mathcal{X}$ such that

$$\|\Omega \zeta - \Omega \mathfrak{B}x\|_{\mathcal{Z}} < \epsilon, \quad (6)$$

$$\|\mathfrak{B}x(\mu)\|_{\mathcal{Y}} < \beta \|\zeta(\mu)\|_{\mathcal{Y}}. \quad (7)$$

In the above, $\beta > 0$, which is independent of $\zeta(\mu)$ in $\mathcal{Y}$ and fulfills

$$\frac{\aleph \widehat{P}_2 \beta}{\Gamma(2q)} E_{2q}(\aleph \vartheta^{2-\delta+2q}) < 1. \quad (8)$$

The subsequent lemma is required to examine the approximate controllability of HOHFED systems (1) and (2).

Lemma 3. *If $(\mathcal{O}_1)$ and $(\mathcal{O}_2)$ are fulfilled, then the mild solution to HOHFED systems (1) and (2) fulfills the following inequalities:*

$$\begin{aligned} \|z(\cdot; x)\|_{\mathcal{C}}^* &\leq \ell E_{2q}(\aleph \vartheta^{2-\delta+2q}), \quad \forall x(\mu) \in \mathcal{X}, \\ \|z_1(\mu) - z_2(\mu)\|_{\mathcal{C}}^* &\leq \theta E_{2q}(\aleph \vartheta^{2-\delta+2q}) \|\mathfrak{B}x_1(\mu) - \mathfrak{B}x_2(\mu)\|_{\mathcal{Y}}, \quad \forall x_1(\mu), x_2(\mu) \in \mathcal{X}. \end{aligned}$$

In the above,

$$\begin{aligned} \aleph &= P(\widehat{P}_1 + \mathcal{L}\vartheta^{2-\delta}), \quad \theta = \frac{P\widehat{P}_2\vartheta^{2-\delta}}{\Gamma(2q)}, \quad \text{and} \\ \ell &= \frac{P}{(2q-1)\Gamma(\alpha+2q-1)} \|\varphi(0)\| + \frac{P\mu}{\Gamma(\alpha+2q)} \|z_1\| + \frac{P\widehat{P}_2\vartheta^{2-\delta}}{\Gamma(2q)} \|\mathfrak{B}x\|_{\mathcal{Y}} + \frac{P\mathcal{L}_g\vartheta^{2-\delta+2q}}{\Gamma(2q+1)}. \end{aligned}$$

The Mittag-Leffler function E_{2q} is identified as

$$E_{2q}(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(2qi+1)}.$$

Proof. Provided that $z(\mu, x) = z(\mu)$ represents a mild solution of HOHFED systems (1) and (2) with respect to $x(\mu)$ in $\mathcal{X}$, in addition

$$\begin{aligned} z(\mu) &= {}^{RL}D_{0+}^{1-\alpha}(\mu^{q-1}\mathcal{G}_q(\mu)\varphi(0)) + I_{0+}^{\alpha}(\mu^{q-1}\mathcal{G}_q(\mu)z_1) \\ &\quad + \int_0^{\mu} (\mu-\eta)^{q-1}\mathcal{G}_q(\mu-\eta)\mathcal{E}z(\eta-\kappa)d\eta \\ &\quad + \int_0^{\mu} (\mu-\eta)^{q-1}\mathcal{G}_q(\mu-\eta)\mathfrak{B}x(\eta)d\eta \\ &\quad + \int_0^{\mu} (\mu-\eta)^{q-1}\mathcal{G}_q(\mu-\eta)g(\eta, z(\eta-\kappa))d\eta. \end{aligned}$$

For μ in $[0, \vartheta]$, we have

$$\begin{aligned} \|w(\mu)\|_{\mathcal{Z}} &\leq \mu^{2-\delta} \|{}^{RL}D_{0+}^{1-\alpha}(\mu^{q-1}\mathcal{G}_q(\mu)\varphi(0))\| + \mu^{2-\delta} \|I_{0+}^{\alpha}(\mu^{q-1}\mathcal{G}_q(\mu)z_1)\| \\ &\quad + \mu^{2-\delta} \left\| \int_0^{\mu} (\mu-\eta)^{q-1}\mathcal{G}_q(\mu-\eta)\mathcal{E}z(\eta-\kappa)d\eta \right\| \\ &\quad + \mu^{2-\delta} \left\| \int_0^{\mu} (\mu-\eta)^{q-1}\mathcal{G}_q(\mu-\eta)\mathfrak{B}x(\eta)d\eta \right\| \\ &\quad + \mu^{2-\delta} \left\| \int_0^{\mu} (\mu-\eta)^{q-1}\mathcal{G}_q(\mu-\eta)g(\eta, z(\eta-\kappa))d\eta \right\| \\ &\leq \frac{P}{(2q-1)\Gamma(\alpha+2q-1)} \|\varphi(0)\| + \frac{P\mu}{\Gamma(\alpha+2q)} \|z_1\| \\ &\quad + \frac{P\vartheta^{2-\delta}}{\Gamma(2q)} \int_0^{\mu} (\mu-\eta)^{2q-1} \|\mathcal{E}z(\eta-\kappa)\|_{\mathcal{Z}} d\eta \\ &\quad + \frac{P\vartheta^{2-\delta}}{\Gamma(2q)} \int_0^{\mu} (\mu-\eta)^{2q-1} \|\mathfrak{B}x(\eta)\| d\eta \end{aligned}$$

$$\begin{aligned}
& + \frac{P\vartheta^{2-\delta}}{\Gamma(2q)} \int_0^\mu (\mu - \eta)^{2q-1} [\|g(\eta, z(\eta - \varkappa)) - g(\eta, 0)\|_{\mathcal{Z}} + \|g(\eta, 0)\|_{\mathcal{Z}}] d\eta \\
& \leq \frac{P}{(2q-1)\Gamma(\alpha+2q-1)} \|\varphi(0)\| + \frac{P\mu}{\Gamma(\alpha+2q)} \|z_1\| \\
& + \frac{P\hat{P}_1\vartheta^{2-\delta}}{\Gamma(2q)} \int_0^\mu (\mu - \eta)^{2q-1} \|z(\eta - \varkappa)\|_{\mathcal{Z}} d\eta + \frac{P\hat{P}_2\vartheta^{2-\delta}}{\Gamma(2q)} \|\mathfrak{B}x\|_{\mathcal{Y}} \\
& + \frac{P\mathcal{L}\vartheta^{2-\delta}}{\Gamma(2q)} \int_0^\mu (\mu - \eta)^{2q-1} \eta^{2-\delta} \|z(\eta - \varkappa)\|_{\mathcal{Z}} d\eta + \frac{P\mathcal{L}_g\vartheta^{2-\delta+2q}}{\Gamma(2q+1)} \\
& \leq \ell + \frac{\aleph}{\Gamma(2q)} \int_0^\mu (\mu - \eta)^{2q-1} \eta^{2-\delta} \|z(\eta - \varkappa)\|_{\mathcal{Z}} d\eta.
\end{aligned}$$

Thus, by referring to [42],

$$\|w\|_{\mathcal{C}}^* = \sup_{\mu \in [0, \vartheta]} e^{-\iota\mu} \vartheta^{2-\delta} \|z(\mu)\|_{\mathcal{Z}} \leq \ell E_{2q}(\aleph \vartheta^{2-\delta+2q}).$$

Similarly, we have

$$\begin{aligned}
& \|w_1(\mu) - w_2(\mu)\|_{\mathcal{Z}} \leq \mu^{2-\delta} \|z_1(\mu) - z_2(\mu)\|_{\mathcal{Z}} \\
& \leq \mu^{2-\delta} \left\| \int_0^\mu (\mu - \eta)^{q-1} \mathcal{G}_q(\mu - \eta) \mathcal{E}[z_1(\eta - \varkappa) - z_2(\eta - \varkappa)] d\eta \right\| \\
& + \mu^{2-\delta} \left\| \int_0^\mu (\mu - \eta)^{q-1} \mathcal{G}_q(\mu - \eta) [\mathfrak{B}x_1(\eta) - \mathfrak{B}x_2(\eta)] d\eta \right\| \\
& + \mu^{2-\delta} \left\| \int_0^\mu (\mu - \eta)^{q-1} \mathcal{G}_q(\mu - \eta) [g(\eta, z_1(\eta - \varkappa)) - g(\eta, z_2(\eta - \varkappa))] d\eta \right\| \\
& \leq \frac{P\hat{P}_1\vartheta^{2-\delta}}{\Gamma(2q)} \int_0^\mu (\mu - \eta)^{2q-1} \|z_1(\eta - \varkappa) - z_2(\eta - \varkappa)\|_{\mathcal{Z}} d\eta \\
& + \frac{P\vartheta^{2-\delta}}{\Gamma(2q)} \int_0^\mu (\mu - \eta)^{2q-1} \|\mathfrak{B}x_1(\eta) - \mathfrak{B}x_2(\eta)\|_{\mathcal{Y}} d\eta \\
& + \frac{P\vartheta^{2-\delta}}{\Gamma(2q)} \int_0^\mu (\mu - \eta)^{2q-1} \|g(\eta, z_1(\eta - \varkappa)) - g(\eta, z_2(\eta - \varkappa))\|_{\mathcal{Z}} d\eta \\
& \leq \frac{P\hat{P}_1\vartheta^{2-\delta}}{\Gamma(2q)} \int_0^\mu (\mu - \eta)^{2q-1} \|z_1(\eta - \varkappa) - z_2(\eta - \varkappa)\|_{\mathcal{Z}} d\eta \\
& + \frac{P\hat{P}_2\vartheta^{2-\delta}}{\Gamma(2q)} \|\mathfrak{B}x_1(\mu) - \mathfrak{B}x_2(\mu)\|_{\mathcal{Y}} \\
& + \frac{P\mathcal{L}\vartheta^{2-\delta}}{\Gamma(2q)} \int_0^\mu (\mu - \eta)^{2q-1} \eta^{2-\delta} \|z_1(\eta - \varkappa) - z_2(\eta - \varkappa)\|_{\mathcal{Z}} d\eta \\
& \leq \theta \|\mathfrak{B}x_1(\mu) - \mathfrak{B}x_2(\mu)\| + \frac{\aleph}{\Gamma(2q)} \int_0^\mu (\mu - \eta)^{2q-1} \eta^{2-\delta} \|z_1(\eta - \varkappa) - z_2(\eta - \varkappa)\|_{\mathcal{Z}} d\eta.
\end{aligned}$$

As a result, referring to [42], we have

$$\|w_1(\mu) - w_2(\mu)\|_{\mathcal{C}}^* \leq \theta E_{2q}(\aleph \vartheta^{2-\delta+2q}) \|\mathfrak{B}x_1(\mu) - \mathfrak{B}x_2(\mu)\|_{\mathcal{Y}}.$$

Hence, the proof is finished. $\square$

4.2. Proof of Approximate Controllability via a Sequence Approach

Theorem 2. *If the assumptions $(\mathcal{O}_1) - (\mathcal{O}_3)$ are fulfilled, then HOHFED systems (1) and (2) are approximately controllable.*

Proof. For $\overline{D(\mathcal{A})} = \mathcal{Z}$, it is sufficient to verify $D(\mathcal{A}) \subset \overline{\Sigma_\vartheta(g)}$, which means that for every $\epsilon > 0$ and ξ in $D(\mathcal{A})$, there exists $u_\epsilon(\mu)$ in $\mathcal{X}$ such that

$$\|\xi - \mathcal{G}_q(\vartheta)\varphi(0) - \mathcal{G}_q(\vartheta)z_1 - \Omega\mathcal{E}z_\epsilon^\vartheta - \Omega\mathbb{G}z_\epsilon^\vartheta - \Omega\mathfrak{B}x_\epsilon\| < \epsilon. \quad (9)$$

First, for every ξ provided in $D(\mathcal{A})$ and every $\varphi(0) \in \mathcal{C}([-\kappa, 0], \mathcal{Z})$, $z_1 \in \mathcal{Z}$, there is a function $\zeta(\mu)$ in $\mathcal{Y}$ such that $\Omega\zeta = \xi - \mathcal{G}_q(\vartheta)\varphi(0) - \mathcal{G}_q(\vartheta)z_1$.

Further, we prove that there is a control $x_\epsilon(\mu)$ in $\mathcal{X}$ such that the inequality (9) holds. We now proceed to formulate the control sequence approach for the subsequent analysis. Assume that $\epsilon > 0$ and $x_1(\mu)$ in $\mathcal{X}$ is arbitrary. As a result $(\mathcal{O}_3)$, there is an $x_2(\mu)$ in $\mathcal{X}$ such that

$$\|\xi - \mathcal{G}_q(\vartheta)\varphi(0) - \mathcal{G}_q(\vartheta)z_1 - \Omega\mathcal{E}z_1^\vartheta - \Omega\mathbb{G}z_1^\vartheta - \Omega\mathfrak{B}x_2\| < \frac{\epsilon}{2^2}. \quad (10)$$

In the above, $z_1(\mu) = z(\mu; x_1)$ for $0 \leq \mu \leq \vartheta$. $z_2(\mu) = z(\mu; x_2)$ for $0 \leq \mu \leq \vartheta$. From $(\mathcal{O}_3)$ we know that there is $v_2(\mu)$ in $\mathcal{X}$, such that

$$\|\Omega[\mathcal{E}z_2^\vartheta - \mathcal{E}z_1^\vartheta + \mathbb{G}z_2^\vartheta - \mathbb{G}z_1^\vartheta] - \Omega\mathfrak{B}v_2\| < \frac{\epsilon}{2^3},$$

and

$$\begin{aligned} \|\mathfrak{B}v_2(\mu)\|_{\mathcal{Y}} &\leq \beta\|(\mathcal{E}z_2^\vartheta - \mathcal{E}z_1^\vartheta) + (\mathbb{G}z_2^\vartheta - \mathbb{G}z_1^\vartheta)\| \\ &\leq \beta(\hat{P}_1 + \mathcal{L}\vartheta^{2-\delta})\|z_2(\mu) - z_1(\mu)\| \\ &\leq \frac{\aleph\hat{P}_2\beta}{\Gamma(2q)}E_{2q}(\aleph\vartheta^{2-\delta+2q})\|\mathfrak{B}x_1(\mu) - \mathfrak{B}x_2(\mu)\|_{\mathcal{Y}}. \end{aligned}$$

We indicate $x_3(\mu) = x_2(\mu) - v_2(\mu)$, $x_3(\mu) \in \mathcal{X}$, and we obtain

$$\begin{aligned} &\|\xi - \mathcal{G}_q(\vartheta)\varphi(0) - \mathcal{G}_q(\vartheta)z_1 - \Omega\mathcal{E}z_2^\vartheta - \Omega\mathbb{G}z_2^\vartheta - \Omega\mathfrak{B}x_3\| \\ &= \|\xi - \mathcal{G}_q(\vartheta)\varphi(0) - \mathcal{G}_q(\vartheta)z_1 - \Omega\mathcal{E}z_1^\vartheta - \Omega\mathbb{G}z_1^\vartheta - \Omega\mathfrak{B}x_2 + \Omega\mathfrak{B}v_2 \\ &\quad - \Omega[\mathbb{G}z_2^\vartheta - \mathbb{G}z_1^\vartheta + \mathcal{E}z_2^\vartheta - \mathcal{E}z_1^\vartheta]\| \\ &\leq \|\xi - \mathcal{G}_q(\vartheta)\varphi(0) - \mathcal{G}_q(\vartheta)z_1 - \Omega\mathcal{E}z_1^\vartheta - \Omega\mathbb{G}z_1^\vartheta - \Omega\mathfrak{B}x_2\| \\ &\quad + \|\Omega\mathfrak{B}v_2 - \Omega[\mathcal{E}z_2^\vartheta - \mathcal{E}z_1^\vartheta + \mathbb{G}z_2^\vartheta - \mathbb{G}z_1^\vartheta]\| \\ &\leq \left(\frac{1}{2^2} + \frac{1}{2^3}\right)\epsilon. \end{aligned}$$

We know through induction that there is a sequence $\{x_k(\mu)\} \subset \mathcal{X}$; consequently,

$$\|\xi - \mathcal{G}_q(\vartheta)\varphi(0) - \mathcal{G}_q(\vartheta)z_1 - \Omega\mathcal{E}z_k^\vartheta - \Omega\mathbb{G}z_k^\vartheta - \Omega\mathfrak{B}x_{k+1}\| < \left(\frac{1}{2^2} + \dots + \frac{1}{2^k}\right)\epsilon.$$

In the above, $z_k(\mu) = z(\cdot; x_k)$ for $0 \leq \mu \leq \vartheta$, and

$$\|\mathfrak{B}x_{k+1} - \mathfrak{B}x_k\|_{\mathcal{Y}} < \frac{\aleph\hat{P}_2\beta}{\Gamma(2q)}E_{2q}(\aleph\vartheta^{2-\delta+2q})\|\mathfrak{B}x_k(\mu) - \mathfrak{B}x_{k-1}(\mu)\|_{\mathcal{Y}}.$$

By referring to (8), we know that $\{\mathfrak{B}x_k : k = 1, 2, \dots\}$ represents a Cauchy sequence on $\mathcal{Y}$. Hence, there is a $\chi(\mu)$ in $\mathcal{Y}$ such that

$$\lim_{k \rightarrow \infty} \mathfrak{B}x_k(\mu) = \chi(\mu) \text{ in } \mathcal{Y}.$$

Thus, for every $\epsilon > 0$, there is a positive integer number N such that

$$\|\Omega \mathfrak{B}x_{N+1} - \Omega \mathfrak{B}x_N\| < \frac{\epsilon}{2}.$$

Hence, we obtain

$$\begin{aligned} & \|\xi - \mathcal{G}_q(\vartheta)\varphi(0) - \mathcal{G}_q(\vartheta)z_1 - \Omega \mathcal{E}z_N^\chi - \Omega \mathbb{G}z_N^\chi - \Omega \mathfrak{B}x_N\| \\ & \leq \|\xi - \mathcal{G}_q(\vartheta)\varphi(0) - \mathcal{G}_q(\vartheta)z_1 - \Omega \mathcal{E}z_N^\chi - \Omega \mathbb{G}z_N^\chi - \Omega \mathfrak{B}x_{N+1}\| + \|\Omega \mathfrak{B}x_{N+1} - \Omega \mathfrak{B}x_N\| \\ & \leq \left(\frac{1}{2^2} + \dots + \frac{1}{2^k}\right)\epsilon + \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

As a result of the above reasoning, it is simple to deduce that HOHFED systems (1) and (2) are approximately controllable. $\square$

Remark 5. This study examines the Hilfer fractional derivative of order $q \in (1, 2)$, which is a fractional extension of the traditional second-order evolution equations generated by a strongly continuous cosine family. The parameter q determines the degree to which memory effects impact the system. When q approaches 1^+ , the fractional operator rapidly loses its second-order hereditary structure. In the limit $q = 1$, the Hilfer derivative transforms into the classical first-order derivative (in the sense of either R-L or Caputo, depending on the type parameter $\wp$). The abstract model no longer represents a cosine-family system but rather an ordinary first-order differential equation driven by a C_0 -semigroup. Thus, the fractional formulation with $1 < q < 2$ represents intermediate memory effects between first- and second-order dynamics; however, the integer-order case $q = 1$ corresponds to the conventional memory-free model.

5. Application

Let the accompanying partial Hilfer fractional evolution equations be as follows:

$$\begin{cases} \partial_\mu^{\frac{3}{2}, \wp} z(\mu, \kappa) = \frac{\partial^2}{\partial \kappa^2} z(\mu, \kappa) + z(\mu - \varkappa, \kappa) + \mathfrak{B}x(\mu, \kappa) + g(\mu, z(\mu - \varkappa, \kappa)), & \mu \in (0, 1], \\ (I^{(2-\delta)}z)(\mu, \kappa) = \varphi(\mu), & \mu \in [-\varkappa, 0], \quad (I^{(2-\delta)}z)'(0, \kappa) = z_1(\kappa), & \kappa \in [0, \pi], \\ z(\mu, 0) = z(\mu, \pi) = 0, & \mu \in [0, 1]. \end{cases} \quad (11)$$

In the above, $\partial_\mu^{\frac{3}{2}, \wp}$ indicates the Hilfer fractional partial differential equations of order $\frac{3}{2}$ and type $\wp \in [0, 1]$, and $\delta = \varrho + \wp(2 - \varrho)$. Suppose that $\mathcal{Z} = L^2([0, \pi])$. We determine that $\mathcal{A}$ fulfills $\mathcal{A}z = \frac{d^2}{dt^2}z$, with the domain $D(\mathcal{A})$ provided by

$$D(\mathcal{A}) = z \in \mathcal{Z} : z(0) = z(\pi) = 0, z'' \in \mathcal{Z}, z', z'', \text{ which are absolutely continuous.}$$

Thus, $\mathcal{A}$ indicates the infinitesimal generator of cosine operator $\{\mathbb{K}(\mu)\}_{\mu \geq 0}$. Assume that $\omega_k(z) = \sqrt{(2/\pi)} \sin(k\pi z)$, implying that $\{-k^2, k \in \mathbb{N}\}$ are eigenvalues of $\mathcal{A}$ and that $\{\omega_k\}_{k=1}^\infty$ indicates an orthonormal basis of $\mathcal{Z}$. In addition,

$$\mathcal{A}z = - \sum_{k=1}^{\infty} k^2 \langle z, \omega_k \rangle \omega_k, \quad z \text{ in } D(\mathcal{A}).$$

From [38], the cosine operator $\mathbb{K}(\mu)$ is determined by

$$\mathbb{K}(\mu)z = \sum_{k=1}^{\infty} \cos(k\pi\mu) \langle z, \omega_k \rangle \omega_k, \quad z \in \mathcal{Z},$$

which is connected along with the sine operator $\{\mathbb{S}(\mu), \mu \geq 0\}$, presented as

$$\mathbb{S}(\mu)z = \sum_{k=1}^{\infty} \frac{1}{k} \sin(k\pi\mu) \langle z, \omega_k \rangle \omega_k, \quad z \in \mathcal{Z}.$$

Consequently, we introduce the operator:

$$\mathcal{G}_q(\mu)z = \sum_{l=1}^{\infty} \mu^{\frac{q}{2}} E_{q,q}(-k^2 \mu^q) \langle z, \omega_l \rangle \omega_l, \quad q = \frac{q}{2}.$$

In the above, $E_{q,q}(u) = \sum_{k=0}^{\infty} \frac{u^k}{\Gamma(q(k+1))}$ is the Mittag-Leffler function. For any $x(\mu)$ in $\mathcal{X} = L^2([0, 1], \mathcal{H})$, we obtain

$$x(\mu) = \sum_{k=1}^{\infty} x_k(\mu) \omega_k, \quad x_k(\mu) = \langle x(\mu), \omega_k \rangle.$$

We determine the operator $\mathfrak{B}$ by

$$\mathfrak{B}x(\mu) = \sum_{k=1}^{\infty} \bar{x}_k(\mu) \omega_k.$$

In the above, for $k = 1, 2, \dots$,

$$\bar{x}_k(\mu) = \begin{cases} 0, & 0 \leq \mu < 1 - \frac{1}{k^2}, \\ x_k(\mu), & 1 - \frac{1}{k^2} \leq \mu \leq 1. \end{cases}$$

Thus, we simply determine that $\|\mathfrak{B}x(\mu)\| \leq \|x(\mu)\|$.

To begin, the definition of $\mathfrak{B}$ defines the equivalent linear system of (11).

$$\begin{cases} D_{\mu}^{\frac{3}{2}, q} z_k(\mu) + k^2 z_k(\mu) = x_k(\mu), & 1 - \frac{1}{k^2} < \mu < 1, \\ (I^{(2-\delta)} z_k)(\mu) = \varphi(\mu), & \mu \in [-\varkappa, 0], \quad (I^{(2-\delta)} z_k)'(0) = z_1. \end{cases} \quad (12)$$

Now, we verify that the $(\mathcal{O}_3)$ is fulfilled. We will examine these statements to determine their validity,

$$\varkappa = \int_0^1 (1-\eta)^{-\frac{1}{4}} \mathcal{G}_{\frac{3}{4}}(1-\eta) g(\eta) d\eta = \sum_{i=1}^{\infty} \varkappa_i \omega_i, \quad \varkappa_i = \langle \varkappa, \omega_i \rangle,$$

for every $g(\mu)$ in $L^2(V, \mathcal{Z})$. Indeed, we have a choice $\hat{x}_k(\mu)$ as a consequence of

$$\hat{x}_k(\mu) = \frac{2k^2}{1-e^{-2}} \varkappa_k e^{-k^2(1-\mu)}, \quad 1 - \frac{1}{k^2} \leq \mu \leq 1,$$

and

$$\varkappa_k = \int_{1-\frac{1}{k^2}}^1 \int_0^{\infty} (1-\eta)^{-\frac{1}{4}} \zeta \mathbb{N}_{\frac{3}{4}}(\zeta) e^{-k^2 \zeta (1-\eta)^{\frac{3}{4}}} \hat{x}_k(\eta) d\zeta d\eta.$$

As a result, we determine $x(\mu) = \sum_{k=1}^{\infty} x_k(\mu) \omega_k$. Then,

$$x_k(\mu) = \begin{cases} 0, & 0 \leq \mu < 1 - \frac{1}{k^2}, \\ \hat{x}_k(\mu), & 1 - \frac{1}{k^2} \leq \mu \leq 1. \end{cases}$$

Hence, for any supplied $g(\mu)$ in $L^2(V, \mathcal{Z})$, there exists $x(\mu) \in \mathcal{X}$ such that

$$\int_0^1 (1-\eta)^{-\frac{1}{4}} \mathcal{G}_{\frac{3}{4}}(1-\eta) \mathfrak{B}x(\eta) d\eta = \int_0^1 (1-\eta)^{-\frac{1}{4}} \mathcal{G}_{\frac{3}{4}}(1-\eta) g(\eta) d\eta.$$

As a consequence, the condition (7) of $(\mathcal{O}_3)$ is fulfilled. Furthermore, we have

$$\begin{aligned} \|\mathfrak{B}x(\mu)\|^2 &= \sum_{k=1}^{\infty} \int_{1-\frac{1}{k^2}}^1 |\widehat{x}_k(\mu)|^2 d\mu \\ &= \frac{4}{3} (1-e^{-2})^{-1} \sum_{k=1}^{\infty} (1-e^{-2k^2}) \int_0^1 |g_k(\mu)|^2 d\mu \\ &\leq \frac{4}{3} (1-e^{-2})^{-1} |g(\mu)|^2. \end{aligned}$$

Consequently, it is shown that if $(\mathcal{O}_3)$ meets its requirements, then for $q = \frac{3}{4}$, the system Equation (11) is approximately controllable on $[0, \vartheta]$, if

$$\frac{4(1+\mathcal{L})}{3\Gamma(\frac{3}{2})} \frac{1}{\sqrt{2}} (1-e^{-2})^{-1} E_{\frac{3}{4}}(1+\mathcal{L}) < 1,$$

is fulfilled.

6. Conclusions and Future Research

This study investigated the approximate controllability of Hilfer fractional evolution equations of order ϱ in $(1, 2)$ in the presence of a time delay. We derived the existence and uniqueness of mild solutions using fractional calculus, cosine families, and the fixed-point technique. Furthermore, we established the approximate controllability results of the system using a sequence approach. The example provided effectively demonstrates the application of our theoretical findings. Future research must investigate the impact of nonlocal and memory-based initial conditions on numerical simulation results. Such an analysis, potentially supported by simulations, would enhance the practical significance of the theoretical results, particularly for the impulsive Hilfer fractional delay differential inclusions under finite dimensional operators. We will also use Martelli's fixed-point theorem to apply the existence and approximate controllability results to Atangana-Baleanu fractional differential inclusions with infinite delay.

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