

Structural Reparameterization of the Complex Variable s and the Fixation of the Critical Line

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Abstract

This paper explains why the critical line sits at the real part equal to one-half by treating it as an intrinsic boundary of a reparametrized complex plane (“z-space”), not a mere artifact of functional symmetry. In z-space the real part is defined by a geometric-series map that gives rise to a rulebook for admissible analytic operations. Within this setting we rederive the classical toolkit—the eta–zeta relation, Gamma reflection and duplication, theta–Mellin identity, functional equation, and the completed zeta—without importing analytic continuation from the usual s -variable. We show that access to the left half-plane occurs entirely through formulas written on the right, with boundary matching only along the line with the real part equal to one-half. A global Hadamard product confirms the consistency and fixed location of this boundary, and a holomorphic change of variables transports these conclusions into the classical setting.

Keywords: critical line; geometric series; Maclaurin series; complex variable; Dirichlet series; zeta function; completed L-function; symmetry

MSC: 30D20

1. Introduction

Reflection symmetries in the complex plane arise frequently in analytic number theory. Among these, one of the most persistent is the appearance of a vertical axis of symmetry at $\Re s = 1/2$. This line emerges across a range of contexts, most famously in the study of zeta and L -functions, where analytic continuation and functional equations balance values at s and $1 - s$ [1]. For this reason, the line $\Re s = 1/2$ is commonly referred to as the critical line [2].

The presence of such symmetry is usually attributed to an associated functional equation. For example, the completed Riemann zeta function $\xi(s)$ satisfies

$$\xi(s) = \xi(1 - s),$$

from which reflection symmetry about $\Re s = 1/2$ follows immediately [3].

While correct, this explanation is descriptive: it identifies the axis at $1/2$ but does not reveal why the symmetry occurs precisely on this boundary rather than along some other vertical line.

The goal of this paper is to supply a structural explanation for the location of the line $\Re s = 1/2$. We introduce a reparameterization of the complex variable s based on a Maclaurin-type expansion [4] that exposes geometric and analytic constraints that naturally

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select $1/2$ as the unique axis of symmetry. This perspective separates the mechanism that fixes the critical line from the familiar fact of symmetry encoded by a functional equation.

For clarity, the material is arranged as follows.

- Section 1 states the motivation, the main structural statement (Theorem 1), and the plan of the argument.
- Section 2 reviews the standard analytic framework for the Riemann zeta function and its completion: Dirichlet and alternating series, the Gamma factor, the functional equation, and the Hadamard product. This provides the standard context against which the later reparameterization is to be compared.
- Sections 3 and 4 introduce the geometric reparameterization,

$$z = \zeta(r) + it, \quad \zeta(r) = \frac{1}{1-r}, \quad r \in [-1, 1),$$

which forces $\Re z \geq 1/2$, and formulate the rulebook that governs admissible analytic operations in this half-plane. These sections explain why the z -space framework is adopted, how access to the reflected side is to be interpreted, and provide a heuristic motivation for the reparameterization.

- Sections 5–8 reconstruct, entirely within the constraints of z -space, the analytic tools needed for the main theorem: Dirichlet/eta relations, Gamma identities, theta–Mellin representation of the functional equation, completed zeta function, and the Hadamard factorization. Although these are classical constructions in the usual s -variable, they are rederived here under the geometric constraint $\Re z \geq 1/2$ imposed from the outset to confirm that each step remains valid in this setting. The components invoked in Theorem 1 are taken from these sections.
- Section 9 collects the consequences of the preceding sections to complete the proof of Theorem 1 and, moreover, shows that the z -variable is not specific to ζ but arises as a generic geometric reparameterization. The concluding remarks indicate that the same z -space architecture applies to any function satisfying the same access and symmetry conditions, and, when available, order-one Hadamard factorization, so that the critical line appears as an intrinsic boundary of the construction rather than as a special feature of the Riemann zeta function.

In this way, Theorem 1 isolates the statements that are structurally responsible for fixing the critical line, while the surrounding sections supply z -space versions of the standard analytic apparatus needed to justify them.

Although one could attempt to restrict classical s -space tools to the region $\Re z \geq 1/2$, that approach would import assumptions validated only in the unrestricted s -variable. By contrast, the arguments here begin directly in z -space and adhere to its rulebook from the outset, so that every identity and limit is justified within the working half-plane, without prior appeal to the full s -domain.

Theorem 1 (Structural fixation of the critical line). *Let $z = \zeta(r) + it$ with*

$$\zeta(r) = \sum_{k=0}^{\infty} r^k = \frac{1}{1-r}, \quad r \in [-1, 1).$$

Then the vertical line $\Re z = 1/2$ is the unique boundary that controls analytic access between the right and left half-planes in z -space: the region $\Re z < 1/2$ is inaccessible except through the reflection $z \mapsto 1 - z$.

Under the holomorphic change of variables

$$z \mapsto s := s(r) = \frac{1}{1-r} + it,$$

this boundary appears in the classical variable as the critical line $\Re s = 1/2$.

Why this boundary is unique:

Access boundary (z-space)

- Geometric constraint (Section 3): The parametrization forces $\zeta(r) \geq 1/2$ with the maximal boundary value $\zeta(-1) = 1/2$ (Abel/Cesàro).
- Functional access only by reflection (Corollaries 2 and 4): The left half-plane is reached only through formulas written entirely in terms of right-half-plane data: $\zeta(1-z)$ expressed via $\zeta(z)$, and $\xi(1-z)$ by $\xi(z)$, consistent with the z-space rulebook (R3)–(R6) of Section 4.
- Boundary matching (Corollary 5): Non-tangential limits from both sides coincide only on $\Re z = 1/2$, yielding exact boundary matching for $\xi(z) = \xi(1-z)$.

Together, these show that the critical line is an impassable analytic boundary except through functional symmetry.

Structural fixation via zeros (Hadamard)

- Global factorization (Corollary 6): The completed zeta function ξ is entire of order one and admits a Hadamard product.
- z-space consistency (Lemma 5): The right-hand and reflected z-space Hadamard factorizations can represent the same entire function only if all admissible zeros satisfy $\Re \rho = 1/2$. Any off-line zero would destroy internal consistency; the zero structure is forced onto the critical line.
- Structural transport (Lemma 6): The holomorphic change of variables $z \mapsto s$ preserves analytic identities and zero sets. Therefore, the zero structure fixed by z-space governs the analytic structure of s-space. The classical Hadamard product inherits this constraint, and the same critical line emerges as $\Re s = 1/2$.

Proof. Section 3 fixes $\Re z \geq 1/2$ as the maximal admissible region and identifies $\Re z = 1/2$ as the unique access boundary.

Corollaries 2 and 4 show that values in the left half-plane are reachable only through reflection from the right, and Corollary 5 establishes boundary matching of nontangential limits precisely on $\Re z = 1/2$.

Corollary 6 provides a global Hadamard factorization. Lemma 5 then forces all admissible zeros onto the boundary $\Re \rho = 1/2$ in order for the two z-space products to represent the same function.

Finally, Lemma 6 shows that this constraint is structural, not merely representational: the same zero set and boundary are inherited by s-space under holomorphic transport.

Hence, the critical line is not a descriptive artifact of the functional equation but a structural necessity of the underlying analytic framework. $\square$

Remark 1 (Section-level references). Sections 2–4 cite sources individually where used. Sections 5–8 rely on standard results treated comprehensively in the references listed at each heading. These support the section as a whole rather than individual items.

2. The Analytic Structure of the Dirichlet Functions

We recall the standard analytic framework for the Riemann zeta function and its completion.

For $\Re s > 1$, the Riemann zeta function [1] is defined by the absolutely convergent Dirichlet series

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s}. \quad (1)$$

That may be rewritten in Euler product form:

$$\zeta(s) := \prod_p \frac{1}{1 - p^{-s}}. \quad (2)$$

The Dirichlet eta function is given by the alternating series

$$\eta(s) := \sum_{n=1}^{\infty} (-1)^{n-1} n^{-s}, \quad \Re(s) > 0, \quad (3)$$

which converges by Dirichlet's test and hence defines a holomorphic function in the half-plane $\Re s > 0$ [5]. In the region $\Re s > 1$, the two series are related by

$$\eta(s) = (1 - 2^{1-s}) \zeta(s), \quad (4)$$

which provides a meromorphic continuation of $\zeta(s)$ to $\Re s > 0$, with a single singularity at $s = 1$.

The Gamma function is defined for $\Re s > 0$ by Euler's integral,

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt, \quad (5)$$

and satisfies the standard analytic continuation and functional identities [6], which are employed in deriving the classical functional equation:

$$\zeta(s) = 2^s \pi^{s-1} \sin(\pi s/2) \Gamma(1-s) \zeta(1-s). \quad (6)$$

This reflection formula extends $\zeta(s)$ meromorphically to the entire complex plane $\mathbb{C}$, with a single simple pole at $s = 1$ [1,3].

Symmetry is most naturally expressed through the completed zeta function

$$\xi(s) := \frac{1}{2} s(s-1) \pi^{-s/2} \Gamma(s/2) \zeta(s). \quad (7)$$

The prefactors cancel the pole at $s = 1$ and the trivial zeros at $s = 0, -2, -4, \dots$, so ξ is entire and satisfies the symmetric functional equation [1,7]

$$\xi(s) = \xi(1-s). \quad (8)$$

Because $\xi(s)$ is entire of order one, its global growth can be described completely by a Hadamard factorization [1,8],

$$\xi(s) = e^{A+Bs} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) \exp\left(\frac{s}{\rho}\right), \quad (9)$$

where the product extends over all nontrivial zeros ρ of $\zeta(s)$ and $A, B \in \mathbb{C}$ are constants.

This product converges absolutely and uniformly on compact subsets of $\mathbb{C}$ and encodes both the distribution of zeros and the global analytic structure of ξ .

The normalization $\xi(0) = 1/2$ determines $e^A = 1/2$, and the symmetry (8) implies $B = 0$, yielding the symmetric form

$$\xi(s) = \frac{1}{2} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) \exp\left(\frac{s}{\rho}\right), \quad (10)$$

which manifests the reflectional symmetry of ξ about the critical line $\Re s = 1/2$.

The structural origin of this axis will be developed in the following sections. For additional historical and expository context, see Edwards (1974) [2] or Hardy (1949) [9].

3. Reparametrizing the Real Part of Complex Numbers via Geometric Series

Consider the geometric series

$$\zeta(r) := \sum_{k=0}^{\infty} r^k, \quad |r| < 1, \quad (11)$$

which converges to the closed form

$$\sigma(r) := \frac{1}{1-r}. \quad (12)$$

For real $r \in (-1, 1)$, the range of $\zeta(r)$ covers the interval $(1/2, \infty)$, since

$$\lim_{r \rightarrow -1^+} \sigma(r) = \frac{1}{1 - (-1)} = \frac{1}{2}, \quad \lim_{r \rightarrow 1^-} \sigma(r) = +\infty. \quad (13)$$

Hence there is a bijection,

$$r \in (-1, 1) \leftrightarrow \zeta \in (1/2, \infty). \quad (14)$$

By the Abel and Cesàro Summations (see Remark 2), the endpoint $r = -1$ is interpreted consistently as $\zeta(-1) = 1/2$, thereby extending the domain of the geometric series to

$$r \in [-1, 1) \leftrightarrow \zeta \in [1/2, \infty). \quad (15)$$

We embed this into the complex plane by introducing a vertical parameter $t \in \mathbb{R}$ and defining

$$z := z(r) = \zeta(r) + it. \quad (16)$$

Thus z lies on the closed half-plane $\Re z = \zeta(r) \geq 1/2$.

The power-series identity

$$\frac{1}{1-r} = \sum_{k=0}^{\infty} r^k, \quad r \in [-1, 1), \quad (17)$$

suggests a natural analytic continuation of $\zeta(r)$ to all $r \in \mathbb{R} \setminus \{1\}$ via

$$\sigma(r) = \frac{1}{1-r}.$$

We then define the corresponding complex parameter

$$s(r) := \sigma(r) + it, \quad (18)$$

interpreting $s(r)$ as the Dirichlet variable used in Section 2 and as the analytically continued image of the geometric-series parameter r .

The closed form $\sigma(r)$ provides an analytic continuation of the original geometric series $\zeta(r)$ beyond the disk of convergence [4]. The shared interval of convergence for (17) and the region where only the closed form remains valid are illustrated in Figure 1.

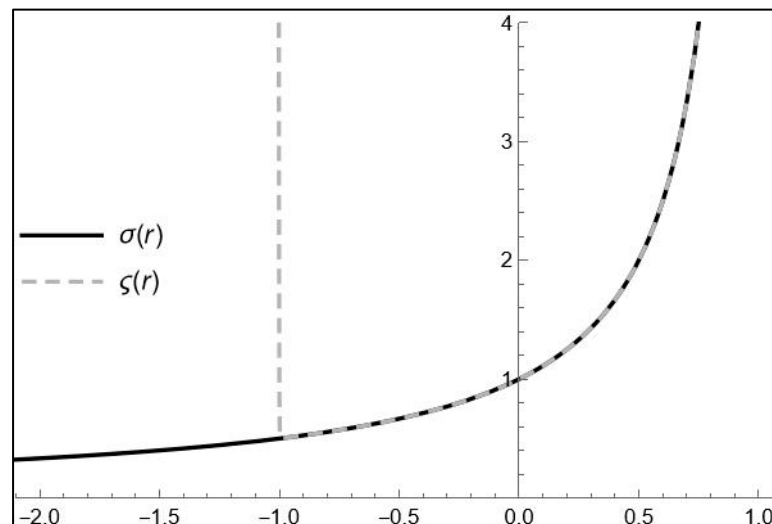


Figure 1. Plot of the convergent domain for the geometric series $\zeta(r)$ over the analytically continued domain of $\sigma(r)$. The series and closed form agree for $|r| < 1$. Outside of this interval the closed form assigns analytic values where the series diverges.

Henceforth, all structural definitions and analytic arguments will be formulated in z -space, without importing analytic continuation from the unrestricted s -domain. This enables a purely geometric framework in which the fixed location and emergence of the critical line $\Re(s) = 1/2$ can be explained.

Remark 2 (Endpoint summability at $r = -1$ [9]). *Abel Summation:* For $0 \leq x < 1$,

$$\sum_{k=0}^{\infty} (-1)^k x^k = \frac{1}{1+x} \Rightarrow \lim_{x \rightarrow 1^-} \sum_{k=0}^{\infty} (-1)^k x^k = \frac{1}{2}. \quad (19)$$

Cesàro Summation: Let $S_N = \sum_{k=0}^N (-1)^k$ so $S_N = 1$ for even N and $S_N = 0$ for odd N . The Cesàro means

$$\sigma_N := \frac{1}{N+1} \sum_{n=0}^N S_n \quad (20)$$

satisfy $\sigma_N \rightarrow 1/2$.

Therefore, both the Abel and Cesàro Summations assign the endpoint value $\zeta(-1) = 1/2$, justifying the extension in (15).

Remark 3 (No further extension by Abel/Cesàro on the real line [9]). *Abel Summation:* For $a_k = r^k$, the Abel sum is

$$\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} (rx)^k = \frac{1}{1-rx}, \quad |rx| < 1. \quad (21)$$

If $|r| > 1$, then for all x sufficiently close to 1^- we have $|rx| < 1$, so the defining power series fails to converge in any neighborhood of $x = 1$. Hence the Abel limit $\lim_{x \rightarrow 1^-} \sum (rx)^k$ is undefined.

Cesàro Summation: A necessary condition for Cesàro $(C, 1)$ summability is $a_k \rightarrow 0$. For $a_k = r^k$ with $|r| \geq 1$ and $r \neq -1$, we have $a_k \not\rightarrow 0$, so the series is not Cesàro-summable.

Hence, no further real values of r beyond -1 yield a consistent Abel or Cesàro limit, and the extended interval $r \in [-1, 1)$ is maximal for real summability.

Remark 4 (Heuristic motivation of reparameterization). Within the region of absolute convergence $\Re(s) > 1$, Euler's product (2) may be written as

$$\zeta(s) := \prod_p \left(\sum_{k=0}^{\infty} p^{-sk} \right), \quad (22)$$

by substituting the geometric identity (17)

$$\frac{1}{1-r} = \sum_{k=0}^{\infty} r^k,$$

with $r = p^{-s}$, into the product. Expanding the product of these geometric series over all primes and regrouping terms by descending rational factors recovers the Dirichlet series representation of $\zeta(s)$. Classically, Euler's product is derived from the Dirichlet series and is valid only in the half-plane $\Re(s) > 1$; by itself, it does not support analytic continuation. Analytic continuation of ζ instead relies on summation and integral representations such as the Dirichlet series, the eta function, or the theta–Mellin formula.

This rewriting highlights that geometric summation is a persistent structural component of the Dirichlet–Euler description and its continuation to the entire complex plane. Motivated by this observation, we reverse the usual perspective: instead of viewing the geometric series as derived from the closed form, we treat the geometric-series parameter as the primary coordinate and introduce the reparameterization (16)

$$z = \varsigma(r) + it, \quad \varsigma(r) = \sum_{k=0}^{\infty} r^k.$$

This remark is purely heuristic and does not replace the classical analytic constructions. Rather, it provides qualitative intuition for the z -space framework developed in the subsequent sections.

4. Rulebook for z -Space Operations

The following rulebook specifies the analytic operations permitted in z -space and governs all subsequent arguments. Its purpose is to ensure that every derivation is carried out exclusively within the restricted domain $\Re z \geq 1/2$, so that no step relies on properties that hold only in the unrestricted s -variable framework.

Lemma 1 (Rulebook). Let $U \subset \mathbb{C}$ be a domain. The following principles govern analytic operations in z -space.

(R1) *Series of holomorphic functions.* If $\sum f_n$ converges uniformly on compact subsets of U , then $\sum f_n$ is holomorphic on U . Finite algebraic operations and termwise differentiation are valid on U [10].

(R2) *Identity theorem/ uniqueness.* If two holomorphic (or meromorphic) functions agree on a nonempty open subset of a connected domain U , then they agree on all U where both are defined [8].

(R3) *Locality.* A derivation carried out inside U is valid provided every intermediate expression is defined and holomorphic on U . Evaluation outside U is neither required nor permitted [8].

(R4) *Restriction of global identities.* If $F(w) = G(w)$ holds as an identity of meromorphic functions on $\mathbb{C}$, then, for any open $U \subset \mathbb{C}$ and any holomorphic map $h: U \rightarrow \mathbb{C}$ avoiding poles,

$$F \circ h = G \circ h \text{ on } U. \quad (23)$$

Ref. [11]

(R5) *Boundary domains and extensions.* Let

$$H_{1/2} = \{z \in \mathbb{C}: \Re z > 1/2\}, \bar{H}_{1/2} = \{z \in \mathbb{C}: \Re z \geq 1/2\}. \quad (24)$$

All analytic operations governed by (R1)–(R4) are confined to open domains such as $H_{1/2}$ where the relevant functions are holomorphic.

The closure $\bar{H}_{1/2}$ is used only for limit evaluation or boundary continuity, as justified by the Abel–Cesàro extension (Remarks 2 and 3). No differentiation, contour deformation, or analytic continuation may be performed on the boundary itself.

Hence $H_{1/2}$ is the working analytic domain, while $\bar{H}_{1/2}$ serves only for evaluating limiting values such as $\xi(1/2 + it)$.

(R6) *Compatibility of convergence with the working domain (access rule).* Let $U \subset \mathbb{C}$ be the working analytic domain in z -space (typically an open set with $U \subset H_{1/2}$), and let f be specified on some open set $V \subset \mathbb{C}$ by a convergent representation (e.g., a Dirichlet/geometric series, integral, or canonical product) that makes f holomorphic on V .

1. *Operative region.* All valid manipulations with f in z -space are confined to

$$E := U \cap V. \quad (25)$$

Conclusions must be stated on E or on its boundary via limits as in (R5).

2. *When U is larger than V :* If $U \not\subseteq V$, the working domain is automatically restricted to E . No evaluation or analytic operation is permitted at points of $U \setminus V$ unless a meromorphic/analytic continuation of f is explicitly established there.
3. *When V is larger than U :* If $V \not\subseteq U$, the part of V outside U is inaccessible within the z -space framework by (R3) and the construction of z -space from the geometric-series parameter, cf. Section 3. Only values in E and boundary limits per (R5) may be used in derivations.
4. *Boundary use and continuation:* Boundary values may be taken as non-tangential limits from E as in (R5). Analytic continuation of f may be invoked only insofar as the continued function remains within U . Values outside U are accessible only by relating them to values inside U solely via global identities valid on $\mathbb{C}$ (e.g., functional symmetries) composed as in (R4), followed—if needed—by boundary limiting in the sense of (R5).
5. *Effective boundary:* For proofs in z -space, the effective boundary for f is $\bar{E} \cap \bar{U} \subset \bar{H}_{1/2}$; that is, the smaller boundary determined by the intersection of the domain of convergence of f and the working domain of U .

Notation. When extending a z -space identity for a function $f(z)$ to a domain D larger than the working region $H_{1/2}$ (or to its closure $\bar{H}_{1/2}$ when invoking (R5)), we do not write $z \in \mathbb{C}$, since z is evaluated only on $H_{1/2}$. Instead, when (R2) or (R4) is used to appeal to an identity valid beyond $z \in H_{1/2}$, we denote this by

$$z \in D[H_{1/2}] \quad \left(\text{or } z \in D[\bar{H}_{1/2}] \text{ when using (R5)} \right) \quad (26)$$

to mean: “the identity holds on the ambient domain D , while access/evaluation is restricted to $H_{1/2}$ (respectively $\bar{H}_{1/2}$).”

Notation. To indicate the provenance of a constraint, we place a small label directly above a relation or membership symbol. The marker “ V ” (e.g., “ $\overset{V}{<}$ ”) signals that the bound comes from the convergence domain of the function f . The marker “ U ” denotes a bound imposed by the working domain. The absence of a marker indicates that the bound is independent of domain constraints.

This annotation is purely declarative: it does not change the underlying logical statement; it simply records the source of the constraint for reference.

5. Zeta and Eta Definitions and Identity in z -Space [5]

The classical definitions recalled in Section 2 must be reformulated to comply with the z -space rulebook of Section 4. Although the underlying functions remain unchanged, the interplay between the domain of convergence and the working domain must now be made explicit within the z -space framework.

Definition 1 (Dirichlet functions in z -space). *For complex z , the Riemann zeta function is defined as*

$$\zeta(z) := \sum_{n=1}^{\infty} n^{-z} \quad (27)$$

which is an absolutely convergent Dirichlet series on $\Re z \geq 1$.

For $\Re(z) \geq 1/2$, the Dirichlet eta function is defined as

$$\eta(z) := \sum_{n=1}^{\infty} (-1)^{n-1} n^{-z} \quad (28)$$

which converges by Dirichlet's test and is holomorphic on $\Re z \geq 1/2$.

In lieu of invoking analytic continuation, note that the reflection $z \mapsto 1 - z$ maps η to a region confined to the strip

$$0 \leq \Re(1 - z) \leq 1/2,$$

Lemma 2 (η - ζ relation on the right half-plane). *For $\Re z > 1$,*

$$\eta(z) = \sum_{n=1}^{\infty} (-1)^{n-1} n^{-z} = (1 - 2^{1-z}) \zeta(z). \quad (29)$$

Hence the meromorphic identity

$$\frac{\eta(z)}{1 - 2^{1-z}} = \zeta(z) \quad (30)$$

extends to the half-plane $\Re z \geq 1/2$.

Proof. On $\Re z > 1$, the Dirichlet series for ζ and η converge uniformly on compact sets, so termwise algebra and regrouping are justified by (R1), yielding (29). Since η is holomorphic on $\Re z \geq 1/2$ by (28), the quotient (30) extends to a meromorphic function there via (R2), with the following qualification:

The denominator vanishes precisely at

$$z = 1 - \frac{2\pi i k}{\log 2}, \quad k \in \mathbb{Z}, \quad (31)$$

which lies on the vertical line $\Re z = 1$. On $\Re z > 1$, (29) gives $\eta(z) = (1 - 2^{1-z}) \zeta(z)$ where ζ is holomorphic. Hence $\eta(z) = 0$ at every point (31).

Thus, the potential singularities of (31) are removable, except at $z = 1$ where ζ has a simple pole. Consequently, (30) defines a meromorphic function on $\Re z \geq 1/2$ with at most a simple pole at $z = 1$, and the identity extends to the boundary $\Re z = 1/2$ by (R5).

□

6. Gamma Definition and Identities in z -Space [4,6]

We continue this process of rederivation with the Gamma function and the identities required for the functional equation, paying particular attention to how the reparameterization affects their derivation. The purpose of this section is not to introduce new analytic results, but to verify that the classical tools needed for Theorem 1 remain valid when all arguments are carried out within the z -space framework introduced in Section 3.

Definition 2 (Gamma in z -space). Define the Gamma function by Euler's integral in z -space to be

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad (32)$$

which converges and is holomorphic on $\Re z \stackrel{U}{>} 1/2$.

By reflection $z \mapsto 1 - z$, admissible use of $\Gamma(1 - z)$ is confined to

$$0 \stackrel{V}{<} \Re(1 - z) \stackrel{U}{<} 1/2,$$

or equivalently

$$1/2 \stackrel{U}{<} \Re z \stackrel{V}{<} 1,$$

and, by (R5), the non-tangential boundary limits $\Re(1 - z) = \Re z = 1/2$ are permitted.

Proposition 1 (Beta–Gamma identity on the working strip). Let $x, y \in \mathbb{C}$ with

$$\Re x \stackrel{U}{>} 1/2, \quad \Re y \stackrel{U}{>} 1/2.$$

Define the beta integral as

$$B(x, y) := \int_0^1 t^{x-1} (1-t)^{y-1} dt. \quad (33)$$

Then

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \quad (34)$$

Proof. First, $B(x, y)$ converges absolutely on the working strip $H_{1/2}$: near $t = 0$ the integrand is $t^{\Re x - 1}$ and near $t = 1$ it is $(1-t)^{\Re y - 1}$, both integrable when $\Re x, \Re y > 0$.

For $\Re x \stackrel{U}{>} 1/2$ and $\Re y \stackrel{U}{>} 1/2$, both $\int_0^\infty u^{x-1} e^{-u} du$ and $\int_0^\infty v^{y-1} e^{-v} dv$, converge by Definition 2, so using Fubini/Tonelli:

$$\Gamma(x)\Gamma(y) = \int_0^\infty \int_0^\infty u^{x-1} v^{y-1} e^{-(u+v)} du dv.$$

Making the change of variables of

$$u = rt, \quad v = r(1-t), \quad r \in (0, \infty), \quad t \in (0, 1),$$

whose Jacobian is r . Then

$$\begin{aligned} \Gamma(x)\Gamma(y) &= \int_0^\infty \int_0^\infty (rt)^{x-1} (r(1-t))^{y-1} e^{-r} r dt dr \\ &= \int_0^\infty r^{x+y-1} e^{-r} dr \int_0^1 t^{x-1} (1-t)^{y-1} dt \\ &= \Gamma(x+y) B(x, y), \end{aligned}$$

which rearranging yields (34). All interchanges and substitutions are justified by absolute convergence in the working domain and by (R1) and (R3), extending to $\mathbb{C}[H_{1/2}]$ by (R2). $\square$

Proposition 2 (Euler reflection and meromorphic continuation of Γ). For $1/2 \leq \Re z < 1$,

$$\Gamma(z) \Gamma(1-z) = \pi / \sin(\pi z). \quad (35)$$

By (R2), the identity extends meromorphically to all $\mathbb{C} \setminus \mathbb{Z}$. In particular, Γ extends meromorphically with simple poles at $z = 0, -1, -2, \dots$

Proof. On the open strip $1/2 \leq \Re z < 1$, the Beta integral,

$$B(z, 1-z) = \int_0^1 t^{z-1} (1-t)^{-z} dt \quad (36)$$

converges absolutely since near $t = 0$ the integrand behaves like $t^{\Re z - 1}$ ($\Re z \geq 1/2$) and near $t = 1$ it behaves like $(1-t)^{-\Re z}$ ($\Re z < 1$).

By Proposition 1,

$$B(z, 1-z) = \Gamma(z) \Gamma(1-z). \quad (37)$$

Evaluate $B(z, 1-z)$ via the substitution

$$t = \frac{u}{1+u}, \quad u \in (0, \infty),$$

so that

$$dt = (1+u)^{-2} du, \quad 1-t = (1+u)^{-1}, \quad t/(1-t) = u.$$

Substituting into (36) gives

$$B(z, 1-z) = \int_0^\infty \left(\frac{u}{1+u} \right)^{z-1} \left(\frac{1}{1+u} \right)^{-z} (1+u)^{-2} du = \int_0^\infty \frac{u^{z-1}}{1+u} du. \quad (38)$$

To compute the last integral directly for $1/2 \leq \Re z < 1$, split $(0, \infty)$ at $u = 1$, set $u \mapsto 1/u$ on $(1, \infty)$, and combine to obtain the following:

$$\int_0^\infty \frac{u^{z-1}}{1+u} du = \int_0^1 \frac{u^{z-1} + u^{-z}}{1+u} du = \sum_{n \in \mathbb{Z}} \frac{(-1)^n}{z+n} = \frac{\pi}{\sin(\pi z)}, \quad (39)$$

where the last equality is the standard partial fraction expansion of $\pi/\sin(\pi z)$. All interchanges are justified by absolute convergence on $1/2 \leq \Re z < 1$.

Combining (37) and (39) yields (35) on $1/2 \leq \Re z < 1$. Since both sides are meromorphic in z and agree on a nonempty open strip, (R2) extends the identity to all $\mathbb{C} \setminus \mathbb{Z}$.

Because $\sin(\pi z)$ vanishes simply at each integer, Γ has simple poles at $z = 0, -1, -2, \dots$ $\square$

Corollary 1 (Nonvanishing). $\Gamma(z) \neq 0$ for all $z \in \mathbb{C} \setminus \mathbb{Z}$. Equivalently, $1/\Gamma$ is entire.

Proof. If $z \notin \mathbb{Z}$, then $\sin(\pi z) \neq 0$ and (35) shows $\Gamma(z)\Gamma(1-z)$ is finite and non-zero, so neither factor vanishes. If $z \in \mathbb{Z}$, then $\pi/\sin(\pi z)$ has a pole, hence $\Gamma(z)\Gamma(1-z)$ cannot be zero. Hence Γ has no zeros anywhere.

Since Γ has only simple poles (at $z = 0, -1, -2, \dots$) and no zeros, the reciprocal $1/\Gamma$ has simple zeros at those points and is holomorphic elsewhere, and the poles of Γ become removable for $1/\Gamma$. Therefore $1/\Gamma$ extends to an entire function. $\square$

Proposition 3 (Legendre's duplication). For $\Re z > 1/2$,

$$\Gamma(z) \Gamma(z + 1/2) = 2^{1-2z} \sqrt{\pi} \Gamma(2z). \quad (40)$$

By (R2), this identity extends meromorphically to all $\mathbb{C} \setminus \mathbb{Z}$.

Proof. On the open strip $1/2 \leq \Re z < 1$, by Proposition 1 and the trigonometric form of the Beta function (obtained by the substitution $t = \sin^2 \theta$, valid under (R1)–(R3)), we have the following:

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt = 2 \int_0^{\pi/2} \sin^{2x-1} \theta \cos^{2y-1} \theta d\theta, \quad \Re x, \Re y \geq 1/2.$$

Apply this with $x = z$, $y = 1/2$ (interpreted as a non-tangential boundary value via (R5)), and $x = y = z$. Then

$$B(z, 1/2) = 2 \int_0^{\pi/2} \sin^{2z-1} \theta d\theta, \quad B(z, z) = 2 \int_0^{\pi/2} \sin^{2z-1} \theta \cos^{2z-1} \theta d\theta. \quad (41)$$

Set $\varphi = 2\theta$ in the second integral and use $\sin \varphi = 2 \sin \theta \cos \theta$ and $d\varphi = 2 d\theta$. A direct calculation (permitted by (R1)–(R3)) gives the trigonometric identity

$$B(z, 1/2) = 2^{2z-1} B(z, z), \quad 1/2 \leq \Re z < 1. \quad (42)$$

Invoking the Beta–Gamma identity (34) on the same strip:

$$B(z, 1/2) = \frac{\Gamma(z)\Gamma(1/2)}{\Gamma(z+1/2)}, \quad B(z, z) = \frac{\Gamma(z)^2}{\Gamma(2z)}. \quad (43)$$

Substitute (43) into (42) and use $\Gamma(1/2) = \sqrt{\pi}$ to obtain

$$\frac{\Gamma(z)\sqrt{\pi}}{\Gamma(z+1/2)} = 2^{2z-1} \frac{\Gamma(z)^2}{\Gamma(2z)},$$

which rearranges to

$$\Gamma(z) \Gamma(z+1/2) = 2^{1-2z} \sqrt{\pi} \Gamma(2z),$$

establishing (40) for $1/2 \leq \Re z < 1$. Since both sides are meromorphic in z and agree on the nonempty open strip, (R2) extends the identity meromorphically to all $z \in \mathbb{C}[H_{1/2}]$. $\square$

Proposition 4 (Scaling law for the Gamma kernel). *Let $\alpha > 0$ and z satisfy $\Re z \geq 1/2$. Then*

$$\int_0^\infty e^{-\alpha t} t^{z-1} dt = \alpha^{-z} \Gamma(z) \quad (44)$$

and the identity is holomorphic in $z \in H_{1/2}$.

Proof. For $t \rightarrow \infty$, $e^{-\alpha t}$ gives exponential decay; for $t \downarrow 0$, the integrand behaves like $t^{\Re z-1}$ with $\Re z \geq 1/2$. Hence the integral converges absolutely, and by dominated convergence on compact subsets of $H_{1/2}$, it defines a holomorphic function of z (R1)–(R3).

Set $u = \alpha t$ (so $t = u/\alpha$, $dt = du/\alpha$). Then we obtain (44):

$$\int_0^\infty e^{-\alpha t} t^{z-1} dt = \int_0^\infty e^{-u} \left(\frac{u}{\alpha}\right)^{z-1} \frac{du}{\alpha} = \alpha^{-z} \int_0^\infty e^{-u} u^{z-1} du = \alpha^{-z} \Gamma(z),$$

where $\Gamma(z)$ is Euler's integral (Definition 2) on $H_{1/2}$. All steps are justified by the absolute convergence already noted and the legality of substitutions in (R3). $\square$

Section 6 Summary (Gamma framework insensitivity to choice of complex variable).

Within z -space, we have rederived the Gamma function (Definition 2) together with the classical identities required for the functional equation, including the Beta–Gamma relation, Euler's reflection formula, Legendre's duplication formula, and the scaling law.

A key observation is that no step in these derivations depends on fixing the complex variable as either z or s . Each identity is first established on a region of absolute convergence common to both frameworks and is then extended by the identity theorem (R2). Thus, the analytic content of

the Gamma framework is insensitive to the underlying complex variable: the reparameterization introduced in Section 3 neither alters nor obstructs these classical constructions.

This confirms that the Gamma apparatus required for Theorem 1 is compatible with the constraints of z -space and is not structurally altered by them.

7. Functional Equation and Completed Zeta Function in z -Space [1,6]

The purpose of this section is to establish the fixed nature of the critical line. By deriving the functional equation and the completed zeta function entirely within the z -space framework, we show that the line $\Re z = 1/2$ constitutes an intrinsic analytic boundary, where the half-plane $\Re z < 1/2$ is inaccessible except through functional symmetry.

We begin by deriving the functional equation in z -space via the theta–Mellin identity.

Lemma 3 (Theta–Mellin identity and symmetry on the strip). *Let*

$$\theta(t) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t}, \quad (45)$$

and for $1 < \Re z < 2$ define

$$I(z) := \frac{1}{2} \int_0^\infty (\theta(t) - 1) t^{\frac{z}{2}} \frac{dt}{t}, \quad (46)$$

then

$$I(z) = \frac{1}{2} \int_1^\infty (\theta(t) - 1) t^{\frac{z}{2}} \frac{dt}{t} + \frac{1}{2} \int_1^\infty (\theta(t) - 1) t^{\frac{1-z}{2}} \frac{dt}{t} = I(1-z) \quad (47)$$

and

$$I(z) = \pi^{-z/2} \Gamma(z/2) \zeta(z) \quad (48)$$

on $\mathbb{C}[H_{1/2}] \setminus \{1\}$

Proof. As $t \rightarrow \infty$, $\theta(t) - 1 = O(e^{-\pi t})$. As $t \downarrow 0$, the modular relation

$$\theta(t) = t^{-1/2} \theta(1/t) \quad (49)$$

implies

$$\theta(t) - 1 = t^{-1/2} + O(e^{-\pi/t})$$

Hence

$$\int_0^\infty |\theta(t) - 1| t^{\frac{\Re z}{2}-1} dt < \infty \text{ for } 1 < \Re z < 2, \quad (50)$$

so (47) converges absolutely and defines a holomorphic function on the strip by dominated convergence (R1) and (R3).

Split at $t = 1$ and use (49). On $(0,1)$, substitute $t \mapsto 1/t$. For $1 < \Re z < 2$ this gives

$$I(z) = \frac{1}{2} \int_1^\infty (\theta(t) - 1) t^{\frac{z}{2}} \frac{dt}{t} + \frac{1}{2} \int_1^\infty (\theta(t) - 1) t^{\frac{1-z}{2}} \frac{dt}{t} = I(1-z),$$

which is (47).

Since $\theta(t) - 1 = 2 \sum_{n \geq 1} e^{-\pi n^2 t}$, absolute convergence on $1 < \Re z < 2$ permits exchanging sum and integral via (R1), (R3), yielding

$$I(z) = \sum_{n \geq 1} \int_0^\infty e^{-\pi n^2 t} t^{\frac{z}{2}-1} dt = \sum_{n \geq 1} (\pi n^2)^{-\frac{z}{2}} \Gamma\left(\frac{z}{2}\right) = \pi^{-\frac{z}{2}} \Gamma\left(\frac{z}{2}\right) \sum_{n \geq 1} n^{-z} = \pi^{-\frac{z}{2}} \Gamma\left(\frac{z}{2}\right) \zeta(z),$$

by the scaling law (Proposition 4) with $\alpha = \pi n^2$ and $z \mapsto z/2$. Thus, we prove (48) on $1 < \Re z < 2$, and extend it meromorphically to $z \in \mathbb{C}[H_{1/2}] \setminus \{1\}$ by (R2). $\square$

Lemma 4 (Classical functional equation in z -space). For $z \in \mathbb{C}[H_{1/2}] \setminus \{1\}$,

$$\zeta(z) = 2^z \pi^{z-1} \sin(\pi z/2) \Gamma(1-z) \zeta(1-z). \quad (51)$$

Proof. From (47) and (48) we have on $1 < \Re z < 2$,

$$\pi^{-\frac{z}{2}} \Gamma\left(\frac{z}{2}\right) \zeta(z) = \pi^{-\frac{1-z}{2}} \Gamma\left(\frac{1-z}{2}\right) \zeta(1-z). \quad (52)$$

Rearrange to obtain

$$\zeta(z) = \pi^{z-\frac{1}{2}} \frac{\Gamma\left(\frac{1-z}{2}\right)}{\Gamma(z/2)} \zeta(1-z). \quad (53)$$

Now use the global Gamma identities from Section 6, admissible by (R4), with arguments avoiding poles on $1 < \Re z < 2$:

1. Duplication with $w = (1-z)/2$ gives

$$\Gamma\left(\frac{1-z}{2}\right) \Gamma\left(1-\frac{z}{2}\right) = 2^{1-2w} \sqrt{\pi} \Gamma(2w) = 2^z \sqrt{\pi} \Gamma(1-z). \quad (54)$$

2. Reflection with $w = z/2$ gives

$$\Gamma(z/2) \Gamma(1-z/2) = \pi / \sin(\pi z/2). \quad (55)$$

Divide (54) by (55) to eliminate $\Gamma(1-z/2)$:

$$\frac{\Gamma\left(\frac{1-z}{2}\right)}{\Gamma(z/2)} = \frac{2^z \sqrt{\pi} \Gamma(1-z)}{\pi / \sin(\pi z/2)} = 2^z \pi^{-1/2} \sin(\pi z/2) \Gamma(1-z). \quad (56)$$

Insert (56) into (53):

$$\zeta(z) = \pi^{z-1/2} [2^z \pi^{-1/2} \sin(\pi z/2) \Gamma(1-z)] \zeta(1-z) = 2^z \pi^{z-1} \sin(\pi z/2) \Gamma(1-z) \zeta(1-z),$$

which is (51) on $1 < \Re z < 2$.

Both sides of (51) are meromorphic in z and agree on a nonempty open strip, so by (R2) together with the composition rule (R4), the identity extends to $\mathbb{C}[H_{1/2}] \setminus \{1\}$. $\square$

The interaction between the convergence of the factors $\Gamma(1-z)$ and $\zeta(1-z)$ with the restricted working domain prevents direct access to the left half-plane $\Re z < 1/2$. We resolve this obstruction by algebraically solving the functional equation for $\zeta(1-z)$, thereby expressing it entirely in terms of $\Gamma(z)\zeta(z)$ and elementary factors defined on the right half-plane.

Analogously, the left half-plane values for the completed zeta function are accessed solely through its defining symmetry.

In both cases, the reflected region is reached without invoking analytic continuation via the identity theorem (R2), but purely through algebraic processes within the admissible z -space framework.

Corollary 2 (Functional continuation without the Identity Theorem). Beginning with the functional Equation (51), already valid for $\mathbb{C}[H_{1/2}] \setminus \{1\}$.

Then, by algebraic rearrangement and identities admissible under (R4), we obtain

$$\zeta(1-z) = 2^{1-z} \pi^{-z} \cos(\pi z/2) \Gamma(z) \zeta(z). \quad (57)$$

Thus ζ admits a meromorphic continuation to $\mathbb{C}[\overline{H}_{1/2}] \setminus \{1\}$ within the z -space framework by (R4) and (R5).

Proof. Starting from (51), on $\mathbb{C}[H_{1/2}] \setminus \{1\}$,

$$\zeta(z) = 2^z \pi^{z-1} \sin(\pi z/2) \Gamma(1-z) \zeta(1-z),$$

and solving for $\zeta(1-z)$:

$$\zeta(1-z) = \frac{\zeta(z)}{2^z \pi^{z-1} \sin(\pi z/2) \Gamma(1-z)}. \quad (58)$$

Apply Euler's reflection Formula (35),

$$\Gamma(z) \Gamma(1-z) = \pi / \sin(\pi z)$$

and the trigonometric identity

$$\sin(\pi z) = 2 \sin(\pi z/2) \cos(\pi z/2),$$

both admissible by (R4). These eliminate $\Gamma(1-z)$ from (58) and yield (57).

Every step uses global identities composed with z -space variables, so (R4) preserves validity on $\mathbb{C}[H_{1/2}] \setminus \{1\}$ and (R5) allows passage to the boundary $\mathbb{C}[\overline{H}_{1/2}] \setminus \{1\}$. $\square$

Corollary 3 (Access to the entire complex plane realized in z -space). *By Definition 1, $\zeta(z)$, via the η -quotient, is meromorphic of $\Re z \geq 1/2$ with at most a simple pole at $z = 1$. And $\zeta(1-z)$ converges for $1/2 \leq \Re z < 1$ (equivalently $0 \leq \Re(1-z) < 1/2$).*

Formula (57),

$$\zeta(1-z) = 2^{1-z} \pi^{-z} \cos\left(\frac{\pi z}{2}\right) \Gamma(z) \zeta(z),$$

expresses $\zeta(1-z)$ entirely in terms of data evaluated on the half-plane $\Re z > 1/2$, extending to the reflected region $\Re(1-z) < 1/2$ using only operations permitted by (R4). Thus (57) satisfies the access rule (R6), and reaches $z \in \mathbb{C}[\overline{H}_{1/2}] \setminus \{1\}$ completely within the z -space framework by (R4) and (R5), without invoking analytic continuation in s -space.

Definition 3 (Completed zeta in z -space). *For $\Re z > 1$ define*

$$\xi(z) := \frac{1}{2} z(z-1) \pi^{-z/2} \Gamma(z/2) \zeta(z). \quad (59)$$

Then ξ extends to be an entire function on $\mathbb{C}[\overline{H}_{1/2}]$ and satisfies the functional identity

$$\xi(z) = \xi(1-z). \quad (60)$$

Justification. The condition $\Re z > 1$ ensures that $\zeta(z)$ and $\Gamma(z/2)$ are holomorphic on the working half-plane $H_{1/2}$, so the product in (59) is well defined there by (R1)–(R3).

From Lemma 3, we have on $1 < \Re z < 2$,

$$\pi^{-\frac{z}{2}} \Gamma\left(\frac{z}{2}\right) \zeta(z) = \pi^{-\frac{1-z}{2}} \Gamma\left(\frac{1-z}{2}\right) \zeta(1-z).$$

Multiplying both sides by $\frac{1}{2} z(z-1)$, which cancels the simple pole at $z = 1$ coming from ζ and the pole at $z = 0$ from $\Gamma(z/2)$, we obtain

$$\xi(z) = \xi(1-z).$$

By (R2) and (R5) this symmetry extends meromorphically to all $z \in \mathbb{C}[\overline{H}_{1/2}]$. The trivial zeros of ζ at $z = 0, -2, -4, \dots$ cancel the further poles of $\Gamma(z/2)$ at these points, so ξ is entire.

Corollary 4 (Access to the reflected side from the right half-plane). Let $\xi(z)$ be defined as in Definition 3. Then for all $z \in \mathbb{C}[\overline{H}_{1/2}]$,

$$\xi(1-z) = \xi(z).$$

In particular, the value of ξ at the reflected point $1-z$ does not require any additional analytic information beyond what is already available in the right half-plane $\Re z \geq 1/2$. The completed zeta function evaluated at $1-z$ is already expressed in terms of the data used to define $\xi(z)$ on $\Re z > 1/2$, and the passage to the boundary $\Re z = 1/2$ is justified by (R5). Thus, within the z -space framework, the reflected half-plane is accessed entirely through formulas written on the right, in accordance with (R6), without any appeal to analytic continuation in the s -variable.

Corollary 5 (Restatement of Boundary equality on the critical line (R5)). Let $\xi(z) = \xi(1-z)$ be defined as in Definition 3. Then taking non-tangential limits from the right and from the left half-plane gives

$$\lim_{\Re z \downarrow 1/2} \xi(z) = \lim_{\Re(1-z) \uparrow 1/2} \xi(1-z) = \xi(1/2 + it), \quad t \in \mathbb{R}. \quad (61)$$

Thus, the analytic continuations from $\Re z > 1/2$ and from $\Re(1-z) < 1/2$ agree on the boundary, adjoining the two analytic pieces at the critical line, such that

$$\xi(z) = \xi(1-z), \quad \Re z = 1/2. \quad (62)$$

Proof. By Corollary 4, the equality $\xi(z) = \xi(1-z)$ holds on $\mathbb{C}[H_{1/2}]$.

Approaching the line $\Re z = 1/2$ from either side yields equal non-tangential boundary values on the whole line by (R5). Hence, ξ is continuous across the critical line. $\square$

Up to this point, the only visible effect of the z -space framework on the classical theory has been the controlled rearrangement of the functional equation in Corollary 2. This rearrangement alters the manner in which the divergent region is accessed, but it does not modify the underlying analytic structure of the theory. By the identity theorem, the analytic content remains unchanged and is therefore insensitive to the particular choice of complex variable.

Corollary 5 shows that the completed zeta function ξ is constructed precisely to accommodate this rearrangement. The next section goes further, revealing structural consequences for ξ through its Hadamard factorization. The following corollary and accompanying remark establish that ξ is entire of order one, thereby supplying the final prerequisite for the Hadamard factorization developed in Section 8.

Corollary 6 (Order one via theta–Mellin within z -space). Let $z = \varsigma(r) + it$ with $\varsigma(r) \geq 1/2$, then

$$\xi(z) = \frac{1}{2} z(z-1) \int_1^\infty (\theta(t) - 1) \left(t^{\frac{z}{2}} + t^{\frac{(1-z)}{2}} \right) \frac{dt}{t}. \quad (63)$$

is entire by Definition 3, and is accessible entirely within the z -space framework by (R4)–(R6).

The right side defines a holomorphic function for $\varsigma(r) > 1/2$, with non-tangential boundary limits at $\varsigma(r) = 1/2$ by (R5). Consequently, there exist constants $A, B > 0$ such that for every $z = \varsigma(r) + it$ with $\varsigma(r) \geq 1/2$,

$$|\xi(z)| \leq \exp(A(|\varsigma(r)| + |t|) \log(2 + |\varsigma(r)| + |t|) + B). \quad (64)$$

Using $\xi(1-z) = \xi(z)$ from (60), and generalizing both sides as $\xi(w)$, this bound holds for any $w \in \mathbb{C}$ by applying (63) to either $w = z$ if $\Re w \geq 1/2$ or to $w = 1-z$ if $\Re w < 1/2$. Hence ξ is of order one on $\mathbb{C}$.

Proof. Write $z = \varsigma + it$ with $\varsigma := \varsigma(r) \geq 1/2$. From (63) and $|\theta(t) - 1| \ll e^{-\pi t}$,

$$|\xi(z)| \leq \frac{1}{2} |z(z-1)| \int_1^\infty e^{-\pi t} \left(t^{\frac{\varsigma}{2}} + t^{\frac{(1-\varsigma)}{2}} \right) \frac{dt}{t}. \quad (65)$$

Make the change $t = e^{2u}$ (so $dt/t = 2 du$) and define

$$\Phi(u) := \frac{1}{2} (\theta(e^{2u}) - 1), \quad (66)$$

so that $|\Phi(u)| \ll 1$ as $u \downarrow 0$ and $|\Phi(u)| \ll \exp(-\pi e^{2u})$ as $u \rightarrow \infty$. Then (65) becomes

$$|\xi(z)| \leq |z(z-1)| \int_0^\infty |\Phi(u)| |e^{uz} + e^{u(1-z)}| du. \quad (67)$$

Since $\varsigma \geq 1/2$,

$$|e^{uz} + e^{u(1-z)}| \leq e^{u(\varsigma+|t|)} + e^{u(1-\varsigma+|t|)} \leq 2 e^{u(\max\{1, \varsigma\}+|t|)}.$$

Let

$$M := \log(2 + |t|). \quad (68)$$

On $0 \leq u \leq M$, we have $|\Phi(u)| \ll 1$, so

$$\int_0^M |\Phi(u)| e^{u(\max\{1, \varsigma\}+|t|)} du \ll \exp(M(\max\{1, \varsigma\} + |t|)) = \exp((\max\{1, \varsigma\} + |t|) \log(2 + |t|)). \quad (69)$$

On $u \geq M$: with $|\Phi(u)| \ll \exp(-\pi e^{2u})$,

$$\int_M^\infty |\Phi(u)| e^{u(\max\{1, \varsigma\}+|t|)} du \ll \int_M^\infty e^{u(\max\{1, \varsigma\}+|t|)-\pi e^{2u}} du \ll 1, \quad (70)$$

since for $u \geq M$ we have $e^{2u} \geq (2 + |t|)^2$ and the term $-\pi e^{2u}$ dominates.

Combining both ranges and using $|z(z-1)| \ll (|\varsigma| + |t|)^2$ we obtain (64).

Finally, for any $w \in \mathbb{C}$: if $\Re w \geq 1/2$, apply this bound directly with $w = z$; if $\Re w < 1/2$, apply (67) with $w = 1 - z$ and use $\xi(z) = \xi(1 - z)$. Thus ξ obeys an order-one growth bound on all $w \in \mathbb{C}$. $\square$

Remark 5 (Standard nature of the auxiliary estimates). *Every inequality used in the proof of Corollary 6 is independent of the z -space parameterization and of the access rules (R1)–(R6). Particularly the following examples:*

- The theta tail bound on $[1, \infty)$, $|\theta(t) - 1| \ll e^{-\pi t}$;
- The change of variables $t = e^{2u}$ with $dt/t = 2 du$ and the definitions $\Phi(u) = \frac{1}{2} (\theta(e^{2u}) - 1)$, together with the bounds $|\Phi(u)| \ll 1$ as $u \downarrow 0$ and $|\Phi(u)| \ll e^{-\pi e^{2u}}$ as $u \rightarrow \infty$;
- The elementary envelope $|e^{uz} + e^{u(1-z)}| \leq e^{u(\Re z + |\Im z|)} + e^{u(1-\Re z + |\Im z|)}$;
- The quadratic estimate $|z(z-1)| \ll (|\Re z| + |\Im z|)^2$.

These are all classical and hold for an arbitrary complex variable z . Thus, the growth bound in Corollary 6 rests only on standard analytic estimates, applied verbatim in the present setting.

Section 7 Summary (A structural explanation for the critical line via reflection-based access).

The rederivation of the classical analytic apparatus within the z -space framework makes it possible to construct the theta–Mellin identity and to obtain the functional properties of the completed zeta function entirely within this restricted setting.

This leads to a structural explanation for the location of the critical line: in z -space, the line $\Re z = 1/2$ emerges as an intrinsic analytic boundary. The function $\zeta(z)$ is confined to the half-plane $\Re z > 1/2$, and access to the region $\Re z < 1/2$ is permitted only through the reflective structure encoded in the completed zeta function and the functional equation (cf. Corollary 2).

Consequently, within z -space the critical line is not merely a descriptive feature of the functional equation, but a genuine analytic boundary, crossed only through the symmetry mechanism built into the theory.

8. The Completed Zeta Function for Order-One Use [1,6,8]

A second mechanism fixing the location of the critical line is developed in this section through the Hadamard factorization of the completed zeta function. As in Section 7, the critical line again emerges as an intrinsic boundary that cannot be crossed except through reflection. The Hadamard representation, however, reveals this constraint at a deeper structural level: the admissible configuration of zeros itself enforces the location of the boundary.

Since the Hadamard factorization is an explicit product over the nontrivial zeros of ξ , we begin by defining these zeros within the z -space framework. We then construct the corresponding z -space Hadamard representations and show that the zero set of the product consists only of nontrivial zeros.

Notation. The symbol ϱ (Greek rho-variant) denotes a nontrivial zero of ζ (equivalently of ξ) in z -space,

$$\xi(\varrho) = 0, \quad \Re \varrho \geq 1/2, \quad (71)$$

where “nontrivial” means that ϱ is not one of the zeros forced by the normalizing prefactor in Definition 3 (i.e., not $z = 1$ from ζ , nor $z = 0, -2, -4, \dots$ coming from the interaction of $\Gamma(z/2)$ with the trivial zeros of ζ). In other words, ϱ denotes a genuine zero of ξ arising from the analytic content of ζ , not from normalizing factors.

Write

$$\varrho := \varsigma(q) + i\gamma, \gamma \in \mathbb{R}, \quad (72)$$

so that its real part can be expressed in the geometric parameter of Section 3 by

$$\varsigma(q) = \sum_{k=0}^{\infty} q^k = \frac{1}{1-q}, q \in [-1, 1), \quad (73)$$

and the real parts of zeros may be indexed by the geometric variable q , with $q = -1$ corresponding to the boundary value $\varsigma(-1) = 1/2$ obtained via (R5).

The symbol ρ (standard rho) denotes the corresponding zero in s -space obtained by the analytic change of variables $z \mapsto s$ introduced in Section 3.

Definition 4 (The Hadamard form of the completed zeta function in z -space). *Since $\xi(z)$ is entire of order one (Corollary 6), Hadamard’s factorization theorem gives a global product for ξ . Within the z -space framework, where zeros are indexed only when they are accessible by $\Re \varrho \geq 1/2$, we write*

$$\xi(z) = \frac{1}{2} \prod_{\varrho} \left(1 - \frac{z}{\varrho}\right) \exp\left(\frac{z}{\varrho}\right), \quad \Re z, \Re \varrho \geq \frac{1}{2}. \quad (74)$$

This means we are evaluating the global Hadamard product on the right half of z -space, indexing zeros by those ϱ that lie in the accessible half-plane determined by the geometric parametrization of Section 3.

By the symmetry $\xi(1-z) = \xi(z)$ from (60), the same product can be written on the reflected side as

$$\xi(1-z) = \frac{1}{2} \prod_{\varrho} \left(1 - \frac{1-z}{\varrho}\right) \exp\left(\frac{1-z}{\varrho}\right), \quad \Re(1-z), \Re(1-\varrho) \leq \frac{1}{2}. \quad (75)$$

Justification. Because $\xi(z)$ is entire of order one, the general Hadamard's factorization has the form

$$\xi(z) = e^{A+Bz} \prod_q \left(1 - \frac{z}{q}\right) \exp\left(\frac{z}{q}\right), \quad (76)$$

with absolute and locally uniform convergence. The normalization $\xi(0) = 1/2$ gives $e^A = 1/2$, and the symmetry $\xi(z) = \xi(1-z)$ forces $B = 0$, yielding (74). The reflected form (75) is obtained via functional symmetry (60). By (R5), both representations admit non-tangential limits to $\Re z = 1/2$, as in Corollary 5.

Corollary 7 (Nontrivial zeros and the z -space Hadamard product). *Let $\xi(z)$ be defined as in Definition 3,*

$$\xi(z) := \frac{1}{2} z(z-1) \pi^{-z/2} \Gamma(z/2) \zeta(z), \quad \Re z > 1.$$

Then every zero that appears in the Hadamard factorization of $\xi(z)$ in Definition 4 is a non-trivial zero of ζ ; conversely, every nontrivial zero of ζ appears as a zero of ξ . Thus, the zero set of the z -space Hadamard product coincides only with the nontrivial zeros.

Proof. By Definition 3, the prefactor

$$\frac{1}{2} z(z-1) \pi^{-z/2} \Gamma(z/2)$$

is chosen so that the following properties are satisfied:

1. The simple pole of $\zeta(z)$ at $z = 1$ and the pole of $\Gamma(z/2)$ at $z = 0$ are both removed by the factor $z(z-1)$;
2. The poles of $\Gamma(z/2)$ at $z = -2, -4, -6, \dots$ are canceled by the trivial zeros of $\zeta(z)$ at those same negative even integers;
3. The factor $\pi^{-z/2}$ never vanishes and therefore introduces no new zeros.

Hence $\xi(z)$ has no zeros coming from the normalizing prefactor; forced singularities and trivial zero are cancelled at the level of ξ . What remains are the zeros coming from $\zeta(z)$ that are not among its trivial zeros.

In Definition 4 we express $\xi(z)$ by its Hadamard product,

$$\xi(z) = \frac{1}{2} \prod_q \left(1 - \frac{z}{q}\right) \exp\left(\frac{z}{q}\right), \quad \Re z, \Re q \geq \frac{1}{2},$$

so this product enumerates only the zeros of $\xi(z)$, which are the zeros of $\zeta(z)$. Therefore, any statements about the zeros of the z -space Hadamard product are equivalent to statements about the nontrivial zeros of ζ . $\square$

The remainder of the section develops the governing principles that shape both the z -domain and the classical s -domain, and records the structural consequences of the reparameterization introduced in Section 3. We examine the internal consistency of the Hadamard factorization imposed by the functional symmetry of the completed zeta function, and we analyze the effect of transporting this structure from z -space to s -space through a holomorphic change of variables.

Definition 5 (Zero set in z -space). *Let*

$$Z_z(\xi) := \{q \in \mathbb{C} : \xi(q) = 0 \text{ and } \Re q \geq 1/2\} \quad (77)$$

denote the set of admissible nontrivial zeros of ξ indexed in z -space. The subscript z indicates that the set is indexed according to the z -space access rules.

Lemma 5 (The z -space Hadamard structure on the location of admissible zeros). *The Hadamard representations (74) and (75) are two canonical factorizations that both represent the same entire function ξ , each written using zeros indexed according to accessibility from the right and left half-planes of z -space (right: $\Re z \geq 1/2$, left: $\Re(1 - z) \leq 1/2$). Under this z -space framework, consistency of the two representations forces all admissible nontrivial zeros to lie on the common boundary*

$$\Re \rho = 1/2. \quad (78)$$

Accordingly, the shared countable set of factors appearing in both products is

$$Z_z(\xi) := \{\rho: \xi(\rho) = 0 \text{ and } \Re \rho = 1/2\}. \quad (79)$$

Proof. Because ξ is entire of order one, it admits a Hadamard factorization that is unique up to normalization, and the product forms (74) and (75) represent this same entire function on overlapping analytic domains.

Assume, for contradiction, that there exists an admissible nontrivial zero ρ with $\Re \rho > 1/2$. By functional symmetry $\xi(z) = \xi(1 - z)$ from (60), the reflected point $1 - \rho$ is also a zero.

The right-hand Hadamard product (74) includes only zeros that are accessible in the half-plane $\Re \rho \geq 1/2$, in accordance with the geometric parametrization $\varsigma(q) = \sum_{k=0}^{\infty} q^k$. Conversely, the reflected product (75) includes only those zeros accessible from $\Re(1 - \rho) \leq 1/2$ ($\Re \rho \geq 1/2$). The two Hadamard products represent the same entire function ξ ; yet, if any zero ρ lay strictly off the critical line, the two products would enumerate non-matching zero sets, contradicting uniqueness under the identity theorem (R2).

Therefore, structural consistency is maintained only if the two products coincide term-by-term, which occurs precisely when all admissible nontrivial zeros lie on the shared boundary line $\Re \rho = 1/2$. Hence,

$$Z_z(\xi) = \{\rho: \xi(\rho) = 0 \text{ and } \Re \rho = 1/2\}.$$

□

Lemma 5 alone does not assert the nonexistence of off-line zeros in the classical s -plane. Rather, it shows that such zeros cannot be indexed consistently within the z -space Hadamard framework: any zero with $\Re \rho \neq 1/2$ would prevent the two z -space Hadamard representations from defining the same entire function ξ . Hence the z -space Hadamard product enumerates only boundary zeros.

The next lemma strengthens this conclusion by showing that this boundary zero set is in fact the global zero set of ξ in z -space, by linking it directly to the classical zero set in the s -variable. The subsequent corollary then presents the same globality property in a more intuitive form.

Lemma 6 (Globality of the z -space Hadamard product under holomorphic transport). *Let the Hadamard product in z -space be defined as in (74) and (75). By Lemma 5, (79) is the admissible zero set in z -space*

$$Z_z(\xi) := \{\rho: \xi(\rho) = 0 \text{ and } \Re \rho = 1/2\}.$$

Define the classical zero set in the s -variable by

$$Z_s(\xi) := \{\rho: \xi(\rho) = 0\}, \quad (80)$$

that is, the set of all nontrivial zeros of $\xi(s)$ enumerated by its Hadamard product.

Using (73), let

$$z \mapsto s = s(q) := \frac{1}{1-q} + it$$

be the holomorphic change of variables introduced in Section 3. Composing the z -space Hadamard factorization with this map yields

$$\xi(s) = \frac{1}{2} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) \exp\left(\frac{s}{\rho}\right),$$

which coincides with the classical Hadamard product (10) for $\xi(s)$.

Then the transported product is global, and the zero sets coincide:

$$Z_s(\xi) = Z_z(\xi). \quad (81)$$

In particular, every nontrivial zero satisfies (78),

$$\Re \rho = 1/2.$$

Proof. Lemma 5 shows that factorizations (74) and (75) are mutually consistent only when the admissible zero set is exactly $Z_z(\xi)$, with all zeros satisfying $\Re \rho = 1/2$.

The map

$$z \mapsto s(q) = \frac{1}{1-q} + it$$

is holomorphic on the domain of the geometric parameter. By Rule (R4), composing an identity of holomorphic functions with a holomorphic change of variables preserves the identity and therefore preserves the zero set with multiplicity. In particular, a holomorphic change of variables cannot introduce, remove, relocate, or change the multiplicity of zeros.

Since the classical Hadamard product for $\xi(s)$ enumerates all nontrivial zeros of ξ , the image of the z -space Hadamard product under this holomorphic transport must already be global. Consequently,

$$Z_s(\xi) = Z_z(\xi),$$

refs. [2,12] and every nontrivial zero ρ of ξ satisfies

$$\Re \rho = 1/2.$$

□

Corollary 8 (Intuitive globality of the z -space Hadamard product). *Within the z -space framework, every complex input*

$$w \in \mathbb{C} \quad (82)$$

is evaluable using only right-half-plane data. Consequently, the Hadamard product constructed in z -space (Definition 4) represents the completed zeta function ξ globally on $\mathbb{C}$.

Proof. Let $w \in \mathbb{C}$ be arbitrary. The following conditions apply:

- If $w \in \overline{H}_{1/2}$, then $\xi(w)$ is evaluated directly by the z -space construction.
- If $w \notin \overline{H}_{1/2}$, then there exists $z \in \overline{H}_{1/2}$ such that $w = 1 - z$.

By the functional symmetry of $\xi(z) = \xi(1 - z)$ (Corollary 4) and the boundary rule (R5), we obtain

$$\xi(w) = \xi(1 - z) = \xi(z). \quad (83)$$

Thus, every value $\xi(w)$ is computed through formulas written entirely in terms of right-half-plane data, consistent with the access rule (R6). Although the z -space rulebook

restricts where computations are performed, it does not restrict which inputs are ultimately evaluable.

Now, by Definition 4, the Hadamard product constructed in z -space represents ξ on the nonempty open domain $H_{1/2}$. Since ξ is entire, the identity theorem (R2) implies that this representation uniquely determines ξ on all of $\mathbb{C}$. Hence the z -space Hadamard product is global: it encodes the same entire function ξ on the whole complex plane, even with its construction and admissible manipulations being carried out on $H_{1/2}$. $\square$

Section 8 Summary. (How z -space fixes the zero structure of s -space).

Section 8 establishes that the zero structure of the classical variable s is not free, but is governed by the structural rules of z -space.

First, Lemma 5 shows that within z -space the Hadamard product of ξ can be internally consistent only if all admissible nontrivial zeros lie on the boundary

$$\Re z = 1/2.$$

Any zero away from this line would render the right-hand and left-hand z -space factorizations incompatible, and therefore impossible to represent the same entire function. Thus, in z -space the location of the nontrivial zeros is not descriptive but enforced.

Second, Lemma 6 shows that this enforced zero set is global. The holomorphic change of variables $z \mapsto s$ transports the z -space Hadamard product into the classical Hadamard product for $\xi(s)$. Because holomorphic changes of variables preserve analytic identities and zero sets, no zeros can be created, removed, relocated, or have their multiplicities altered in passing from z -space to s -space. Consequently, the complete nontrivial zero set of $\xi(s)$ is identical to the zero set fixed in z -space.

Therefore, the analytic structure of s -space is constrained by the rules that become explicit in z -space. Although this constraint is invisible when working directly in the classical variable, it is structurally unavoidable once the underlying geometry is made clear.

In this way, Section 8 shows that all nontrivial zeros are globally forced onto the critical line in both z -space and s -space, and the critical line $\Re s = 1/2$ is not an incidental feature of the functional equation, but the only location compatible with the global analytic structure of ξ .

9. Conclusions

Sections 3–8 Synthesis. (Structural interpretation within z -space and conclusion of Theorem 1). *We now combine Sections 3–8 to prove Theorem 1.*

Proof. On the working domain $\Re z \geq 1/2$, the identity $\xi(z) = \xi(1-z)$ from (60) gives reflection symmetry about the vertical line $\Re z = 1/2$. By the geometric reparameterization of Section 3, this line is not arbitrary, it is exactly the boundary value $\varsigma(-1) = 1/2$ of (15),

$$\varsigma(r) = \sum_{k=0}^{\infty} r^k = \frac{1}{1-r}, \quad r \in [-1, 1),$$

so $\Re z = 1/2$ is the unique vertical boundary coming from the image of the geometric parameter.

Corollaries 2 and 4 show that the reflected half-plane is accessed solely through formulas written in terms of right-half-plane data, in accordance with the access rule (R6). Thus $\Re z = 1/2$ is the sharp cutoff, impassable except by functional symmetry.

Lemma 5 shows that the z -space Hadamard structure of ξ enforces this same axis analytically: compatibility of the right- and left-half Hadamard products on their overlap requires all admissible zeros to satisfy $\Re \rho = 1/2$. Lemma 6 then confirms that this constraint is preserved under the holomorphic change of variables $z \mapsto s$: the z -space access

rules (R6) and boundary matching (R5) are inherited by s -space, even though the geometric origin of the symmetry is no longer explicit there.

Hence, the geometric and analytic structures coincide, and the critical line $\Re z = 1/2$ is fixed uniquely as the axis of symmetry in z -space, manifesting in s -space as the critical line.

This completes the proof of Theorem 1. $\square$

Remark 6 (Generality of z -space and scope of the framework). *The complex variable z used in this paper is not tied to any single function. It is produced by geometric reparameterization of the real part*

$$z = \zeta(r) + it, \quad \zeta(r) = \sum_{k=0}^{\infty} r^k = \frac{1}{1-r}, \quad r \in [-1, 1),$$

which forces $\Re z \geq 1/2$ with boundary value $\zeta(-1) = 1/2$ (Abel/Cesàro). All the analytic arguments in Sections 3–8 use only this geometric origin of z together with the access rules of Section 4, not any special property of ζ . Thus, the same scheme applies to any function Ξ satisfying the conditions below.

Geometric parameterization: There is a representation $z = \zeta(r) + it$ with $\zeta(r) = 1/(1-r)$, $r \in [-1, 1)$, so that $\Re z \geq 1/2$ and $\Re z = 1/2$ is a genuine endpoint obtained by Abel/Cesàro. Analytic work is conducted in $H_{1/2} = \{\Re z > 1/2\}$; the boundary $\Re z = 1/2$ is used only for non-tangential limits, as in (R5).

Reflection accessed from the right: The function Ξ satisfies a symmetry of the form

$$\Xi(z) = \varepsilon \Xi(1-z), \quad |\varepsilon| = 1,$$

and the term $\Xi(1-z)$ is obtained only by writing it in terms of right-half data (values of $\Xi(z)$, admissible Gamma factors, and global identities) in accordance with (R1)–(R6). This “access-from-the-right” condition makes $\Re z = 1/2$ the non-movable boundary of the construction; replacing $1-z$ by $\alpha-z$ does not change the geometric cutoff $\Re z \geq 1/2$.

Optional Hadamard strengthening: If, in addition, Ξ is entire of order one and admits a canonical Hadamard product, then the right-hand and reflected products can coincide on their overlap only when they enumerate the same zeros. Under the z -space admissibility restriction, this forces all nontrivial zeros (i.e., zeros not coming from normalizing prefactors) to lie on $\Re z = 1/2$.

Preservation under the s -variable: For the holomorphic change of variables

$$z \mapsto s(r) = \frac{1}{1-r} + it,$$

allowed by (R4), the boundary $\Re z = 1/2$ is carried to $\Re s = 1/2$, so both the access property and, when present, the zero-location property persist in s -space.

In this sense, “ z -space” is a general analytic device for functions whose real part can be written via the geometric series. Within this device, $\Re z = 1/2$ is an intrinsic access boundary, and—when order-one growth and Hadamard factorization are available—the same boundary is enforced by the zero structure. Thus, we may apply the framework to any function that satisfies the same geometric access and symmetry conditions, so that the critical line appears as an intrinsic boundary rather than a merely descriptive one [3].

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