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Weyl-Type Symmetry and Subalgebra Rigidity in von Neumann Algebras

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Abstract

We propose and develop a unified framework for *Weyl-type symmetry* in von Neumann algebras. Motivated by recent automorphism-rigidity phenomena that identify finite Weyl groups inside automorphism groups of crossed products arising from lattice actions on homogeneous spaces, we introduce the *Weyl group of an inclusion* $W(M; B) := \text{Aut}_B(M) / \text{Inn}_B(M)$, for a unital inclusion $B \subset M$ of von Neumann algebras, and investigate its structure across several rigidity regimes. Our main results (1) prove finiteness or triviality of $W(M; B)$ for large classes of nonamenable crossed products, including hyperbolic and product-type actions with spectral gap and malleability; (2) establish a subgroup-normalizer rigidity principle for inclusions $L(\Lambda) \subset L(\Gamma)$ that identifies $\text{Aut}_{L(\Lambda)}(L(\Gamma))$ with a discrete group controlled by $N_\Gamma(\Lambda)$; (3) show that permutation-type symmetry for product/tensor decompositions is the only possible nontrivial symmetry of the underlying group subalgebras; and (4) extend the analysis to type III factors via Maharam extensions and unique-Cartan phenomena, proving that $W(M; B)$ is discrete and often trivial, leaving only modular flows as outer symmetries. Consequences include new computations of outer automorphism groups, constraints on intermediate subalgebras, and classification consequences for crossed products and amalgamated free products. The methods combine Popa's intertwining-by-bimodules, spectral-gap and s-malleable deformations, boundary/ucp-map rigidity, and groupoid/Cartan techniques.

Keywords: von Neumann algebra; automorphism rigidity; Weyl group; Cartan subalgebra; crossed product; deformation/rigidity; outer automorphism group; type III factor; measured groupoid

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1. Introduction

Automorphism rigidity has emerged as a central theme in the structure theory of von Neumann algebras generated by group actions. A recent breakthrough by Houdayer and Ioana shows that for a noncompact semisimple algebraic group G with maximal k -split torus $S < G$ and centralizer $H = Z_G(S)$, and for an irreducible lattice $\Gamma < G$ with $\Lambda = \Gamma / Z(\Gamma)$, the inclusion

$$L(\Lambda) \subset L(\Lambda \curvearrowright G/H)$$

retains the *Weyl group* $W_G = N_G(S) / Z_G(S)$ in the precise sense that the canonical homomorphism

$$\rho : W_G \longrightarrow \text{Aut}_{L(\Lambda)}(L(\Lambda \curvearrowright G/H)), \quad w \mapsto \Theta_w,$$

is an isomorphism; in particular, $\text{Aut}_{L(\Lambda)}(M) \cong W_G$ and (by irreducibility) $\text{Out}_{L(\Lambda)}(M) \cong W_G$ [1]. This provides a noncommutative analogue of the Bader–Furman–Gorodnik–Weiss rigidity for nonsingular automorphisms of G/H [2].

Beyond its intrinsic interest, this result interacts with the noncommutative factor theorem of Boutonnet–Houdayer (which describes intermediate algebras $L(\Lambda) \subset L(\Lambda \curvearrowright G/P)$ under higher-rank hypotheses) and yields new evidence toward Connes’s rigidity conjecture for higher-rank lattices [3]. In particular, any isomorphism $L(\Lambda_1 \curvearrowright G_1/H_1) \simeq L(\Lambda_2 \curvearrowright G_2/H_2)$ that maps $L(\Lambda_1)$ onto $L(\Lambda_2)$ induces an isomorphism $W_{G_1} \simeq W_{G_2}$, highlighting that W_G is an operator-algebraic invariant of the pair [1].

Motivated by these developments, we introduce and systematically study a *Weyl-type symmetry* attached to a general unital inclusion $B \subset M$ of von Neumann algebras,

$$W(M; B) := \text{Aut}_B(M) / \text{Inn}_B(M),$$

and we develop a deformation/rigidity framework that yields finiteness (often triviality) of $W(M; B)$ in broad classes of nonamenable crossed products. Our viewpoint unifies and extends the identification $\text{Aut}_{L(\Lambda)}(L(\Lambda \curvearrowright G/H)) \cong W_G$ in the homogeneous setting [1] with rigidity phenomena for Cartan pairs, subgroup inclusions $L(\Lambda) \subset L(\Gamma)$, and product/tensor decompositions.

Given a free ergodic (p.m.p. or strongly ergodic) action $\Gamma \curvearrowright (X, \mu)$ and $M = L^\infty(X) \rtimes \Gamma$, two natural choices of B are $B = L^\infty(X)$ (Cartan) and $B = L(\Gamma)$ (group subalgebra). The homogeneous model of [1] identifies $W(M; L(\Lambda))$ with W_G by exploiting three pillars: (i) rigidity of equivariant normal UCP maps reducing to convex combinations of Weyl translates; (ii) a noncommutative convexity/maximality argument reminiscent of Popa’s intertwining-by-bimodules; and (iii) essential freeness of the $\Lambda \times W_G$ -action on G/H to force unitaries into the Cartan and kill residual symmetry. We ask to what extent analogous mechanisms constrain $\text{Aut}_B(M)$ for other rigid actions and inclusions; cf. the deformation/rigidity toolkit surveyed in [4].

In the current paper, we prove the following results; precise statements appear in Sections 3–6.

- A. (*Finiteness/triviality for nonamenable crossed products*). For large classes of nonamenable, spectrally gapped, malleable actions (e.g., Bernoulli/Gaussian and many hyperbolic or product actions), we show $W(M; B)$ is finite for $B \in \{L^\infty(X), L(\Gamma)\}$, and often $W(M; B) = \{e\}$. Our argument abstracts the UCP-map rigidity in [1] and then imports Popa’s convexity/intertwining method to upgrade convex decompositions into algebraic normalizers [4].
- B. (*Subgroup-normalizer rigidity*). For malnormal (or almost malnormal) $\Lambda < \Gamma$ in a Popa-rigid regime, we prove

$$\text{Aut}_{L(\Lambda)}(L(\Gamma)) \cong N_\Gamma(\Lambda) / \Lambda,$$

so $W(L(\Gamma); L(\Lambda))$ is discrete and often trivial, paralleling the Weyl computation in the homogeneous case [1].

- C. (*Permutation rigidity for product/tensor decompositions*). For product actions and tensor products, any automorphism preserving the set of group subalgebras can only implement the obvious finite permutation (typically a transposition); hence, W is either trivial or $\mathbb{Z}_2$. This rules out exotic factor-swapping beyond the visible symmetry.
- D. (*Type III extensions via Maharam*). For nonsingular actions with unique Cartan and spectral gap on the Maharam extension, we show $W(M; B)$ is discrete for $B \in \{L^\infty(X), L(\Gamma)\}$; in many cases $W(M; B) = \{e\}$. Consequently, $\text{Out}(M)$ is gener-

ated by the modular flow together with a finite group from $W(M; B)$ —often reducing to $\mathbb{R}$ or the trivial group, in line with the strategy initiated in [1].

Two technical ingredients deserve emphasis. First, we formulate an *equivariant UCP-map rigidity schema* that upgrades to finite convex hulls of explicit symmetries, generalizing the homogeneous-space case [1,2]. Second, we devise a *maximality-by-convexity* device in Popa’s spirit to manufacture partial isometries implementing the desired conjugacies and to force normalizers to live where essential freeness allows—mirroring the proof architecture behind [1,4].

Our results complement the noncommutative factor theorem of [3] (which retains the poset of parabolics inside $L(\Lambda \curvearrowright G/P)$ in higher rank) by showing that the larger inclusion $L(\Lambda) \subset L(\Lambda \curvearrowright G/H)$ retains *Weyl-type* symmetry without rank assumptions [1]. They also feed into structural constraints on $\text{Out}(M)$ and the fundamental group and into diagonal-preserving classification of crossed products; see Sections 8–10.

Section 2 recalls preliminaries on Weyl groups, group measure-space factors, and Popa’s deformation/rigidity toolkit. Section 3 introduces $W(M; B)$ and proves finiteness/triviality results for spectrally gapped malleable actions. Section 4 establishes subgroup-normalizer rigidity in group factors. Section 5 treats permutation rigidity for product/tensor situations. Section 6 develops the type III extension via Maharam, leading to outer automorphism computations. In Section 7, we study the groupoid/Cartan viewpoint and computations. Section 8 discusses consequences and applications to classification. Section 9 is allocated to proof strategies, and Section 10 presents examples, counterexamples, and open problems.

We write $\text{Aut}_B(M) = \{\Theta \in \text{Aut}(M) \mid \Theta|_B = \text{id}\}$ and $\text{Inn}_B(M) = \{\text{Ad}(u) \mid u \in \mathcal{U}(B)\}$. For a locally compact group R and $L < R$, we use $N_R(L)$ for the normalizer. For a measurable action $G \curvearrowright Z$, we denote by $\text{Prob}(Z)$ the Borel probability measures on Z .

We notice that many of the structural results in this paper for nonsingular actions and outer automorphism groups are obtained by passing to the Maharam (or core) semifinite lift and then applying semifinite techniques. While this is a standard and powerful method, it has an important limitation: certain phenomena that are intrinsically tied to the purely type III modular data need not persist in the semifinite lift, so they may be invisible to arguments that rely solely on the Maharam extension.

Concretely, automorphisms which act nontrivially on the flow of weights (or on other modular invariants) can produce distinctions in $\text{Out}(M)$ that the core fails to detect, and conversely there exist modular effects (for instance arising from almost periodic states or nontrivial scaling behavior) that are collapsed or regularized when passing to the semifinite picture. For this reason, the Maharam-based results presented here should be understood as describing the semifinite shadow of outer structure; a complementary, intrinsic type III analysis is in some cases necessary to reach sharper conclusions about $\text{Out}(M)$ in the nonsingular realm. We discuss below several routes to address these issues and indicate examples and hypotheses under which our Maharam-method conclusions do (or do not) extend.

2. Preliminaries

This section collects background and fixes notation on Weyl groups and homogeneous spaces, group measure-space constructions, deformation/rigidity tools, equivariant UCP maps, and the Maharam extension. Throughout, all von Neumann algebras are σ -finite, and all actions are measurable and standard.

Let k be a local field of characteristic 0 (i.e., $k = \mathbb{R}, \mathbb{C}$ or a finite extension of $\mathbb{Q}_p$). Let G be a Zariski-connected, simply connected, k -isotropic, almost k -simple algebraic

k -group [5,6]. Let $S < G$ be a maximal k -split torus and $H = Z_G(S)$ its centralizer. The Weyl group of G (relative to k) is

$$W_G := N_G(S)/Z_G(S).$$

When $G = \mathrm{SL}_n$ ($n \geq 2$) and S the diagonal torus, $W_G \simeq \mathfrak{S}_n$. The homogeneous space G/H carries a G -invariant σ -finite infinite measure whose class is the unique G -invariant class (both G and H are unimodular here), see Corollary B.1.7 of [7]. For an irreducible lattice $\Gamma < G$, Moore's ergodicity implies that $\Gamma \curvearrowright G/H$ is ergodic, and $\Lambda := \Gamma/Z(\Gamma)$ acts essentially freely in the settings considered in [1].

Remark 1. A key structural fact is that

$$\mathrm{Aut}_G(G/H) \simeq W_G,$$

and, more generally, any Γ -equivariant nonsingular automorphism of G/H must be G -equivariant when $\Gamma < G$ is an irreducible lattice in a product of almost simple groups; see Theorem 1.6 of [2]. This identification is at the core of the Weyl-symmetry phenomenon in the operator-algebraic setting developed in [1].

Let $\Gamma \curvearrowright (X, \mu)$ be a free ergodic nonsingular action on a standard measure space. The (right) Koopman action on $A := L^\infty(X)$ is denoted by $\sigma : \Gamma \rightarrow \mathrm{Aut}(A)$, and the group measure-space (or crossed product) von Neumann algebra is

$$M := A \rtimes_\sigma \Gamma = \left\{ \sum_{\gamma \in \Gamma} a_\gamma u_\gamma : a_\gamma \in A, \sum_{\gamma} \|a_\gamma\|_2^2 < \infty \right\}''$$

with multiplication $(au_\gamma)(bu_\delta) = a\sigma_\gamma(b)u_{\gamma\delta}$ and u_γ the canonical unitaries. There is a canonical faithful normal conditional expectation $E_A : M \rightarrow A$ given by $E_A(\sum a_\gamma u_\gamma) = a_e$. When μ is finite and the action is p.m.p., M is a II_1 factor if Γ is icc. For nonsingular actions, M is typically type III; one can still realize E_A with respect to a reference σ -finite measure and use semifinite weights.

Throughout, we write $A = L^\infty(X, \mu)$ for the abelian von Neumann algebra of essentially bounded measurable functions on the standard measure space (X, μ) . The algebra A acts canonically on the Hilbert space $L^2(X, \mu)$ by multiplication,

$$(a\xi)(x) = a(x)\xi(x), \quad a \in L^\infty(X), \xi \in L^2(X),$$

and all L^2 -norms $\|\cdot\|_2$ appearing below are taken with respect to this representation. Thus, the use of the L^2 -norm is intrinsic to the standard Hilbert space realization of the group measure space construction and is compatible with the underlying algebra $A = L^\infty(X)$.

Definition 1. For a unital inclusion $B \subset M$, set

$$\mathrm{Aut}_B(M) := \{\Theta \in \mathrm{Aut}(M) \mid \Theta|_B = \mathrm{id}\}, \quad \mathrm{Inn}_B(M) := \{\mathrm{Ad}(u) \mid u \in \mathcal{U}(B)\}.$$

We define the Weyl-type symmetry group of the inclusion $(B \subset M)$ by

$$W(M; B) := \mathrm{Aut}_B(M)/\mathrm{Inn}_B(M).$$

In the homogeneous setting of [1], with $M = L(\Lambda \curvearrowright G/H)$ and $B = L(\Lambda)$, one has a canonical isomorphism $W(M; B) \simeq W_G$, see Theorem A of [1].

We recall two pillars of Popa's deformation/rigidity theory [4].

For subalgebras $A, B \subset M$, we say that A embeds into B inside M , and we write $A \preceq_M B$, if there exist $n \in \mathbb{N}$, a nonzero partial isometry $v \in \mathbf{M}_{1,n}(\mathbb{C}) \overline{\otimes} M$, and a normal $*$ -homomorphism $\phi : A \rightarrow \mathbf{M}_n(\mathbb{C}) \overline{\otimes} B$ such that $avs. = vs.\phi(a)$ for all $a \in A$. Equivalently, no sequence of unitaries in A can asymptotically centralize B through E_B .

A subalgebra $B \subset M$ has *spectral gap* relative to a deformation $(\alpha_t)_{t \in \mathbb{R}} \subset \text{Aut}(M)$ if, for every bounded net $(x_i) \subset M$ with $\|x_i - \alpha_t(x_i)\|_2 \rightarrow 0$ uniformly on compact t -sets, one has $\|x_i - E_{B' \cap M}(x_i)\|_2 \rightarrow 0$. In practice, malleable deformations (e.g., Bernoulli s -malleable) provide spectral gap for rigid subalgebras [4].

These principles allow one to convert analytic control (via deformations) into algebraic control of normalizers and relative commutants.

We will use a standard correspondence between equivariant normal ucp maps and measurable fields of probability measures (Markov kernels).

Lemma 1. *Let L be an lcsc group acting nonsingularly on standard spaces (X, ν_X) and (Y, ν_Y) . For any normal unital completely positive map $\Phi : L^\infty(Y) \rightarrow L^\infty(X)$ that is L -equivariant ($\Phi \circ \sigma_\ell^Y = \sigma_\ell^X \circ \Phi$), there exists an essentially unique measurable map (kernel) $\beta : X \rightarrow \text{Prob}(Y)$, $x \mapsto \beta_x$, such that*

$$\Phi(f)(x) = \int_Y f(y) d\beta_x(y) \quad \text{for a.e. } x \in X, \forall f \in C_0(Y),$$

and $\beta_{\ell \cdot x} = (\ell \cdot)_* \beta_x$ for all $\ell \in L$. Conversely, any such kernel yields a normal L -equivariant ucp map.

Proof. Let Z be the one-point compactification of Y with point ∞ and let ν_Z be the measure extending ν_Y by $\nu_Z(\{\infty\}) = 0$. Then $L^\infty(Y) = L^\infty(Z)$ and $C_0(Y) \subset C(Z)$ canonically.

By normality, Φ is determined by its values on $C(Z)$. By Riesz representation, for a.e. $x \in X$ there is a Borel probability $\tilde{\beta}_x \in \text{Prob}(Z)$ so that $\Phi(g)(x) = \int_Z g d\tilde{\beta}_x$ for every $g \in C(Z)$, and the assignment is measurable.

Choose an increasing approximate unit $\{f_n\} \subset C_0(Y)$ with $0 \leq f_n \leq 1$ and $f_n \uparrow 1_Y$. Normality gives $\Phi(f_n) \uparrow 1_X$ in L^∞ ; hence, for a conull set of x , one has $\int f_n d\tilde{\beta}_x \uparrow 1$. Since $f_n \leq 1_Y$ and $\mathbf{1}_{\{\infty\}} = 1_Z - \mathbf{1}_Y$ is lower semicontinuous, we conclude $\tilde{\beta}_x(\{\infty\}) = 0$ a.e. Thus, $\tilde{\beta}_x$ restricts to a probability β_x on Y with the stated formula for $f \in C_0(Y)$.

L -equivariance of Φ implies, for $g \in C(Z)$ and $\ell \in L$,

$$\int_Z g d\tilde{\beta}_{\ell \cdot x} = \Phi(g)(\ell \cdot x) = \Phi(g \circ \ell^{-1})(x) = \int_Z g \circ \ell^{-1} d\tilde{\beta}_x,$$

hence $\tilde{\beta}_{\ell \cdot x} = (\ell \cdot)_* \tilde{\beta}_x$ and therefore $\beta_{\ell \cdot x} = (\ell \cdot)_* \beta_x$ on Y .

Uniqueness follows since $C_0(Y)$ separates points and determines measures. Conversely, the formula defines a normal ucp map that is L -equivariant by construction. $\square$

In the homogeneous context, this correspondence combines with [2] to show that any Λ -equivariant normal ucp endomorphism of $L^\infty(G/H)$ is a finite convex combination of Weyl translates; see Theorem 2.5 of [1] for a precise formulation and its operator-algebraic exploitation.

Let $\Gamma \curvearrowright (X, \mu)$ be nonsingular. Define the Radon–Nikodym cocycle $\omega : \Gamma \times X \rightarrow \mathbb{R}$ by

$$\frac{d\mu \circ \gamma^{-1}}{d\mu}(x) = e^{-\omega(\gamma, x)}.$$

For each $\gamma \in \Gamma$ we denote by

$$\omega(\gamma, x) = \frac{d(\mu \circ \gamma^{-1})}{d\mu}(x)$$

the associated Radon–Nikodym derivative. The map $\omega : \Gamma \times X \rightarrow \mathbb{R}_+^\times$ satisfies the cocycle identity

$$\omega(\gamma_1 \gamma_2, x) = \omega(\gamma_1, \gamma_2 x) \omega(\gamma_2, x), \quad \gamma_1, \gamma_2 \in \Gamma, x \in X, \quad (1)$$

which follows from the chain rule for Radon–Nikodym derivatives.

The *Maharam extension* is the p.m.p. action $\Gamma \curvearrowright (X \times \mathbb{R}, \mu \otimes e^{-t} dt)$ given by

$$\gamma \cdot (x, t) := (\gamma \cdot x, t + \omega(\gamma, x)).$$

This allows one to transfer II_1 -type deformation/rigidity arguments to the type III case by working in $M \overline{\otimes} B(L^2(\mathbb{R}))$ and then projecting back along modular theory. In many classes (e.g., when the extension is mixing with spectral gap), uniqueness of Cartan and control of outer automorphisms can be achieved; see, e.g., [1,4] for the strategy in related contexts.

We record for later use two facts from the homogeneous setting of [1]. First, $\Lambda \curvearrowright G/H$ is ergodic by Moore’s theorem when H is noncompact and $\Gamma < G$ is irreducible [6]. Second, the product action $(\Lambda \times W_G) \curvearrowright G/H$ is essentially free; this is verified via an algebraic subvariety argument using Zariski density and the connectedness of G (see Lemma 2.3 of [1]). Essential freeness will be invoked to force implementing unitaries into prescribed subalgebras in our convexity/maximality arguments.

We use $A \subset M$ for the Cartan inclusion in a group measure-space factor, E_A for the canonical expectation, and u_γ for the canonical unitaries. For a pair $B \subset M$, we denote by $\mathcal{N}_M(B) = \{u \in \mathcal{U}(M) : uBu^* = B\}$ the normalizer and by $B' \cap M$ the relative commutant. When referencing the homogeneous model, we keep the notation of [1].

3. Finiteness and Triviality of $W(M; B)$ for Spectrally Gapped Malleable Actions

In this section, we establish finiteness, and in many cases triviality, of the Weyl-type symmetry group

$$W(M; B) := \text{Aut}_B(M) / \text{Inn}_B(M)$$

for broad classes of nonamenable crossed products $M = A \rtimes \Gamma$ arising from actions with spectral gap and malleability. Our proofs abstract the *ucp-map rigidity* $\Rightarrow$ *finite convex hull of explicit symmetries* mechanism and the *maximality-by-convexity* argument that were crucial in the homogeneous case (Theorem 2.5 and proof of Theorem A of [1]), and couple them with Popa’s intertwining technology [4].

The action of Γ on $L^2(X, \mu)$ is implemented by unitaries

$$(U_\gamma \xi)(x) = \omega(\gamma^{-1}, x)^{1/2} \xi(\gamma^{-1}x), \quad \xi \in L^2(X),$$

and the cocycle identity (1) ensures that $U_{\gamma_1} U_{\gamma_2} = U_{\gamma_1 \gamma_2}$.

Let $\Gamma \curvearrowright (X, \mu)$ be a free ergodic action and $A = L^\infty(X)$, $M = A \rtimes \Gamma$.

(H1) (Malleability and spectral gap). There exists an s -malleable deformation $(\alpha_t)_{t \in \mathbb{R}}$ of M (e.g., Bernoulli, Gaussian) such that A (or $L(\Gamma)$) has spectral gap relative to (α_t) .

(H2) (Unique Cartan when $B = A$). A is the unique Cartan subalgebra of M up to unitary conjugacy (holds for many nonamenable classes, including all free ergodic actions

of non-elementary hyperbolic groups, also in the nonsingular realm; cf. the strategy in [4] and related developments).

(H3) (Equivariant ucp-map rigidity). There exists a finite group $K \leq \text{Aut}(A)$ of Γ -equivariant automorphisms such that any Γ -equivariant normal ucp map $\Phi : A \rightarrow A$ admits a decomposition

$$\Phi = \sum_{k \in K} \alpha_k \theta_k, \quad \alpha_k \geq 0, \quad \sum_{k \in K} \alpha_k = 1,$$

where $\theta_k := k|_A$. (For homogeneous spaces, $K = W_G$ by Theorem 2.5 of [1]; for Bernoulli and many mixing actions, one expects $K = \{e\}$, see Lemma 5 below.)

(H4) (Essential freeness of the K -extension). The action $(\Gamma \times K) \curvearrowright X$ given by $(\gamma, k) \cdot x := k(\gamma \cdot x)$ is essentially free. (This holds in the homogeneous setting with $K = W_G$ (Lemma 2.3 of [1]).)

Remark 2.

- (i) For the homogeneous model of [1], (H1)–(H4) hold with $K = W_G$ and $B = L(\Lambda)$ or $B = A = L^\infty(G/H)$.
- (ii) For Bernoulli shifts $\Gamma \curvearrowright (X_0^\Gamma, \mu_0^\Gamma)$ of ICC nonamenable Γ with non-atomic base, (H1)–(H2) hold and (H3)–(H4) hold with $K = \{e\}$ (see Lemma 5); hence, $W(M; B)$ is trivial for $B \in \{A, L(\Gamma)\}$.

We isolate the operator-algebraic consequence of (H3).

Proposition 1. Assume (H3). Let $\Theta \in \text{Aut}_B(M)$ for $B \in \{A, L(\Gamma)\}$. Then

$$E_A \circ \Theta|_A = \sum_{k \in K} \alpha_k \theta_k \quad (\alpha_k \geq 0, \quad \sum_k \alpha_k = 1).$$

Proof. The map $\Phi := E_A \circ \Theta|_A : A \rightarrow A$ is a normal unital completely positive map and is Γ -equivariant because E_A is Γ -equivariant and, for $B = L(\Gamma)$, Θ fixes the implementing $\{u_\gamma\}$; for $B = A$, Θ fixes A pointwise. Apply (H3). $\square$

We adapt the convexity/minimization device from proof of Theorem A of [1].

Lemma 2. Assume (H1) and let $\Theta \in \text{Aut}_B(M)$ with $B \in \{A, L(\Gamma)\}$. Let $p \in A$ be a finite-trace projection (if M is II_1 , take $p = 1$). Consider in $M^{\oplus|K|}$ the set

$$\mathcal{S} := \left\{ (\Theta(u)pu^*, (\Theta \circ \text{Ad } u_k)(u)pu^*)_{k \in K \setminus \{e\}} : u \in \mathcal{U}(A) \right\}.$$

Let $\overline{\text{co}}^{\|\cdot\|_2}$ be the closure of the convex hull in the L^2 -norm. Then $\overline{\text{co}}^{\|\cdot\|_2}(\mathcal{S})$ has a unique element of minimal L^2 -norm, say, $(c, (c_k)_{k \neq e})$, and it satisfies

$$\text{Ad } \Theta(u)(c, c_k) = (c, c_k) \quad \forall u \in \mathcal{U}(A).$$

Proof. As in [1], $\mathcal{S}$ is bounded in $\|\cdot\|_2$, hence the convex closure admits a (unique) minimizer by strict convexity of the L^2 -norm. Averaging over the compact unitary group of A (in the strong operator topology) preserves $\mathcal{S}$ and the convex closure, so the minimizer is fixed by $\text{Ad } \Theta(u)$ for all $u \in \mathcal{U}(A)$. $\square$

Lemma 3. Assume (H1), (H3) and (H4). With notation as in Lemma 2, one has $c_k = 0$ for all $k \in K \setminus \{e\}$, and $c \neq 0$. Writing $c = v|c|$ is the polar decomposition, we get a nonzero partial isometry $v \in M$ such that

$$\Theta(a)v = va \quad \forall a \in A,$$

and $vv^*, v^*v \in A$.

Proof. By Proposition 1, $E_A(\Theta(u)pu^*) = \sum_{k \in K} \alpha_k \theta_k(u)pu^*$. Tracking expectations along the convex closure and using that $(\Gamma \times K) \curvearrowright X$ is essentially free (H4), any nonzero component in a nontrivial $k \neq e$ would contradict the uniqueness/minimality of the convex minimizer (the argument parallels (proof of Theorem A of [1]) verbatim once K replaces W_G). Thus, $c_k = 0$ for $k \neq e$. Moreover, $E_A(c) = \alpha_e p \neq 0$, so $c \neq 0$. Let $c = v|c|$ be the polar decomposition. From invariance under $\text{Ad } \Theta(u)$, we deduce $\Theta(u)vs. = vs.u$ for $u \in \mathcal{U}(A)$, hence for all $a \in A$ by linearity. Since A is maximal abelian in M (Cartan case) or $A' \cap M = A$ (group-measure-space case), projections vv^*, v^*v lie in A . $\square$

Lemma 4. Assume (H1) and that $A \curvearrowright M$ is ergodic under Γ . From the partial isometry v in Lemma 3, one constructs a unitary $u \in \mathcal{U}(M)$ such that

$$\Theta(a) = uau^* \quad \forall a \in A.$$

Proof. As in proof of Theorem A of [1]: enumerate $\Gamma = \{\gamma_n\}$; define

$$p_n := \bigvee_{k \leq n} \sigma_{\gamma_k}(v^*v), \quad q_n := \bigvee_{k \leq n} \sigma_{\gamma_k}(vv^*),$$

build an increasing sequence of partial isometries u_n by extending u_n on $p_{n+1} - p_n$ with $\sigma_{\gamma_{n+1}}(v)$, preserving the intertwining relation $\Theta(a)u_n = u_n a$. Ergodicity implies $\bigvee_{\gamma} \sigma_{\gamma}(v^*v) = \bigvee_{\gamma} \sigma_{\gamma}(vv^*) = 1$, so $u := s\text{-}\lim u_n$ is unitary and $\Theta(a) = uau^*$ for $a \in A$. $\square$

Theorem 1. Let $M = A \rtimes \Gamma$ and suppose (H1)–(H4). Then for $B \in \{A, L(\Gamma)\}$ the natural map

$$\Xi : K \longrightarrow W(M; B), \quad k \mapsto [\Theta_k],$$

is surjective, where $\Theta_k \in \text{Aut}_B(M)$ extends θ_k and fixes B . In particular $W(M; B)$ is finite. If $K = \{e\}$, then $W(M; B) = \{e\}$.

Proof. Let $\Theta \in \text{Aut}_B(M)$. By Lemma 4, there exists $u \in \mathcal{U}(M)$ with $\Theta(a) = uau^*$ for all $a \in A$. Set $\tilde{\Theta} := \text{Ad}(u^*) \circ \Theta \in \text{Aut}_B(M)$; then $\tilde{\Theta}|_A = \text{id}$.

By Proposition 1, $E_A \circ \tilde{\Theta}|_A = \sum_k \alpha_k \theta_k$. Since $\tilde{\Theta}$ acts trivially on A , $\sum_k \alpha_k \theta_k = \text{id}_A$, hence $\alpha_k = 0$ for $k \neq e$ by essential freeness (H4) and the same disintegration argument as in proof of Theorem A of [1]: the kernel β_x attached to $E_A \circ \tilde{\Theta}$ decomposes as $\sum_k \alpha_k \delta_{k^{-1}x}$, and essential freeness forces $\alpha_k = 0$ for $k \neq e$. Thus, $E_A \circ \tilde{\Theta} = \text{id}_A$.

Since $\tilde{\Theta}$ fixes A pointwise and $B \in \{A, L(\Gamma)\}$, we have $\tilde{\Theta} \in \text{Inn}_B(M)$ by Lemma 3 (applied with $\Theta = \tilde{\Theta}$, which yields a partial isometry in A and then a unitary in A implementing $\tilde{\Theta}$). Hence, $[\Theta] = [\Theta_k]$ in $W(M; B)$ for some k (indeed, $k = e$ for the normalized $\tilde{\Theta}$), proving surjectivity of Ξ and finiteness of $W(M; B)$. If $K = \{e\}$, then $W(M; B)$ is trivial. $\square$

Corollary 1. For the setting of [1] with $B = L(\Lambda)$ or $B = A = L^\infty(G/H)$, one has $W(M; B) \cong W_G$.

Proof. All hypotheses (H1)–(H4) hold with $K = W_G$ by [1]. Theorem 1 gives a surjection $W_G \rightarrow W(M; B)$. Injectivity follows since the canonical lift $w \mapsto \Theta_w$ is faithful on W_G (as in Theorem A of [1]). $\square$

We record a general criterion ensuring $K = \{e\}$ in (H3).

Lemma 5. *Let $\Gamma \curvearrowright (X, \mu)$ be free ergodic and mixing, and suppose $A = L^\infty(X)$ has no nontrivial Γ -equivariant automorphisms (e.g., Bernoulli shift over a non-atomic base). Then (H3) holds with $K = \{e\}$, i.e., any Γ -equivariant normal ucp map $\Phi : A \rightarrow A$ equals id_A .*

Proof. By Lemma 4, Φ corresponds to an equivariant kernel $\beta : X \rightarrow \text{Prob}(X)$. Mixing and ergodicity imply that the only Γ -equivariant measurable maps $X \rightarrow X$ are a.e. the identity (or null), and extreme points in the convex set of equivariant kernels are precisely Dirac kernels at such maps. Since Φ is unital, integrating the identity function yields $\int f d\beta_x = f(x)$ a.e., hence $\beta_x = \delta_x$ a.e. and $\Phi = \text{id}_A$. (For Bernoulli shifts, Γ -equivariant automorphisms of A are trivial by standard arguments on tail σ -algebras.) $\square$

Corollary 2. *If Γ is ICC, is nonamenable, and acts by a Bernoulli shift on a non-atomic base, then $W(M; B) = \{e\}$ for $B \in \{A, L(\Gamma)\}$.*

Proof. Bernoulli shifts admit an s -malleable deformation with spectral gap for both A and $L(\Gamma)$ in the sense of [4], A is unique Cartan (for the classes covered by the deformation/rigidity framework), and Lemma 5 gives (H3) with $K = \{e\}$. Essential freeness is automatic. Apply Theorem 1. $\square$

In Section 6, we will extend Theorem 1 to nonsingular actions by working on the Maharam extension. The role of (H1)–(H4) is played by the corresponding properties on the II_1 Maharam extension, and essential freeness is inherited along the skew-product. The proof then follows the same pattern: ucp-map rigidity on the base, convex minimization, annihilation of nontrivial components, and patching to a unitary.

4. Subgroup-Normalizer Rigidity in Group Factors

In this section, we establish a von Neumann algebraic normalizer rigidity principle for subgroup inclusions inside group factors. Roughly speaking, we show that if $\Lambda < \Gamma$ satisfies a mild malnormality-type hypothesis, then any $*$ -automorphism of $L(\Gamma)$ that fixes $L(\Lambda)$ pointwise is implemented by a unitary coming from the *group-theoretic* normalizer $N_\Gamma(\Lambda)$; consequently,

$$\text{Aut}_{L(\Lambda)}(L(\Gamma)) \cong N_\Gamma(\Lambda)/\Lambda,$$

so the Weyl-type symmetry $W(L(\Gamma); L(\Lambda))$ is discrete (often finite, e.g., if $N_\Gamma(\Lambda)/\Lambda$ is finite).

Throughout, Γ is a countable discrete ICC group, and $\Lambda < \Gamma$ is a subgroup. We write u_γ for the canonical group unitaries inside $L(\Gamma)$, and $E_{L(\Lambda)} : L(\Gamma) \rightarrow L(\Lambda)$ for the trace-preserving conditional expectation.

We consider the following standing assumptions:

(N1) (Almost malnormality). For every $g \in \Gamma \setminus N_\Gamma(\Lambda)$, the intersection $g\Lambda g^{-1} \cap \Lambda$ is finite.

(N2) (No finite normal subgroup). Λ has a trivial finite normal subgroup in Γ (e.g., Λ is torsion-free or ICC in its conjugation action).

Remark 3. Hypothesis (N1) is satisfied, for instance, if Λ is malnormal in Γ (i.e., $g\Lambda g^{-1} \cap \Lambda = \{e\}$ for $g \notin N_\Gamma(\Lambda)$), or when Γ is hyperbolic and Λ is a maximal virtually cyclic subgroup. These are standard regimes where conjugates intersect Λ in small (finite) sets.

We begin with a structural description of the normalizer algebra of $L(\Lambda)$ inside the group factor $L(\Gamma)$.

Lemma 6. Under (N1), the von Neumann algebra generated by the normalizer of $L(\Lambda)$ in $L(\Gamma)$ is

$$\mathcal{N}_{L(\Gamma)}(L(\Lambda))'' = L(N_\Gamma(\Lambda)).$$

Proof. The inclusion $L(N_\Gamma(\Lambda)) \subset \mathcal{N}_{L(\Gamma)}(L(\Lambda))''$ is immediate: for $g \in N_\Gamma(\Lambda)$, $u_g L(\Lambda) u_g^* = L(\Lambda)$.

For the reverse inclusion, let $x \in L(\Gamma)$ be a normalizing unitary: $xL(\Lambda)x^* = L(\Lambda)$. Write the Fourier expansion $x = \sum_{\gamma \in \Gamma} a_\gamma u_\gamma$ with $a_\gamma \in \mathbb{C}$ and $\sum_\gamma |a_\gamma|^2 = 1$. Fix $\lambda \in \Lambda$. Since $xu_\lambda x^* \in L(\Lambda)$, we have

$$xu_\lambda = \left(\sum_\gamma a_\gamma u_\gamma \right) u_\lambda = \sum_\gamma a_\gamma u_{\gamma\lambda} \in L(\Gamma) x L(\Lambda).$$

Taking the conditional expectation $E_{L(\Lambda)}$ and using $E_{L(\Lambda)}(u_\eta) = 0$ unless $\eta \in \Lambda$, we obtain

$$E_{L(\Lambda)} \left(\sum_\gamma a_\gamma u_{\gamma\lambda} x^* \right) = E_{L(\Lambda)}(xy) = E_{L(\Lambda)}(xy) \in L(\Lambda)$$

for some $y \in L(\Lambda)$, hence

$$\sum_{\gamma, \eta} a_\gamma \overline{a_\eta} E_{L(\Lambda)}(u_{\gamma\lambda\eta^{-1}}) \in L(\Lambda).$$

The only nonzero terms correspond to $\gamma\lambda\eta^{-1} \in \Lambda$, equivalently $\eta \in \gamma\lambda\Lambda$. Consequently, for each fixed λ ,

$$\sum_\gamma a_\gamma \overline{a_{\gamma\lambda\mu}} u_{\gamma\lambda\mu} (\in \Lambda) \in L(\Lambda) \quad (\mu \in \Lambda).$$

Orthogonality in $\ell^2(\Gamma)$ forces that the support set

$$\text{Supp}(x) := \{\gamma \in \Gamma \mid a_\gamma \neq 0\}$$

is contained in a finite union of double cosets $\Lambda\gamma\Lambda$ with the property that $\gamma\Lambda\gamma^{-1} \cap \Lambda$ is infinite (since otherwise only finitely many μ contribute and the above sums cannot saturate $L(\Lambda)$). By (N1), the only way $\gamma\Lambda\gamma^{-1} \cap \Lambda$ is infinite is $\gamma \in N_\Gamma(\Lambda)$. Hence, $\text{Supp}(x) \subset \Lambda N_\Gamma(\Lambda) = N_\Gamma(\Lambda)$. Since the set of normalizing unitaries is strongly closed, it follows that every normalizer lies in $L(N_\Gamma(\Lambda))$, hence $\mathcal{N}_{L(\Gamma)}(L(\Lambda))'' \subset L(N_\Gamma(\Lambda))$. $\square$

Corollary 3. Every $*$ -automorphism $\Theta \in \text{Aut}_{L(\Lambda)}(L(\Gamma))$ is implemented by a unitary $v \in L(N_\Gamma(\Lambda))$, i.e., $\Theta = \text{Ad}(v)$.

Proof. Since Θ fixes $L(\Lambda)$ pointwise, Θ normalizes $L(\Lambda)$. By Lemma 6, Θ belongs to the von Neumann algebra generated by the normalizer and hence is inner by a unitary in $L(N_\Gamma(\Lambda))$. $\square$

We can now identify the group $\text{Aut}_{L(\Lambda)}(L(\Gamma))$ explicitly.

Theorem 2. Let Γ be a countable discrete group and $\Lambda \leq \Gamma$. Consider the inclusion $L(\Lambda) \subset L(\Gamma)$ with canonical group unitaries $\{u_g : g \in \Gamma\}$. Assume:

(AR) Automorphism rigidity. For every $\theta \in \text{Aut}(L(\Gamma))$ with $\theta(L(\Lambda)) = L(\Lambda)$, there exist $g \in N_\Gamma(\Lambda)$ and $v \in \mathcal{U}(L(\Lambda))$ such that

$$\theta = \text{Ad}(v u_g).$$

Define

$$\Phi : N_\Gamma(\Lambda) \longrightarrow \text{Aut}_{L(\Lambda)}(L(\Gamma))/\text{Inn}_{L(\Lambda)}(L(\Gamma)), \quad g \longmapsto [\text{Ad}(u_g)],$$

where $\text{Aut}_{L(\Lambda)}(L(\Gamma)) := \{\theta \in \text{Aut}(L(\Gamma)) : \theta(L(\Lambda)) = L(\Lambda)\}$ and $\text{Inn}_{L(\Lambda)}(L(\Gamma)) := \{\text{Ad}(w) : w \in \mathcal{U}(L(\Lambda))\}$.

Then Φ is a surjective group homomorphism with

$$\ker \Phi = \Lambda Z(\Gamma).$$

In particular, if $Z(\Gamma) = \{e\}$, then

$$\text{Aut}_{L(\Lambda)}(L(\Gamma))/\text{Inn}_{L(\Lambda)}(L(\Gamma)) \cong N_\Gamma(\Lambda)/\Lambda.$$

Proof. If $g \in N_\Gamma(\Lambda)$, then $\text{Ad}(u_g)(u_\lambda) = u_{g\lambda g^{-1}} \in L(\Lambda)$ for all $\lambda \in \Lambda$, hence $\text{Ad}(u_g) \in \text{Aut}_{L(\Lambda)}(L(\Gamma))$ and Φ is well-defined. The map $g \mapsto \text{Ad}(u_g)$ is a group homomorphism, and passing to the quotient by $\text{Inn}_{L(\Lambda)}(L(\Gamma))$ preserves multiplicativity, so Φ is a group homomorphism.

Let $\theta \in \text{Aut}_{L(\Lambda)}(L(\Gamma))$. By (AR) there exist $g \in N_\Gamma(\Lambda)$ and $v \in \mathcal{U}(L(\Lambda))$ with $\theta = \text{Ad}(v u_g)$. Thus, $[\theta] = [\text{Ad}(u_g)] = \Phi(g)$, proving surjectivity.

First, $\Lambda Z(\Gamma) \subset \ker \Phi$: for $\lambda \in \Lambda$, $\text{Ad}(u_\lambda) \in \text{Inn}_{L(\Lambda)}(L(\Gamma))$; for $z \in Z(\Gamma)$, u_z is central in $L(\Gamma)$, so $\text{Ad}(u_z) = \text{id}$. Hence, $\Phi(\lambda z) = [\text{Ad}(u_{\lambda z})] = [\text{Ad}(u_z) \circ \text{Ad}(u_\lambda)] = 1$.

Conversely, take $g \in \ker \Phi$. Then there exists $v \in \mathcal{U}(L(\Lambda))$ with

$$\text{Ad}(u_g) = \text{Ad}(v) \quad \text{on } L(\Gamma). \quad (2)$$

Restricting (2) to $L(\Lambda)$ gives, for all $\lambda \in \Lambda$,

$$u_{g\lambda g^{-1}} = v u_\lambda v^*. \quad (3)$$

Let $\varphi : \Lambda \rightarrow \Lambda$ be the automorphism $\varphi(\lambda) = g\lambda g^{-1}$. Equation (3) says that the $*$ -automorphism of $L(\Lambda)$ induced by φ is inner. We now use the following lemma.

Lemma 7. If $v \in \mathcal{U}(L(\Lambda))$ satisfies $v u_\lambda v^* = u_{\varphi(\lambda)}$ for all $\lambda \in \Lambda$, then $\varphi = \text{Ad}(\mu)$ for some $\mu \in \Lambda$ and $v \in \mathbb{T} u_\mu$.

Proof. Write $v = \sum_{s \in \Lambda} a_s u_s$ (Fourier series, converging in $\|\cdot\|_2$). From $v u_\lambda = u_{\varphi(\lambda)} v$ we get

$$\sum_s a_s u_{s\lambda} = \sum_s a_s u_{\varphi(\lambda)s}, \quad \forall \lambda \in \Lambda.$$

By uniqueness of Fourier coefficients, $a_{t\lambda^{-1}} = a_{\varphi(\lambda)^{-1}t}$ for all $t, \lambda \in \Lambda$. Fix $t = e$ to get $a_{\lambda^{-1}} = a_{\varphi(\lambda)^{-1}}$, hence the function $s \mapsto a_s$ is constant on right cosets of $\varphi(\Lambda)$. Varying λ and using unitarity of v one deduces that $\text{supp}(v)$ is a single right coset $\mu \varphi(\Lambda)$ and $|a_s|$ is constant on that coset. The intertwining relation then forces $\varphi(\lambda) = \mu \lambda \mu^{-1}$ and $v \in \mathbb{T} u_\mu$. $\square$

Applying Lemma 7 to (3), there is $\mu \in \Lambda$ with

$$g\lambda g^{-1} = \mu\lambda\mu^{-1} \quad (\forall \lambda \in \Lambda) \quad \text{and} \quad v \in \mathbb{T}u_\mu.$$

Thus, $c := \mu^{-1}g \in C_\Gamma(\Lambda)$. Using (2) and $v \in \mathbb{T}u_\mu$ we get

$$\text{Ad}(u_g) = \text{Ad}(v) = \text{Ad}(u_\mu) \quad \text{on } L(\Gamma),$$

hence $\text{Ad}(u_c) = \text{id}$ on $L(\Gamma)$, i.e., $c \in Z(\Gamma)$. Therefore, $g = \mu c \in \Lambda Z(\Gamma)$, proving $\ker \Phi \subset \Lambda Z(\Gamma)$.

Combining the two inclusions yields $\ker \Phi = \Lambda Z(\Gamma)$, as claimed. In particular, if $Z(\Gamma) = \{e\}$, then

$$\text{Aut}_{L(\Lambda)}(L(\Gamma))/\text{Inn}_{L(\Lambda)}(L(\Gamma)) \cong N_\Gamma(\Lambda)/\Lambda.$$

□

Remark 4. If $Z(\Gamma) = \{e\}$ (e.g., Γ is ICC), then $\ker \Phi = \Lambda$ and

$$\text{Aut}_{L(\Lambda)}(L(\Gamma))/\text{Inn}_{L(\Lambda)}(L(\Gamma)) \cong N_\Gamma(\Lambda)/\Lambda.$$

Corollary 4. Under (N1)–(N2), the Weyl-type symmetry group $W(L(\Gamma); L(\Lambda))$ is canonically isomorphic to the discrete group $N_\Gamma(\Lambda)/\Lambda$. In particular, if $N_\Gamma(\Lambda)/\Lambda$ is finite (e.g., Λ self-normalizing), then $W(L(\Gamma); L(\Lambda))$ is finite (often trivial).

Proof. This proof is immediate from Theorem 2. □

The proof of Lemma 6 only uses that for $\gamma \notin N_\Gamma(\Lambda)$, the intersection $\gamma\Lambda\gamma^{-1} \cap \Lambda$ is finite, so that the L^2 -mass of any normalizer cannot spread across infinitely many Λ -double cosets without violating the Fourier orthogonality constraints forced by $E_{L(\Lambda)}$. If one assumes instead that these intersections are merely *small* in a suitable sense (e.g., uniformly finite index in a fixed family), the same argument yields an inclusion $\mathcal{N}_{L(\Gamma)}(L(\Lambda))'' \subset L(\text{Comm}_\Gamma(\Lambda))$, where $\text{Comm}_\Gamma(\Lambda) = \{g \in \Gamma : [\Lambda : \Lambda \cap g\Lambda g^{-1}] < \infty, [g\Lambda g^{-1} : \Lambda \cap g\Lambda g^{-1}] < \infty\}$ is the commensurator of Λ in Γ . One then obtains $\text{Aut}_{L(\Lambda)}(L(\Gamma)) \cong \text{Comm}_\Gamma(\Lambda)/\Lambda$ under a mild strengthening of (N2).

While the above argument stays within the combinatorics of Fourier coefficients, it is compatible with Popa's deformation/rigidity philosophy: one can detect that any $\Theta \in \text{Aut}_{L(\Lambda)}(L(\Gamma))$ intertwines $L(\Lambda)$ into itself, whence Θ belongs to the normalizer algebra, and then finish as above; see [4] for intertwining-by-bimodules criteria that justify such reductions in more general contexts.

In the homogeneous setting of [1], the analogue of Theorem 2 identifies $\text{Aut}_{L(\Lambda)}(L(\Lambda \curvearrowright G/H))$ with the algebraic Weyl group W_G . The present theorem provides a parallel statement in the *pure group factor* case, with W_G replaced by $N_\Gamma(\Lambda)/\Lambda$. Both illustrate the general principle that “relative automorphisms are governed by visible discrete symmetries of the inclusion”.

Example 1. Let Γ be a non-elementary hyperbolic ICC group, and let $\Lambda < \Gamma$ be a maximal infinite cyclic subgroup. Then Λ is malnormal and $N_\Gamma(\Lambda) = \Lambda$; hence, $\text{Aut}_{L(\Lambda)}(L(\Gamma))$ is trivial and $W(L(\Gamma); L(\Lambda)) = \{e\}$.

Example 2. Let $\Gamma = \Gamma_1 \times \Gamma_2$ with ICC, and $\Lambda = \Gamma_1 \times \{e\}$. Then $N_\Gamma(\Lambda) = \Gamma$, and $N_\Gamma(\Lambda)/\Lambda \cong \Gamma_2$. Thus, $\text{Aut}_{L(\Lambda)}(L(\Gamma)) \cong \Gamma_2$, implemented by $\{\text{Ad}(u_{(e, \gamma_2)}) : \gamma_2 \in \Gamma_2\}$, while inner automorphisms from $L(\Lambda)$ are modded out.

Example 3. Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ and Λ equal the subgroup generated by a primitive parabolic. Then $\mathrm{Comm}_\Gamma(\Lambda)$ is strictly larger than $N_\Gamma(\Lambda)$, and the variant in the remark above yields $\mathrm{Aut}_{L(\Lambda)}(L(\Gamma)) \cong \mathrm{Comm}_\Gamma(\Lambda)/\Lambda$.

5. Permutation Rigidity for Tensor Products

In this section, we establish a permutation rigidity statement for tensor products of crossed-product II_1 factors coming from actions of (at least) non-elementary hyperbolic groups. The point is to work in a regime where both *uniqueness of Cartan* and a *no cross-intertwining* principle are theorems so that the permutation conclusion follows by Popa’s intertwining-by-bimodules method.

Throughout, for a unital inclusion $P \subset M$ of tracial von Neumann algebras, we write $P \preceq_M Q$ (resp. $P \not\preceq_M Q$) for Popa’s intertwining-by-bimodules relation (see Section 2 of [8]). We call $A \subset M$ a Cartan subalgebra if A is diffuse abelian and $\mathcal{N}_M(A)'' = M$.

For $i = 1, \dots, n$, let Γ_i be a non-elementary hyperbolic group (*more generally*, weakly amenable and bi-exact) acting freely and ergodically pmp on (X_i, μ_i) . Put

$$A_i := L^\infty(X_i), \quad M_i := A_i \rtimes \Gamma_i, \quad M := M_1 \bar{\otimes} \cdots \bar{\otimes} M_n.$$

Each M_i is a II_1 factor and $A_i \subset M_i$ is Cartan.

Theorem 3. For each i , any Cartan subalgebra of M_i is unitarily conjugate to A_i .

Proof. This is due to Popa–Vaes for actions of non-elementary hyperbolic groups [9]; see also extensions to weakly amenable bi-exact groups. We will only use the conclusion and do not reproduce the proof. $\square$

We also need the dichotomy/intertwining principle that forbids “crossing” a Cartan from one tensor leg into another.

Lemma 8. With notation as above, fix $i \neq j$. Then $A_i \not\preceq_M M_j$.

Proof. Assume $A_i \preceq_M M_j$. By [9] (see also [10]), subalgebras of $M_j = A_j \rtimes \Gamma_j$ satisfy a dichotomy: for any diffuse $P \subset M_j$, either $P \preceq_{M_j} A_j$, or $\mathcal{N}_{M_j}(P)''$ is amenable relative to A_j . Transporting a corner of A_i into M_j and compressing, we obtain a diffuse $P \subset M_j$ stably conjugate to a corner of A_i . Since A_i is Cartan in M_i , the normalizer of that corner inside M contains a corner of M_i , which is non-amenable relative to A_j ; thus, the dichotomy forces $P \preceq_{M_j} A_j$. Unwinding the intertwining gives $A_i \preceq_M A_j$, which contradicts freeness/ergodicity together with Theorem 3 (the tensor legs are independent). Therefore, $A_i \not\preceq_M M_j$. $\square$

The proof uses Popa’s intertwining-by-bimodules and the relative amenability dichotomy of Popa–Vaes for the hyperbolic/bi-exact class; see [8–10].

We can now state and prove the permutation rigidity.

Theorem 4. Let $M_i = A_i \rtimes \Gamma_i$ be as above and $M = \bar{\otimes}_{i=1}^n M_i$. Assume M also admits a tensor decomposition

$$M \cong N_1 \bar{\otimes} \cdots \bar{\otimes} N_m,$$

where each N_k is a II_1 factor of the same hyperbolic/bi-exact crossed-product type (so Theorem 3 and the dichotomy hold for each N_k as well), and let $B_k \subset N_k$ be (their unique) Cartan subalgebras.

Then $m = n$ and there exists a permutation $\sigma \in S_n$ and a unitary $u \in \mathcal{U}(M)$ such that

$$u M_i u^* = N_{\sigma(i)} \quad \text{and} \quad u A_i u^* \text{ is Cartan in } N_{\sigma(i)}, \quad \forall i.$$

Proof. Let $\theta : M \rightarrow N_1 \bar{\otimes} \cdots \bar{\otimes} N_m$ be an isomorphism and identify M with the right-hand side via θ . Fix i . Since $A_i \subset M$ is diffuse amenable with expectation and the N_k 's are mutually commuting subfactors whose product generates M , Popa's intertwining criterion implies that $A_i \preceq_M N_k$ for some k (otherwise A_i would be too small relative to each leg, contradicting $\mathcal{N}_M(A_i)'' = M$). By Lemma 8, $A_i \not\preceq_M N_{k'}$ for all $k' \neq k$, so k is *unique*. Using Corollary 2.3 of [8], there exist nonzero projections $p \in A_i$, $q \in B_k$, a $*$ -homomorphism $\phi : pA_i p \rightarrow qN_k q$ and a partial isometry $v \in qMp$ intertwining A_i into N_k :

$$\phi(x)v = vx, \quad x \in pA_i p.$$

By Theorem 3 inside N_k , the subalgebra $\phi(pA_i p) \subset qN_k q$ is unitarily conjugate to a corner of B_k . Conjugating inside N_k and absorbing the unitary into v , we may assume $\phi(pA_i p) \subset qB_k q$. Because A_i is Cartan in M_i , its normalizer in M contains $M_i \bar{\otimes} 1$, hence the von Neumann algebra generated by v and the normalizer of $pA_i p$ intertwines into N_k . By the usual "normalizer amplification" argument (see Section 2 of [8]), we conclude $M_i \preceq_M N_k$.

Applying the same reasoning to the inverse isomorphism, we get $N_k \preceq_M M_i$. Since M_i and N_k are II_1 factors, the mutual intertwining yields a unitary $u \in \mathcal{U}(M)$ with $uM_i u^* \subset N_k$ and, by symmetry, equality $uM_i u^* = N_k$ (Proposition 2.3 of [8]). Repeating for each i , we obtain an injective map $i \mapsto k(i)$ with $u_i M_i u_i^* = N_{k(i)}$. As the $N_{k(i)}$'s are pairwise commuting tensor legs and the M_i 's generate M , cardinality forces $m = n$ and k is a permutation $\sigma \in S_n$. Finally, since $uM_i u^* = N_{\sigma(i)}$ and A_i is Cartan in M_i , Theorem 3 implies $uA_i u^*$ is a Cartan subalgebra of $N_{\sigma(i)}$, as required. $\square$

Corollary 5. *With the notation of Theorem 4, the same conclusion holds for amplifications: if $M^t \cong \bar{\otimes}_{k=1}^m N_k$ for some $t > 0$ and N_k 's as above, then $m = n$ and, up to unitary conjugacy and permutation, each N_k is an amplification of some M_i .*

The hyperbolic/bi-exact hypothesis can be replaced by any class for which (i) uniqueness of Cartan holds for free ergodic pmp actions, and (ii) the Popa–Vaes dichotomy for subalgebras applies to rule out cross-intertwining as in Lemma 8. The proof of Theorem 4 then goes through verbatim.

6. Type III Extensions via Maharam and Discreteness of $W(M; B)$

We extend the finiteness/triviality results of Section 3 to *nonsingular* actions and type III crossed-product factors. Our strategy is classical: lift the problem to the II_∞ Maharam extension where deformation/rigidity arguments apply verbatim, prove the desired statement there, and then descend to the original type III factor using modular theory.

Let $\Gamma \curvearrowright (X, \mu)$ be a free ergodic nonsingular action on a standard measure space. Set $A = L^\infty(X)$ and $M = A \rtimes \Gamma$, a type III factor in general. Let $\omega : \Gamma \times X \rightarrow \mathbb{R}$ be the Radon–Nikodym cocycle

$$\frac{d\mu \circ \gamma^{-1}}{d\mu}(x) = e^{-\omega(\gamma, x)}.$$

The *Maharam extension* is the p.m.p. action $\Gamma \curvearrowright (X \times \mathbb{R}, \mu \otimes e^{-t} dt)$ given by

$$\gamma \cdot (x, t) = (\gamma \cdot x, t + \omega(\gamma, x)).$$

We write $\tilde{A} = L^\infty(X \times \mathbb{R})$ and $\tilde{M} = \tilde{A} \rtimes \Gamma$. Then $\tilde{M}$ is a II_∞ factor, equipped with a semifinite trace obtained by integrating the conditional expectation onto $\tilde{A}$ against $\mu \otimes e^{-t} dt$. The *modular translation* flow $(\sigma_s)_{s \in \mathbb{R}}$ acts on $\tilde{A}$ by $(\sigma_s f)(x, t) = f(x, t + s)$ and commutes with the Γ -action; thus, it extends to a flow by trace-preserving automorphisms of $\tilde{M}$. As

usual, M is recovered from $(\tilde{M}, \sigma)$ by taking a crossed product or a fixed-point algebra depending on the normalization (see, e.g., Connes's flow of weights framework).

We fix:

- (T1) (Unique Cartan on the Maharam extension).** $\tilde{A}$ is the unique Cartan subalgebra of $\tilde{M}$ up to unitary conjugacy for the Maharam action (this holds for broad classes including free ergodic actions of non-elementary hyperbolic groups; see the unique-Cartan paradigm initiated for nonsingular actions in the deformation/rigidity framework).
- (T2) (Spectral gap and malleability on the Maharam extension).** The Maharam action admits an s -malleable deformation $(\alpha_t)_{t \in \mathbb{R}}$ of $\tilde{M}$ such that each of the subalgebras $\tilde{A}$ and $L(\Gamma)$ has spectral gap relative to (α_t) .
- (T3) (Equivariant ucp-map rigidity on the base).** Any Γ -equivariant normal ucp map $\Phi : A \rightarrow A$ arises as a finite convex combination of a fixed finite family of Γ -equivariant automorphisms of A , indexed by a finite group $K \leq \text{Aut}(A)$ (in many mixing cases, $K = \{e\}$).
- (T4) (Essential freeness of the K -extension).** The action $(\Gamma \times K) \curvearrowright X$ given by $(\gamma, k) \cdot x = k(\gamma \cdot x)$ is essentially free.

Assumptions (T1)–(T2) are natural II_∞ analogues of (H1)–(H2) and are known for many nonsingular actions by adapting the II_1 arguments to the Maharam extension. Assumptions (T3)–(T4) hold with $K = W_G$ in the homogeneous setting and with $K = \{e\}$ for many mixing actions (e.g., Bernoulli-type bases).

We first show that *relative* automorphisms of M canonically lift to relative automorphisms of $\tilde{M}$ that commute with the modular translation flow.

Lemma 9. *Let $B \in \{A, L(\Gamma)\}$. Every $\Theta \in \text{Aut}_B(M)$ admits a lift $\tilde{\Theta} \in \text{Aut}_{\tilde{B}}(\tilde{M})$ such that $\tilde{\Theta} \circ \sigma_s = \sigma_s \circ \tilde{\Theta}$ for all $s \in \mathbb{R}$, where $\tilde{B}$ denotes the canonical copy of B inside $\tilde{M}$ ($\tilde{A} \supset A$ and $L(\Gamma) \subset \tilde{M}$ canonically). Moreover, if Θ is inner by a unitary in B , then $\tilde{\Theta}$ is inner by a unitary in $\tilde{B}$.*

Proof. Identify M with the fixed-point algebra of $\tilde{M}$ under the modular translation flow (σ_s) in the standard implementation (or, equivalently, identify $\tilde{M} \cong M \rtimes_{\sigma^\varphi} \mathbb{R}$ for an appropriate weight φ on M). Then M embeds into $\tilde{M}$ as the σ -fixed-point algebra, and A embeds into $\tilde{A}$ as the t -constant functions.

Define $\tilde{\Theta}$ on the algebraic crossed product $\tilde{A} \rtimes_{\text{alg}} \Gamma$ by

$$\tilde{\Theta}\left(\sum a_\gamma u_\gamma\right) := \sum \tilde{\Theta}_0(a_\gamma) u_\gamma,$$

where $\tilde{\Theta}_0$ is the unique extension of $\Theta|_A$ to $\tilde{A}$ acting trivially on the $\mathbb{R}$ -coordinate (hence commuting with σ_s). This is well-defined and extends to a normal automorphism $\tilde{\Theta}$ of $\tilde{M}$ because Θ preserves the covariance relations between A and $\{u_\gamma\}_{\gamma \in \Gamma}$ and because $\tilde{\Theta}_0$ is Γ -equivariant if Θ is A -fixing; in the case $B = L(\Gamma)$, Θ fixes u_γ 's and thus $\tilde{\Theta}$ acts as identity on the group part.

By construction, $\tilde{\Theta}|_{\tilde{B}} = \text{id}$ if $\Theta|_B = \text{id}$, and $\tilde{\Theta} \circ \sigma_s = \sigma_s \circ \tilde{\Theta}$ because $\tilde{\Theta}_0$ acts trivially on the $\mathbb{R}$ -coordinate. If $\Theta = \text{Ad}(u)$ with $u \in \mathcal{U}(B)$, then $\tilde{\Theta} = \text{Ad}(u)$ as well (viewing u in $\tilde{B}$), completing the proof. $\square$

The ucp-map/kernel correspondence upgrades to the Maharam extension with an extra invariance under the translation flow.

Lemma 10. Let $\Phi : \tilde{A} \rightarrow \tilde{A}$ be a normal ucp map commuting with both the Γ -action and the translation flow $(\sigma_s)_{s \in \mathbb{R}}$. Then Φ is determined by a measurable field of probabilities $\beta_{(x,t)}$ on $X \times \mathbb{R}$ that is product-like in the second coordinate:

$$\beta_{(x,t)} = \sum_{k \in K} \alpha_k \delta_{(k \cdot x, t)} \quad (\alpha_k \geq 0, \sum_k \alpha_k = 1),$$

i.e., it does not smear mass in the t -direction and reduces to the base-kernel from (T3).

Proof. Apply Lemma 4 to the lcsc group $L = \Gamma \times \mathbb{R}$ acting on $X \times \mathbb{R}$, with Γ acting nonsingularly and $\mathbb{R}$ by translations in t . Equivariance under the $\mathbb{R}$ -action says $\beta_{(x,t+s)} = (\text{id}_X \times \tau_s)_* \beta_{(x,t)}$ for all s , where $\tau_s(t) = t + s$. This forces the conditional distributions on the t -coordinate to be Dirac at t (otherwise translation-invariance would create a nontrivial absolutely continuous component contradicting normality against bounded functions supported in small t -intervals). Projecting to X , the Γ -equivariance reduces to the base case (T3), yielding the announced convex combination. $\square$

Lemma 11. Let $(A \subset M)$ be a Cartan inclusion, with $A \cong L^\infty(X)$ and $M \cong L(R, c)$ the Feldman–Moore construction of a countable pmp equivalence relation R with a $\mathbb{T}$ -valued 2-cocycle c . If $\theta \in \text{Aut}(M)$ satisfies $\theta|_A = \text{id}$, then the class of θ in

$$\text{Aut}_A(M) / \text{Inn}_A(M)$$

identifies with a class in $H^1(R, \mathbb{T})$. In particular, if the associated $\mathbb{T}$ -valued 1-cocycle is a coboundary, then $\theta \in \text{Inn}_A(M)$.

Proof. By the Feldman–Moore model, any $\theta \in \text{Aut}(M)$ with $\theta(A) = A$ corresponds to a measurable automorphism of the measured groupoid (R, c) together with a $\mathbb{T}$ -valued 1-cocycle; restricting to $\theta|_A = \text{id}$ yields precisely $\mathbb{T}$ -valued 1-cocycles on R . Moreover,

$$\text{Aut}_A(M) / \text{Inn}_A(M) \cong H^1(R, \mathbb{T}),$$

see Sections 2 and 3 of [11,12]. Under this identification, θ is A -inner iff its cocycle is cohomologous to the trivial cocycle, i.e., a coboundary. $\square$

Corollary 6. In the set-up of Lemma 11, suppose the Cartan comes from a free ergodic pmp action with a cocycle superrigidity property that kills $\mathbb{T}$ -valued cocycles (e.g., Popa’s cocycle superrigidity for malleable, weakly mixing actions of w -rigid groups [13], or Ioana’s result for profinite actions of property (T) groups [14]; see also the Gaussian case [15]). Then every $\theta \in \text{Aut}(M)$ with $\theta|_A = \text{id}$ lies in $\text{Inn}_A(M)$.

Proof. By Lemma 11, it suffices to show the associated $\mathbb{T}$ -valued 1-cocycle is a coboundary. The cited cocycle superrigidity results imply any $\mathbb{T}$ -valued cocycle is cohomologous to a character; when the acting group has no nontrivial characters in the relevant setting (as is the case in our hypotheses), the cocycle is cohomologous to the trivial one and is hence a coboundary. Therefore, $\theta \in \text{Inn}_A(M)$. $\square$

Theorem 5. Let $M = A \rtimes \Gamma$ be the type III crossed product of a free ergodic nonsingular action. Assume (T1)–(T4). Then for $B \in \{A, L(\Gamma)\}$ the group $W(M; B) = \text{Aut}_B(M) / \text{Inn}_B(M)$ is finite. If, moreover, $K = \{e\}$ in (T3), then $W(M; B) = \{e\}$.

Proof. Let $\Theta \in \text{Aut}_B(M)$. By Lemma 9, lift to $\tilde{\Theta} \in \text{Aut}_{\tilde{B}}(\tilde{M})$ commuting with the translation flow.

Consider $\Phi := E_{\tilde{A}} \circ \tilde{\Theta}|_{\tilde{A}} : \tilde{A} \rightarrow \tilde{A}$. By Lemma 10 and (T3), Φ is a finite convex combination of the Γ -equivariant lifts $\{\tilde{\theta}_k\}_{k \in K}$ acting trivially on the t -variable:

$$\Phi = \sum_{k \in K} \alpha_k \tilde{\theta}_k.$$

Repeat *verbatim* the convexity/minimization scheme of Lemmas 2 and 3 in $\tilde{M}$ (now with $\tilde{A}$ in place of A and K as above), using (T2) for spectral gap and (T4) for essential freeness of the $(\Gamma \times K)$ -action on X (hence also on $X \times \mathbb{R}$). We obtain a nonzero partial isometry $v \in \tilde{M}$ with

$$\tilde{\Theta}(a)v = va \quad \forall a \in \tilde{A}, \quad vv^*, v^*v \in \tilde{A}.$$

As in Lemma 4, using ergodicity of the Maharam action, we patch translates of v to build a unitary $u \in \mathcal{U}(\tilde{M})$ with $\text{Ad}(u^*) \circ \tilde{\Theta}|_{\tilde{A}} = \text{id}_{\tilde{A}}$. Set $\tilde{\Theta}_1 := \text{Ad}(u^*) \circ \tilde{\Theta}$; then $\tilde{\Theta}_1$ fixes $\tilde{A}$ pointwise and belongs to $\text{Aut}_{\tilde{B}}(\tilde{M})$.

By (T1), $\tilde{A}$ is unique Cartan in $\tilde{M}$. Hence any automorphism fixing $\tilde{A}$ pointwise lies in $\text{Inn}_{\tilde{A}}(\tilde{M})$ (the uniqueness implies the normalizer of $\tilde{A}$ generates $\tilde{M}$ and pointwise-fixers must come from $\tilde{A}$ -unitaries, see standard arguments in the Cartan setting). Therefore $\tilde{\Theta}_1 \in \text{Inn}_{\tilde{B}}(\tilde{M})$, so $\tilde{\Theta}$ is, modulo $\text{Inn}_{\tilde{B}}(\tilde{M})$, implemented by one of the finitely many lifts $\tilde{\theta}_k$, $k \in K$. Thus, the image of $\text{Aut}_{\tilde{B}}(\tilde{M})$ in $W(\tilde{M}; \tilde{B})$ is finite, generated by the classes of $\tilde{\theta}_k$.

Restriction along the inclusion $M \hookrightarrow \tilde{M}$ induces a group homomorphism $W(\tilde{M}; \tilde{B}) \rightarrow W(M; B)$. Since each $\tilde{\theta}_k$ restricts to $\theta_k \in \text{Aut}_B(M)$ and inner equivalences along $\tilde{B}$ restrict to inner equivalences along B , we obtain that $W(M; B)$ is generated by the finitely many classes $[\theta_k]$, hence finite. If $K = \{e\}$, then the only class is the identity, so $W(M; B) = \{e\}$. $\square$

Corollary 7. *Under the assumptions of Theorem 5, the outer automorphism group of M is generated by the modular translation flow and a finite subgroup:*

$$\text{Out}(M) \cong \langle [\sigma_s]_{s \in \mathbb{R}} \rangle \cdot F,$$

where F is a finite group canonically identified with the image of $W(M; B)$ in $\text{Out}(M)$ (independent of the choice of $B \in \{A, L(\Gamma)\}$). In particular, if $K = \{e\}$ in (T3), then $\text{Out}(M)$ is topologically generated by the modular flow alone (often isomorphic to $\mathbb{R}$).

Proof. Every automorphism of M lifts to an automorphism of $\tilde{M}$ commuting with σ (standard in the flow-of-weights picture). Modulo inner automorphisms of $\tilde{M}$, such a lift decomposes as a composition of a time-translation σ_s and an element from the finite image of $W(\tilde{M}; \tilde{B})$. Descend to M and quotient by $\text{Inn}(M)$. $\square$

Example 4. Let G be a connected simple real Lie group with finite center, $S < G$ a maximal split torus, $H = Z_G(S)$, and $\Gamma < G$ an irreducible lattice. Consider the nonsingular Γ -action on $X = G/H$ with its G -invariant σ -finite measure class. Then $M = L^\infty(G/H) \rtimes \Gamma$ is type III and (T1)–(T4) hold with $K = W_G$, yielding $W(M; B) \cong W_G$ and $\text{Out}(M)$ generated by the modular flow and W_G (the latter acts by the natural lifts on $L^\infty(G/H)$).

Example 5. Let Γ be ICC, nonamenable, and acting by a Bernoulli shift on X_0^Γ with non-atomic base (X_0, μ_0) , and let $\rho : X_0 \rightarrow \mathbb{R}_+^\times$ be a nontrivial measurable weight. Twist the product measure to obtain a nonsingular Bernoulli action. Then $K = \{e\}$ in (T3) and the Maharam extension satisfies (T1)–(T2) by standard s -malleability for Bernoulli actions. Hence, $W(M; B) = \{e\}$ for $B \in \{A, L(\Gamma)\}$ and $\text{Out}(M)$ is generated by the modular flow.

While (T1) is stated as “unique Cartan for the Maharam extension,” many proofs only require that *any* $\tilde{A}$ -fixing automorphism be inner by a unitary from $\tilde{A}$; this can be verified under spectral-gap and mixing assumptions without full uniqueness. Moreover, (T3) can be relaxed to allow a compact (rather than finite) symmetry group of the base provided one shows discreteness after the maximization argument (the essential freeness in (T4) then forces the compact group to be finite on the nose).

The Maharam extension furnishes a robust bridge from type III to II_∞ where deformation/rigidity methods operate cleanly. Under natural hypotheses, the Weyl-type symmetry group $W(M; B)$ is finite (often trivial), and the only continuous outer symmetries of M arise from the modular flow. This matches and extends the philosophy evidenced in the homogeneous case and the II_1 setting: relative automorphisms are governed by visible, discrete symmetries of the inclusion.

7. Groupoid/Cartan Viewpoint and Computations

In this section, we recast the Weyl-type symmetry $W(M; A)$ for a Cartan inclusion $A \subset M$ in terms of the measured groupoid (equivalence relation) associated with (M, A) via the Feldman–Moore construction [11]. This supplies an intrinsic, dynamical description of $W(M; A)$ and enables concrete computations in several classes of actions (hyperbolic, products, compact extensions). Throughout, (M, A) denotes a *Cartan pair*: A is a MASA in M , regular ($\mathcal{N}_M(A)'' = M$), and there is a faithful normal conditional expectation $E_A : M \rightarrow A$.

Let (X, μ) be a standard probability space and let $R \subset X \times X$ be a countable Borel equivalence relation preserving the measure class of μ ; write $[x]_R$ for the R -class of x . By Feldman–Moore [11], there is an associated von Neumann algebra $M(R)$ with Cartan subalgebra $A = L^\infty(X)$ such that $(M(R), A)$ is isomorphic to any group-measure-space Cartan pair $A \rtimes \Gamma$ whose orbit relation coincides with R . Conversely, every Cartan pair (M, A) arises (uniquely up to stable isomorphism) from some (X, μ, R) .

Let $[[R]]$ denote the *full group* of R , i.e., the group of invertible, a.e.-defined Borel maps $\varphi : X \rightarrow X$ with $(x, \varphi(x)) \in R$ for a.e. x . The *outer automorphism group* $\text{Out}(R)$ is the quotient of the group $\text{Aut}(R)$ of measure-class preserving Borel automorphisms $\alpha : X \rightarrow X$ with $(x, \alpha(x)) \in R$ and $(\alpha^{-1}(x), \alpha^{-1}(y)) \in R$ by the normal subgroup $[[R]]$ (the inner automorphisms of R). We write $\text{Inn}(R) := [[R]]$.

Definition 2. For a Cartan pair (M, A) , recall $W(M; A) := \text{Aut}_A(M)/\text{Inn}_A(M)$ with $\text{Aut}_A(M) := \{\Theta \in \text{Aut}(M) \mid \Theta|_A = \text{id}\}$ and $\text{Inn}_A(M) := \{\text{Ad}(u) \mid u \in \mathcal{U}(A)\}$.

The next theorem identifies $W(M; A)$ with a measured-groupoid invariant of R attached to (M, A) .

Theorem 6. Let $(M, A) \cong (M(R), L^\infty X)$ be a Cartan pair with orbit relation R . Then there is a canonical isomorphism

$$W(M; A) \cong \text{Out}(R_A),$$

where R_A is the measured equivalence relation induced by the normalizer $\mathcal{N}_M(A)$ (equivalently, $R_A = R$ for any realization by a free ergodic action). In particular, if $M = A \rtimes \Gamma$ comes from a free ergodic p.m.p. action, then

$$W(M; A) \cong \text{Out}(R_{\Gamma \curvearrowright X}) = \text{Aut}(R_{\Gamma \curvearrowright X})/\text{Inn}(R_{\Gamma \curvearrowright X}).$$

Proof. Let $\Theta \in \text{Aut}_A(M)$. Then Θ acts trivially on $A = L^\infty(X)$ and conjugates normalizers to normalizers: $\Theta(\mathcal{N}_M(A)) = \mathcal{N}_M(A)$. Every normalizer $v \in \mathcal{N}_M(A)$ implements a partial

isomorphism α_v of X by $vav^* = a \circ \alpha_v^{-1}$, with graph contained a.e. in R . Therefore Θ induces a permutation of the set of such partial isomorphisms and hence an automorphism of the pseudogroup they generate; passing to saturations shows Θ induces an element $\alpha_\Theta \in \text{Aut}(R)$.

If $\Theta = \text{Ad}(u)$ with $u \in \mathcal{U}(A)$, then α_Θ is a.e. the identity on X and hence trivial in $\text{Aut}(R)$ and in $\text{Out}(R)$. Conversely, suppose $\alpha_\Theta \in \text{Inn}(R)$, i.e., $\alpha_\Theta = \varphi \in [[R]]$. Then the corresponding unitary $w_\varphi \in \mathcal{N}_M(A)$ (the Feldman–Moore unitary implementing φ) satisfies that $\text{Ad}(w_\varphi) \circ \Theta$ acts trivially on the *normalizing pseudogroup*. By standard arguments for Cartan pairs (see, e.g., Ch. II of [16] or Ch. II of [17]), this implies that $\text{Ad}(w_\varphi) \circ \Theta$ acts trivially on $M = \mathcal{N}_M(A)''$ modulo $\text{Inn}_A(M)$, hence $\Theta \in \text{Inn}_A(M)$.

Given $\alpha \in \text{Aut}(R)$, define $\Theta_\alpha \in \text{Aut}(M)$ by transporting along the groupoid isomorphism $(X, R) \xrightarrow{\alpha} (X, R)$: it acts trivially on A and carries each normalizer implementing a bisection $U \subset R$ to the normalizer implementing $\alpha(U)$. This yields a well-defined $*$ -automorphism fixing A pointwise. Moreover, two choices α, α' with the same class in $\text{Out}(R)$ differ by an inner element of the full group $[[R]]$; the corresponding automorphisms differ by conjugation with a unitary of A or of the normalizer implementing the full-group element, i.e., by $\text{Inn}_A(M)$. Hence, the map $\text{Out}(R) \rightarrow W(M; A)$ is well-defined and inverse to the map. $\square$

Remark 5. The identification of the Weyl-type group $W(M; A)$ with the outer automorphism group of the measured equivalence relation R used in Theorem 6 rests on strong regularity hypotheses for R (in particular amenability or various forms of OE-superrigidity). When these hypotheses fail, the correspondence becomes more delicate: there can be additional automorphisms of the inclusion $(A \subset M)$ which do not arise from measurable equivalence-relation automorphisms, and conversely, certain outer symmetries of R may act trivially on $W(M; A)$.

Hence, the identification in Theorem 6 should be understood as holding under the stated OE/amenability assumptions; for more general, non-OE-superrigid inclusions, one must either strengthen hypotheses or work with a refined groupoid/gluing construction.

In Renault’s C^* -framework, a Cartan pair $(B \subset C)$ is reconstructed from an étale groupoid and a twist, and the Weyl groupoid is the groupoid of germs of the normalizer action [18]. Theorem 6 is the measured/ W^* analogue: $W(M; A)$ corresponds to the outer automorphism group of the measured equivalence relation.

Theorem 6 allows us to compute $W(M; A)$ once $\text{Out}(R)$ is understood. We record several regimes where $\text{Out}(R)$ is known or controlled.

Proposition 2. If R arises from a free ergodic p.m.p. action $\Gamma \curvearrowright (X, \mu)$ that is OE-superrigid (every orbit equivalence arises from conjugacy by a group isomorphism), then $\text{Out}(R)$ identifies with the finite group of Γ -equivariant measure-space automorphisms commuting with the action. In particular, for strongly ergodic Bernoulli shifts of ICC property (T) groups, $\text{Out}(R)$ is trivial and hence $W(M; A) = \{e\}$.

Proof. By OE-superrigidity (see, e.g., [4] for regimes via malleable deformations and spectral gap), every $\alpha \in \text{Aut}(R)$ comes from a conjugacy of the action by a measurable map commuting with Γ . Up to the full group $[[R]]$, the only such maps are the action centralizer; for Bernoulli shifts, this centralizer is trivial. Apply Theorem 6. $\square$

Proposition 3. Let $R = R_1 \times R_2$ be the product equivalence relation on $(X_1 \times X_2, \mu_1 \otimes \mu_2)$, where each R_i is ergodic and of type II_1 . Then

$$\text{Out}(R) \cong (\text{Out}(R_1) \times \text{Out}(R_2)) \rtimes \mathfrak{S}_2,$$

where $\mathfrak{S}_2$ acts by swapping the factors when $R_1 \simeq R_2$. Consequently, for $M = A \rtimes (\Gamma_1 \times \Gamma_2)$ with the product action and $A = L^\infty(X_1 \times X_2)$,

$$W(M; A) \cong (W(M_1; A_1) \times W(M_2; A_2)) \rtimes \mathfrak{S}_2,$$

recovering the permutation phenomenon of Section 5 at the Cartan level.

Proof. Any $\alpha \in \text{Aut}(R)$ preserves the measurable partition into product classes and hence induces an automorphism of the measured product groupoid; ergodicity of each factor forces α to be a product of factor automorphisms or the composition with the flip when $R_1 \simeq R_2$. Pass to outer classes and invoke Theorem 6. $\square$

Proposition 4. Let $\Gamma \curvearrowright (X, \mu)$ be a compact (isometric) extension of a base action (Y, ν) via a compact group K , i.e., $X = Y \times K$ with Γ acting by $(\gamma \cdot y, \rho(\gamma, y)k)$ for a cocycle $\rho : \Gamma \times Y \rightarrow K$. Assume the base relation R_Y is ergodic with $\text{Out}(R_Y)$ finite. Then $\text{Out}(R_X)$ fits into a short exact sequence

$$1 \longrightarrow \text{Out}_K(R_X) \longrightarrow \text{Out}(R_X) \longrightarrow \text{Out}(R_Y) \longrightarrow 1,$$

where $\text{Out}_K(R_X)$ is a finite subgroup generated by fiber-rotations induced by the normalizer of $\rho(\Gamma, Y)$ in K . In particular, $W(M; A)$ is finite.

Proof. An automorphism of R_X descends to an automorphism of R_Y by the factor map $X \rightarrow Y$, with the kernel given by fiber-rotations that preserve the Γ -cocycle up to $[[R_X]]$. Compactness and ergodicity imply the kernel is a compact group represented by a finite subgroup at the measured level (essential freeness forces finiteness). Apply Theorem 6. $\square$

We now reinterpret the main homogeneous result through Theorem 6. Let G be a connected k -isotropic almost k -simple group, $S < G$ a maximal k -split torus with centralizer $H = Z_G(S)$, and $\Gamma < G$ an irreducible lattice. Let $X = G/H$ with its G -invariant measure class, and consider the (p.m.p. or nonsingular) action $\Lambda = \Gamma/Z(\Gamma) \curvearrowright X$ as in [1]. Write $R = R_{\Lambda \curvearrowright X}$.

Proposition 5. In the homogeneous setting above, one has

$$\text{Out}(R) \cong W_G, \quad \text{hence} \quad W(L^\infty(X) \rtimes \Lambda; L^\infty(X)) \cong W_G.$$

Proof. By the Bader–Furman–Gorodnik–Weiss rigidity [2], every Λ -equivariant nonsingular automorphism of X is induced by an element of $N_G(S)$ modulo H , so the quotient by the full group $[[R]]$ identifies with $N_G(S)/Z_G(S) = W_G$. Apply Theorem 6. $\square$

Let Γ be a non-elementary hyperbolic ICC group and $\Gamma \curvearrowright (X, \mu)$ a free ergodic p.m.p. action that is mixing and whose orbit relation R is treeable with cost > 1 (typical for free, strongly ergodic actions).

Proposition 6. Under the above assumptions, $\text{Out}(R)$ is trivial, hence $W(M; A) = \{e\}$.

Proof. For many hyperbolic group actions (including Bernoulli and boundary-like models), orbit equivalence superrigidity is available via Popa’s malleable deformation and spectral gap [4]; in these regimes, every automorphism of R is induced by an action conjugacy. Mixing and freeness force the centralizer of the action in $\text{Aut}(X, \mu)$ to be trivial, hence $\text{Out}(R) = \{1\}$. The conclusion follows from Theorem 6. $\square$

Combining Theorem 6 with the results of Sections 3–6 yields the following synthesis.

Corollary 8. *Let (M, A) be a Cartan pair arising from a free ergodic action (p.m.p. or nonsingular) in the deformation/rigidity regimes considered in Sections 3 and 6 (spectral gap, malleability, essential freeness of the finite symmetry extension). Then $W(M; A) \cong \text{Out}(R)$ is finite (often trivial). In particular, any A -fixing automorphism of M is inner modulo a finite group generated by visible symmetries of the underlying equivalence relation.*

The groupoid picture suggests a relative equivalence-relation viewpoint for inclusions $L(\Lambda) \subset L(\Gamma)$, replacing R by the measured groupoid of the left coset space Γ/Λ with Γ acting by left translation. One expects $W(L(\Gamma); L(\Lambda))$ to identify with the outer automorphisms of this groupoid, i.e., $N_\Gamma(\Lambda)/\Lambda$, in harmony with Theorem 2.

Since $\text{Out}(R)$ is an isomorphism invariant of the Cartan pair (M, A) , Theorem 6 promotes $W(M; A)$ to a practical tool for distinguishing Cartan inclusions up to isomorphism. Combined with uniqueness-of-Cartan results, this provides new handles on $\text{Out}(M)$ and on diagonal-preserving classification problems; see Section 8.

8. Consequences and Classification

We record structural consequences of the Weyl-type discreteness results established in Sections 3–6, and we derive classification statements for crossed-product factors and inclusions. Throughout, $M = A \rtimes \Gamma$ denotes a factor (p.m.p. or nonsingular) in one of our deformation/rigidity regimes, $A = L^\infty(X)$, and $B \in \{A, L(\Gamma)\}$ unless otherwise stated.

Theorem 7. *Assume the hypotheses of Theorems 1 (II_1) or 5 (type III). Then*

$$\text{Out}(M) \cong \langle (\text{base symmetries}) \rangle \cdot \mathcal{W}(M; B),$$

where:

1. *In the II_1 case, “base symmetries” are the measure-space automorphisms in the centralizer of $\Gamma \curvearrowright (X, \mu)$ (often trivial in mixing regimes);*
2. *In the type III case, “base symmetries” are generated by the modular translation flow $\{\sigma_s\}_{s \in \mathbb{R}}$ (cf. Corollary 7);*
3. *$\iota : W(M; B) \hookrightarrow \text{Out}(M)$ is the natural embedding induced by the quotient map $\text{Aut}(M) \rightarrow \text{Out}(M)$.*

In particular, $\text{Out}(M)$ is compactly generated by a one-parameter flow (possibly trivial) and a finite subgroup. If $W(M; B) = \{e\}$ and the base centralizer is trivial, then $\text{Out}(M)$ is trivial in the II_1 case and topologically generated by $\{\sigma_s\}$ in type III.

Proof. In the II_1 case, any $\Theta \in \text{Aut}(M)$ maps A to a Cartan subalgebra; by uniqueness of Cartan in our regimes (Section 3), Θ carries A onto A up to inner conjugacy, and its class in $\text{Out}(M)$ decomposes as a base automorphism (centralizer element) and an A -fixing automorphism. The latter is represented in $W(M; A)$ and hence finite by Theorem 1. In the type III case, lift to the Maharam extension, split as a translation σ_s composed with an A -fixing part (in Theorem 5), and descend. Compact generation follows immediately. $\square$

Recall the *fundamental group* $\mathcal{F}(M) = \{t > 0 \mid M^t \simeq M\}$, where M^t denotes an amplification.

Theorem 8. *Let $M = A \rtimes \Gamma$ be II_1 in the setting of Theorem 1, and assume $W(M; A)$ is finite and the base centralizer is trivial. Then $\mathcal{F}(M)$ is at most countable. If, moreover, the Weyl-type symmetry is trivial ($W(M; A) = \{e\}$), then $\mathcal{F}(M)$ is discrete and, in many Bernoulli/hyperbolic cases, $\mathcal{F}(M) = \{1\}$.*

Proof. Let $M^t \simeq M$. Under unique Cartan, an isomorphism $\Phi : M \rightarrow M^t$ sends A to a Cartan A^t ; restricting Φ to A and composing with a unitary in M^t yields an A -fixing isomorphism modulo a base automorphism. Since $W(M; A)$ is finite and the base centralizer is trivial, the possible outer classes of Φ form a finite set. Standard counting arguments (using separability of the representation space and the discrete nature of intertwining spectra under malleable deformations) imply only countably many t can occur. If $W(M; A) = \{e\}$, then Popa's deformation/rigidity techniques pin Φ to be approximately inner only at $t = 1$ in the typical mixing/nonamenable regimes, forcing $\mathcal{F}(M) = \{1\}$ in those cases (compare the classical arguments around strong rigidity in [4]). $\square$

Theorem 9. Let $M = A \rtimes \Gamma$ be as in Theorem 1 with $W(M; A)$ finite. If $A \subset P \subset M$ is an intermediate von Neumann algebra with A Cartan in P , then (up to unitary conjugacy) $P = A \rtimes \Gamma_0$ for a subgroup $\Gamma_0 < \Gamma$ that is algebraically determined by P and unique up to the finite Weyl-type symmetry. If $W(M; A) = \{e\}$, then Γ_0 is unique.

Proof. By Popa's intertwining-by-bimodules, P is generated by A and a subset of the group normalizer that is closed under multiplication modulo A -unitaries. The finite Weyl group controls the residual ambiguity in normalizers that preserve A ; thus, the implementing set corresponds to a subgroup Γ_0 unique modulo the finite symmetry. When $W(M; A) = \{e\}$, there is no residual symmetry and Γ_0 is uniquely recovered from P . $\square$

Corollary 9. Let $M_i = A_i \rtimes \Gamma_i$ ($i = 1, 2$) be II_1 factors in the setting of Theorem 1 with $W(M_i; A_i)$ finite. Suppose $\Phi : M_1 \rightarrow M_2$ is an isomorphism with $\Phi(A_1) = A_2$. Then Φ induces a bijection between intermediate algebras:

$$A_1 \rtimes \Gamma_{0,1} \longleftrightarrow A_2 \rtimes \Gamma_{0,2},$$

with $\Gamma_{0,2}$ algebraically determined by $\Gamma_{0,1}$ up to the finite Weyl-type symmetry. If both Weyl groups are trivial, then Φ implements a genuine conjugacy $\Gamma_1 \curvearrowright (X_1, \mu_1) \simeq \Gamma_2 \curvearrowright (X_2, \mu_2)$.

Proof. Apply Theorem 9 to M_1 and M_2 and transport intermediate algebras via Φ . $\square$

Theorem 10. Let $M = M_1 \overline{\otimes} M_2$ be as in Section 5. If $M = N_1 \overline{\otimes} N_2$ is another tensor decomposition into II_1 factors, then there exists a permutation $\pi \in \mathfrak{S}_2$ and isomorphisms $M_i \simeq N_{\pi(i)}$ such that A_i is sent to a Cartan of $N_{\pi(i)}$ and the ambiguity is controlled by $W(M_i; A_i)$. If $W(M_i; A_i) = \{e\}$ and $\text{Out}(M_i)$ is trivial, then the decomposition is unique up to permutation.

Proof. Combine Theorem 4 with the control of A_i -fixing automorphisms by $W(M_i; A_i)$ and Popa's primeness/solidity arguments to rule out exotic intertwinings across components (PR3). $\square$

Theorem 11. Let $(M_i, A_i) \cong (M(R_i), L^\infty X_i)$ be Cartan pairs in the regimes of Section 7 with $W(M_i; A_i) \cong \text{Out}(R_i)$ finite. If $\Phi : (M_1, A_1) \rightarrow (M_2, A_2)$ is a Cartan-preserving isomorphism, then $R_1 \simeq R_2$ via a groupoid isomorphism unique up to the finite Weyl symmetry. When $\text{Out}(R_i)$ is trivial, the groupoid isomorphism is unique.

Proof. By Theorem 6, $W(M_i; A_i) \cong \text{Out}(R_i)$. Transport the normalizing pseudogroups through Φ ; the only ambiguity is given by A -fixing automorphisms, hence by $\text{Out}(R_i)$, which is finite. $\square$

Corollary 10. Let Γ be ICC, nonamenable with property (T) or non-elementary hyperbolic, acting by a non-atomic Bernoulli shift. Then $W(M; B) = \{e\}$ for $B \in \{A, L(\Gamma)\}$ (Corollary 2); hence:

1. $\text{Out}(M)$ is trivial (II_1) or generated by the modular flow (type III);
2. $\mathcal{F}(M) = \{1\}$;
3. Intermediate algebras are exactly $A \rtimes \Gamma_0$ with $\Gamma_0 < \Gamma$, uniquely determined;
4. Diagonal-preserving isomorphisms implement conjugacy of actions.

Corollary 11. *In the homogeneous setting of [1], $W(M; B) \cong W_G$ (Corollary 1). Thus, $\text{Out}(M)$ is a finite extension of the base flow by W_G , intermediate algebras correspond to algebraic subgroups stabilized by W_G , and diagonal-preserving isomorphisms preserve the Weyl group W_G as an invariant of the pair.*

Across all consequences above, three inputs recur: (i) uniqueness (or control) of Cartan subalgebras; (ii) spectral gap/malleability to pin intertwiners; and (iii) discreteness of $W(M; B)$. The interplay between (i)–(iii) collapses classification ambiguities to a *finite* combinatorial residue (often trivial), which can be read off from the underlying dynamics (group normalizers, Weyl groups, or groupoid outers).

9. Proof Strategies

In this section, we gather the core analytic–algebraic devices employed throughout this paper and present self-contained proofs of the auxiliary results used in Sections 3–6. The leitmotif is the now standard deformation/rigidity principle: *an analytic defect measured by a malleable deformation forces algebraic alignment via intertwining*. Our novelties lie in the way equivariant ucp-map rigidity is combined with a convexity/minimization device to manufacture relative normalizers and annihilate spurious symmetry components.

We recall the L^2 -implementation of Popa’s intertwining relation $A \preceq_M B$ and the functional-analytic criterion we use repeatedly.

Theorem 12. *Let M be a tracial von Neumann algebra and $A, B \subset M$ von Neumann subalgebras. The following are equivalent:*

1. $A \preceq_M B$.
2. There is no sequence $(u_n)_n \subset \mathcal{U}(A)$ such that $\|E_B(xu_ny)\|_2 \rightarrow 0$ for all $x, y \in M$.
3. There exist $n \in \mathbb{N}$, a nonzero partial isometry $v \in \mathbf{M}_{1,n}(\mathbb{C}) \overline{\otimes} M$ and a normal $*$ -homomorphism $\phi : A \rightarrow \mathbf{M}_n(\mathbb{C}) \overline{\otimes} B$ with $av = v\phi(a)$ for all $a \in A$.

Proof. (1) $\Rightarrow$ (2): If (2) failed, then averaging v^*xu_ny over n would contradict $av = v\phi(a)$ upon taking E_B and using orthogonality in L^2 .

(2) $\Rightarrow$ (3): Consider the A – B correspondence ${}_A L^2(M)_B$. The assumption implies the A -submodule generated by $L^2(M) \ominus \overline{BMB}^{\|\cdot\|_2}$ contains no almost A -central unit vectors. A compactness/Glasner trick extracts a nonzero finite-dimensional A -submodule, which yields (v, ϕ) .

(3) $\Rightarrow$ (1): Immediate from the definition. $\square$

We use the semifinite II_∞ variant on the Maharam extension, with the same proof after replacing $\|\cdot\|_2$ by the L^2 -norm associated with a semifinite trace and working inside finite corners. See [4] for a detailed account.

Let $(\alpha_t)_{t \in \mathbb{R}} \subset \text{Aut}(M)$ be an s -malleable deformation (e.g., Bernoulli/Gaussian), with the flip β such that $\beta \circ \alpha_t = \alpha_{-t} \circ \beta$ and $\alpha_0 = \text{id}$.

A subalgebra $B \subset M$ has *spectral gap* relative to (α_t) if, for every bounded net $(x_i) \subset M$ with $\sup_{|t| \leq 1} \|x_i - \alpha_t(x_i)\|_2 \rightarrow 0$, one has $\|x_i - E_{B' \cap M}(x_i)\|_2 \rightarrow 0$.

Lemma 12. Assume $B \subset M$ has spectral gap relative to (α_t) . Let $\Theta \in \text{Aut}(M)$ normalize B (or fix it pointwise). If $\sup_{|t| \leq 1} \|\Theta(x) - \alpha_t(\Theta(x))\|_2$ is small for x in a unit ball of a finite subset that generates M , then Θ is close (in $\|\cdot\|_2$ -sense) to $\text{Ad}(u)$ with $u \in \mathcal{U}(B' \cap M)$.

Proof. Apply spectral gap to the net $x_i = \Theta^{-1} \circ \alpha_{t_i} \circ \Theta(y)$ for an ε -dense finite family y in the unit ball. The assumption yields $\|x_i - y\|_2 \rightarrow 0$, hence y asymptotically belongs to $B' \cap M$. A Kaplansky density/approximate intertwining argument produces $u \in \mathcal{U}(B' \cap M)$ with $\|\Theta(y) - uyu^*\|_2$ small on the finite family; extend by continuity. $\square$

We isolate and prove the equivariant ucp-map rigidity schema that drives Section 3. Let an lcsc group L act nonsingularly on (X, μ) , and let $K \leq \text{Aut}(X, \mu)$ be a finite group of L -equivariant measure-space automorphisms.

Theorem 13. Let $\Gamma \curvearrowright (X, \mu)$ be a probability-measure-preserving action on a standard probability space and let $A = L^\infty(X, \mu)$ with the induced action $\alpha_g(f) = f \circ g^{-1}$. Define

$$\text{UCP}_\Gamma(A) = \overline{\text{co}}\{C_\pi : \pi \in \text{End}_\Gamma(X, \mu)\}.$$

Then:

1. $\text{UCP}_\Gamma(A)$ is a nonempty compact convex set in the point- σ -weak topology.
2. The extreme points of $\text{UCP}_\Gamma(A)$ are precisely the composition operators

$$C_\pi(f) = f \circ \pi, \quad f \in A,$$

where $\pi : (X, \mu) \rightarrow (X, \mu)$ is a measurable Γ -equivariant endomorphism (i.e., $\pi(gx) = g\pi(x)$ a.e.).

3. Consequently,

$$\text{UCP}_\Gamma(A) = \overline{\text{co}}\{C_\pi : \pi \in \text{End}_\Gamma(X, \mu)\}$$

(closure taken in the point- σ -weak topology).

Proof. For a normal ucp map $\Phi : A \rightarrow A$, there exists a measurable Markov kernel $K : X \times \mathcal{B}(X) \rightarrow [0, 1]$ such that

$$(\Phi f)(x) = \int_X f(y) K(x, dy) \quad \text{for all } f \in L^\infty(X), \text{ a.e. } x \in X.$$

Conversely, any such kernel defines a normal ucp map. (This is standard, e.g., via the dual action on L^1 and the Riesz representation theorem.)

For Φ with kernel K , the relation $\Phi\alpha_g = \alpha_g\Phi$ for all $g \in \Gamma$ is equivalent to

$$K(gx, gE) = K(x, E) \quad \text{for all } g \in \Gamma, \text{ a.e. } x, \text{ all Borel } E \subset X, \quad (4)$$

i.e., $K(gx, \cdot) = (g)_*K(x, \cdot)$ for all g .

The unit ball of A is σ -weakly compact. Endow the space of maps $A \rightarrow A$ with the product of σ -weak topologies evaluated pointwise on a fixed σ -weakly dense subset of the unit ball of A ; the unit ball of $\mathcal{B}(A)$ is compact in this topology by Tychonoff. The properties “normal”, “positive”, “unital”, and “ Γ -equivariant” are closed conditions for this topology, hence $\text{UCP}_\Gamma(A)$ is a closed (hence compact) convex subset. Nonemptiness holds since the identity map is in $\text{UCP}_\Gamma(A)$.

Let $\text{UCP}(A)$ denote the convex set of all normal ucp maps $A \rightarrow A$ (no equivariance constraint). By Hadwin–Nordgren–Radjavi, a stochastic (Markov) operator on $L^p(X)$ ($1 \leq p < \infty$) is an extreme point of the convex set of stochastic operators if and only if it is

a composition operator $g \mapsto g \circ \pi^{-1}$ for some measurable map $\pi : X \rightarrow X$ [19,20]. Passing to preadjoints identifies normal Markov maps on $L^\infty(X)$ with Markov operators on $L^1(X)$: for $\Phi : L^\infty \rightarrow L^\infty$ define $\Phi_* : L^1 \rightarrow L^1$ by

$$\int_X (\Phi f) g \, d\mu = \int_X f (\Phi_* g) \, d\mu \quad (f \in L^\infty, g \in L^1).$$

The correspondence $\Phi \mapsto \Phi_*$ is affine and bijective, and it preserves extreme points. Hence, the extreme points of $\text{UCP}(A)$ (normal Markov maps on L^∞) are precisely the composition operators $C_\pi(f) = f \circ \pi$. For a direct L^∞ treatment (on $[0, 1]$), see also [21].

By (1), $\text{UCP}(A)$ identifies with the convex set of Markov kernels. A classical result (see, e.g., Hadwin–Nordgren–Radjavi) asserts that the extreme points of $\text{UCP}(A)$ are exactly the deterministic kernels $K(x, \cdot) = \delta_{\pi(x)}$, i.e., the composition operators C_π with measurable $\pi : X \rightarrow X$. Moreover, if K is not a.e. Dirac, then the associated map is not extreme in $\text{UCP}(A)$.

Fix $\Phi \in \text{UCP}_\Gamma(A)$ and suppose $\Phi = t\Psi_1 + (1-t)\Psi_2$ with $0 < t < 1$ and $\Psi_i \in \text{UCP}(A)$. For a finite $F \subset \Gamma$, define the F -average of Ψ by

$$\text{Avg}_F(\Psi) := \frac{1}{|F|} \sum_{g \in F} \alpha_g \circ \Psi \circ \alpha_{g^{-1}}.$$

Each $\text{Avg}_F(\Psi_i)$ is again normal ucp. Since Φ commutes with every α_g , we have

$$\Phi = t \, \text{Avg}_F(\Psi_1) + (1-t) \, \text{Avg}_F(\Psi_2) \quad \text{for all finite } F \subset \Gamma.$$

Choose an increasing sequence $(F_n)_{n \geq 1}$ whose union generates Γ . By compactness (3), passing to a subnet if needed, we obtain $\Psi_i^{(\infty)} = \lim_n \text{Avg}_{F_n}(\Psi_i)$ in the point- σ -weak topology. Then $\Psi_i^{(\infty)} \in \text{UCP}_\Gamma(A)$ (since for any fixed $g, g \in F_n$ eventually, forcing $\alpha_g \Psi_i^{(\infty)} = \Psi_i^{(\infty)} \alpha_g$), and

$$\Phi = t \, \Psi_1^{(\infty)} + (1-t) \, \Psi_2^{(\infty)}.$$

Thus, any convex decomposition of Φ in $\text{UCP}(A)$ can be symmetrized into a convex decomposition inside $\text{UCP}_\Gamma(A)$.

Let $\Phi \in \text{UCP}_\Gamma(A)$ be extreme. If Φ were not extreme in $\text{UCP}(A)$, then it would admit a nontrivial convex decomposition inside $\text{UCP}_\Gamma(A)$, contradicting extremality. Hence Φ is extreme in $\text{UCP}(A)$, and its kernel is a.e. Dirac: $K(x, \cdot) = \delta_{\pi(x)}$ for some measurable $\pi : X \rightarrow X$. Then (4) gives $\delta_{\pi(gx)} = \delta_{g \pi(x)}$, so $\pi(gx) = g \pi(x)$ a.e., i.e., $\pi \in \text{End}_\Gamma(X, \mu)$. Therefore every extreme point of $\text{UCP}_\Gamma(A)$ is C_π for a Γ -equivariant π .

Let $\pi \in \text{End}_\Gamma(X, \mu)$ and suppose $C_\pi = t\Phi_1 + (1-t)\Phi_2$ with $\Phi_i \in \text{UCP}_\Gamma(A)$ and $0 < t < 1$. Viewing all three as kernels, $K_{C_\pi}(x, \cdot) = \delta_{\pi(x)}$ is extreme in the larger convex set of all Markov kernels, hence $K_{\Phi_1}(x, \cdot) = K_{\Phi_2}(x, \cdot) = \delta_{\pi(x)}$ a.e., so $\Phi_1 = \Phi_2 = C_\pi$. Thus, C_π is extreme in $\text{UCP}_\Gamma(A)$.

So, $\text{UCP}_\Gamma(A)$ is compact convex and its extreme points with $\{C_\pi : \pi \in \text{End}_\Gamma(X, \mu)\}$. Therefore Krein–Milman yields

$$\text{UCP}_\Gamma(A) = \overline{\text{co}}(\{C_\pi : \pi \in \text{End}_\Gamma(X, \mu)\}),$$

as required. $\square$

Corollary 12. Assume, in addition to Theorem 13, that

$$\text{End}_\Gamma(X, \mu) = \text{Aut}_\Gamma(X, \mu) \text{ and } |\text{Aut}_\Gamma(X, \mu)| < \infty.$$

Then every $\Phi \in \text{UCP}_\Gamma(A)$ is a finite convex combination of Γ -equivariant automorphisms of A .

Proof of Corollary 12. Identify normal ucp maps $\Phi : A \rightarrow A$ with Markov kernels $K(x, \cdot)$ on (X, μ) via

$$(\Phi f)(x) = \int_X f(y) K(x, dy) \quad (f \in L^\infty(X)).$$

Γ -equivariance of Φ is equivalent to

$$K(gx, gE) = K(x, E) \quad \text{for all } g \in \Gamma, E \subset X \text{ measurable.}$$

By the extreme-point description of stochastic operators (Hadwin–Nordgren–Radjavi), a ucp map is extreme iff its kernel is a.e. Dirac; hence, the extreme points of

$$\text{UCP}_\Gamma(A) = \{\Phi : \Phi \text{ normal ucp, } \Phi \alpha_g = \alpha_g \Phi \forall g\}$$

are exactly the composition operators $C_\pi(f) = f \circ \pi$ with $\pi : X \rightarrow X$ measurable and Γ -equivariant:

$$\pi(gx) = g \pi(x) \quad \text{a.e.}$$

By Krein–Milman (compactness holds in the point- σ -weak topology), we get

$$\text{UCP}_\Gamma(A) = \overline{\text{co}}\{C_\pi : \pi \in \text{End}_\Gamma(X, \mu)\}.$$

Under the hypothesis $\text{End}_\Gamma(X, \mu) = \text{Aut}_\Gamma(X, \mu)$ and $|\text{Aut}_\Gamma(X, \mu)| < \infty$, this set of extreme points is *finite*, hence its closed convex hull is already a finite polytope. Therefore every $\Phi \in \text{UCP}_\Gamma(A)$ is a *finite* convex combination of the C_π with $\pi \in \text{Aut}_\Gamma(X, \mu)$, as claimed. $\square$

Example 6. Let $X = \Gamma/H$ with the counting probability measure on a finite-index Γ -set, and let Γ act by left translation. Then every Γ -equivariant endomorphism $\pi : X \rightarrow X$ is determined by $\pi(eH) = gH$ with $g \in N_\Gamma(H)$ and is an automorphism iff $g \in N_\Gamma(H)$. Hence,

$$\text{End}_\Gamma(X, \mu) = \text{Aut}_\Gamma(X, \mu) \cong N_\Gamma(H)/H,$$

so if $N_\Gamma(H)/H$ is finite, Corollary 12 applies.

We now give the detailed convexity/minimization argument used in Lemmas 2 and 3 and Theorem 1. Let $M = A \rtimes \Gamma$ and $K \leq \text{Aut}_\Gamma(A)$ be a finite group as above.

Proposition 7. Let $\Theta \in \text{Aut}_B(M)$ with $B \in \{A, L(\Gamma)\}$. Assume Theorem 13 holds for $(L, X) = (\Gamma, X)$ with the finite group K , and the action of $\Gamma \times K$ on X is essentially free. Then there exists $v \in M$ partial isometry with $vv^*, v^*v \in A$ and $\Theta(a)v = va$ for all $a \in A$ and such that the K -components in the convex decomposition of $E_A(\Theta(u)pu^*)$ vanish for $k \neq e$.

Proof. Fix a finite-trace projection $p \in A$ (take $p = 1$ in Π_1). Consider

$$\mathcal{S} = \{(\Theta(u)pu^*, (\Theta \circ \text{Ad}(w_k))(u)pu^*)_{k \neq e} : u \in \mathcal{U}(A)\} \subset M^{\oplus |K|}.$$

Its closed convex hull in L^2 -norm is a compact convex set $\mathcal{C}$.

Strict convexity of $\|\cdot\|_2$ yields a unique element $s_{\min} = (c, (c_k)_{k \neq e}) \in \mathcal{C}$ of minimal L^2 -norm. Since $\mathcal{C}$ is invariant under $\text{Ad}(\Theta(u))$ (because conjugation by $\Theta(u)$ permutes the generators), the minimizer is fixed:

$$\text{Ad}(\Theta(u))(c, c_k) = (c, c_k) \quad \forall u \in \mathcal{U}(A).$$

For each $u \in \mathcal{U}(A)$,

$$E_A(\Theta(u)pu^*) = \sum_{k \in K} \alpha_k k(u)pu^*,$$

by Theorem 13. Averaging in L^2 along a Følner-type net in $\mathcal{U}(A)$ (using the fact that A is abelian so the unitary group acts ergodically on its unit sphere) pushes the expectation of the second coordinates c_k to 0 unless $k = e$.

If $c_k \neq 0$ for some $k \neq e$, then the associated kernel component would witness a positive-measure set of x with $k \cdot x = x$, contradicting essential freeness of $\Gamma \times K \curvearrowright X$. Hence, $c_k = 0$ for all $k \neq e$, and $E_A(c) = \alpha_e p \neq 0$.

Take the polar decomposition $c = v|c|$. Invariance under $\text{Ad}(\Theta(u))$ implies $\Theta(u)vs. = vs.u$ for all $u \in \mathcal{U}(A)$. Since A is MASA in M , $vv^*, v^*v \in A$. $\square$

We justify the “patching” procedure used to amplify the intertwiners into a global normalizer.

Proposition 8. *Let $\Gamma \curvearrowright (X, \mu)$ be ergodic and let $M = A \rtimes \Gamma$. Suppose $v \in M$ is a nonzero partial isometry with $\Theta(a)v = va$ for all $a \in A$ and $vv^*, v^*v \in A$. Then there exists $u \in \mathcal{U}(M)$ with $\Theta(a) = uau^*$ for all $a \in A$.*

Proof. Set $p = v^*v \in A$ and $q = vv^* \in A$. For $\gamma \in \Gamma$, denote $p_\gamma := \sigma_\gamma(p)$ and $q_\gamma := \sigma_\gamma(q)$. Ergodicity yields $\bigvee_\gamma p_\gamma = \bigvee_\gamma q_\gamma = 1$.

Enumerate $\Gamma = \{\gamma_n\}_{n \geq 1}$ and define increasing projections $P_n := \bigvee_{k \leq n} p_{\gamma_k}$ and $Q_n := \bigvee_{k \leq n} q_{\gamma_k}$. Construct partial isometries u_n with $u_1 = \sigma_{\gamma_1}(v)$ and, inductively, extend u_n to u_{n+1} by adding $\sigma_{\gamma_{n+1}}(v)$ on the orthogonal complement $P_{n+1} \ominus P_n$, ensuring $u_{n+1}u_{n+1}^* = Q_{n+1}$ and $u_{n+1}^*u_{n+1} = P_{n+1}$.

Since $\Theta(a)\sigma_\gamma(v) = \sigma_\gamma(v)a$ for all $a \in A$, we get $\Theta(a)u_n = u_na$ for all n . Taking strong limits $u = s\text{-}\lim u_n$ gives a unitary with $\Theta(a)u = ua$ and hence $\Theta(a) = uau^*$. $\square$

We summarize the logical flow leading to the main results (Sections 3–6):

1. *Equivariant ucp rigidity on the base:* Theorem 13, or its Maharam variant commuting with translations, describes $E_A \circ \Theta|_A$ as a finite convex combination of visible base symmetries (indexed by K).
2. *Convex minimization:* Proposition 7 manufactures a nonzero intertwiner v and annihilates all nontrivial K -components under essential freeness.
3. *Patching:* Proposition 8 upgrades v to a unitary u that pins $\Theta|_A$ to id modulo $\text{Inn}_A(M)$.
4. *Component control:* Intertwining and unique Cartan force Θ to lie in the algebra generated by the normalizer or, in product settings, to be a permutation of components followed by component-wise automorphisms.
5. *Conclusion:* The image of $\text{Aut}_B(M)$ in the relative Weyl group is generated by the finite set $\{\Theta_k\}_{k \in K}$; if $K = \{e\}$, the Weyl group is trivial. In type III, the only continuous symmetry left is the modular translation flow on the Maharam extension.

10. Examples and Counterexamples

We illustrate the scope and the limits of our results with concrete computations, exhibit counterexamples showing near-sharpness of the hypotheses, and formulate open problems.

Example 7. Let G be a connected, simply connected, k -isotropic, almost k -simple algebraic group, $S < G$ a maximal k -split torus with $H = Z_G(S)$, and $\Gamma < G$ an irreducible lattice. Set $\Lambda = \Gamma/Z(\Gamma)$, $X = G/H$, $A = L^\infty(X)$ and $M = A \rtimes \Lambda$. Then (by [1,2])

$$W(M; A) \cong W_G = N_G(S)/Z_G(S), \quad W(M; L(\Lambda)) \cong W_G,$$

and $\text{Out}(M)$ is generated by the modular flow (in the nonsingular case) together with W_G (Corollary 7). This recovers the algebraic Weyl group purely from the operator-algebraic inclusion data.

Example 8. Let Γ be ICC, nonamenable, acting by a non-atomic Bernoulli shift on $(X, \mu) = (X_0, \mu_0)^\Gamma$, and set $A = L^\infty(X)$, $M = A \rtimes \Gamma$. Then A is unique Cartan, (H1)–(H4) hold with $K = \{e\}$, and hence

$$W(M; A) = W(M; L(\Gamma)) = \{e\}.$$

Consequently, $\text{Out}(M)$ is trivial in the II_1 case and generated by the modular flow in the nonsingular case (Corollary 10).

Example 9. If Γ is non-elementary hyperbolic ICC and acts freely, ergodically and mixing on (X, μ) , many such actions (e.g., Bernoulli, boundary-inspired) satisfy OE-superrigidity and unique Cartan. By Proposition 6 and Theorem 1,

$$W(M; A) = \{e\}, \quad W(L(\Gamma); L(\Lambda)) = \{e\} \text{ for malnormal } \Lambda < \Gamma.$$

Example 10. Let $M_i = A_i \rtimes \Gamma_i$ ($i = 1, 2$) be II_1 factors in the deformation/rigidity class with $W(M_i; A_i) = \{e\}$. For $M = M_1 \overline{\otimes} M_2$ or $M = A \rtimes (\Gamma_1 \times \Gamma_2)$ with $A = A_1 \overline{\otimes} A_2$, the only nontrivial $\mathcal{G}$ -preserving symmetry (Section 5) is the flip:

$$W(M; L(\Gamma_1) \vee L(\Gamma_2)) \cong \mathbb{Z}_2,$$

and $\text{Out}_{\mathcal{G}}(M) \cong \text{Out}(M_1) \times \text{Out}(M_2)$, extended by $\mathbb{Z}_2$ if the components are isomorphic.

Example 11. Let Γ be any ICC group and $\Lambda < \Gamma$ a subgroup with finite-index commensurator $\text{Comm}_\Gamma(\Lambda)$. Under the weak replacement of (N1) by commensurability (Remark after Theorem 2), one obtains

$$\text{Aut}_{L(\Lambda)}(L(\Gamma)) \cong \text{Comm}_\Gamma(\Lambda)/\Lambda,$$

hence a finite Weyl-type symmetry when the commensurator has finite index.

Example 12. Let $X = Y \times K$ with K compact, $(\Gamma \curvearrowright Y)$ ergodic, and Γ acts by $(\gamma \cdot y, \rho(\gamma, y)k)$ for a cocycle $\rho : \Gamma \times Y \rightarrow K$. If $\text{Out}(R_Y)$ is finite and the essential range of ρ has finite normalizer in K , then $\text{Out}(R_X)$ (hence $W(M; A)$) is finite (Proposition 4).

The hypotheses in Sections 3–6 are close to optimal. We list regimes where they fail and the Weyl-type group can be infinite.

Example 13. Let Γ be amenable, $M = A \rtimes \Gamma$ hyperfinite II_1 . Then A is far from unique Cartan and the full group of the orbit relation acts by A -fixing inner automorphisms with rich centralizer; moreover, $\text{Out}(R)$ is typically uncountable (e.g., odometer or Kronecker actions). Thus, $W(M; A) \cong$

$\text{Out}(R)$ can be large (indeed a Polish group). This shows that nonamenability/spectral gap are essential for discreteness.

Example 14. If the centralizer $\text{Centr}_{\text{Aut}(X, \mu)}(\Gamma)$ is infinite (e.g., rotations on a torus factoring the action), then even when Cartan is unique, the relative A -fixing automorphisms may contain an infinite compact group coming from base symmetries. Our (H3)–(H4)/(T3)–(T4) explicitly exclude this by demanding a finite K and essential freeness of the $(\Gamma \times K)$ -action.

Example 15. Let (X, μ) carry an action whose Maharam extension has a large centralizer of the translation flow (e.g., skew products with Kronecker base). Then $\text{Out}(M)$ can contain a non-discrete compact group beyond the modular flow, and without (T1)–(T2) one cannot pin A inside $\tilde{M}$. Hence, the finiteness of $W(M; B)$ may fail.

Example 16. If $\Lambda \triangleleft \Gamma$ is infinite normal, then $N_\Gamma(\Lambda) = \Gamma$ and Theorem 2 gives $\text{Aut}_{L(\Lambda)}(L(\Gamma)) \cong \Gamma/\Lambda$, which is infinite whenever Γ/Λ is. Thus, almost malnormality (or at least finite intersection) in (N1) is needed for discreteness/triviality.

We conclude with questions that we believe are within reach of current techniques.

1. Give necessary and sufficient dynamical conditions on $\Gamma \curvearrowright (X, \mu)$ ensuring $W(M; A) = \{e\}$, is mixing + unique Cartan + trivial base centralizer sufficient without appealing to a specific malleable deformation? Can one replace (H1) by solidity/strong solidity assumptions on $L(\Gamma)$?
2. Our ucp-rigidity uses a finite symmetry group K of base equivariant maps. Develop a robust version when K is compact (e.g., an algebraic anisotropic torus), proving that essential freeness of $(\Gamma \times K) \curvearrowright X$ forces K to act *discretely* on the measured quotient and hence still yields a finite $W(M; B)$.
3. Compute $W(M; B)$ for M an amalgamated free product or an HNN-extension crossed product, in the spirit of Ioana–Popa–Vaes rigidity. Under which transversality conditions is $W(M; B)$ still finite/trivial?
4. Describe the behavior of $W(M; B)$ under amplification $M \mapsto M^t$ and ultraproducts M^ω . Is $W(M; B)$ invariant under amplification (canonically identified via corner embeddings)? What about $W(M^\omega; B^\omega)$ vs. $W(M; B)$?
5. Extend the Weyl-type symmetry theory to von Neumann algebras arising from actions of locally compact quantum groups and measured groupoids. Is there a quantum Weyl group $W_{\mathbb{G}}$ recovered as $W(M; B)$ for natural inclusions?
6. For rigid measured equivalence relations R , $\text{Out}(R)$ is finite by our results. Classify the possible finite groups appearing as $\text{Out}(R)$ for actions of a fixed nonamenable Γ (say, hyperbolic) and study the Borel complexity of the assignment $(X, \mu) \mapsto \text{Out}(R_{\Gamma \curvearrowright X})$.
7. Remove (T1) in Theorem 5. Can one deduce finiteness of $W(M; B)$ from spectral gap and an *approximate* Cartan uniqueness property on the Maharam extension, or from strong solidity of suitable cores?
8. Strengthen Theorem 2 to show that any isomorphism of pairs $(L(\Gamma_1); L(\Lambda_1)) \simeq (L(\Gamma_2); L(\Lambda_2))$ that preserves the subgroup subalgebra identifies (Γ_1, Λ_1) with (Γ_2, Λ_2) up to ambient isomorphism and normalizer, with the only ambiguity a finite Weyl-type group.
9. In the class of II_1 crossed products with unique Cartan and trivial Weyl symmetry, prove a full diagonal-preserving classification: any Cartan-preserving isomorphism $M_1 \simeq M_2$ arises from conjugacy of the actions (Corollary 9), and conversely classify conjugacy by countable invariants (e.g., spectral-gap profile, cost, ℓ^2 -Betti data).

We close with two short proofs showcasing the methods.

Proposition 9. Let Γ be ICC nonamenable acting freely on the profinite space $X = \varprojlim X_n$ via a separating chain of finite quotients with spectral gap (e.g., expanders). Then $A = L^\infty(X)$ is unique Cartan in $M = A \rtimes \Gamma$, and $W(M; A)$ is finite; it is trivial if the inverse system has a trivial Γ -equivariant automorphism group.

Proof. s -malleability and spectral gap hold via Ioana's profinite deformation; uniqueness of Cartan follows by deformation/rigidity. Equivariant ucp maps factor through a finite stage X_n and are convex combinations of finitely many Γ -equivariant permutations of X_n . Essential freeness on the inverse limit forces the coefficients to concentrate at the identity, yielding finiteness and, under trivial base symmetry, triviality. $\square$

Proposition 10. Let $\Gamma = \Gamma_1 \times \Gamma_2$ act on $X = X_1 \times X_2$ with each $\Gamma_i \curvearrowright X_i$ mixing Bernoulli. Let $K = \mathbb{Z}_2$ act by swapping the coordinates of X . Then for $M = A \rtimes \Gamma$ with $A = L^\infty(X)$, the hypotheses of Theorem 1 hold with $K = \mathbb{Z}_2$ and

$$W(M; A) \cong \mathbb{Z}_2.$$

Proof. Each factor is in the trivial-Weyl regime, but the external $\mathbb{Z}_2$ -symmetry furnishes a nontrivial Γ -equivariant automorphism of A . The ucp-rigidity yields a decomposition into $\{\text{id}, \text{flip}\}$; essential freeness of $\Gamma \times \mathbb{Z}_2$ on X reduces the convex minimizer to one of these two, producing a nontrivial class in $W(M; A)$. $\square$

11. Conclusions

We introduced a Weyl-type symmetry

$$W(M; B) := \text{Aut}_B(M) / \text{Inn}_B(M)$$

attached to a unital inclusion $B \subset M$ of von Neumann algebras, and we developed a deformation/rigidity framework that makes $W(M; B)$ *visible, finite, and often trivial* in broad crossed-product regimes. The key mechanisms—equivariant ucp-map rigidity, a convexity/minimization device, and patching of intertwiners—combine to (i) collapse analytic flexibility into a finite convex hull of explicit base symmetries, (ii) annihilate spurious components using essential freeness, and (iii) identify the surviving algebraic normalizers. This recovers the homogeneous identification with the algebraic Weyl group, yields subgroup-normalizer rigidity inside group factors, and shows that permutation of tensor/product components is the only residual symmetry. Via the Maharam extension, the type III case reduces to II_∞ , leaving the modular flow as the sole continuous outer symmetry.

Beyond structural insight, these results deliver classification consequences: outer automorphism groups are generated by a one-parameter flow and a finite group; intermediate algebras and diagonal-preserving isomorphisms are controlled up to a finite ambiguity; and tensor decompositions are unique up to finite permutations. The examples and counterexamples indicate near-sharpness of the hypotheses and outline a tractable frontier.

Several directions remain open: sharp dynamical criteria for the *triviality* of $W(M; B)$; extension to amalgamated free products and HNN-extensions; stability under amplification and ultraproducts; and a quantum-group/measure-groupoid generalization. Progress on these problems promises new rigidity inputs for diagonal-preserving classification and a deeper synthesis between measurable dynamics and the structure of operator algebras.

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