

Article

Modulation Spaces with Variable Smoothness and Integrability

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Abstract

This paper introduces modulation spaces with variable smoothness and integrability, defined via frequency-uniform decomposition operators and mixed Lebesgue-sequence spaces. Since the conventional dyadic decomposition is replaced by a uniform one, a new theoretical foundation is required. Therefore, we first introduce a new sequence of functions and establish some important results related to these functions, which are fundamental to our analysis. We then demonstrate that the definition of these modulation spaces is independent of the choice of basis functions. Furthermore, we establish several embedding theorems and prove the completeness properties of these spaces.

Keywords: modulation space; variable smoothness; variable integrability**MSC:** 42B35; 46E35

1. Introduction

The theory of function spaces is pivotal to harmonic analysis and provides indispensable tools for investigating both ordinary and partial differential equations. Central to the modern theory of function spaces is the technique of frequency localization, which typically relies on two primary partitions of the Euclidean space $\mathbb{R}^n$: the dyadic decomposition $\mathbb{R}^n = \{x : |x| < 1\} \cup (\cup_{j=1}^{\infty} \{x : |x| \in [2^{j-1}, 2^j)\})$, and the uniform decomposition $\mathbb{R}^n = \cup_{k \in \mathbb{Z}^n} (k + [-1/2, 1/2]^n)$. These partitions naturally induce the corresponding dyadic and uniform decomposition operators. Then, together with the definition of $\ell^q(L^p)$, the Besov space $B_{p,q}^s$ and the modulation space $M_{p,q}^s$ can be defined.

First introduced by Feichtinger [1] in 1983 via short-time Fourier transform on locally compact Abelian groups, the modulation spaces $M_{p,q}^s$ were conceived as a novel function space theory, offering a viable alternative to the established Besov spaces. In the beginning, the investigations of modulation spaces were focused on fundamental theories as well as the analogy between Besov spaces and modulation spaces. However, in recent times, there has been a growing recognition of their utility in characterizing the time-frequency behavior of functions. Consequently, researchers have demonstrated significant interest in modulation spaces, α -modulation spaces and their applications, as seen in the works [2–9] and the references cited therein. In particular, from a PDE perspective, Wang and collaborators [10–12] demonstrated the crucial role of combining frequency-uniform decomposition operators with Banach function spaces of the form $\ell^q(X(\mathbb{R}^n))$ in the derivation of nonlinear estimates, where X denotes a Banach function space on $\mathbb{R}^n$.

In parallel, variable exponent function spaces have attracted considerable interest in the recent literature. While the investigation of variable Lebesgue spaces dates back to



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the seminal work of Orlicz [13,14], their modern resurgence began with the influential paper by O. Kováčik and J. Rákosník in 1991 [15]. Subsequently, the results presented in [15] were re-established in [16] via the methodology of Musielak–Orlicz spaces. These foundations have led to the extensive exploration of variable Lebesgue and Sobolev spaces, as documented in works such as [17–21].

The Besov spaces with variable smoothness $B_{p,p}^{s(\cdot)}(\mathbb{R}^n)$ were introduced in [22,23] by Leopold and Schrohe when they studied pseudo-differential operators, and Besov generalized them to the case $B_{p,q}^{s(\cdot)}(\mathbb{R}^n)$ and $F_{p,q}^{s(\cdot)}(\mathbb{R}^n)$ in [24–26]. Xu began the study of variable exponent Besov and Triebel–Lizorkin spaces, specifically the spaces $B_{p(\cdot),q}^{s(\cdot)}(\mathbb{R}^n)$ and $F_{p(\cdot),q}^{s(\cdot)}(\mathbb{R}^n)$, in [27,28], later establishing their atomic decompositions in [29]. A significant advancement occurred in [30] when Diening, Hästö and Roudenko introduced the function spaces featuring both variable smoothness and integrability, specifically the Triebel–Lizorkin space $F_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$. This was followed by the introduction of the corresponding Besov space $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ by Almeida and Hästö in [31]. Since these foundational works, the field has expanded significantly, with numerous contributions such as [32–34]. Furthermore, Yang, Zhuo, and Yuan generalized these concepts to Triebel–Lizorkin-type and Besov-type spaces with variable exponents, $F_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$ and $B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$ in [35,36]. In recent years, the landscape of variable exponent spaces has continued to grow, encompassing weighted versions like $B_{p(\cdot),q(\cdot)}^{s(\cdot),\omega}(\mathbb{R}^n)$ and $F_{p(\cdot),q(\cdot)}^{s(\cdot),\omega}(\mathbb{R}^n)$, Hardy spaces $H^{*,p(\cdot)}(\mathcal{X})$ on RD-spaces, variable Besov spaces associated with heat kernels, as well as variable matrix-weighted Besov spaces; see [37–41]. In [42], the authors studied weighted variable modulation spaces $M_{p(\cdot),q(\cdot),a}(\omega)$ by some maximal functions. Additionally, variable exponent function spaces have demonstrated significant utility in a broad range of scientific fields. Notable applications can be found in the study of fluid dynamics [43], the realm of image processing [44], and within the framework of partial differential equations [45].

Motivated by the preceding research, this paper introduces a novel family of modulation spaces characterized by variable smoothness $s(\cdot)$ and variable integrability exponents $p(\cdot), q(\cdot)$, denoted by $M_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$, which serves as a generalization of classical modulation spaces. In [42], the function spaces $M_{p(\cdot),q(\cdot),a}(\omega)$ is defined by means of certain maximal functions in the case where $s = 1$. By contrast, in this paper we introduce a more general framework, defining the function spaces $M_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$ via frequency-uniform decomposition operators under the assumption that $s(\cdot)$ is locally log-Hölder continuous. Moreover, the introduction of $M_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$ opens up a natural pathway for investigating its connections with $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$, enhancing our understanding of the structure of such function classes. In this paper, we also derive several fundamental properties of these spaces, such as embeddings and completeness. Additionally, we characterized the duals of these variable exponent modulation spaces in [46].

This paper is organized as follows. Section 2 provides a comprehensive review of the fundamental concepts and notations pertaining to semimodular spaces, variable Lebesgue spaces, and the framework of mixed Lebesgue-sequence spaces. Section 3 presents several technical lemmas that play a crucial role in this paper. A key distinction in our approach is the utilization of a uniform decomposition of $\mathbb{R}^n$ during the definition of $M_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$, rather than the traditional dyadic decomposition. This necessitates the development of new techniques, as the standard η -function methods employed in references such as [30,31,35–38] are no longer directly applicable. To address this issue, we first introduce a new sequence of functions, namely θ -functions. Subsequently, we establish several useful lemmas concerning these functions. In Section 4, with the aid of frequency-uniform decomposition operators and mixed Lebesgue-sequence spaces, we introduce the spaces $M_{p(\cdot),q(\cdot)}^{s(\cdot)}$. A key result demonstrated therein is that the definition of these spaces does

not depend on the specific choice of $\{\varphi_k\}_{k \in \mathbb{Z}^n} \in \mathcal{Y}$ or the associated frequency-uniform decomposition operators. In Section 5, we prove several elementary embedding properties about modulation spaces with variable exponents. These properties serve as the foundation for deriving the embeddings among modulation spaces $M_{p(\cdot),q(\cdot)}^{s(\cdot)}$, Schwartz function space $\mathcal{S}$ and its dual space $\mathcal{S}'$. Finally, we show the completeness of $M_{p(\cdot),q(\cdot)}^{s(\cdot)}$.

2. Preliminaries

This section is devoted to recalling essential definitions, establishing notation, and presenting fundamental results. The constant C , appearing repeatedly throughout the paper, is independent of the main variables and may assume different values in different contexts. The relation $A \sim B$, implies that A and B are comparable, meaning there exists a constant $C > 0$ such that $1/C \leq A/B \leq C$. The notation $A \lesssim B$ is used to express the inequality $A \leq CB$. For a real number s , $[s]$ denotes the floor function of s . We adopt the standard conventions $\mathbb{N} \equiv \{1, 2, \dots\}$ and $\mathbb{Z}_+ \equiv \mathbb{N} \cup \{0\}$. For $x \in \mathbb{R}^n$, we define the functions $\langle x \rangle = (1 + |x|^2)^{1/2}$ and $\langle x \rangle_o = 1 + |x_1| + |x_2| + \dots + |x_n|$, noting that $\langle x \rangle \sim \langle x \rangle_o$. Given a multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, we use the standard notation $D^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_n^{\alpha_n}$. Furthermore, for $k = (k_1, k_2, \dots, k_n) \in \mathbb{R}^n$, $|k|_\infty = \max_{i=1, \dots, n} |k_i|$ denotes the supremum norm of k .

Let $\mathcal{S} := \mathcal{S}(\mathbb{R}^n)$ denote the Schwartz space of rapidly decreasing functions, and let $\mathcal{S}' := \mathcal{S}'(\mathbb{R}^n)$ be its strong dual, consisting of all tempered distributions. For $f \in \mathcal{S}$, the Fourier transform $\mathcal{F}f$ and its inverse $\mathcal{F}^{-1}f$ are defined respectively as follows:

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx, \quad \mathcal{F}^{-1}f(x) = \int_{\mathbb{R}^n} f(\xi) e^{2\pi i x \cdot \xi} d\xi.$$

Throughout this paper, ℓ^p stands for the Lebesgue sequence space, while $L^p := L^p(\mathbb{R}^n)$ represents the Lebesgue space of measurable functions on $\mathbb{R}^n$, with norm $\|\cdot\|_p$.

2.1. Modular Spaces

In the subsequent discussion, let X denote a vector space defined over the real or complex field. Our analysis of function spaces is carried out within the context of semi-modular spaces. We direct the reader to [19] for a thorough and detailed exposition of the underlying theory.

Definition 1. Let $\varrho : X \rightarrow [0, \infty]$ be a function, and it is a semimodular on X provided it fulfills the following conditions:

- (i) $\varrho(0) = 0$;
- (ii) For all $f \in X$ and all scalars $\lambda \in \mathbb{R}$ or $\mathbb{C}$ with $|\lambda| = 1$, $\varrho(\lambda f) = \varrho(f)$;
- (iii) If $\varrho(\lambda f) = 0$ for all $\lambda > 0$, then $f = 0$;
- (iv) For any $f \in X$, the mapping $\lambda \mapsto \varrho(\lambda f)$ is left-continuous on the interval $[0, \infty)$.

A semimodular ϱ is referred to as a modular when $\varrho(f) = 0$ forces $f = 0$. It is said to be continuous provided $\lambda \mapsto \varrho(\lambda f)$ is continuous on the interval $[0, \infty)$ for all $f \in X$. Additionally, a semimodular ϱ is said to possess the (quasi)convexity property if for any $f, g \in X$ and $\theta \in [0, 1]$, the inequality

$$\varrho(\theta f + (1 - \theta)g) \leq A[\theta \varrho(f) + (1 - \theta)\varrho(g)]$$

holds for some constant A . Specifically, the case $A = 1$ is convexity, while $A \in [1, \infty)$ is quasiconvexity (see [31] (p. 1631)).

Remark 1. Let ϱ be a convex semimodular on X and $f \in X$. Then

- (i) $\varrho(\lambda f) = \varrho(|\lambda|f) \leq |\lambda|\varrho(f)$ for all $|\lambda| \leq 1$.
- (ii) $\varrho(\lambda f) = \varrho(|\lambda|f) \geq |\lambda|\varrho(f)$ for all $|\lambda| \geq 1$.

A semimodular gives rise to a (semi)modular space, defined as follows:

Definition 2. Let ϱ be a (semi)modular defined on X . The (semi)modular space associated with ϱ is defined as

$$X_\varrho := \{f \in X : \exists \lambda > 0, \text{ s.t. } \varrho(\lambda f) < \infty\}.$$

The following conclusions can be founded in [19,31].

Lemma 1 (see [31] (Theorem 2.3)). For a (quasi)convex semimodular ϱ on X , the space X_ϱ becomes a (quasi)normed space when endowed with the Luxemburg (quasi)norm, which is given by

$$\|f\|_\varrho := \inf\{\lambda > 0 : \varrho(f/\lambda) \leq 1\}.$$

By definition, the infimum over the empty set is equal to infinity.

Lemma 2 (see [19] (Lemma 2.1.9)). Suppose ϱ is a (quasi)convex semimodular on X , and let $\{x_k\}$ be a sequence in X_ϱ . Then, the convergence $\|x_k\|_\varrho \rightarrow 0$ as $k \rightarrow \infty$ is equivalent to the condition that

$$\lim_{k \rightarrow \infty} \varrho(\lambda x_k) = 0$$

holds for every $\lambda > 0$.

By virtue of the definition and the left-continuity property, we derive the subsequent relationship, which facilitates the treatment of certain intricate norm definitions.

Lemma 3 (see [19] (Lemma 2.1.14). Norm-modular unit ball property). For a semimodular ϱ on X and $f \in X$, $\|f\|_\varrho \leq 1$ is equivalent to $\varrho(f) \leq 1$. Moreover, when ϱ is continuous, the strict inequality $\|f\|_\varrho < 1$ is equivalent to $\varrho(f) < 1$, and equality $\|f\|_\varrho = 1$ is equivalent to $\varrho(f) = 1$.

Corollary 1 (see [19] (Corollary 2.1.15)). Given a semimodular ϱ on X and $f \in X$, the following properties hold:

- (i) $\|f\|_\varrho \leq 1$ implies $\varrho(x) \leq \|f\|_\varrho$.
- (ii) $1 < \|f\|_\varrho$ implies $\|f\|_\varrho \leq \varrho(f)$.
- (iii) $\|f\|_\varrho \leq \varrho(f) + 1$.

2.2. Function Spaces of Variable Exponents

In this section, we begin with the variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$. A measurable function $p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty]$ is classified as a variable exponent function if it satisfies $p(x) > c$ for some $c > 0$. The family of all such functions is denoted by $\mathcal{P}_0$, and we use $\mathcal{P}$ to signify the subclass where $p(\cdot)$ takes values in $[1, \infty]$. Let $p(\cdot)$ be a measurable function and $\Omega \subset \mathbb{R}^n$ a measurable set. We introduce the notation:

$$p_\Omega^- := \operatorname{ess\,inf}_\Omega p(x) \quad \text{and} \quad p_\Omega^+ := \operatorname{ess\,sup}_\Omega p(x).$$

When the domain is the entire space $\mathbb{R}^n$, we simply write p^- and p^+ for p_Ω^- and p_Ω^+ , respectively.

Let f be a locally integrable function on $\mathbb{R}^n$. We define its Hardy-Littlewood maximal function as follows:

$$Mf(x) := \sup_{x \in B} \frac{1}{|B|} \int_B |f(y)| dy, \quad \forall x \in \mathbb{R}^n,$$

where the supremum is taken over all balls B centered at x , and $|B|$ denotes the Lebesgue measure of B . To ensure the boundedness of the Hardy-Littlewood maximal function on variable exponent Lebesgue spaces, it is necessary to impose additional constraints on the exponent function. Specifically, this requires the exponent to satisfy the log-Hölder continuity condition, a concept initially introduced in [21].

Definition 3. Let $p(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}$ be a given function.

- (i) The function $p(\cdot)$ is said to be locally log-Hölder continuous, denoted by $p \in C_{\text{loc}}^{\log}$, provided that

$$|p(x) - p(y)| \leq \frac{c_{\log}}{\log(e + 1/|x - y|)} \quad (1)$$

for some $c_{\log} > 0$ and all $x, y \in \mathbb{R}^n$.

- (ii) If, in addition to being locally log-Hölder continuous, $p(\cdot)$ satisfies

$$|p(x) - p_{\infty}| \leq \frac{c_{\log}}{\log(e + |x|)} \quad (2)$$

for some $p_{\infty} \in \mathbb{R}$ and all $x \in \mathbb{R}^n$, then it is referred to as globally log-Hölder continuous, written as $p \in C^{\log}$.

Remark 2.

- (i) For brevity, in this article, we will use abbreviations to denote assumptions about relevant functions. For example, when referring to a variable exponent function $p(\cdot)$, we simplify the notation by writing $p \in \mathcal{P}$ instead of $p(\cdot) \in \mathcal{P}$.
- (ii) Clearly, the membership $p \in C_{\text{loc}}^{\log}$ implies that $p \in L^{\infty}$.
- (iii) For $p \in \mathcal{P}_0$ and $p^+ \in (0, \infty)$, $p \in C^{\log}$ is equivalent to $1/p \in C^{\log}$. Moreover, the validity of (2) ensures that $p_{\infty} = \lim_{|x| \rightarrow \infty} p(x)$.

We define the class $\mathcal{P}^{\log}$ as the collection of all variable exponents $p \in \mathcal{P}$ for which $\frac{1}{p} \in C^{\log}$. A corresponding definition applies to $\mathcal{P}_0^{\log}$.

We introduce the function $\varphi_p(t)$ as follows:

$$\varphi_p(t) = \begin{cases} t^p & \text{if } p \in (0, \infty), \\ 0 & \text{if } p = \infty \text{ and } t \leq 1, \\ \infty & \text{if } p = \infty \text{ and } t > 1, \end{cases}$$

with the understanding that $1^{\infty} = 0$ which guarantees the left-continuity of φ_p . For a measurable function f on $\mathbb{R}^n$, its variable exponent modular is defined as

$$\varrho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} \varphi_{p(x)}(|f(x)|) dx.$$

Following Definition 2, the associated semimodular space $L^{p(\cdot)}(\mathbb{R}^n)$, which is referred to as the variable exponent Lebesgue space, is thereby defined.

Remark 3. Let $p \in \mathcal{P}_0$.

(i) The Luxemburg (quasi)norm on $L^{p(\cdot)}(\mathbb{R}^n)$ is given by

$$\|f\|_{p(\cdot)} := \inf \left\{ \lambda > 0 : \varrho_{p(\cdot)} \left(\frac{f}{\lambda} \right) \leq 1 \right\}.$$

This space forms a Banach space provided that $p^- \in [1, \infty]$.

(ii) For a measurable set $\Omega \subset \mathbb{R}^n$ with positive measure. $L^{p(\cdot)}(\Omega)$ is the set of measurable functions f on Ω for which $\varrho_{p(\cdot)}(\lambda f)$ is finite for some $\lambda > 0$:

$$L^{p(\cdot)}(\Omega) := \left\{ f \text{ is measurable on } \Omega : \varrho_{p(\cdot)}(\lambda f) < \infty \text{ for some } \lambda > 0 \right\}.$$

Lemma 4 (Generalized Hölder inequality, see [18,19]). Suppose $p \in \mathcal{P}(\mathbb{R}^n)$. Then, for every pair of functions $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, the following inequality holds:

$$\int_{\mathbb{R}^n} |f(x)g(x)| dx \leq \left(1 + \frac{1}{p^-} - \frac{1}{p^+} \right) \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)}.$$

The exponent $p'(\cdot)$ is the conjugate of $p(\cdot)$, characterized by the identity $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ for all $x \in \mathbb{R}^n$, and we adopt the convention $1/\infty := 0$.

Remark 4.

- (i) Membership of $p(\cdot)$ in $\mathcal{C}^{\log}$ implies that its conjugate exponent $p'(\cdot)$ is also an element of $\mathcal{C}^{\log}$.
- (ii) (See [19] (Lemma 4.6.3)) Assume $p \in \mathcal{P}^{\log}$. Then, for any $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and any nonnegative, radially decreasing $g \in L^1(\mathbb{R}^n)$, the following convolution inequality holds:

$$\|f * g\|_{p(\cdot)} \leq C \|f\|_{p(\cdot)} \|g\|_1,$$

where $C > 0$ is a constant that does not depend on the functions f and g .

We now turn our attention to the mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$, a construction due to Almeida and Hästö [31].

Definition 4. Given $p, q \in \mathcal{P}_0$ and a measurable domain $\Omega \subset \mathbb{R}^n$, the mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)}(\Omega))$ consists of those sequences $\{f_j\}_{j \in \mathbb{N}}$ of $L^{p(\cdot)}(\Omega)$ -valued functions satisfying

$$\left\| \{f_j\}_j \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(\Omega))} := \inf \left\{ \lambda > 0 : \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})} \left(\{f_j \chi_\Omega / \lambda\}_j \right) \leq 1 \right\} < \infty.$$

The corresponding modular $\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ is defined via the expression

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}(\{f_j\}_j) := \sum_{j \in \mathbb{N}} \inf \left\{ \mu_j > 0 : \varrho_{p(\cdot)} \left(\frac{f_j}{\mu_j^{1/q(\cdot)}} \right) \leq 1 \right\}$$

under the assumption that $\lambda^{1/\infty} = 1$ for any $\lambda > 0$.

Remark 5. Suppose $p, q \in \mathcal{P}_0$.

(i) When $q^+ < \infty$, the following identity holds:

$$\inf \left\{ \lambda > 0 : \varrho_{p(\cdot)} \left(f / \lambda^{\frac{1}{q(\cdot)}} \right) \leq 1 \right\} = \left\| |f|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}}.$$

Following the convention established in [31], we denote the left-hand side by this expression even in the case $q^+ = \infty$. Consequently, we frequently utilize the simplified form:

$$\mathcal{Q}_{\ell q(\cdot)(L^{p(\cdot)})}(\{f_j\}_j) = \sum_j \left\| |f_j|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}}.$$

- (ii) According to Proposition 3.3 in [31], if $q \in (0, \infty]$ is a constant exponent, then the norm satisfies:

$$\left\| \{f_j\}_j \right\|_{\ell q(L^{p(\cdot)})} = \left\| \left\{ \|f_j\|_{p(\cdot)} \right\}_j \right\|_{\ell q}.$$

- (iii) Suppose $q \in C_{\text{loc}}^{\log}$ with $q_+ = \infty$, then condition (1) readily implies that $q(x) = \infty$ everywhere on $\mathbb{R}^n$. This means the norm $\|\cdot\|_{\ell q(\cdot)(L^{p(\cdot)})}$ coincides with $\|\cdot\|_{\ell^\infty(L^{p(\cdot)})}$.
- (iv) As shown in [31] (Proposition 3.5), $\mathcal{Q}_{\ell q(\cdot)(L^{p(\cdot)})}$ defines a semimodular. It is worth noting that it is a modular provided that $p^+ < \infty$, and it exhibits continuity whenever $p^+, q^+ < \infty$.
- (v) Almeida and Hästö [31] established that $\|\cdot\|_{\ell q(\cdot)(L^{p(\cdot)})}$ defines a quasi-norm for arbitrary $p(\cdot), q(\cdot) \in \mathcal{P}$. A norm structure is obtained if the pointwise inequality $\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} \leq 1$ is satisfied or if q is constant. Subsequently, Kempka and Vybíral [34] extended this result by proving that the quasi-norm becomes a norm under the conditions that $p(\cdot), q(\cdot) \in \mathcal{P}$ and either $1 \leq q(x) \leq p(x) \leq \infty$ a.e. or $p(x) \geq 1$ with q being a constant in $[1, \infty)$ almost everywhere.

3. Some Technical Lemmas

The boundedness of the Hardy–Littlewood maximal operator is well known to play a key role in classical function space theory. However, within the framework of variable exponent function spaces, this property does not generally hold. Specifically, see [31] (Example 4.1), the Hardy–Littlewood maximal operator often lacks boundedness on the mixed Lebesgue-sequence space. To address this, convolution operators involving radially decreasing kernels provide a natural fit; such kernels are precisely the η -functions referred to in many studies.

However, since we adopt uniform decomposition over dyadic decomposition in this paper, we can no longer use η -functions. This means we have to develop a new class of functions, which lead us to introduce θ -functions. In this section, we will present several lemmas and remarks related to the θ -functions, such as some properties concerning convolution and property akin to the r -trick. These properties play a crucial role in the study of the modulation spaces with variable exponents.

In this paper, we introduce a family of functions on $\mathbb{R}^n$, referred to as θ -functions, defined by

$$\theta_{k,m}(x) := \frac{r_k^n}{(1 + r_k|x|)^m}, \quad k \in \mathbb{Z}^n, \quad m > 0,$$

where $r_k := \sqrt{n}(1 + |k|_\infty)$. It is worth noting that for $m > n$, $\theta_{k,m} \in L^1$ and the norm $\|\theta_{k,m}\|_1 = c_m$ is a constant that does not depend on the index k . In [30,31], η -functions were defined by $\eta_m(x) = (1 + |x|)^{-m}$ and $\eta_{v,m}(x) = 2^{nv}\eta_m(2^v x)$, $v \in \mathbb{N}$. We can find that $\theta_{k,m}(x) = r_k^n \eta_m(r_k x)$ which are different from $\eta_{v,m}(x)$. For consistency, we write $\theta_m(x) = (1 + |x|)^{-m}$ in our paper.

Next we will give some useful results about θ -functions. These results are based on the lemmas about η -functions which were presented in [30,31].

Lemma 5. For $k_1, k_2 \in \mathbb{Z}^n$, let $r_{k_1} \geq r_{k_2}$, $m > n$, and $x \in \mathbb{R}^n$. Then

- (i) $\theta_{k_2,m}(x) \leq 2^m \theta_{k_1,m}(x)$, if $|x| \leq r_{k_1}^{-1}$,

(ii) $\theta_{k_1,m}(x) \leq 2^m \theta_{k_2,m}(x)$, if $|x| \geq r_{k_2}^{-1}$.

Proof. Let $|x| \leq r_{k_1}^{-1}$, then $1 + r_{k_1}|x| \leq 2$, and we have

$$\frac{\theta_{k_2,m}(x)}{\theta_{k_1,m}(x)} = \frac{r_{k_2}^n (1 + r_{k_1}|x|)^m}{r_{k_1}^n (1 + r_{k_2}|x|)^m} \leq \frac{r_{k_2}^n \cdot 2^m}{r_{k_1}^n} \leq 2^m.$$

If $|x| \geq r_{k_2}^{-1}$, then $1 + r_{k_2}|x| \leq 2r_{k_2}|x|$, and we can get

$$\frac{\theta_{k_1,m}(x)}{\theta_{k_2,m}(x)} = \frac{r_{k_1}^n (1 + r_{k_2}|x|)^m}{r_{k_2}^n (1 + r_{k_1}|x|)^m} \leq \frac{r_{k_1}^n (2r_{k_2}|x|)^m}{r_{k_2}^n (r_{k_1}|x|)^m} \leq \frac{r_{k_2}^{m-n} \cdot 2^m}{r_{k_1}^{m-n}} \leq 2^m.$$

□

Remark 6. If $|k_2 - k_1|_\infty \leq 1$ and $m > n$, then $\theta_{k_1,m}(x) \sim \theta_{k_2,m}(x)$. In fact, if $|k_2 - k_1|_\infty \leq 1$, then $r_{k_1} \leq 2r_{k_2}$ and $r_{k_2} \leq 2r_{k_1}$, thus we have

$$\frac{\theta_{k_2,m}(x)}{\theta_{k_1,m}(x)} = \frac{r_{k_2}^n (1 + r_{k_1}|x|)^m}{r_{k_1}^n (1 + r_{k_2}|x|)^m} \leq \frac{(2r_{k_1})^n (1 + 2r_{k_2}|x|)^m}{r_{k_1}^n (1 + r_{k_2}|x|)^m} \leq 2^n \cdot 2^m,$$

that is $\theta_{k_2,m}(x) \lesssim \theta_{k_1,m}(x)$. By the same way, we can get $\theta_{k_1,m}(x) \lesssim \theta_{k_2,m}(x)$. Therefore, we obtain $\theta_{k_1,m}(x) \sim \theta_{k_2,m}(x)$.

Lemma 6. Let $k_1, k_2 \in \mathbb{Z}^n$, $m > n$ and $x \in \mathbb{R}^n$. If we denote $k \in \{k_1, k_2\}$ satisfying $r_k = \min\{r_{k_1}, r_{k_2}\}$ by $k_\wedge$, then we get the estimate

$$\theta_{k_1,m} * \theta_{k_2,m}(x) \sim \theta_{k_\wedge,m}(x),$$

and the constants involved are determined by m and n alone.

Proof. The proof is based on Lemma A.1 in [30]. By symmetry we may assume that $r_{k_1} \geq r_{k_2}$. The condition $m > n$ ensures that $\|\theta_{k_1,m}\|_1 \leq c_m$ and $\|\theta_{k_2,m}\|_1 \leq c_m$. If $|y| \leq r_{k_1}^{-1} \leq 1$, then $r_{k_2}|y| \leq r_{k_1}|y| \leq 1$ and $1 + r_{k_2}|x - y| \leq 1 + r_{k_2}|x| + r_{k_2}|y| \leq 2(1 + r_{k_2}|x|)$, thus $\theta_{k_2,m}(x - y) \geq C \theta_{k_2,m}(x)$. Therefore, we get

$$\begin{aligned} \theta_{k_1,m} * \theta_{k_2,m}(x) &\geq \int_{\{y: |y| \leq r_{k_1}^{-1}\}} \theta_{k_2,m}(x - y) \theta_{k_1,m}(y) dy \\ &\geq C \theta_{k_2,m}(x) \int_{\{y: |y| \leq r_{k_1}^{-1}\}} r_{k_1}^n (1 + r_{k_1}|y|)^{-m} dy \\ &\geq C 2^{-m} \theta_{k_2,m}(x) \int_{\{y: |y| \leq r_{k_1}^{-1}\}} r_{k_1}^n dy \\ &\geq C 2^{-m} \theta_{k_2,m}(x). \end{aligned}$$

We proceed to establish the remaining assertion.

Define the set $E := \{y \in \mathbb{R}^n : |y| \leq 3r_{k_2}^{-1} \text{ or } |x - y| > |x|/2\}$. For $y \in E$, we have the estimate $1 + r_{k_2}|x - y| \geq \frac{1}{4}(1 + r_{k_2}|x|)$, which yields $\theta_{k_2,m}(x - y) \lesssim \theta_{k_2,m}(x)$. Consequently,

$$\int_E \theta_{k_2,m}(x - y) \theta_{k_1,m}(y) dy \lesssim \theta_{k_2,m}(x) \int_E \theta_{k_1,m}(y) dy \lesssim \theta_{k_2,m}(x).$$

Now consider the case $y \in \mathbb{R}^n \setminus E$, which implies $|y| > 3r_{k_2}^{-1} > r_{k_2}^{-1}$ and $|y| \geq |x|/2$. By Lemma 5, we have $\theta_{k_1,m}(y) \lesssim \theta_{k_2,m}(y) \lesssim \theta_{k_2,m}(x)$. Furthermore, we derive the inequality

$$\int_{\mathbb{R}^n \setminus E} \theta_{k_2,m}(x-y) \theta_{k_1,m}(y) dy \lesssim \theta_{k_2,m}(x) \int_{\mathbb{R}^n \setminus E} \theta_{k_2,m}(x-y) dy \lesssim \theta_{k_2,m}(x).$$

Finally, by summing the contributions from E and $\mathbb{R}^n \setminus E$, we conclude that

$$\theta_{k_1,m} * \theta_{k_2,m}(x) \lesssim \theta_{k_2,m}(x).$$

Therefore, $\theta_{k_1,m} * \theta_{k_2,m}(x) \sim \theta_{k_{\wedge},m}(x)$ and the proof is completed. $\square$

Remark 7. By Remark 6 and Lemma 6, if $|k_2 - k_1|_\infty \leq 1$ and $m > n$, then we have

$$\theta_{k_1,m} * \theta_{k_2,m}(x) \sim \theta_{k_1,m}(x) \sim \theta_{k_2,m}(x).$$

*Epecially, we also have $\theta_m * \theta_m(x) \sim \theta_m(x)$.*

Lemma 7. Let $s(\cdot) \in C_{\text{loc}}^{\log}$ and $k \in \mathbb{Z}^n$. Then, for any $R \geq c_{\log}(s)$, there exists a constant $C > 0$ so that

$$\langle k \rangle^{s(x)} \theta_{k,m+R}(x-y) \leq C \langle k \rangle^{s(y)} \theta_{k,m}(x-y)$$

holds for any $x, y \in \mathbb{R}^n$. Here, $c_{\log}(s)$ denotes the constant associated with $s(\cdot)$ from inequality (1).

Proof. Let $l \in \mathbb{N}$ be as minimal as possible with $r_k|x-y| \leq 2^l$ and $1 + r_k|x-y| \sim 2^l$. This implies

$$\frac{\theta_{k,m+R}(x-y)}{\theta_{k,m}(x-y)} = \frac{1}{(1 + r_k|x-y|)^R} \leq C 2^{-lR}.$$

In addition, it is straightforward to observe that $\langle k \rangle \sim r_k$; then, combining with the log-Hölder continuity of $s(\cdot)$, we have

$$\begin{aligned} \langle k \rangle^{s(y)-s(x)} &\geq \langle k \rangle^{-\frac{c_{\log}(s)}{\log(e+1/|x-y|)}} \geq C r_k^{-\frac{c_{\log}(s)}{\log(e+1/|x-y|)}} \\ &\geq C \left(2^{-l}|x-y| \right)^{\frac{c_{\log}(s)}{\log(e+1/|x-y|)}} \\ &\geq C 2^{-l c_{\log}(s)}. \end{aligned}$$

By choosing $R \geq c_{\log}(s)$ and combining above estimates, we can obtain the conclusion. $\square$

Remark 8.

(i) As the Lemma 2.1 of [42], we also have

$$b^{s(x)} \theta_{m+R}(x-y) \lesssim b^{s(y)} \theta_m(x-y)$$

for any $1 \leq b < \infty$ and $x, y \in \mathbb{R}^n$.

(ii) Lemma 7 allows us to interchange the position of the term within the convolution, resulting in the estimate

$$\langle k \rangle^{s(x)} \theta_{k,m+R} * f(x) \lesssim \theta_{k,m} * \left(\langle k \rangle^{s(\cdot)} f \right)(x).$$

Such a rearrangement facilitates the analysis of variable smoothness in various contexts.

- (iii) For $s(\cdot) \in C_{\text{loc}}^{\log}$, $k \in \mathbb{Z}^n$ and $l \in \mathbb{Z}^n$ with $|l|_{\infty} \leq 1$, we also have a result similar to the previous lemma; that is,

$$\langle k \rangle^{s(x)} \theta_{k+l, m+R}(x-y) \lesssim \langle k \rangle^{s(y)} \theta_{k+l, m}(x-y).$$

In the spirit of the r-trick from [30,31], we use the subsequent lemma to address exponents less than 1.

Lemma 8. Suppose $r > 0$, $k \in \mathbb{Z}^n$ and $m > n$. Then, for any tempered distribution $g \in \mathcal{S}'$ satisfying $\text{supp } \hat{g} \subset \{\xi : |\xi|_{\infty} \leq 1\}$, the inequality

$$|g(x)| \leq C(\theta_m * |g|^r(x))^{1/r}$$

holds for all $x \in \mathbb{R}^n$, where $C = C(r, m, n) > 0$ is a constant.

Proof. As in the proof of Lemma A.6 of [30], for $v = 0$ and a fixed dyadic cube

$$Q = Q_{0,u} := \{x \in \mathbb{R}^n : u_i \leq x_i < (u_i + 1), i = 1, 2, \dots, n\},$$

where $u \in \mathbb{Z}^n$, we obtain the estimate

$$|g(x)|^r \leq \sup_{z \in Q} |g(z)|^r \leq C \sum_{l \in \mathbb{Z}^n} (1 + |l|)^{-m} \int_{Q_{0,u+l}} |g(y)|^r dy.$$

Furthermore, if $x \in Q_{0,u}$ and $y \in Q_{0,u+l}$, then $|x - y| \sim |l|$ for all sufficiently large l . It follows that $1 + |x - y| \sim 1 + |l|$. We thus conclude

$$\begin{aligned} \sup_{z \in Q} |g(z)|^r &\leq C \sum_{l \in \mathbb{Z}^n} \int_{Q_{0,u+l}} (1 + |x - y|)^{-m} |g(y)|^r dy \\ &\leq C \int_{\mathbb{R}^n} (1 + |x - y|)^{-m} |g(y)|^r dy = C \theta_m * |g|^r(x). \end{aligned}$$

Then we get the conclusion by taking the r th root. $\square$

Remark 9. Let $k \in \mathbb{Z}^n$ and let $g \in \mathcal{S}'$ be a tempered distribution whose Fourier transform is supported in $\{\xi : |\xi - k|_{\infty} \leq 1\}$.

- (i) If we take $g_k(x) = e^{-2\pi i x \cdot k} g(x)$, then $\text{supp } \hat{g}_k \subset \{\xi : |\xi|_{\infty} \leq 1\}$; thus, by Lemma 8 we have $|g(x)| \leq C(\theta_m * |g|^r(x))^{1/r}$ as well.
- (ii) On the other hand, for $r_k = \sqrt{n}(1 + |k|_{\infty})$, we can find $v \in \mathbb{N}$ satisfying $2^v \leq r_k < 2^{v+1}$ and $\{\xi : |\xi - k|_{\infty} \leq 1\} \subset \{\xi : |\xi| \leq 2^{v+1}\}$. Choosing a fixed dyadic cube $Q = Q_{v,u} := \{x \in \mathbb{R}^n : 2^{-v}u_i \leq x_i < 2^{-v}(u_i + 1), i = 1, 2, \dots, n\}$, with $u \in \mathbb{Z}^n$, we can proceed analogously to the proof of Lemma 8 to obtain $|g(x)| \lesssim (\theta_{k,m} * |g|^r(x))^{1/r}$.

Now let us prove another lemma about θ -functions, which is useful for $\ell^{q(\cdot)}(L^{p(\cdot)})$ quasi norm.

Lemma 9. Suppose $p, q \in \mathcal{P}^{\log}$. Then, for all $m > n$ and sequences $\{f_k\}_{k \in \mathbb{Z}^n} \subset L_{\text{loc}}^1$, we have

$$\|\{\theta_{k,2m} * f_k\}_k\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leq C \|\{f_k\}_k\|_{\ell^{q(\cdot)}(L^{p(\cdot)})},$$

where $C > 0$ is a constant.

Proof. We follow the ideas in [31] (Lemma 4.7). By a standard scaling argument, we may assume without loss of generality that $\|\{f_k\}_k\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = 1$. This, in turn, implies

$$\mathcal{Q}_{\ell^{q(\cdot)}(L^{p(\cdot)})}(\{f_k\}_k) = \sum_k \left\| |f_k|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq 1$$

by virtue of Lemma 3. It remains to establish the existence of a constant $c_1 > 0$ for which

$$\left\| |c_1 \theta_{k,2m} * f_k|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq \left\| |f_k|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} + \langle k \rangle^{-(n+1)} =: \delta \quad (3)$$

holds. Notably, this is equivalent to

$$\left\| \delta^{-1} |c_1 \theta_{k,2m} * f_k|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq 1,$$

and likewise to

$$\left\| \delta^{-\frac{1}{q(\cdot)}} c_1 \theta_{k,2m} * f_k \right\|_{p(\cdot)} \leq 1. \quad (4)$$

In view of the log-Hölder continuity of $1/q$ and the inclusion $\delta \in [\langle k \rangle^{-(n+1)}, 1 + \langle k \rangle^{-(n+1)}]$, an argument analogous to that used in Lemma 7 gives

$$(\delta^{-1})^{\frac{1}{q(x)}} \theta_{k,2m}(x-y) \leq C (\delta^{-1})^{\frac{1}{q(y)}} \theta_{k,m}(x-y).$$

Consequently, the term $\delta^{-\frac{1}{q(\cdot)}}$ can be shifted inside the convolution, implying

$$\delta^{-\frac{1}{q(\cdot)}} |\theta_{k,2m} * f_k| \leq C \left| \theta_{k,m} * \left(\delta^{-\frac{1}{q(\cdot)}} f_k \right) \right|.$$

Then, according to Remark 4(ii), for an appropriate constant $c_1 > 0$, we have

$$\left\| \delta^{-\frac{1}{q(\cdot)}} c_1 \theta_{k,2m} * f_k \right\|_{p(\cdot)} \leq \left\| c_1 \theta_{k,m} * \left(\delta^{-\frac{1}{q(\cdot)}} f_k \right) \right\|_{p(\cdot)} \leq \left\| \delta^{-\frac{1}{q(\cdot)}} f_k \right\|_{p(\cdot)}. \quad (5)$$

By the definition of δ , it is easy to get

$$\left\| \left| \delta^{-\frac{1}{q(\cdot)}} f_k \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq 1,$$

which is equivalent to

$$\left\| \delta^{-\frac{1}{q(\cdot)}} f_k \right\|_{p(\cdot)} \leq 1. \quad (6)$$

Combining (5) and (6), we obtain (4), which means that (3) is true.

Notice that $\langle k \rangle \sim \langle k \rangle_o$. Furthermore, the number of lattice points $k = (k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$ such that $|k_1| + |k_2| + \dots + |k_n| = i$ is bounded by $O(i^{n-1})$. Consequently, we obtain

$$\begin{aligned} \sum_{k \in \mathbb{Z}^n} \frac{1}{\langle k \rangle^{n+1}} &\lesssim \sum_{k \in \mathbb{Z}^n} \frac{1}{\langle k \rangle_o^{n+1}} \lesssim \sum_{i=0}^{\infty} \sum_{|k_1|+|k_2|+\dots+|k_n|=i} \frac{1}{(1+i)^{n+1}} \\ &\lesssim \sum_{i=0}^{\infty} \frac{i^{n-1}}{(1+i)^{n+1}} < \infty. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{k \in \mathbb{Z}^n} \left\| |c_1 \theta_{k,2m} * f_k|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} &\leq \sum_{k \in \mathbb{Z}^n} \left\| |f_k|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} + \sum_{k \in \mathbb{Z}^n} \frac{1}{\langle k \rangle^{n+1}} \\ &\leq 1 + \sum_{k \in \mathbb{Z}^n} \frac{1}{\langle k \rangle^{n+1}} := c_2 < \infty \end{aligned}$$

in which $c_2 > 1$. Then, combining with Remark 1, we have

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})} \left(\left(\frac{c_1}{c_2} \theta_{k,2m} * f_k \right)_k \right) \leq \frac{1}{c_2} \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})} ((c_1 \theta_{k,2m} * f_k)_k) \leq 1,$$

which implies

$$\|(\theta_{k,2m} * f_k)_k\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leq C$$

with a constant $C > 0$. This completes the proof of lemma. $\square$

Remark 10.

(i) As the Lemma 2.2 in [42], we also have

$$\|(\theta_{2m} * f_k)_k\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leq C \|(f_k)_k\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}.$$

(ii) In certain situations, although the assumption $p^-, q^- \geq 1$ is needed, this condition can be weakened with the help of Lemma 8 and the identity

$$\| \{f_k\}_k \|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = \| \{|f_k|^r\}_k \|_{\ell^{\frac{q(\cdot)}{r}}(L^{\frac{p(\cdot)}{r}})}^{\frac{1}{r}}. \quad (7)$$

4. The Definition of Modulation Spaces

The definition of modulation spaces with variable exponents relies on certain background definitions from the constant exponent theory.

For each $k \in \mathbb{Z}^n$, let Q_k denote the unit cube centered at k . The collection $\{Q_k\}_{k \in \mathbb{Z}^n}$ forms a decomposition of $\mathbb{R}^n$. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be a smooth function mapping $\mathbb{R}^n$ to $[0, 1]$ such that $\phi(\xi) = 1$ for $|\xi|_\infty \leq 1/2$ and $\phi(\xi) = 0$ for $|\xi|_\infty \geq 1$. We define ϕ_k as the translation of ϕ by k :

$$\phi_k(\xi) = \phi(\xi - k), \quad k \in \mathbb{Z}^n.$$

It follows that $\phi_k(\xi) = 1$ on Q_k and $\sum_{k \in \mathbb{Z}^n} \phi_k(\xi) \geq 1$ for any $\xi \in \mathbb{R}^n$. Setting

$$\varphi_k(\xi) = \phi_k(\xi) \left(\sum_{k \in \mathbb{Z}^n} \phi_k(\xi) \right)^{-1}, \quad k \in \mathbb{Z}^n,$$

we obtain

$$\begin{cases} |\varphi_k(\xi)| \geq c, & \forall \xi \in Q_k, \\ \text{supp } \varphi_k \subset \{\xi : |\xi - k|_\infty \leq 1\}, \\ \sum_{k \in \mathbb{Z}^n} \varphi_k(\xi) \equiv 1, & \forall \xi \in \mathbb{R}^n, \\ |D^\alpha \varphi_k(\xi)| \leq C_{|\alpha|}, & \forall \xi \in \mathbb{R}^n, \alpha \in (\mathbb{N} \cup \{0\})^n. \end{cases} \quad (8)$$

We introduce the set

$$Y = \{ \{ \varphi_k \}_{k \in \mathbb{Z}^n} : \{ \varphi_k \}_{k \in \mathbb{Z}^n} \text{ satisfies (8)} \}.$$

It is clear that Y is nonempty. For each $\{\varphi_k\}_{k \in \mathbb{Z}^n} \in Y$, we can define a corresponding family of operators via

$$\square_k := \mathcal{F}^{-1} \varphi_k \mathcal{F}, \quad k \in \mathbb{Z}^n.$$

These operators are referred to as frequency-uniform decomposition operators. For $s \in \mathbb{R}$ and $0 < p, q \leq \infty$, the modulation space is defined by

$$M_{p,q}^s(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{M_{p,q}^s} = \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{sq} \|\square_k f\|_p^q \right)^{1/q} < \infty \right\}.$$

Comprehensive discussions regarding frequency-uniform decomposition methods and their relevance to partial differential equations are available in the monograph [11] as well as the papers [10,12].

Definition 5. Given $\{\varphi_k\}_{k \in \mathbb{Z}^n} \in Y$ and the associated frequency-uniform decomposition operators $\{\square_k\}_{k \in \mathbb{Z}^n}$, for $p, q \in \mathcal{P}_0^{\log}$ and $s \in C_{\text{loc}}^{\log}$, we define the modulation space with variable smoothness and integrability, denoted $M_{p(\cdot), q(\cdot)}^{s(\cdot)}$, as the collection of all tempered distributions $f \in \mathcal{S}'$ satisfying

$$\|f\|_{M_{p(\cdot), q(\cdot)}^{s(\cdot)}}^q := \left\| \left\{ \langle k \rangle^{s(\cdot)} \square_k f \right\}_k \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

Remark 11.

(i) On the basis of the aforementioned modulation spaces, we introduce the following modular:

$$\varrho_{M_{p(\cdot), q(\cdot)}^{s(\cdot)}}^p(f) := \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})} \left(\left\{ \langle k \rangle^{s(\cdot)} \square_k f \right\}_k \right).$$

Utilizing this modular, one can establish the norm of the spaces.

(ii) As a direct consequence of Remark 5(ii), setting q to be constant yields

$$\|f\|_{M_{p(\cdot), q}^{s(\cdot)}}^p = \left\| \left\{ \left\| \langle k \rangle^{s(\cdot)} \square_k f \right\|_{p(\cdot)} \right\}_k \right\|_{\ell^q}.$$

We proceed to demonstrate that the spaces introduced in Definition 5 do not depend on the specific selection of $\{\varphi_k\}_{k \in \mathbb{Z}^n} \in Y$ and the associated frequency-uniform decomposition operators.

Theorem 1 (Equivalent quasinorm). Suppose that $p, q \in \mathcal{P}_0^{\log}$, $s \in C_{\text{loc}}^{\log}$, and let $\{\varphi_k\}_{k \in \mathbb{Z}^n}$ and $\{\psi_k\}_{k \in \mathbb{Z}^n}$ be two families in Y . Then, the quasinorms induced by these families on $M_{p(\cdot), q(\cdot)}^{s(\cdot)}$ are mutually equivalent.

Proof. Symmetry considerations allow us to reduce the problem to proving that

$$\|f\|_{M_{p(\cdot), q(\cdot)}^{s(\cdot)}}^p \lesssim \|f\|_{M_{p(\cdot), q(\cdot)}^{s(\cdot)}}^\psi. \quad (9)$$

For convenience, let us denote

$$\square_k^\varphi := \mathcal{F}^{-1} \varphi_k \mathcal{F}, \quad \square_k^\psi := \mathcal{F}^{-1} \psi_k \mathcal{F},$$

and we have the almost orthogonality of $\square_k^\varphi$:

$$\square_k^\varphi = \sum_{|l|_\infty \leq 1} \square_k^\varphi \square_{k+l}^\psi,$$

that is,

$$\begin{aligned}\mathcal{F}^{-1}\varphi_k\mathcal{F}f &= \sum_{|l|_\infty \leq 1} \mathcal{F}^{-1}\varphi_k\mathcal{F}\mathcal{F}^{-1}\psi_{k+l}\mathcal{F}f \\ &= \sum_{|l|_\infty \leq 1} \mathcal{F}^{-1}\varphi_k * \mathcal{F}^{-1}\psi_{k+l} * f.\end{aligned}$$

Choose $r \in (0, \min\{1, p^-, q^-\})$ and select $m > n$ to be sufficiently large. In accordance with the definition of φ_k , we have $|\mathcal{F}^{-1}\varphi_k| \leq C\theta_{2m/r}$ with C independent of k . Then by Remark 9(i), we have

$$\begin{aligned}|\mathcal{F}^{-1}\varphi_k * \mathcal{F}^{-1}\psi_{k+l} * f| &\lesssim \theta_{2m/r} * |\mathcal{F}^{-1}\psi_{k+l} * f| \\ &\lesssim \theta_{2m/r} * \left(\theta_{2m} * |\mathcal{F}^{-1}\psi_{k+l} * f|^r\right)^{1/r}.\end{aligned}$$

In addition, combining Remark 7 with Minkowski's integral inequality (with exponent $1/r > 1$), we further obtain

$$\begin{aligned}|\mathcal{F}^{-1}\varphi_k * \mathcal{F}^{-1}\psi_{k+l} * f|^r &\lesssim \left(\theta_{2m/r} * \theta_{2m}^{1/r}\right)^r * |\mathcal{F}^{-1}\psi_{k+l} * f|^r \\ &\lesssim \theta_{2m} * |\mathcal{F}^{-1}\psi_{k+l} * f|^r.\end{aligned}$$

Next, by Remarks 8 and 10, we get

$$\begin{aligned}\left\|\left\{\langle k \rangle^{s(\cdot)} \square_k^\varphi f\right\}_k\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} &= \left\|\left\{\langle k \rangle^{s(\cdot)r} \left|\square_k^\varphi f\right|^r\right\}_k\right\|_{\ell^{\frac{q(\cdot)}{r}}(L^{\frac{p(\cdot)}{r}})}^{1/r} \\ &\lesssim \sum_{|l|_\infty \leq 1} \left\|\left\{\langle k \rangle^{s(\cdot)r} \theta_{2m} * |\mathcal{F}^{-1}\psi_{k+l} * f|^r\right\}_k\right\|_{\ell^{\frac{q(\cdot)}{r}}(L^{\frac{p(\cdot)}{r}})}^{1/r} \\ &\lesssim \sum_{|l|_\infty \leq 1} \left\|\left\{\theta_m * \left(\langle k \rangle^{s(\cdot)r} \left|\square_{k+l}^\psi f\right|^r\right)\right\}_k\right\|_{\ell^{\frac{q(\cdot)}{r}}(L^{\frac{p(\cdot)}{r}})}^{1/r} \\ &\lesssim \sum_{|l|_\infty \leq 1} \left\|\left\{\langle k \rangle^{s(\cdot)r} \left|\square_{k+l}^\psi f\right|^r\right\}_k\right\|_{\ell^{\frac{q(\cdot)}{r}}(L^{\frac{p(\cdot)}{r}})}^{1/r} \\ &\lesssim \sum_{|l|_\infty \leq 1} \left\|\left\{\langle k+l \rangle^{s(\cdot)r} \left|\square_{k+l}^\psi f\right|^r\right\}_k\right\|_{\ell^{\frac{q(\cdot)}{r}}(L^{\frac{p(\cdot)}{r}})}^{1/r} \\ &\lesssim \sum_{|l|_\infty \leq 1} \left\|\left\{\langle k+l \rangle^{s(\cdot)} \left|\square_{k+l}^\psi f\right|\right\}_k\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \\ &\lesssim \left\|\left\{\langle k \rangle^{s(\cdot)} \square_k^\psi f\right\}_k\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})},\end{aligned}$$

where we use the shift invariance of the mixed Lebesgue-sequence space as well as the fact that $\langle k \rangle \sim \langle k+l \rangle$ when $|l|_\infty \leq 1$ and $\langle k \rangle^{s(\cdot)r} \lesssim \langle k+l \rangle^{s(\cdot)r}$ for $s(\cdot) \in C_{\text{loc}}^{\log}$. This completes the proof. $\square$

In light of Theorem 1, we may pick $\{\varphi_k\}_{k \in \mathbb{Z}^n} \in \mathcal{Y}$ as needed. Consequently, we will suppress the dependence on φ in the notation for both the norm and the modular.

5. Embeddings

This section is devoted to establishing several fundamental embedding properties and the completeness of the aforementioned modulation spaces.

Theorem 2. Let $p, q_1, q_2 \in \mathcal{P}_0^{\log}$, $s \in C_{\text{loc}}^{\log}$. If $q_1 \leq q_2$, then

$$M_{p(\cdot), q_1(\cdot)}^{s(\cdot)} \hookrightarrow M_{p(\cdot), q_2(\cdot)}^{s(\cdot)}.$$

Proof. If $q_1 \leq q_2$ and $\lambda \leq 1$, we have

$$\lambda^{\frac{1}{q_1(x)}} \leq \lambda^{\frac{1}{q_2(x)}},$$

and it follows that

$$\varrho_{p(\cdot)} \left(\frac{\langle k \rangle^{s(\cdot)} \square_k f}{\lambda_k^{\frac{1}{q_2(\cdot)}}} \right) \leq \varrho_{p(\cdot)} \left(\frac{\langle k \rangle^{s(\cdot)} \square_k f}{\lambda_k^{\frac{1}{q_1(\cdot)}}} \right)$$

for $k \in \mathbb{Z}^n$. Thus, by Definitions 4 and 5, we obtain

$$\varrho_{M_{p(\cdot), q_2(\cdot)}^{s(\cdot)}}(f/\mu) \leq \varrho_{M_{p(\cdot), q_1(\cdot)}^{s(\cdot)}}(f/\mu)$$

for each $\mu > 0$, which implies

$$\|f\|_{M_{p(\cdot), q_2(\cdot)}^{s(\cdot)}} \leq \|f\|_{M_{p(\cdot), q_1(\cdot)}^{s(\cdot)}}.$$

This completes the proof. $\square$

Theorem 3. Let $p, q_1, q_2 \in \mathcal{P}_0^{\log}$, $s_1, s_2 \in C_{\text{loc}}^{\log}$. For the following three conditions,

- (i) $q_1^+ \leq q_2^-$ and $s_1 \geq s_2$;
- (ii) $q_1^+ > q_2^-$ and $(s_1 - s_2)^- > \frac{n}{q_2^-} - \frac{n}{q_1^+} > 0$;
- (iii) $(s_1 - s_2)^- > \frac{n}{q_2^-} > 0$,

if one of them is satisfied, then

$$M_{p(\cdot), q_1(\cdot)}^{s_1(\cdot)} \hookrightarrow M_{p(\cdot), q_2(\cdot)}^{s_2(\cdot)}. \quad (10)$$

Proof. By Theorem 2, we have

$$M_{p(\cdot), q_1(\cdot)}^{s_1(\cdot)} \hookrightarrow M_{p(\cdot), q_1^+}^{s_1(\cdot)} \quad \text{and} \quad M_{p(\cdot), q_2(\cdot)}^{s_2(\cdot)} \hookrightarrow M_{p(\cdot), q_2^-}^{s_2(\cdot)}. \quad (11)$$

For case (i), the assumption $s_2 \leq s_1$, directly implies that

$$\left\| \langle k \rangle^{s_2(\cdot)} \square_k f \right\|_{p(\cdot)} \leq \left\| \langle k \rangle^{s_1(\cdot)} \square_k f \right\|_{p(\cdot)},$$

which implies

$$\left\| \left\{ \left\| \langle k \rangle^{s_2(\cdot)} \square_k f \right\|_{p(\cdot)} \right\}_k \right\|_{\ell^{q_1^+}} \leq \left\| \left\{ \left\| \langle k \rangle^{s_1(\cdot)} \square_k f \right\|_{p(\cdot)} \right\}_k \right\|_{\ell^{q_1^+}}.$$

Hence,

$$M_{p(\cdot), q_1^+}^{s_1(\cdot)} \hookrightarrow M_{p(\cdot), q_1^+}^{s_2(\cdot)}$$

by Remark 11 (ii). In view of $\ell^{q_1^+} \hookrightarrow \ell^{q_2^-}$ when $q_1^+ \leq q_2^-$, we can obtain

$$M_{p(\cdot), q_1^+}^{s_1(\cdot)} \hookrightarrow M_{p(\cdot), q_2^-}^{s_2(\cdot)}, \quad (12)$$

which together with (11) implies (10).

For case (ii), the combination of Hölder's inequality and the inequality $s_1(\cdot) - s_2(\cdot) \geq (s_1 - s_2)^-$ yields

$$\begin{aligned} \left\| \left\{ \left\| \langle k \rangle^{s_2(\cdot)} \square_k f \right\|_{p(\cdot)} \right\}_k \right\|_{\ell^{q_2^-}} &= \left\| \left\{ \left\| \langle k \rangle^{s_2(\cdot)-s_1(\cdot)} \langle k \rangle^{s_1(\cdot)} \square_k f \right\|_{p(\cdot)} \right\}_k \right\|_{\ell^{q_2^-}} \\ &\leq \left\| \left\{ \langle k \rangle^{-(s_1-s_2)^-} \left\| \langle k \rangle^{s_1(\cdot)} \square_k f \right\|_{p(\cdot)} \right\}_k \right\|_{\ell^{q_2^-}} \\ &\leq \left\| \left\{ \left\| \langle k \rangle^{s_1(\cdot)} \square_k f \right\|_{p(\cdot)} \right\}_k \right\|_{\ell^{q_1^+}} \\ &\quad \cdot \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{-(s_1-s_2)^- q_1^+ q_2^- / (q_1^+ - q_2^-)} \right)^{(q_1^+ - q_2^-) / q_1^+ q_2^-}. \end{aligned}$$

Since

$$\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{-(s_1-s_2)^- q_1^+ q_2^- / (q_1^+ - q_2^-)} \lesssim \sum_{i=0}^{\infty} (1+i)^{n-1-(s_1-s_2)^- q_1^+ q_2^- / (q_1^+ - q_2^-)}, \quad (13)$$

and $(s_1 - s_2)^- > \frac{n}{q_2^-} - \frac{n}{q_1^+}$ are sufficient to ensure the convergence of the series on the right-hand side of (13), we thus obtain (12) which makes (10) true.

For case (iii), if $f \in M_{p(\cdot), q_1^+}^{s_1(\cdot)}$, then by $\ell^{q_1^+} \hookrightarrow \ell^\infty$ we can get

$$\begin{aligned} \left\| \left\{ \left\| \langle k \rangle^{s_2(\cdot)} \square_k f \right\|_{p(\cdot)} \right\}_k \right\|_{\ell^{q_2^-}} &= \left\| \left\{ \left\| \langle k \rangle^{s_2(\cdot)-s_1(\cdot)} \langle k \rangle^{s_1(\cdot)} \square_k f \right\|_{p(\cdot)} \right\}_k \right\|_{\ell^{q_2^-}} \\ &\leq \left\| \left\{ \langle k \rangle^{-(s_1-s_2)^-} \left\| \langle k \rangle^{s_1(\cdot)} \square_k f \right\|_{p(\cdot)} \right\}_k \right\|_{\ell^{q_2^-}} \\ &\leq \left\| \left\{ \langle k \rangle^{-(s_1-s_2)^-} \right\}_k \right\|_{\ell^{q_2^-}} \cdot \left\| \left\{ \left\| \langle k \rangle^{s_1(\cdot)} \square_k f \right\|_{p(\cdot)} \right\}_k \right\|_{\ell^\infty} \\ &\lesssim \left\| \left\{ \langle k \rangle^{-(s_1-s_2)^-} \right\}_k \right\|_{\ell^{q_2^-}} \cdot \left\| \left\{ \left\| \langle k \rangle^{s_1(\cdot)} \square_k f \right\|_{p(\cdot)} \right\}_k \right\|_{\ell^{q_1^+}} \\ &\lesssim \left\| \left\{ \left\| \langle k \rangle^{s_1(\cdot)} \square_k f \right\|_{p(\cdot)} \right\}_k \right\|_{\ell^{q_1^+}}, \end{aligned}$$

where the last inequality relies on the observation that

$$\begin{aligned} \left\| \left\{ \langle k \rangle^{-(s_1-s_2)^-} \right\}_k \right\|_{\ell^{q_2^-}}^{q_2^-} &= \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{-(s_1-s_2)^- q_2^-} \\ &\lesssim \sum_{i=0}^{\infty} (1+i)^{n-1-(s_1-s_2)^- q_2^-} < \infty, \end{aligned}$$

when $(s_1 - s_2)^- > \frac{n}{q_2^-} > 0$. Therefore, we get (12) and finish the proof of theorem. $\square$

Remark 12. If $s_1(\cdot) \geq s_2(\cdot)$, then $M_{p(\cdot), q(\cdot)}^{s_1(\cdot)} \hookrightarrow M_{p(\cdot), q(\cdot)}^{s_2(\cdot)}$. In fact, if $s_1(\cdot) \geq s_2(\cdot)$, we have

$$\begin{aligned} \mathfrak{q}_{M_{p(\cdot), q(\cdot)}^{s_2(\cdot)}}(f) &= \mathfrak{q}_{\ell^{q(\cdot)}(L^{p(\cdot)})} \left(\left\{ \left\| \langle k \rangle^{s_2(\cdot)} \square_k f \right\|_{p(\cdot)} \right\}_k \right) \\ &= \sum_{k \in \mathbb{Z}^n} \left\| \left\| \langle k \rangle^{s_2(\cdot)} \square_k f \right\|_{p(\cdot)} \right\|_{q(\cdot)}^{q(\cdot)} \\ &\leq \sum_{k \in \mathbb{Z}^n} \left\| \left\| \langle k \rangle^{s_1(\cdot)} \square_k f \right\|_{p(\cdot)} \right\|_{q(\cdot)}^{q(\cdot)} \\ &= \mathfrak{q}_{M_{p(\cdot), q(\cdot)}^{s_1(\cdot)}}(f), \end{aligned}$$

which implies

$$\|f\|_{M_{p(\cdot),q(\cdot)}^{s_2(\cdot)}} \leq \|f\|_{M_{p(\cdot),q(\cdot)}^{s_1(\cdot)}}.$$

Theorem 4. For $p_1, p_2, q \in \mathcal{P}_0^{\log}$ and $s_1, s_2 \in \mathcal{C}_{\log}^{\log}$ with $s_1(\cdot) \geq s_2(\cdot)$, the condition that $1/q$ and

$$s_1(x) - \frac{n}{p_1(x)} = s_2(x) - \frac{n}{p_2(x)}$$

are locally log-Hölder continuous yields

$$M_{p_1(\cdot),q(\cdot)}^{s_1(\cdot)} \hookrightarrow M_{p_2(\cdot),q(\cdot)}^{s_2(\cdot)}. \quad (14)$$

Proof. We use a similar argument to the proof of [31] (Theorem 6.4). A standard scaling argument allows us to restrict our attention to the normalized case where $\|f\|_{M_{p_1(\cdot),q(\cdot)}^{s_1(\cdot)}} = 1$.

In this case, the Norm-modular unit ball property ensures that

$$\sum_{k \in \mathbb{Z}^n} \left\| \langle k \rangle^{s_1(\cdot)} \square_k f \right\|_{p_1(\cdot)/q(\cdot)}^{q(\cdot)} \leq 1.$$

Next, we will prove that there exists a constant $c_1 > 0$ such that

$$\left\| c_1 \langle k \rangle^{s_2(\cdot)} \square_k f \right\|_{p_2(\cdot)/q(\cdot)}^{q(\cdot)} \leq \left\| \langle k \rangle^{s_1(\cdot)} \square_k f \right\|_{p_1(\cdot)/q(\cdot)}^{q(\cdot)} + \langle k \rangle^{-(n+1)} =: \beta. \quad (15)$$

Denote $\alpha := s_2 - \frac{n}{p_2}$. Notice that $\beta \in [\langle k \rangle^{-(n+1)}, 1 + \langle k \rangle^{-(n+1)}]$; then through a similar argument as in the proof of Lemma 9, we have

$$\begin{aligned} \beta^{-\frac{r}{q(x)}} \langle k \rangle^{r\alpha(x)} |\square_k f(x)|^r &\leq C \beta^{-\frac{r}{q(x)}} \langle k \rangle^{r\alpha(x)} (\theta_{k,2m} * |\square_k f|^r)(x) \\ &\leq C \theta_{k,m} * \left(\beta^{-\frac{1}{q(\cdot)}} \langle k \rangle^{\alpha(\cdot)} |\square_k f| \right)^r(x) \end{aligned}$$

for fixed $r \in (0, p_1^-)$ and large m . Let $s = p_1/r \in \mathcal{P}_0$, the Hölder's inequality implies

$$\begin{aligned} \beta^{-\frac{1}{q(x)}} \langle k \rangle^{\alpha(x)} |\square_k f(x)| &\leq C \left(\int_{\mathbb{R}^n} \langle k \rangle^{-\frac{rn}{p_1(y)}} \theta_{k,m}(x-y) \left(\beta^{-\frac{1}{q(\cdot)}} \langle k \rangle^{\alpha(\cdot) + \frac{n}{p_1(\cdot)}} |\square_k f| \right)^r(y) dy \right)^{1/r} \\ &\leq C \left\| \langle k \rangle^{-\frac{n}{s(\cdot)}} \theta_{k,m}(x-\cdot) \right\|_{s'(\cdot)}^{1/r} \left\| \beta^{-\frac{1}{q(\cdot)}} \langle k \rangle^{\alpha(\cdot) + \frac{n}{p_1(\cdot)}} |\square_k f| \right\|_{p_1(\cdot)}. \end{aligned} \quad (16)$$

For the first norm on the right-hand side of (16), since $\langle k \rangle \sim r_k$ for $k \in \mathbb{Z}^n$, we can investigate the corresponding modular as follows:

$$\begin{aligned} \varrho_{s'(\cdot)} \left(\langle k \rangle^{-\frac{n}{s(\cdot)}} \theta_{k,m}(x-\cdot) \right) &= \int_{\mathbb{R}^n} \langle k \rangle^{-\frac{s'(y)n}{s(y)}} r_k^{s'(y)n} (1 + r_k |x-y|)^{-ms'(y)} dy \\ &\leq C \int_{\mathbb{R}^n} r_k^n (1 + r_k |x-y|)^{-ms'(y)} dy \\ &\leq C \int_{\mathbb{R}^n} (1 + |r_k x - z|)^{-m(s')^-} dz < \infty, \end{aligned}$$

when m is large enough such that $m(s')^- > n$. This establishes the boundedness of the first norm. Furthermore, suppose that $\left\| |f|^{q(\cdot)} \right\|_{p(\cdot)}^{p(\cdot)} < \lambda$, this condition implies that

$$\varrho_{\frac{p(\cdot)}{q(\cdot)}} \left(\frac{|f|^{q(\cdot)}}{\lambda} \right) \leq 1, \text{ which in turn yields } \left\| \lambda^{-\frac{1}{q(\cdot)}} f \right\|_{p(\cdot)} \leq 1. \text{ Hence, the second norm in (16) is}$$

bounded by 1, a result justified by the definition of β and the identity $\alpha + \frac{n}{p_1} = s_2 - \frac{n}{p_2} + \frac{n}{p_1} = s_1$.

Therefore, we can take an appropriate constant $c_1 \in (0, 1]$ such that

$$\begin{aligned} \left(c_1 \beta^{-\frac{1}{q(x)}} \langle k \rangle^{s_2(x)} |\square_k f(x)| \right)^{p_2(x)} &= c_1^{p_1(x)} \left(c_1 \beta^{-\frac{1}{q(x)}} \langle k \rangle^{\alpha(x)} |\square_k f(x)| \right)^{p_2(x)-p_1(x)} \\ &\quad \cdot \left(\beta^{-\frac{1}{q(x)}} \langle k \rangle^{\alpha(x)+\frac{n}{p_1(x)}} |\square_k f(x)| \right)^{p_1(x)} \\ &\leq \left(\beta^{-\frac{1}{q(x)}} \langle k \rangle^{\alpha(x)+\frac{n}{p_1(x)}} |\square_k f(x)| \right)^{p_1(x)}. \end{aligned}$$

Integration of this inequality over $\mathbb{R}^n$, together with the definition of β , yields

$$\varrho_{\frac{p_2(\cdot)}{q(\cdot)}} \left(\frac{\left(c_1 \langle k \rangle^{s_2(\cdot)} |\square_k f| \right)^{q(\cdot)}}{\beta} \right) \leq \varrho_{\frac{p_1(\cdot)}{q(\cdot)}} \left(\frac{\left(\langle k \rangle^{s_1(\cdot)} |\square_k f| \right)^{q(\cdot)}}{\beta} \right) \leq 1,$$

which implies (15).

Then, by the similar discussion in the proof of Lemma 9 and summing on both sides of (15), we obtain

$$\begin{aligned} \sum_{k \in \mathbb{Z}^n} \left\| c_1 \langle k \rangle^{s_2(\cdot)} |\square_k f|^{q(\cdot)} \right\|_{\frac{p_2(\cdot)}{q(\cdot)}} &\leq \sum_{k \in \mathbb{Z}^n} \left\| \langle k \rangle^{s_1(\cdot)} |\square_k f|^{q(\cdot)} \right\|_{\frac{p_1(\cdot)}{q(\cdot)}} + \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{-(n+1)} \\ &\leq 1 + \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{-(n+1)} \leq c_2 < \infty, \end{aligned}$$

where $c_2 > 1$ is a constant. Then, combining with Remark 1, we have

$$\varrho_{\ell q(\cdot)(L^{p_2(\cdot)})} \left(\left\{ \frac{c_1}{c_2} \langle k \rangle^{s_2(\cdot)} |\square_k f| \right\}_k \right) \leq \frac{1}{c_2} \varrho_{\ell q(\cdot)(L^{p_2(\cdot)})} \left(\left\{ c_1 \langle k \rangle^{s_2(\cdot)} |\square_k f| \right\}_k \right) \leq 1,$$

which implies

$$\|f\|_{M_{p_2(\cdot), q(\cdot)}^{s_2(\cdot)}} = \left\| \left\{ \langle k \rangle^{s_2(\cdot)} |\square_k f| \right\}_k \right\|_{\ell q(\cdot)(L^{p_2(\cdot)})} \leq C$$

with a constant $C > 0$. Therefore, we get the embedding (14) and finish the proof. $\square$

Let $\Omega \subset \mathbb{R}^n$ be compact. The space $L_{\Omega}^{p(\cdot)}$ comprises those functions $f \in L^{p(\cdot)}(\mathbb{R}^n)$ for which $\text{supp } f \subset \Omega$. Then, in the following theorem which has been given in [47], we can compare $\|f\|_{p(\cdot)}$ with $\|f\|_{q(\cdot)}$ as in the constant exponent Lebesgue spaces.

Lemma 10 (see [47] (Theorem 4.14)). *For any compact subset Ω of $\mathbb{R}^n$ and $p, q \in \mathcal{P}_0$ satisfying $0 < p(x) \leq q(x) < \infty$, there exists a constant $C > 0$ ensuring that*

$$\|D^{\alpha} f\|_{q(\cdot)} \leq C \|f\|_{p(\cdot)}$$

and

$$\|D^{\alpha} f\|_{\infty} \leq C \|f\|_{p(\cdot)}$$

for all $f \in L_{\Omega}^{p(\cdot)}$ and every multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$.

Remark 13. *In the context of constant exponent modulation spaces, the embedding relation $M_{p_1, q_1}^{s_1} \subset M_{p_2, q_2}^{s_2}$ holds whenever $s_2 \leq s_1, p_1 \leq p_2$ and $q_1 \leq q_2$. If p_1, p_2 are variable exponents,*

we can obtain a similar conclusion, namely $M_{p_1(\cdot),q_1}^{s_1} \subset M_{p_2(\cdot),q_2}^{s_2}$ when $s_2 \leq s_1$, $p_1(x) \leq p_2(x)$ and $q_1 \leq q_2$.

In fact, using Lemma 10, we have $\|\square_k f\|_{p_2(\cdot)} \lesssim \|\square_k f\|_{p_1(\cdot)}$, which implies

$$\langle k \rangle^{s_2} \|\square_k f\|_{p_2(\cdot)} \lesssim \langle k \rangle^{s_1} \|\square_k f\|_{p_1(\cdot)}$$

for $k \in \mathbb{Z}^n$. Then, in view of $\ell^{q_1} \subset \ell^{q_2}$ and Remark 11, we obtain

$$\left\| \left\{ \langle k \rangle^{s_2} \|\square_k f\|_{p_2(\cdot)} \right\}_k \right\|_{\ell^{q_2}} \lesssim \left\| \left\{ \langle k \rangle^{s_1} \|\square_k f\|_{p_1(\cdot)} \right\}_k \right\|_{\ell^{q_1}}$$

and $M_{p_1(\cdot),q_1}^{s_1} \subset M_{p_2(\cdot),q_2}^{s_2}$.

It is well known that $\mathcal{S}(\mathbb{R}^n) \subset M_{p,q}^s(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ for the constant exponent modulation spaces. Similarly, combining the above several embedding properties, we are able to establish the conclusion below:

Theorem 5. Given $p, q \in \mathcal{P}_0^{\log}$ and $s \in C_{\text{loc}}^{\log}$, then

$$\mathcal{S} \hookrightarrow M_{p(\cdot),q(\cdot)}^{s(\cdot)} \hookrightarrow \mathcal{S}'.$$

Proof. Let us prove $M_{p(\cdot),q(\cdot)}^{s(\cdot)} \hookrightarrow \mathcal{S}'$ first. Since $p(x) \leq p^+$, by Theorem 4, we have

$$M_{p(\cdot),q(\cdot)}^{s(\cdot)} \hookrightarrow M_{p^+,q(\cdot)}^{s(\cdot)+n/p^+-n/p(\cdot)}.$$

Taking $\varepsilon > n/q^- > 0$, together with Theorems 2 and 3 and the constant exponent case, we can get

$$M_{p^+,q(\cdot)}^{s(\cdot)+n/p^+-n/p(\cdot)} \hookrightarrow M_{p^+,q(\cdot)}^{(s(\cdot)+n/p^+-n/p(\cdot))^- - \varepsilon} \hookrightarrow M_{p^+, \infty}^{(s(\cdot)+n/p^+-n/p(\cdot))^- - \varepsilon} \hookrightarrow \mathcal{S}'.$$

Hence, $M_{p(\cdot),q(\cdot)}^{s(\cdot)} \hookrightarrow \mathcal{S}'$.

On the other hand, since $p(x) \geq p^-$, by Theorem 4 again, we have

$$M_{p^-,q(\cdot)}^{s(\cdot)+n/p^--n/p(\cdot)} \hookrightarrow M_{p^-,q(\cdot)}^{s(\cdot)}.$$

Taking $s_0 \in \mathbb{R}^n$ satisfying $s_0 > (s(\cdot) + n/p^- - n/p(\cdot))^+ + n/q^-$, we deduce

$$\mathcal{S} \hookrightarrow M_{p^-,q^+}^{s_0} \hookrightarrow M_{p^-,q(\cdot)}^{s(\cdot)+n/p^--n/p(\cdot)}.$$

Therefore, $\mathcal{S} \hookrightarrow M_{p(\cdot),q(\cdot)}^{s(\cdot)}$, and the proof is completed. $\square$

Analogous to the constant exponent case, the completeness of $M_{p(\cdot),q(\cdot)}^{s(\cdot)}$ can be proven as follows, rendering it a (quasi-)Banach space. To prove the completeness, we need the following lemma.

Lemma 11 (see [48]). Given $p, q \in \mathcal{P}_0(\mathbb{R}^n)$ with $q^- < \infty$, consider any sequence $\{f_k\}_k$ of measurable functions. The inequality

$$\|\{f_k\}_k\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leq \max \left\{ \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}(\{f_k\}_k)^{\frac{1}{q^-}}, \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}(\{f_k\}_k)^{\frac{1}{q^+}} \right\}$$

holds if $\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}(\{f_k\}_k) > 0$ or $q^+ < \infty$.

Theorem 6. Let $p, q \in \mathcal{P}_0^{\log}$, $q^+ < \infty$, and $s \in C_{\text{loc}}^{\log}$, then $M_{p(\cdot), q(\cdot)}^{s(\cdot)}$ is complete.

Proof. Let $\{f_l\}_{l=1}^\infty$ be a Cauchy sequence in $M_{p(\cdot), q(\cdot)}^{s(\cdot)}$. Then, Theorem 5 implies that $\{f_l\}_{l=1}^\infty$ is likewise a Cauchy sequence in $\mathcal{S}'$. Owing to the completeness and local convexity of $\mathcal{S}'$, there exists an element $f \in \mathcal{S}'$ with $f_l \rightarrow f$ in the strong topology of $\mathcal{S}'$, which implies $\square_k f_l \rightarrow \square_k f$ in $\mathcal{S}'$ for every $k \in \mathbb{Z}^n$ when $l \rightarrow \infty$.

On the other hand, the fact that $\{f_l\}_{l=1}^\infty$ is a Cauchy sequence in $M_{p(\cdot), q(\cdot)}^{s(\cdot)} \hookrightarrow M_{p(\cdot), q^+}^{s(\cdot)}$ ensures that $\{\square_k f_l\}_{l=1}^\infty$ is a Cauchy sequence in $L^{p(\cdot)}(\mathbb{R}^n)$. Invoking Lemma 10, we deduce that $\{\square_k f_l\}_{l=1}^\infty$ is also a Cauchy sequence in $L^\infty(\mathbb{R}^n)$. As $L^\infty(\mathbb{R}^n)$ is complete, we can find a $g_k \in L^\infty$ satisfying

$$\|\square_k f_l - g_k\|_\infty \rightarrow 0, \quad \text{as } l \rightarrow \infty, \quad (17)$$

Thus, we have $\square_k f_l \rightarrow g_k$ almost everywhere. Combining the convergence $\square_k f_l \rightarrow \square_k f$ in $\mathcal{S}'$ as $l \rightarrow \infty$ with (17) and the inclusion $L^\infty \subset \mathcal{S}'$, we conclude that $g_k = \square_k f$.

Then, by using Fatou's lemma twice, we have

$$\begin{aligned} \varrho_{M_{p(\cdot), q(\cdot)}^{s(\cdot)}}(f_l - f) &= \varrho_{\ell^q(\cdot)(L^{p(\cdot)})} \left(\left\{ \langle k \rangle^{s(\cdot)} \square_k (f_l - f) \right\}_k \right) \\ &= \sum_{k \in \mathbb{Z}^n} \left\| \left| \langle k \rangle^{s(\cdot)} \square_k (f_l - f) \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \\ &= \sum_{k \in \mathbb{Z}^n} \left\| \left| \langle k \rangle^{s(\cdot)} \lim_{m \rightarrow \infty} \square_k (f_l - f_m) \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \\ &\leq \sum_{k \in \mathbb{Z}^n} \lim_{m \rightarrow \infty} \left\| \left| \langle k \rangle^{s(\cdot)} \square_k (f_l - f_m) \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \\ &\leq \lim_{m \rightarrow \infty} \sum_{k \in \mathbb{Z}^n} \left\| \left| \langle k \rangle^{s(\cdot)} \square_k (f_l - f_m) \right|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \\ &= \lim_{m \rightarrow \infty} \varrho_{M_{p(\cdot), q(\cdot)}^{s(\cdot)}}(f_l - f_m), \end{aligned}$$

which, combining with Lemma 2, implies

$$\varrho_{M_{p(\cdot), q(\cdot)}^{s(\cdot)}}(f_l - f) \rightarrow 0, \quad \text{as } l \rightarrow \infty. \quad (18)$$

By Lemma 11, we can obtain $f \in M_{p(\cdot), q(\cdot)}^{s(\cdot)}$ and $\|f_l - f\|_{M_{p(\cdot), q(\cdot)}^{s(\cdot)}} \rightarrow 0$ as $l \rightarrow \infty$. The proof is completed. $\square$

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